«ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА»
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INTEGRABLE SYSTEMS

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On the base of algebraic construction, new discrete symmetry of integrable system is introduced and its applications are studied. Its universality is demonstrated by examples of explicit solutions of many known integrable evolution equations and hierarchies in (1 + 1) and (2 + 1) dimensions.

Предложен алгебраический подход к построению солитонных решений интегрируемых нелинейных систем, основанный на исследовании алгебр их симметрии. Его универсальность продемонстрирована на примерах явных решений многих известных интегрируемых уравнений и их иерархий в (1 + 1) и (2 + 1) измерениях.

1. GENERAL REMARKS [1–7]

In this paper we will consider the dynamical systems which have solution of soliton kind. The origin of the concept of the solitons is connected with the physical problems, in particular, with the study of the waves on the water in the canals. These waves correspond to the processes of propagating a perturbation peak (soliton wave) which brings finite energy. This peak is stable with respect to the outer influences. The soliton wave differs from the usual periodical waves which have many peaks.

From mathematical point of view all these systems are united by the properties of their algebra of inner symmetry. These algebras are infinite dimensional but have the finite dimensional representations with the «spectral parameter», i.e., they are realized by finite dimensional matrices which entries are rational functions of the parameter λ, taking values in complex plane. This case sufficiently differs from the finite-dimensional algebras of inner symmetry of exactly integrable systems.

Now we have generalized the partial problems of the soliton solution behaviour into the following mathematical scheme. We formulate the problem. It is necessary to find an element g, which takes values in some group and depends on the parameter λ. This element is also a function of independent variables ξ and satisfies the relation

$$\frac{\partial g}{\partial \xi} g^{-1} = u. \tag{1.1}$$

The elements u, taking values in corresponding algebra, are postulated to be rational functions of the spectral parameter. In the general case the parameters which determine the position of the poles are the functions of the independent variables ξ.

The Maurer–Cartan identities subject to (1.1) are reduced to the system

$$\frac{\partial u_i}{\partial \xi_j} - \frac{\partial u_j}{\partial \xi_i} = [u_i, u_j]. \tag{1.2}$$

From this identity we can take out the equalities of residues in all the poles of any order. After that we have the system of equations under consideration, i.e., there arises a polycomponent equation which is equivalent to investigated dynamical system. One can say that system (1.2) with respect to spectral parameter is a generating expression (Laurent series) for the equations of the dynamical system.

Originally for the solution of the systems of type (1.2) the wide known inverse scattering method was elaborated. By means of this method many equations of importance for physical applications (such as the Korteveg–de Vries, sin-Gordon, nonlinear Schrödinger and so on) were integrated. This method is described in detail in many known monographs.

In further works the solution of the systems (1.2) was connected with the matrix Riemann problem (the so-called Zacharov–Shabat dressing method). This method provides a possibility of finding the solutions of integrable system, when the solution of the Riemann problem is known from the independent consideration.

In this paper we shall use a purely algebraic construction for finding the system of equations possessing soliton-type solutions together with their explicit form, by passing the stage of investigating and exploring their internal symmetry algebra. The problem of necessity for such approach and the existence of systems having soliton solutions that do not fall within the scope of our construction will not be considered here. In any case all the systems that are integrable by the inverse scattering method fall within the construction which follows below. In all cases it yields explicit formulae for soliton type solutions even when traditional methods grow so cumbersome that it becomes impossible from the purely technical standpoint to produce the result.

The initial (input) elements of the construction are specially coded data of the structure of internal symmetry algebra of the system which are used to express, by several algebraically operations, soliton-type solutions together with the system of equations they satisfy. Here, the solution of sin–Gordon, Korteveg–de Vries, nonlinear Schrödinger and other wave and evolution equations is described by common formulae distinguished only by the parameters related to the internal symmetry algebra.

2. INFINITE-DIMENSIONAL RATIONAL FUNCTIONS ALGEBRA [8,9]

In this section we will describe the construction of a special type of infinite-dimensional algebras, which is an essential point in the following approach to the consideration of the dynamical systems in question. We will call these as the algebras of the rational functions, because up to now any terminology on this subject is absent.

In order to explain the construction of these algebras we will examine simple algebraic identities for the decomposition over the simple fractions, which are well known:

$$\frac{1}{(\lambda - a)^m}\frac{1}{(\lambda - b)^n} = \sum_{i=1}^{m} \frac{f_i\,(a, b, m, n)}{(\lambda - a)^i} + \sum_{j=1}^{n} \frac{g_j\,(a, b, m, n)}{(\lambda - b)^j}.$$

The explicit form of the functions f_i, g_j may be determined by many independent methods and it may be found in any book on this subject.

Consider some Lie algebra with its generators L_i. Let the finite ambiguity of the arbitrary parameters $a = (a_1, a_2, \ldots a_r)$ be denoted by one symbol; and the generators of some infinite-dimensional algebra $L_i^{a,k}$ are determined by relations

$$L_i^{a,k} = (\lambda - a)^{-k} L_i.$$

The new generators are labeled with an integer index k and continuous-complex number a; λ — the complex parameter. In the case of the negative k we introduce additional generators $L_i^s = \lambda^s L_i$. The most significant fact is that the variety of these generators is a closed infinite-dimensional Lie algebra. Indeed, we calculate the commutator

$$[L_i^{a,k}, L_j^{b,l}] = (\lambda - a)^{-k} (\lambda - b)^{-l} \sum_m C_{i,j}^m L_m.$$

By virtue of the previous identity we represent the right-hand side of the last equality in the form of linear combination of the generators constructed above. Continuing the last equality, we have

$$\left(\sum_{k'=1}^{k}\frac{f_{k'}(a,b,k,l)}{(\lambda-a)^{k'}}+\sum_{l'=1}^{l}\frac{g_{k'}(a,b,k,l)}{(\lambda-b)^{l'}}\right)\sum_m C_{ij}^{m}L_m,$$

or saving the first and the last terms of the written equality we obtain

$$[L_i^{a,k},L_j^{b,l}]=\sum_m C_{i,j}^{m}\left(\sum_{k'=1}^{k}(f_{k'}(a,b,k,l)\,L_m^{a,k'}+\sum_{l'=1}^{l}g_{k'}(a,b,k,l)\,L_m^{b,l'}\right).$$

It may be considered as the commutation relations of abstract infinite-dimensional algebra. We note that in the last sum there is only the finite number of the generators. It may be connected with the filtration properties of the constructed algebras. We will call them as the algebras of the rational functions.

The algebras of the inner symmetry of the integrable systems have a direct connection to the subject of this section.

3. THE STATEMENT OF THE PROBLEM AND ITS NONLINEAR SYMMETRIES

Here, the problem, which transistory was stayed in section 1, will be formulated more carefully and its symmetry properties will be described. With the help of these symmetries it will be possible to construct the whole hierarchy of solutions of the problem, if some solution of it is known.

The formulation of the problem is the following: It is needed to find such element g, taking values in some group, which depends on complex parameter λ and arguments ξ: $(\xi_1 ... \xi_n)$, in such a way that constructed from g elements $u=\frac{\partial g}{\partial \xi}g^{-1}$ from corresponding algebra are rational functions in complex λ plane.

The known under this consideration are the positions of the poles and their multiplicity for each of the elements u_i. In what follows we shall call the totality of these data a spectral structure of the element u_i. The unknown are the residues in all poles of the elements u_i, as the functions of ξ or, and this is the same, matrix elements of g as the functions of λ and ξ.

The problem possesses the remarkable symmetry properties, which we shall describe now.

Let g_0 be some solution. For definiteness let g_0 belong to the group $SL(k, c)$. This means that g_0 is $(k+1, k+1)$ matrix with determinant equal to 1. Let us introduce the matrix of the polynomials P, which has the following structure:

$$\begin{pmatrix} \tilde{P}_{n+1}^{11} & P_n^{12} & P_n^{13} & \dots & P_n^{1,k+1} \\ P_n^{21} & \tilde{P}_{n+1}^{22} & P_n^{23} & \dots & P_n^{2,k+1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ P_n^{k+1,1} & P_n^{k+1,2} & \dots & \dots & \tilde{P}_n^{k+1,k+1} \end{pmatrix} \tag{3.1}$$

where the $P_n^{\alpha,\beta}$ is the polynomial of degree n on λ; and $\tilde{P}_{n+1}^{\alpha,\alpha}$, the polynomial of degree $n + 1$ with the coefficient 1 at λ^{n+1}. The equal degrees of the polynomials are chosen for simplicity and in future this limitation will be taken away as some other restriction which takes place in the definition of the polynomial matrix.

The coefficients of the entries of the polynomial matrix are defined from requirement that in $(k + 1)$ $(n + 1)$ different points of the λ plane there is a linear dependence between the $(k + 1)$ column of the matrix Pg^0. That means

$$\sum_{s=1}^{k+1} (P_n(\lambda_i)\, g_0(\lambda_i))_{\alpha,\beta} c_\beta(\lambda_i) = 0,$$

$$\alpha, \beta = 1, 2, \dots (k+1), \quad i = 1, 2, \dots (k+1)(n+1), \tag{3.2}$$

where $c_\beta(\lambda_i)$ are totalities of arbitrary c-number parameters. These conditions determine all the coefficient functions of the entries of the matrix P (3.2).

Indeed, let us take $\alpha = 1$ in the last equality. We have

$$\sum_{s=1}^{k+1} P_n^{1\beta}(\lambda_i)\, (g_0(\lambda_i)\, c(\lambda_i))_\beta = 0, \quad i = 1, 2, \dots (k+1)(n+1). \tag{3.3}$$

The last equality is the system of $(n + 1)(k + 1)$ linear algebraic equations, in which the $(n + 1)(k + 1)$ coefficient functions of the polynomials of the first line of the matrix P are unknowns. (Each polynomial has exactly $(n + 1)$ coefficients, the number of polynomials is equal to $(k + 1)$).

From the definition of the matrix P it follows that its determinant is the polynomial of the degree $(n + 1)(k + 1)$ with the coefficient 1 at the highest power of λ. From (3.2) we conclude that determinant P vanishes exactly in the $(n + 1)(k + 1)$ points of the λ plane. In these points, the columns of the matrix Pg_0 (Det $g_0 = 1$) are linearly dependent. So we obtain:

$$\text{Det } P = \prod_{1}^{(n+1)(k+1)} (\lambda - \lambda_i). \tag{3.4}$$

From definition of the entries of the inverse matrix g^{-1} via the ratio of the k-order minors to the determinant of matrix g, we have for entries of the matrix u:

$$u_{\alpha,\beta} = \left(\frac{\partial g}{\partial \xi} g^{-1} \right)_{\alpha,\beta} = \frac{\mathrm{Det}\,\|(\beta \to \dot{\alpha})\|}{\mathrm{Det}\,(\|g\|)}, \tag{3.5}$$

where by symbol $\|\beta \to \dot{\alpha}\|$ we denote the matrix arising from the matrix g when its β line changes over the derivatives with respect to ξ from its α line.

Let us illustrate these formulae by the examples of the second order matrix.

$$u_{11} = \frac{\mathrm{Det}\begin{pmatrix} \dot{g}_{11} & \dot{g}_{12} \\ g_{21} & g_{22} \end{pmatrix}}{\mathrm{Det}\, g}, \quad u_{21} = \frac{\mathrm{Det}\begin{pmatrix} \dot{g}_{21} & \dot{g}_{22} \\ g_{21} & g_{22} \end{pmatrix}}{\mathrm{Det}\, g}, \quad u_{12} = \frac{\mathrm{Det}\begin{pmatrix} g_{11} & g_{12} \\ \dot{g}_{11} & \dot{g}_{12} \end{pmatrix}}{\mathrm{Det}\, g}, \tag{3.6}$$

where $\dot{g} = \frac{\partial g}{\partial \xi}$. To determine the analytical properties of entries of u, as functions of the parameter λ we use the general formulae to the case of $g = Pg_0$. For matrix $\dot{g}$ we have:

$$\dot{g} = \dot{P}g_0 + P\dot{g}_0 = (\dot{P} + P\dot{g}_0 g_0^{-1})\, g_0.$$

For the entries of u we obtain:

$$u_{\alpha,\beta} = \frac{\mathrm{Det}\,\|(P)_\beta \to (\dot{P} + Pu_0)_\alpha\|}{\prod_{1}^{(k+1)(n+1)} (\lambda - \lambda_i)}. \tag{3.7}$$

The coefficients of $c_\beta(\lambda_i)$ in (3.2) are independent of ξ. Thus the columns of matrix (Pg_0) are linearly dependent. By this reason the numerator in the expression for matrix elements of u (the matrix u_0 has the rational dependence on λ) will contain the multiplier, which cancels with the denominator, and analytic properties of matrix u repeat the same of u_0. The position and maximal multiplicities of the poles are the same for matrices u and u_0.

Some separate treatment is needed to understand the behaviour u in the limit $\lambda \to \infty$. Let us at first consider this situation on the example of the second order matrix. The numerator of the element u_{11} is the determinant of the matrix:

$$\begin{pmatrix} \dot{P}^{11}_{n+1} + P^{11}_{n+1}u^0_{11} + P^{12}_n u^0_{21} & \dot{P}^{12}_{n+1} + P^{11}_{n+1}u^0_{12} + P^{12}_n u^0_{22} \\ P^{21}_n & P^{22}_{n+1} \end{pmatrix}.$$

As $\lambda \to \infty$, the denominator of the matrix element u_{11} has the asymptotics $\lambda^{2(n+1)}$. The maximal power on λ in the numerator may be contained in the product $P^{11}_{n+1} P^{22}_{n+1} U^{0}_{11}$. This power is not more than $2(n + 1) + s$, where by s we denote the maximal power of the matrix elements u_0 at infinity.

Let us make some resume. It has been shown that if there is some solution g_0 of the problem, then the element Pg_0, which is constructed by the rules of this section, is also the solution of the same problem. So we obtain the entire hierarchy of the solutions (n is arbitrary). The problem possesses some nonlinear symmetry. This symmetry is usually regarded as the Backlund transformation. The Backlund transformation plays the most important role in the theory of integrable systems. It will become clear from the next sections how these transformations are used for the construction of the exact solutions of the integrable system.

4. THE SPECTRAL EQUATION [8,11]

In this section the connection between the matrix equation (1.1) and the theory of the ordinary differential equations will be considered. Equation (1.1), being written in the form

$$\dot{g} = ug, \tag{4.1}$$

is the equation for unknown g, taking values in some group, under the assumption that element u, which takes values in corresponding algebra, is known. This equation is the system of equations on the parameters of the group element g (it is assumed that they are functions of an independent argument and differentiation is carried out with respect to it), and in this sense it is invariant with respect to the choice of any representation of an algebra (or group). On the other hand, we can consider it in some fixed representation when g and u are certain finite-dimensional matrices. We shall use the Dirac notation for the basis vectors $\| \alpha\rangle$, $\langle\beta\|$ (α, β extend their values from 1 to N, where N is the dimension of the representation). (4.1) makes it possible to calculate the serial derivatives of the element g:

$$\ddot{g} \equiv g^{[.2]} = (\dot{u} + u^2)\, g \equiv u_2 g, \quad g^{[.s]} = u_s\, g, \quad u_{s+1} = \dot{u}_s + u_s u_1, \quad u_0 \equiv 1.$$

Writing the matrix elements for the first N derivatives by using the basis vectors $\langle 1\|$ and $\|\alpha\rangle$, we have:

$$\langle 1\| g_{[.s]}\| \alpha\rangle = \langle 1\| u_s\| 1\rangle \langle 1\| g\| \alpha\rangle + \sum_{2}^{N} \langle 1\| u_s\| \beta\rangle \langle \beta\| g\| \alpha\rangle. \tag{4.2}$$

Eliminating the $N-1$ matrix elements $\langle\beta\| g\| \alpha\rangle$ ($\beta = 2,..(k+1)$) from the last system of N equations, we derive an ordinary N-order differential equation for the function $\psi_\alpha \equiv \langle 1\| g\| \alpha\rangle$:

$$\mathrm{Det}\, \| \psi_\alpha^{[.s]} - \langle 1\| u_s\| 1\rangle\psi_\alpha,\ \ \langle 1\| u_s\| 2\rangle,\ldots\langle 1\| u_s\| \beta\rangle,\ldots\| = 0, \qquad (4.3)$$

where the index s labels the lines and takes values from 1 to N; the index β labels the columns $2,\ldots N$, except of the first one. Thus all the elements in the «first» line $\langle 1\| g\| \alpha\rangle$ satisfy the same N-order differential equation (i.e., they are its fundamental solutions), whose coefficients are expressed explicitly through the entries of the matrix u and its derivatives up to the $(N-1)$th order. We shall call this equation a spectral equation. The matrix elements $\langle\beta\| g\| \alpha\rangle$ may be found from a linear system of $(N-1)$ equations (4.2) ($1 \le s \le N$), and in this manner the matrix g is explicitly expressed through N fundamental solutions of the spectral equation, the matrix elements u and their derivatives up to the $(N-1)$th order inclusively. The coefficient functions of the spectral equation may be expressed through a set of its fundamental solutions by known relations. Let us rewrite the spectral equation in the form:

$$\psi^{[.N]} - \ln V\, \psi^{[.N-1]} + \sum_{k=0}^{N-2} (-1)^k a_k \psi^{[.k]} = 0. \qquad (4.4)$$

We introduce the notation $\| \psi^{[.s_1]}, \psi^{[.s_2]} \ldots\ldots \psi^{[.s_N]}\|$ for the determinant of the matrix whose first column consists of the derivatives of the s_1 order of fundamental solutions of the spectral equation, the second column from the derivatives of the s_2 order from fundamental solutions and so on. In this notation we have:

$$V = \| \psi, \psi^{[.1]}, \ldots\ldots \psi^{[.(N-1)]}\| ,$$

$$Va_k = - \| \psi,..\psi^{[.k-1]}, \psi^{[.N]}, \psi^{[.k+1]} \ldots\ldots \psi^{[.(N-1)]}\| . \qquad (4.5)$$

Comparing these expressions with the coefficients of the spectral equation we see, that to obtain them it is necessary to make the substitution $\psi_\alpha^{[.S]} \to \to \langle 1\| u_s\| \alpha\rangle$ in the last formulae. This relation will become useful in constructing the matrix elements via the known set of the fundamental solutions of the spectral equation.

5. CONSTRUCTION OF THE SOLUTIONS IF THE ELEMENT g_0 BELONGS TO DIAGONAL (COMMUTATIVE) SUBGROUP [8,10–13]

The general construction of the third section will be used now so as to get the whole class of solutions of integrable systems for the algebra $SL(k, c)$. Let us pay attention to the case in which the background equation has a trivial solution if one assumes that element g takes values in commutative (Cartan in the semisimple case) subgroup

$$g = \exp \sum_{s=1}^{r} h_s \tau_s, \quad [h_i, h_j] = 0, \tag{5.1}$$

where r is the dimension of the commutative subgroup. From the background equation (in our case $\frac{\partial g}{\partial \xi} g^{-1} = \sum_{s=1}^{r} h_s \frac{\partial \tau_s}{\partial \xi}$) it follows that the functions τ_s must be rational functions of the argument λ, whose analytical properties are determined by the spectral structure of the elements u. It means, that the residues in the poles of τ and the coefficient functions of its Laurent expansion near the infinite point of the λ plane, must the functions of the single argument ξ_i. So

$$\tau_s = \sum_{i=1}^{t} \tau_s^i (\xi_i, \lambda) + \tau_s^0(\xi_i),$$

i.e., τ_s is the sum of the rational functions of one argument, the number of which coincides with the number of independent parameters ξ in the problem, and some function τ_s^0 which depends on all parameters except λ. We will call this function the null mode of τ and the whole function τ_s — the source function. Let us take this solution in the capacity g_0 of the general construction of section 3, and use the quadratic $(k + 1, k + 1)$ matrix of the polynomials P, in

$$\begin{pmatrix} \tilde{P}^{11}_{n_1+1} & \tilde{P}^{12}_{n} & \cdots & \tilde{P}^{1,k+1}_{n_{k+1}} \\ P^{21}_{n_1} & P^{22}_{n+1} & \cdots & P^{2,k+1}_{n_{k+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P^{k+1,1}_{n_1} & P^{k+1,2}_{n} & \cdots & \tilde{P}^{k+1,k+1}_{n} \end{pmatrix} \tag{5.2}$$

containing as before the notation $P_n(\lambda)$ for arbitrary polynomial of the degree n and $\tilde{P}_n(\lambda)$ for the same polynomial with the coefficient 1 at the highest power of λ.

The difference as compared with the general scheme consists, at first, in distinguishing the degrees of the polynomials of the columns. Second, all polynomials of the first line have sign $\tilde{P}$. The last modification is necessary for determination of the null modes of the functions τ_s, as it will be later. It is assumed, as before, that there is a linear dependence between the columns of the matrix $g = Pg_0 = P \exp \sum_{s=1}^{r} h_s \tau_s$ in $\sum_{\alpha=1}^{k+1} (n_\alpha + 1)$ different points of the λ plane,

$$\sum_{\beta=1}^{k+1} P^{\alpha,\beta}_{n_{\beta+\delta_{\alpha,\beta}}} (\lambda_i)\, c_\beta(\lambda_i) \exp(\tau_\beta - \tau_{\beta-1}) = 0,$$

$$\alpha = 1, 2, \ldots (k+1), \quad i = 1, 2, \ldots \sum_{\alpha=1}^{k+1} (n_\alpha + 1), \tag{5.3}$$

where $c_\beta(\lambda_i)$ is the totality of arbitrary c-number parameters, $\tau_0 = \tau_{k+1} = 0$. The last equality is the system of $\sum_{\alpha=1}^{k+1} (n_\alpha + 1)$ linear algebraic equations, where the coefficient functions of the polynomial matrix P and the null mode components of the τ functions are unknown. The number of equations equals the number of the unknown variables. Indeed, let us take $\alpha = 1$ in (5.3). The number of the coefficient functions of the first column of the polynomial matrix P is equal to $\sum_{\alpha=1}^{k+1} n_\alpha + 1$ and the k null mode components of τ functions are unknown. So the whole number of the unknown variables equals $\sum (n_\alpha + 1)$, i.e., exact number of equations. The same situation takes place for the other columns and consequently the system of equations (5.3) explicitly determines all parameters of the polynomial matrix and the null mode components of τ functions.

Det P = Det (Pg_0) and from definition of the polynomial matrix P it follows that its determinant is a polynomial of the degree $\sum (n_\alpha + 1)$ with the coefficient 1 at the highest power of λ. From (5.3) we know that Det P vanishes in the $\sum (n_\alpha + 1)$ points of the complex plane. Thus we get:

$$\text{Det } P = \prod_{i=1}^{\sum_{\alpha=1}^{k+1}(n_\alpha+1)} (\lambda - \lambda_i). \tag{5.4}$$

The general formulae of section 3, for the entries of the matrix u remain true after obvious substitution $(u_0)_{\alpha,\beta} = \delta_{\alpha,\beta}(\tau_\beta - \tau_{\beta-1})$

$$u_{\alpha,\beta} = \frac{\text{Det } \| P_\beta \to \dot{P}_\alpha + \dot{P}_\alpha(\tau_\alpha - \tau_{\alpha-1})\|}{\prod(\lambda - \lambda_i)}. \tag{5.5}$$

From the last expression we get convinced, as in section 3, that all the peculiarities of the matrix elements of the matrix u, including the infinite point, are determined by the analytic properties of the source functions.

6. THE CASE OF THE ALGEBRA $SL(2,C)$ [10,13–15]

The results of sections 3 and 5 will be specified here for the case of an algebra $SL(2, C)$, which has many physical applications. In that case, it is possible to write all the expressions in the form convenient for practical calculations.

Let the polynomial matrix P (5.2) be rewritten in the form;

$$P = \begin{pmatrix} \prod_{i=1}^{n_1+1} (\lambda - a_i) & \prod_{j=1}^{n_2} (\lambda - b_j) \\ P_{n_1} & P_{n_2+1} \end{pmatrix}. \tag{6.1}$$

The polynomials of the first row $\tilde{P}_{n_1+1}$, $\tilde{P}_{n_2}$ are decomposed on the systems of their roots (a_i, b_j). In our case $\tau_1 = -\tau_2 = \sum \tau_s(\xi_s, \lambda) + \tau_0 \equiv \tau$. The coefficient functions of the polynomials and null mode τ_0 are determined from the linear system of the algebraic equations

$$\exp 2\tau(\lambda_s) \exp 2\tau_0 P_{n_1+1}(\lambda_s) + c(\lambda_s)\, P_{n_2}(\lambda_s) = 0,$$

$$s = 1, 2, \ldots (n_1 + n_2 + 1),$$

$$\exp 2\tau(\lambda_s) \exp 2\tau_0 P_{n_1}(\lambda_s) + c(\lambda_s)\, P_{n_2+1}(\lambda_s) = 0. \tag{6.2}$$

The number of unknown quantities in the first system of equations is equal to $(n_1 + 1)$ symmetrical combinations, composed of the roots a_i (coefficient functions of the polynomials P_{n_1+1}), n_2 symmetrical combinations, composed of the roots b_j (the coefficient functions of the polynomials P_{n_1}), and null mode component in the form $\exp 2\tau_0$. This exactly equals the number of equations. The same situation takes place for the second system (6.2). As in (5.5) for $u_{12} \equiv u_+$, we obtain:

$$u_+ = \left(\frac{\prod_{i=1}^{n_1+1} (\lambda - a_i) \prod_{j=1}^{n_2} (\lambda - b_j)}{\prod_{s=1}^{n_1+n_2+2} (\lambda - \lambda_s)} \right) \left(-\sum_{j=1}^{n_2} \frac{\dot{b}_j}{\lambda - b_j} + \sum_{i=1}^{n_1+1} \frac{\dot{a}_i}{\lambda - a_i} + 2\dot{\tau} \right). \quad (6.3)$$

As it follows from the explicit expression for u_+, the analytical dependence $\dot{\tau}$: the positions of poles, their multiplicity, behaviour at infinity are the same as for the u_+ as for $\dot{\tau}$. Rewriting (6.3) in an equivalent form we have

$$\left(-\sum_{j=1}^{n_2} \frac{\dot{b}_j}{\lambda - b_j} + \sum_{i=1}^{n_1+1} \frac{\dot{a}_i}{\lambda - a_i} + 2\dot{\tau} \right) = \frac{\prod_{s=1}^{n_1+n_2+2} (\lambda - \lambda_s)}{\prod_{i=1}^{n_1+1} (\lambda - a_i) \prod_{j=1}^{n_2} (\lambda - b_j)} u_+ . \quad (6.4)$$

The last equality is the definition of u_+ and the examples of the next section will show how it may be used. By decomposition of the right side of the last equation over the simple fraction we obtain the expressions for the derivatives

$$\dot{b}_j = - u_+(b_j) \frac{\pi(b_j)}{P_{n_1+1}(b_j) \hat{P}_{n_2}(b_j)}, \qquad \dot{a}_i = - u_+(a_i) \frac{\pi(a_i)}{\hat{P}_{n_1+1}(a_i) P_{n_2}(a_i)}. \quad (6.5)$$

In the equality the symbol $\wedge$ over the imliex of the polynomial means this polynomial without multiplier, which goes to zero at the present significance of its argument, $\pi(\lambda) = \prod_{s=1}^{n_1+n_2+2} (\lambda - \lambda_s)$.

For u_0 we have the equivalent representations

$$u_0 = \pi^{-1}(\lambda)\ \mathrm{Det}\begin{pmatrix} \dot{P}_{n_1+1} + \dot{\tau}P_{n_1+1} & \dot{P}_{n_2} - \dot{\tau}P_{n_2} \\ P_{n_1} & \tilde{P}_{n_2+1} \end{pmatrix} =$$

$$= \frac{P_{n_1+1}P_{n_2}}{\pi(\lambda)}\ \mathrm{Det}\begin{pmatrix} -\sum \frac{\dot{a}_i}{\lambda - a_i} + \dot{\tau} & -\sum \frac{\dot{b}_j}{\lambda - b_j} - \dot{\tau} \\ \frac{P_{n_1}}{\tilde{P}_{n_1+1}} & \frac{\tilde{P}_{n_2+1}}{\tilde{P}_{n_2}} \end{pmatrix} =$$

$$= \frac{\tilde{P}_{n_1+1}\tilde{P}_{n_2}}{\pi(\lambda)}\ \mathrm{Det}\begin{pmatrix} -\sum \frac{\dot{a}_i}{\lambda - a_i} + \dot{\tau} & \frac{\pi(\lambda)}{\tilde{P}_{n_1+}\tilde{P}_{n_2}}u_+ \\ \frac{P_{n_1}}{\tilde{P}_{n_1+1}} & \frac{\pi(\lambda)}{\tilde{P}_{n_1+1}\tilde{P}_{n_2}} \end{pmatrix} =$$

$$= \mathrm{Det}\begin{pmatrix} -\sum \frac{\dot{a}_i}{\lambda - a_i} + \dot{\tau} & u_+ \\ \frac{P_{n_1}}{\tilde{P}_{n_1+1}} & 1 \end{pmatrix}. \tag{6.6}$$

In the last transformation we have used definitions of $\pi(\lambda)$ and u_+. Then

$$\pi(\lambda) = \mathrm{Det}\begin{pmatrix} \tilde{P}_{n_1+1} & \tilde{P}_{n_2} \\ P_{n_1} & \tilde{P}_{n_2+1} \end{pmatrix} = \tilde{P}_{n_1+1}\tilde{P}_{n_2}\left(\frac{\tilde{P}_{n_2+1}}{\tilde{P}_{n_2}} - \frac{P_{n_1}}{\tilde{P}_{n_1+1}}\right) \tag{6.7}$$

or

$$\frac{\pi(\lambda)}{\tilde{P}_{n_1+1}\tilde{P}_{n_2}} = \lambda - \delta + \sum \frac{B_j}{(\lambda - b_j)} - \sum \frac{A_i}{(\lambda - a_i)}.$$

Comparing the residues at the poles $\lambda = a_i$ and $\lambda = b_j$ in both sides of the last equality, we obtain

$$B_j = \frac{\pi(b_j)}{\tilde{P}_{n_1+1}(b_j)\tilde{P}_{n_2}(b_j)} = \frac{\dot{b}_j}{u_+(b_j)}, \quad A_i = \frac{\pi(a_i)}{\tilde{P}_{n_1+1}(a_i)\tilde{P}_{n_2}(a_i)} = \frac{\dot{a}_i}{u_+(a_i)}.$$

Now we continue the interrupted calculation (6.7),

$$u_0 = \text{Det}\begin{pmatrix} -\sum \frac{\dot a_i}{\lambda - a_i} + \dot\tau & u_+ \\ -\sum \frac{\dot a_i}{(\lambda - a_i)\, u_+(a_i)} & 1 \end{pmatrix} = \dot\tau + \sum_{i=1}^{n_1+1} \frac{\dot a_i}{u_+(a_i)} \frac{u_+(\lambda) - u_+(a_i)}{\lambda - a_i}. \qquad (6.8)$$

Let us finally calculate $u\_(\lambda) \equiv u_{12}$.

$$u\_ = \pi^{-1}(\lambda)\,\text{Det}\begin{pmatrix} \dot g_{21} & \dot g_{22} \\ g_{21} & g_{22} \end{pmatrix} = \pi^{-1}(\lambda)\, g_{11}g_{12}\,\text{Det}\begin{pmatrix} \frac{\dot g_{21}}{g_{11}} & \frac{\dot g_{22}}{g_{12}} \\ \frac{g_{21}}{g_{11}} & \frac{g_{22}}{g_{12}} \end{pmatrix} =$$

$$= \pi^{-1}(\lambda)\, g_{11}g_{12}\,\text{Det}\begin{pmatrix} \frac{\dot g_{21}}{g_{11}} & \left(\frac{g_{22}}{g_{12}} - \frac{g_{21}}{g_{11}}\right)^{\cdot} + \frac{g_{22}}{g_{12}}\frac{\dot g_{12}}{g_{12}} - \frac{g_{21}}{g_{11}}\frac{\dot g_{11}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{g_{22}}{g_{12}} - \frac{g_{21}}{g_{11}} \end{pmatrix} =$$

$$= g_{11}g_{12}\,\text{Det}\begin{pmatrix} \frac{\dot g_{21}}{g_{11}} & \left(\frac{1}{g_{12}g_{11}}\right)^{\cdot} + \frac{\dot g_{12}}{g_{12}^2 g_{11}} + \frac{g_{21}}{g_{11}^2 g_{12}}\, u_+(\lambda) \\ \frac{g_{21}}{g_{11}} & \frac{1}{g_{12}g_{11}} \end{pmatrix} =$$

$$= \text{Det}\begin{pmatrix} \left(\frac{g_{21}}{g_{11}}\right)^{\cdot} + \dot g_{11}\frac{g_{21}}{g_{11}^2} & -\frac{\dot g_{11}}{g_{11}} + \frac{g_{21}}{g_{11}}\, u_+(\lambda) \\ \frac{g_{21}}{g_{11}} & 1 \end{pmatrix} =$$

$$= \left(\frac{g_{21}}{g_{11}}\right)^{\cdot} + 2\,\frac{\dot g_{11}}{g_{11}}\frac{g_{21}}{g_{11}} - u_+(\lambda)\left(\frac{g_{21}}{g_{11}}\right)^2 =$$

$$= \left(\frac{g_{21}}{g_{11}}\right)^{\cdot} + 2\,\frac{1}{u_0}\frac{g_{21}}{g_{11}} + u_+(\lambda)\left(\frac{g_{21}}{g_{11}}\right)^2. \qquad (6.9)$$

In the latter transformations we use several times equalities (6.4), (6.7), which in the notations of the previous calculations have the form

$$\frac{g_{22}}{g_{12}} - \frac{g_{21}}{g_{11}} = \frac{\pi(\lambda)}{g_{11}g_{12}}\,\frac{\dot g_{12}}{g_{12}} - \frac{\dot g_{11}}{g_{11}} = \frac{\pi(\lambda)}{g_{11}g_{12}}\, u_+(\lambda).$$

We may pass to expression (6.9) for u_- in more short way if to profit by the spectral equation of section 4. The matrix elements g_{11}, $g_{12} = \psi$ satisfy this equation, and so we have

$$\left(\frac{\ddot{\psi}}{u_+}\right) = \left(u_- + \frac{u_0^2}{u_+}\right)\psi.$$

Substituting the expression for u_0 into this equation (6.8) we once more come to previous formula (6.9). We limit ourselves by the first inference only with the aim of preserving the uniformity of the calculation scheme. Using (6.4) and (6.8) we rewrite (6.9) in the form

$$u_- = \left(\sum \frac{\dot{a}_i}{u_+(a_i)} \frac{1}{(a_i - \lambda)}\right)^{\cdot} + \left(\sum \frac{\dot{a}_i}{u_+(a_i)} \frac{1}{(a_i - \lambda)}\right)^2 +$$

$$+ 2\left(\sum \frac{\dot{a}_i}{u_+(a_i)} \frac{1}{(a_i - \lambda)}\right)\left(\dot{\tau} + \sum \frac{\dot{a}_i}{u_+(a_i)} \frac{u_+(a_i) - u_+(\lambda)}{(a_i - \lambda)}\right). \tag{6.10}$$

Now we go to further transformations of the last expression. As we know, the matrix element u_- has no singularities in the points $\lambda = a_i$. For this reason the residues in these poles must vanish. This results in the system of equations of the second order for functions a_i. We shall reduce these equations somewhat later. Note that the first integrals of them are contained in formulae (6.5), where the constants λ_i play the role of the constants of integration. The terms without singularities in (6.10) give rise to the final expression for u_-

$$u_- = -2\sum \frac{\dot{a}_i}{u_+(a_i)} \frac{\dot{\tau}(a_i) - \dot{\tau}(\lambda)}{(a_i - \lambda)} +$$

$$+ \sum\sum \frac{\dot{a}_i}{u_+(a_i)} \frac{\dot{a}_j}{u_+(a_j)} \frac{1}{(a_i - \lambda)} \left(\frac{u_+(a_i) - u_+(\lambda)}{a_i - \lambda} - \frac{u_+(a_i) - u_+(a_j)}{a_i - a_j}\right). \tag{6.11}$$

Formulae (6.4), (6.8) and (6.11) solve the problem which was imposed in the beginning of this section. The entries of the matrix u for every rational $\tau(\lambda)$ are expressed uniformly and give us the possibility of avoiding the hard operation of the division of the polynomial $\pi(\lambda)$ in general expressions of the previous section. Each of the entries of $u_\pm$, u_0 is expressed in the form of the derivatives of some combinations of symmetrical functions, which are composed from a_i, i.e., the coefficient functions of the polynomial $\tilde{P}_{n+1}$. All these coefficient functions are the solutions of a linear system of algebraic equations (6.2). The form of the matrix u essentially depends on the form of background function $\tau(\lambda)$ and in each concrete case they may be obtained only in direct calculations with the help of the formulae of this section.

To conclude, we write down the second order equations for the functions a_i, b_j. These equations give a possibility of establishing numerous recurrence relations among the symmetrical combinations, which are composed of the «roots» a_i, b_j and their derivatives. These formulae will play an essential role in the next section when we pass to the concrete examples of the integrable systems and their solutions. The equations for functions a_i, b_j are as follows:

$$\ddot{a}_i + 2\dot{\tau}(a_i)\,\dot{a}_i + 2\sum_{i=k}\frac{\dot{a}_i\,\dot{a}_k}{a_k - a_i} = \frac{\dot{u}_+(a_i)}{u_+(a_i)}\,\dot{a}_i,$$

$$\ddot{b}_j - 2\dot{\tau}(b_j)\,\dot{b}_j + 2\sum_{j=k}\frac{\dot{b}_j\,\dot{b}_k}{b_k - b_j} = \frac{\dot{u}_+(b_j)}{u_+(b_j)}\,\dot{b}_j. \qquad (6.12)$$

In the last equations $\dot{\tau}(a_i) \equiv \dot{\tau}(\lambda)$ and only after this $\lambda = a_i$ and so on. The systems of the ordinary differential equations (6.12) are of some special interest. They are exactly integrable and the result of their integration can be found from the solution of the algebraic system of linear equations (6.2), where a_i (b_j) are the roots of the polynomial with known coefficients. The first integrals of the systems are known and are contained in formulae (6.5). Their independent integration can be performed on the background of the solutions of integrable systems. The example connected with sine-Gordon equation will be considered in one of the next sections.

7. CONCRETE EXAMPLES [8,10,13–16]

Now we will examine the most familiar systems and equations related to the algebra of rational functions of the second order matrices. The aim consists in demonstration of the general formulae of the previous section and their application to the concrete examples.

For simplicity, in the first examples we assume that our problem is invariant under transformation $\lambda \to -\lambda$ of the spectral parameter. This means that the second solution of the spectral equation is obtained from the first one by the same transformation, i.e.

$$\tau(-\lambda) = -\tau(\lambda), \quad N_1 = N_2 = N, \quad a_i = -b_i.$$

Having these equations in mind, we obtain from (6.7) and (6.11)

$$v_0 = \lambda^3 + \lambda^2 s_{-1} + \lambda s_0 + s_1 + s_0 s_{-1},$$

$$v_- = -\left(\lambda^2 s_{-1} + \lambda\left(\frac{s_{-1}^2}{2} + s_0\right) + s_1 + s_0 s_{-1}\right).$$

The equations of the second order (6.12) in the case under consideration are the following

$$\ddot{a}_i + 2a_i\dot{a}_i + 2\sum_{k=i}\frac{\dot{a}_i\dot{a}_k}{a_k - a_i} = 0\,.$$

Multiplying each equation by a_i^n and summarizing the result, after some calculations we come to the recurrence relation for the functions s_n. For the case $n > 0$ we have

$$\dot{s}_n + 2s_{n+1} - \sum_{k=0}^{n-1} s_k s_{n-1-k} = 0. \tag{7.3}$$

From the same equations there follow the recurrence relations also for the case when $n < -1$. And on the boundary we have $2s_0 + \dot{s}_{-1} + s_{-1}^2 = 0$. The recurrence relations allow one to express all the functions s_n via function s_{-1} and its derivatives up to the $(n+1)$-order. For the matrix v we have

$$\begin{pmatrix} \lambda^3 + \lambda^2 s_{-1} + \lambda s_0 + s_1 + s_0 s_{-1} & 2\lambda^3 + 2s_0 \\ -\left(\lambda^2 s_{-1} + \lambda\left(\frac{s_{-1}^2}{2} + s_0\right) + s_1 + s_0 s_{-1}\right) & -(\lambda^3 + \lambda^2 s_{-1} + \lambda s_0 + s_1 + s_0 s_{-1}) \end{pmatrix}.$$

We know that the Cartan–Maurer identity or the condition of compatibility is satisfied. It gives us the equation for the function $Q = s_{-1}$:

$$Q_t - \left(\frac{Q_{xx}}{4} - \frac{Q^3}{2}\right)_x = 0, \quad (t \equiv \bar{z},\ x \equiv z). \tag{7.4}$$

This is modified Korteveg–de Vries equation. We know from our construction that its solution is given by the expression $Q = \dfrac{\partial \ln \prod a_i}{\partial z}$. To find it in the explicit form, it is necessary to solve the linear system of algebraic equations. In the case of the Korteveg–de Vries this system is as follows

$$\exp 2(\lambda_s z + \lambda_s^3 \bar{z}) \prod (a_i - \lambda_s) + c(\lambda_s) \prod (a_i + \lambda_s) = 0.$$

An explicit solution of this system leads to the solution of the modified Korteveg–de Vries equation in the form of the derivative of the logarithm of the ratio of two determinants of the n-th order.

In all cases, when τ is odd and has polynomial structure there arise the equations for only one function. These are (modified) Korteveg–de Vries equations of the highest order.

The next example is connected with the simplest case of such equation. Let $\tau = \lambda z + (\nu\lambda^3 + \mu\lambda^5)\,\bar{z}$. The matrix u remains the same as above. For the calculation of the matrix v we need some modifications. From (6.4) we conclude that $v_+ = A\lambda^5 + B\lambda^3 + C\lambda$, and the parameters A, B, C enter to the equality

$$2\left(\nu\lambda^4 + \mu\lambda^2 + \sum \frac{a_i'}{a_i^2 - \lambda^2}\right) = (A\lambda^4 + B\lambda^2 + C\lambda)\left(1 + \sum \frac{\dot{a}_i}{a_i^2 - \lambda^2}\right),$$

which allows one to determine them. We get the entries of v in the form:

$$v_+ = 2\nu\lambda^5 + 2(\mu + \nu s_0)\,\lambda^3 + 2(\nu(s_2 + s_0^2) + \mu s_0)\,\lambda,$$

$$v_0 = \nu\lambda^5 + \nu s_{-1}\lambda^4 + (\nu s_0 + \mu)\,\lambda^3 + (\nu(s_1 + s_0 s_{-1}) + \mu s_{-1})\,\lambda^2 + \\ + (\nu(s_2 + s_0^2) + \mu s_0)\,\lambda + \mu(s_1 + s_0 s_{-1}) + \nu(s_3 + s_0 s_1 + s_{-1}s_2 + s_0^2 s_{-1}),$$

$$-v_- = \nu s_{-1}\lambda^4 + \nu\left(s_0 + \frac{s_{-1}^2}{2}\right)\lambda^3 + \nu(s_1 + s_0 s_{-1}) + \mu s_1)\,\lambda^2 + \\ + \left(\nu\left(s_2 + s_1 s_{-1} + \frac{s_0^2}{2} + \frac{s_0 s_{-1}^2}{2}\right) + \mu\left(s_0 + \frac{s_{-1}^2}{2}\right)\right)\lambda + \\ + \nu(s_3 + s_2 s_{-1} + s_1 s_0 + s_0^2 s_{-1}) + \mu(s_1 + s_0 s_{-1}).$$

We need the following recurrence relations:

$$s_0 = -\frac{\dot{s}_{-1} + s_{-1}^2}{2}, \quad s_1 = -\frac{\dot{s}_0}{2}, \quad s_2 = \frac{s_0^2 - \dot{s}_1}{2}, \quad s_3 = -\frac{\dot{s}_2}{2} + s_0 s_1.$$

Thus we conclude that all s_n may be expressed via function $Q \equiv s_{-1}$ and its derivatives. The condition of compatibility leads to equation

$$Q_t + Q_{xxxxx} - 10Q^2 Q_{xxx} - 40QQ_x Q_{xx} - 10(Q_x)^2 + 30Q^4 Q_x = 0, \qquad (7.5)$$

which is the modified Korteveg–de Vries equation of the fifth order (in our general formulae we put $\nu = 1$, $\mu = 0$).

When the τ takes null values at the point $\lambda = 0$, there is always the second possibility of constructing the background element g, and, as a consequence, some other integrable system. It may be assumed that two fundamental solutions of the spectral equation coincide when $\lambda = 0$. This means that Vronskinian of the spectral equation vanishes at this point and the background equation (6.4) changes to:

$$2\lambda\left(1+\sum\frac{\dot{a}_i}{a_i^2-\lambda^2}\right)=\lambda\frac{\prod(\lambda^2-\lambda_i^2)}{\prod(\lambda^2-a_i^2)}u_+(\lambda),$$

from that we conclude $u_+(\lambda) = 2$. From the equalities (6.8) and (6.11) we find $u_0 = \lambda$, $u_+ = -s_0$ and so the matrix u has the form:

$$u=\begin{pmatrix}\lambda & 2\\ -s_0 & -\lambda\end{pmatrix},$$

where as above $s_0=\sum_{k=1}^{n}\dot{a}_k$. The system of the second order equations is the same as in precéding case. All the recurrence relations remain unchanged and we obtain the entries of v:

$$\begin{pmatrix}\lambda^3+\lambda s_0-\dfrac{\dot{s}_0}{2} & 2\lambda^2+2s_0\\ -\left(\lambda^2 s_0+\lambda\dfrac{\dot{s}_0}{2}+s_0^2-\dfrac{\ddot{s}_0}{4}\right) & -\left(\lambda^3+\lambda s_0-\dfrac{\dot{s}_0}{2}\right)\end{pmatrix}.$$

The Cartan–Maurer identity leads to the equation for the function $U\equiv\dfrac{s_0}{2}$,

$$-U_t+U_{xxx}+6UU_x=0,\quad\left(t=\frac{\bar{z}}{4}\right).\qquad(7.6)$$

This is the Korteveg–de Vries equation in its original form.

If τ_0 is any odd polynomial on λ, we will obtain the higher order Korteveg–de Vries equations. It is not difficult to write down the explicit form for these equations. We limit ourselves by the Korteveg–de Vries equation of the fifth order.

Let $\tau=(\lambda z+\lambda^5)\,\bar{z}$.

The matrix u is the same as before. We obtain the entries of the matrix v by the same technique as in modified case:

$$u_+=2\lambda^4+2s_0\lambda^+2(s_2+s_0^2),$$

$$v_0=\lambda^5+s_0\lambda^3+s_1\lambda^2+(s_2+s_0^2)\,\lambda+s_3+s_0s_1,$$

$$-v_-=s_0\lambda^4+s_1\lambda^3+\left(s_2+\frac{s_0^2}{2}\lambda^2+(s_3+s_1s_0)\,\lambda+\left(s_4+s_2s_0+\frac{s_1^2}{2}+\frac{s_0^3}{2}\right)\right).$$

The condition of the consistensy gives us the Korteveg–de Vries equation for the function $U \equiv -2s_0$:

$$-U_t + U_{xxxxx} - 10UU_{xxx} - 20U_xU_{xx} + 30U^2U_x = 0, \quad \left(t = \frac{\bar{z}}{4} \right). \tag{7.7}$$

As it follows from our construction, the connection between solutions of the Korteveg–de Vries equation and its modified version is given by the equality $U = \dot{Q} + Q^2$, as a consequence of the recurrence relations for s_0, s_{-1} (7.3) of this section.

7.2. $\tau = \lambda z + \lambda^{-1}\bar{z}$ – sin–Gordon Equation. Calculations of the matrix u do not change and we have as before:

$$u = \begin{pmatrix} \lambda + \dot{\rho} & 2\lambda \\ -\dot{\rho} & -(\lambda + \dot{\rho}) \end{pmatrix},$$

where $\dot{\rho} = s_{-1}$. The general equation (6.4) yields the expression for v_+:

$$2\left(\lambda^{-1} + \lambda \sum \frac{a_i'}{a_i^2 - \lambda^2} \right) = \frac{\prod (\lambda^2 - \lambda_i^2)}{\prod (\lambda^2 - a_i^2)} v_+(\lambda).$$

It follows from the last equality that v_+ has a single pole at the point $\lambda = 0$, with the residue equal to $2\exp 2\rho$, where $\exp\rho = \prod \frac{a_i}{\lambda_i}$. Thus $v_+ = 2\lambda^{-1}\exp 2\rho$. With the help of equations (6.4), (6.8) we find $v_0 = \lambda^{-1}\exp 2\rho$, $v_- = -\lambda^{-1}\sinh\rho$. It must be noticed that in these calculations we use the relation $1 + s_2 = \exp(-\rho)$, which follows from equation for u_+ if its both sides are divided by 2λ before setting $\lambda = 0$.

$$v = \lambda^{-1}\begin{pmatrix} \exp 2\rho & 2\exp 2\rho \\ \sinh 2\rho & -\exp 2\rho \end{pmatrix}.$$

The Cartan–Maurer identity yields now the sin-Gordon equation for the function ρ:

$$\frac{\partial^2\rho}{\partial z\partial\bar{z}} = 2\sinh 2\rho. \tag{7.8}$$

As in the case of the Korteveg–de Vries equation, the assumption that τ is an odd order polynomial on λ^{-1} results in the higher order sin-Gordon equations. This is related to the fact that all moments of negative degree $s_{-n} = \sum a_i^{-n}\dot{a}_i$ are connected with each other by the system of the recurrence relations, which follow from the second order equations for the functions a_i.

7.3. $\tau = \lambda z + (\mu\lambda^{-1} + \lambda^3)\,\bar{z}$. This example is connected with the equation, which is in some sense «intermediate» between the sin-Gordon and Korteveg–de Vries equations.

Matrix u is the same as in the previous examples. The element v_+ is given by equation (6.4)

$$2\lambda\left(\lambda^2 + \mu\lambda^{-2} + \sum \frac{a_i'}{a_i^2 - \lambda^2}\right) = +\frac{\prod(\lambda^2 - \lambda_i^2)}{\prod(\lambda^2 - a_i^2)}\, v_+(\lambda).$$

We conclude that the maximal degree of v_+ at infinity is 3, it has the pole at the point $\lambda = 0$ and the residue in it equals $\mu \exp 2\rho$. Using the technique of the previous examples, we find other parameters without difficulties. Finally we have:

$$v_+ = 2(\lambda^3 + s_0\lambda + \mu \exp 2\rho\lambda^{-1}),$$

$$v_0 = \lambda^3 + s_{-1}\lambda^2 + s_0\lambda + s_1 + s_0 s_{-1} + \mu \exp 2\rho\lambda^{-1},$$

$$v_- = -\left(s_1\lambda^2 + \left(\frac{s_{-1}^2}{2} + s_0\right)\lambda + s_1 + s_0 s_{-1} + \mu \sinh 2\rho\lambda^{-1}\right).$$

It must be noticed that in this case any calculations are not necessary. It is sufficient to take the linear combinations of matrices v of Korteveg–de Vries and sin-Gordon equations. The Cartan–Maurer identity leads to the equation for the function ρ,

$$4\frac{\partial^2\rho}{\partial z\partial\bar{z}} = \frac{\partial^4\rho}{\partial z^4} - 6\frac{\partial^2\rho}{\partial z^2}\left(\frac{\partial\rho}{\partial z}\right)^2 + 8\mu \sinh 2\rho = 0. \qquad (7.9)$$

7.4. $-\tau = z\lambda + \bar{z}\,\lambda_{-1} + f$ — **Lund–Pohlmeyer–Regge System.** This is the first example of the general case, in which τ has null mode. General equation (6.4) for u_+ now has the form

$$\left(-\sum_{j=1}^{n_2}\frac{\dot{b}_j}{\lambda - b_j} + \sum_{i=1}^{n_1+1}\frac{\dot{a}_i}{\lambda - a_i} + 2\dot{f} + 2\lambda\right) = \left(\frac{\prod_{s=1}^{n_1+n_2+2}(\lambda - \lambda_s)}{\prod_{i=1}^{n_1+1}(\lambda - a_i)\prod_{j=1}^{n_2}(\lambda - b_j)}\right)u_+.$$

Comparing the highest powers of λ, we conclude $u_+ = 2$. Equations (6.7), (6.8) give $u_0 = \lambda + \dot{f}$, $u_- = -\sum \dot{a}_i = -s_0$. The equations of the second order in the case under consideration

$$\ddot{a}_i + 2(a_i + f)\,\dot{a}_i + 2\sum_{k=i} \frac{\dot{a}_i\,\dot{a}_k}{a_k - a_i} = 0$$

give us the possibility of finding the recurrence relations for s_n. It will be sufficient for our purpose to multiply each equation by a_i^{-1} and sum the result. On this way we obtain

$$\dot{s}_{-1} + 2s_0 + 2\dot{f}s_{-1} + s_{-1}^2 = 0.$$

Equation (6.4)

$$\left(-\sum_{j=1}^{n_2} \frac{b_j'}{\lambda - b_j} + \sum_{i=1}^{n_1+1} \frac{\dot{a}_i}{\lambda - a_i} + 2f' + 2\lambda^{-1}\right) = \left(\frac{\prod_{s=1}^{n_1+n_2+2} (\lambda - \lambda_s)}{\prod_{i=1}^{n_1+1} (\lambda - a_i) \prod_{j=1}^{n_2} (\lambda - b_j)}\right) v_+$$

determines $v_+ = 2f'\lambda^{-1}$.

It follows directly from (6.7), (6.8) that

$$v_0 = f' + (1 - f's_{-1})\,\lambda^{-1}, \qquad v_- = \frac{1}{2}\,(2s_{-1} - f's_{-1}^2)\,\lambda^{-1}.$$

Let us now perform gauge transformation (see section 4) with $g_0 = \exp Hf$. The aim of such transformation is to take away the derivatives f from the entries of u_0, v_0. After the introduction of new variables

$$x = \exp - 2f, \qquad y = s_{-1}\exp 2f$$

the matrices u, v acquire a simple and suitable form

$$u = \begin{pmatrix} \lambda & 2x \\ \dfrac{\dot{y} + y^2 x}{2} & -\lambda \end{pmatrix}, \qquad v = \lambda^{-1}\begin{pmatrix} 1 + \dfrac{yx'}{2} & -x' \\ y + \dfrac{y^2 x'}{2} & -\left(1 + \dfrac{yx'}{2}\right) \end{pmatrix}.$$

The consistency condition leads to the integrable system

$$\frac{\partial^2 x}{\partial z \partial \bar{z}} - 4x - 2(xy)\,\frac{\partial x}{\partial \bar{z}} = 0, \qquad \frac{\partial^2 y}{\partial z \partial \bar{z}} - 4y + 2(xy)\,\frac{\partial y}{\partial \bar{z}} = 0. \qquad (7.10)$$

If one executes the transformation, which is motivated by the form of the matrix v:

$$1 + \frac{yx'}{2} = \cos\alpha, \qquad x' = -\sin\alpha \exp\theta,$$

i.e., passes from the pair of variables $(x, y) \to (\alpha, \theta)$ and after this to the pair (α, β) in accordance with the formulae

$$\theta_z = \frac{\beta_z \cos\alpha}{1 + \cos\alpha}, \qquad \theta_{\bar z} = \frac{\beta_{\bar z}}{1 + \cos\alpha},$$

then in variables (α, β) the system under investigation takes the form

$$\alpha_{z\bar z} + 4\sin\alpha - \frac{\sin\frac{\alpha}{2}}{2\left(\cos\frac{\alpha}{2}\right)^3}\beta_z\beta_{\bar z} = 0, \qquad \beta_{z\bar z} + \frac{\alpha_z\beta_{\bar z} + \alpha_{\bar z}\beta_z}{\sin\alpha} = 0.$$

This is the Lund–Pohlmeyer–Regge system in its canonical form. It is connected with some geometrical construction.

7.5. $\tau = -(\lambda z + \lambda^2\bar z + f)$ — Nonlinear Schrödinger Equation. The matrix u is evidently the same as in the last case. From general equation (6.4) we conclude (analysing its behaviour at infinity) that $v_+ = 2\lambda + c$. To find c, it is suitable to take the ratio of the two polynomials in the expression of v_+ from the definition of u_+, i.e.,

$$\left(-\sum\frac{b_j'}{\lambda - b_j} + \sum\frac{a_i'}{\lambda - a_i} + 2f' + 2\lambda^2\right) =$$

$$= \left(-\sum\frac{\dot b_j}{\lambda - b_j} + \sum\frac{\dot a_i}{\lambda - a_i} + 2\dot f + 2\lambda\right)\left(\lambda + \frac{c}{2}\right).$$

Comparing the highest degree of λ, we conclude that $c = -2\dot f$. No difficulties arise in calculations of the entries of v_0, v_-, and finally we have

$$v_+ = 2\lambda - 2\dot f, \qquad v_0 = \lambda^2 + f' + s_0, \qquad v_- = -\lambda s_0 + \frac{\dot s_0}{2} + \dot f s_0.$$

Executing the gauge transformation, which removes derivatives f from the matrix elements v_0, u_0, and introducing new variables

$$r = 2\exp-(2f), \qquad q = -s_0\exp(2f)$$

we bring the matrix u, v to the form:

$$u = \begin{pmatrix} \lambda & r \\ q & -\lambda \end{pmatrix}, \qquad v = \begin{pmatrix} \lambda^2 - \frac{qr}{2} & \lambda r + \frac{\dot r}{2} \\ \lambda q - \frac{\dot q}{2} & -\left(\lambda^2 - \frac{qr}{2}\right) \end{pmatrix}.$$

The consistency condition gives us the integrable system

$$r' - \frac{\ddot r}{2} + (qr)\, r = 0, \quad q' - \frac{\ddot q}{2} - (qr)\, q = 0. \tag{7.11}$$

This is a nonlinear Schrödinger equation without derivatives. In the next section we will need an explicit form of its solutions.

Following the rules by Kramer, we get from the linear system of algebraic equations

$$\exp(-2f) = \frac{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1+1}; 1, \lambda, \ldots, \lambda^{n_2-1})}{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1}; 1, \lambda, \ldots, \lambda^{n_2})},$$

$$\sum a_i = \frac{(\exp 2\tau, \ldots, \exp 2\tau\lambda^{n_1-1}, \exp 2\tau\lambda^{n_1+1}; 1, \lambda, \ldots, \lambda^{n_2})}{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1}; 1, \lambda, \ldots, \lambda^{n_2})}.$$

By induction, one can get from the last equality the expression for $s_0 = \sum \dot a_i$,

$$s_0 = -2\frac{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1-1}; 1, \lambda, \ldots, \lambda^{n_2+1})}{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1}; 1, \lambda, \ldots, \lambda^{n_2})} \exp - 2f.$$

By $(a, b, \ldots, c)$ in the previous expressions (as in section 4) we denote the determinant of the n-th order (n is the number of elements $a, b, \ldots, c$), whose s-th line consists of the elements $(a_s, b_s, \ldots, c_s)$. The index s numerates the points of λ plane in which the columns of the polynomial matrix are linearly dependent. At last, for the quantities of interest we obtain

$$r = 2\,\frac{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1+1}; 1, \lambda, \ldots, \lambda^{n_2-1})}{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1}; 1, \lambda, \ldots, \lambda^{n_2})},$$

$$q = 2\,\frac{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1-1}; 1, \lambda, \ldots, \lambda^{n_2+1})}{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1}; 1, \lambda, \ldots, \lambda^{n_2})}.$$

7.6. $\tau = \sum \frac{\xi_k}{\lambda - \theta_k}$ **— the Principal Chiral Field Problem in *n*-Dimensions.** The notations in the heading of this subsection should be understood as follows: ξ_k are the coordinates of the n-th order space; θ_k, totality of arbitrary c-number parameters ($\theta_k \neq \theta_s$, if $k = s$). In this case $\tau \to 0$ when $\lambda \to \infty$, and so

τ has no null mode. The matrix of the polynomials in this case must be taken in the form

$$P = \begin{pmatrix} \tilde{P}_{n_1+1} & \tilde{P}_{n_2+1} \\ P_{n_1} & \tilde{Q}_{n_2+1} \end{pmatrix}.$$

Equation (6.4) can be written as

$$\left(- \sum_{j=1}^{n_2+1} \frac{\dot{b}_j}{\lambda - b_j} + \sum_{i=1}^{n_1+1} \frac{\dot{a}_i}{\lambda - a_i} + \frac{2}{\lambda - \theta_k} \right) = \left(\frac{\prod_{s=1}^{n_1+n_2+2} (\lambda - \lambda_s)}{\prod_{i=1}^{n_1+1} (\lambda - a_i) \prod_{j=1}^{n_2} (\lambda - b_j)} \right) u_+$$

$\left(\dot{f} \equiv \frac{\partial f}{\partial \xi_k} \right)$. We conclude from the last equality that $u_+^k = \frac{A_k}{\lambda - \theta_k}$. As $\lambda \to \infty$, one finds

$$A_k = \sum \dot{a}_i - \sum \dot{b}_j + 2 = \frac{\partial}{\partial \xi_k} (\sum a_i - \sum b_j + 2 \sum \xi_r).$$

The calculation of u_0^k, u_-^k is carried out via the general scheme and results in the expressions

$$u_0^k = \frac{1 - s_0}{\lambda - \theta_k} = \frac{\frac{\partial}{\partial \xi_k} (\sum \xi_r - \sum a_i)}{\lambda - \theta_k},$$

$$u_-^k = \frac{2s_0 - s_0^2}{A_k(\lambda - \theta_k)} = \frac{\frac{\partial}{\partial \xi_k} \left(\frac{\theta_k s_0 - s_1}{A_k} \right)}{\lambda - \theta_k}.$$

The last equality needs some explanation. In the case under investigation, the system of second order equations for functions a_i is as follows:

$$\ddot{a}_i + 2 \frac{\dot{a}_i}{a_i - \theta_k} + 2 \sum_{i \neq k} \frac{\dot{a}_i \dot{a}_k}{a_k - a_i} = \frac{\dot{A}_k}{A_k} \dot{a}_i .$$

Multiplying each equation of the system by $(a_i - \theta_i)$ and summarizing the result we have:

$$\dot{s}_i - \frac{\dot{A}_k}{A_k} s_1 - \theta_k \left(\dot{s}_0 - \frac{\dot{A}_k}{A_k} s_0 \right) = s_0^2 - 2s_0$$

or

$$\frac{\partial}{\partial \xi_k}\left(\frac{s_1 - \theta_k s_0}{A_k}\right) = \frac{s_0^2 - 2s_0}{A_k}.$$

This is just the same equality, which was used in the transformation of the expression for u_-.

We propose also another deduction, which permits one to obtain the expressions for u_+, u_-, u_0 in some different form. From the direct definition of the matrix u for the element u_0 we have

$$u_0^k = \frac{\operatorname{Det}\begin{pmatrix} \dot{P}_{n_1+1} + \dfrac{P_{n_1+1}}{\lambda - \theta_k} & \dot{P}_{n_2+1} - \dfrac{P_{n_2}+1}{\lambda - \theta_k} \\ P_{n_1} & Q_{n_2+1} \end{pmatrix}}{\prod\limits_{s=1}^{n_1+n_2+2} (\lambda - \lambda_s)} =$$

$$= \frac{1}{\lambda - \theta_k} \frac{\operatorname{Det}\begin{pmatrix} (\lambda - \theta_k)\dot{P}_{n_1+1} + P_{n_1+1} & (\lambda - \theta_k)\dot{P}_{n_2+1} - P_{n_2} + 1 \\ P_{n_1} & Q_{n_2+1} \end{pmatrix}}{\prod\limits_{s=1}^{n_1+n_2+2} (\lambda - \lambda_s)}.$$

We know that in the last equality the numerator is divided by the denominator and so it is only necessary to find the coefficient at the $\lambda^{n_1+n_2+2}$ in the determinant of the numerator. It can be done easily. The result is

$$u_0 = \frac{1 + (\tilde{P}_{n_1+1}^{n_1})_{\xi_k}}{\lambda - \theta_k}.$$

The upper index at the symbol of the polynomial means the coefficient function at the corresponding degree of λ. $\tilde{P}_{n_1+1}^{n_1} = -\sum a_i$, and so we return to the expression which was obtained above. In the same manner:

$$u_+^k = \frac{(\tilde{P}_{n_2+1}^{n_2})_{\xi_k} - (\tilde{P}_{n_1+1}^{n_1})_{\xi_k} + 2}{\lambda - \theta_k}, \qquad u_-^k = \frac{(P^{n_1})_{\xi_k}}{\lambda - \theta_k}.$$

If the first expression is the same as before, then in the second one the coefficient $P_{n_1}^{n_1}$ arises from the solution of linear system of the algebraic equations for the second line of the polynomial matrix. This is the explicit result of summation in the previous expression for u_-^k. Finally, we have $u^k = \frac{\partial F}{\partial \xi_k}$, and condition of the compatibility gives us the equation for the matrix-valued function F:

$$(\theta_i - \theta_j)\frac{\partial^2 F}{\partial \xi_i\, \partial \xi_j} = \left[\frac{\partial^2 F}{\partial \xi_i}, \frac{\partial^2 F}{\partial \xi_j}\right].$$

This is the system of equations of the principal chiral field problem in the n-dimensional space playing an important role in the general theory of the self-dual equations.

8. THE SOLUTION OF THE GENERAL EQUATION IN THE SOLVABLE CASE AND THE DISCRETE TRANSFORMATION [17,18]

Here we shall show that the equation for the element g has a regular exact solution not only if, as it is proposed, element g belongs to the commutative group, but also if it belongs to the solvable one (diagonal plus upper (lower) triangular matrices).

We shall consider this situation on example of the case $SL(2, R)$ group (algebra):

$$\frac{\partial g}{\partial x_i} = u^i g, \qquad g = \exp x^+ \alpha \exp H\tau$$

or

$$u^i = \tau_{\xi_i} H + (\alpha_{\xi_i} - 2\alpha\tau_{\xi_i})\, X^+,$$

where the element u^i belongs to the solvable algebra and must have deficient analytical properties as a function of the spectral parameter λ in the complex plane. The solution of this problem for τ is the same as in the diagonal case: $\tau(\lambda)$ — the rational function, the residues in each pole of which are the functions of only one variable ξ_i. For α we obtain:

$$\alpha = P(\lambda) \int_C d\lambda' \frac{\alpha(\lambda')}{\lambda - \lambda'} \exp 2\tau(\lambda),$$

where $\alpha(\lambda)$ is an arbitrary function of one variable; and C is some circle in the λ plane, on which the integral has the meaning; $P(\lambda)$ is some polynomial. Indeed:

$$u^i = \alpha_{\xi_i} - 2\alpha\tau_{\xi_i} = P(\lambda) \int d\lambda' \alpha(\lambda') \frac{\tau_{\xi_i}(\lambda) - \tau_{\xi_i}(\lambda')}{\lambda - \lambda'} \exp 2\tau(\lambda').$$

As follows from the last expression u_+ has the same peculiarities in the finite λ plane as $\tau(\lambda)$. At infinity, if $\tau(\lambda) \sim \lambda^s$ and $P(\lambda) \sim \lambda^t$, then $u_+ \sim \lambda^{s+t-1}$. The situation for the solvable groups of higher dimensions is the same and it is not difficult to obtain a solution and explicit formulae in that case. Yet this is not so important for our purposes.

Now we take the solution for the solvable case as the element g_0 in our general construction of section 5. In this way we obtain the solutions, which depend on arbitrary function, the definite choice of which gives us the possibility to find solutions with definite boundary conditions, solve the reduction problem and so on.

Let us consider this question more carefully. We take the matrix of the polynomials in the usual form:

$$P = \begin{pmatrix} \tilde{P}_{n_1+1} & \tilde{P}_{n_2} \\ P_{n_1} & \tilde{P}_{n_2+1} \end{pmatrix}.$$

The condition of the linear dependence of the columns of the matrix $g = Pg_0$ has the form (we write it only for the first line):

$$\tilde{P}_{n_1+1} \exp \tau + c \exp - \tau\, [\tilde{P}_{n_2} + \alpha \tilde{P}_{n_1+1}] = 0$$

or

$$\tilde{P}_{n_1+1}(\exp 2\tau(\lambda) c^{-1}(\lambda) + P(\lambda) \int d\lambda' \frac{\alpha(\lambda')}{\lambda - \lambda'} \exp 2\tau(\lambda')) + \tilde{P}_{n_2} = 0.$$

We see from the last equation that there is only one difference as compared with the diagonal case ($\alpha = 0$). That is the formal change in all formulae:

$$\exp 2\tau\,(\lambda) c^{-1}(\lambda) \to \exp 2\tilde{\tau}(\lambda) \equiv$$

$$\equiv \exp 2\tau(\lambda) c^{-1}(\lambda) + P(\lambda) \int d\lambda' \frac{\alpha(\lambda')}{\lambda - \lambda'} \exp 2\tau(\lambda') \equiv \exp 2\tau'(\lambda). \qquad (8.1)$$

To have the correct behaviour of u^i at infinity one should demand $n_2 > n_1 + 1$. All formulae of the issues 6—7 remain correct under the substitution (8.1). We have the hierarchy solutions of the investigated system, which depend on the arbitrary function $\alpha(\lambda)$.

These solutions can be connected with the solutions of the diagonal case by some limiting process. We explain this on the example of the nonlinear

Schrödinger equation. From the results of the corresponding subsection we have the explicit form of its solutions:

$$r_{n_1, n_2} = \frac{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1+1}; 1, \lambda, \ldots, \lambda^{n_2-1})}{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1}; 1, \lambda, \ldots, \lambda^{n_2})},$$

$$q_{n_1, n_2} = \frac{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1-1}; 1, \lambda, \ldots, \lambda^{n_2+1})}{(\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1}; 1, \lambda, \ldots, \lambda^{n_2})}. \quad (8.2)$$

Let n_1 have fixed values and $n_2 \to \infty$ (more exactly $n_2 = n_1 + 1 + N$, $N \to \infty$). First of all we prove the following equality

$$\exp 2\tau, \exp 2\tau\lambda, \ldots, \exp 2\tau\lambda^{n_1}; 1, \lambda, \ldots, \lambda^{n_2}) = W(\lambda_1, \lambda_2, \ldots, \lambda_{n_1+n_2}),$$

$$\sum_{i,j,..k}^{n_1} \Phi(\lambda_i)\,\Phi(\lambda_j)\ldots\Phi(\lambda_k)\, W^2(\lambda_i, \lambda_j, \ldots \lambda_k), \quad (8.3)$$

where $W(\lambda_l \lambda_n \ldots \lambda_m)$ is a Vandermond determinant and $\Phi(\lambda)$ is determined by the expression:

$$\Phi(\lambda_i) = \exp 2\tau(\lambda_i) \prod_{k \neq i}^{n_1+n_2} (\lambda_k - \lambda_i)^{-1}.$$

Let $n_1 = 1$. Expanding the determinant with respect to the elements of the first column, we obtain:

$$(\exp 2\tau; 1, \lambda, .., \lambda^{n_2}) = \sum_{s=1}^{n_2+1} \exp 2\tau(\lambda_s)\,(-1)^{s+1} W'(\lambda_l \lambda_n \ldots \lambda_m) =$$

$$= W(\lambda_1, \lambda_2, \ldots \lambda_{n_2+1}) \sum_{s=1}^{n_2+1} \Phi(\lambda_s),$$

where W' is the Vandermond determinant, which is constructed from the n_2 values of λ_i except λ_s. Let $n_1 = 2$. Expanding the determinant with respect to the minors of the two first columns, we have:

$$(\exp 2\tau, \exp 2\tau\lambda; 1, \lambda, .., \lambda^{n_2}) = \sum_{s,k}^{n_2+2} (-1)^{s+k} \exp 2\tau(\lambda_s) \exp 2\tau(\lambda_k)(\lambda_s - \lambda_k),$$

$$W''(\lambda_l \lambda_n \ldots \lambda_m) = W(\lambda_1, \lambda_2, \ldots \lambda_{n_2+2}) \sum_{s,k}^{n_2+2} \Phi(\lambda_s)\,\Phi(\lambda_k)\,(\lambda_s - \lambda_k)^2.$$

In the case of arbitrary n_1 the expansion of the determinant over the minors of its n_1 first columns and some regrouping of the multipliers under the symbol of the summation prove the validity of the proposition (Refs.[8,44]).

Now let us return to expressions (8.2), use equality (8) and take the limit $n_2 \to \infty$. Then for the values

$$q_n = q_{n,\infty}, \qquad r_n = r_{n,\infty}$$

we obtain

$$q_n = \frac{\Theta_{n+1}}{\Theta_n}, \qquad r_n = \frac{\Theta_{n-1}}{\Theta_n}, \tag{8.4}$$

where

$$\Theta_n = \int \dots \int d\lambda_1 \dots d\lambda_n \exp 2\tau(\lambda_1) \dots \exp 2\tau(\lambda_n)\, W^2(\lambda_1, \lambda_2, \dots \lambda_{n_2}).$$

This is just the same result, as can be obtained if for g_0 one picks out the element of solvable group of the upper triangle matrix at the beginning of this section.

Now we shall draw some consequences from the last expressions for the solutions of the system of nonlinear Schrödinger equations, which after some evidently performed transformation, will be written in the form:

$$r' - \ddot{r} + 2(qr)\, r = 0, \qquad q' - \ddot{q} - 2(qr)\, q = 0.$$

From (8.4) we see, that

$$r_{n+1} = \frac{\Theta_n}{\Theta_{n+1}} = \frac{1}{q_n}.$$

Let us assume, that the system under investigation is invariant under the substitution $R = \frac{1}{q}$, $Q = ?$. Then from the first equation we get $Q = q(qr - (\ddot{ln q}))$. By direct check we get convinced that the second equation is also satisfied. So we conclude that our system is invariant under transformation:

$$R = \frac{1}{q}, \qquad Q = q(qr - (\ddot{ln q}))$$

which we will call the Discrete transformation for this system.

In the next sections we shall see that the integrable systems under consideration have their own Discrete transformation, solving which one can find the wide class of solutions of the integrable system, including soliton of type one.

So the independent construction of the Discrete transformations opens the new, more direct method for solving the integrable systems.

9. DISCRETE TRANSFORMATION FOR THE MAIN CHIRAL FIELD PROBLEM [19]

In this section by the example of the main chiral problem we propose the direct method for the construction of the Discrete transformation using only the form of equations of the integrable system.

As was mentioned in section 7, the system of the equations of the main chiral field problem in n-dimensional space has the form:

$$(\theta_i - \theta_j)\frac{\partial^2 f}{\partial x_i \partial x_j} = \left[\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}\right], \tag{9.1}$$

where function f takes values in the arbitrary semisimple algebra, θ_i are numerical parameters.

First of all we discribe in detail the calculations for the algebra A_1 which allows us to simplify calculations for the general case. Let $f = X^+ f_+ + Hf_0 + X^- f_-$ (here $[X^+, X^-] = H$, $[H, X^\pm] = \pm 2X^\pm$) be some solution of (9.1) and $F = X^+ F_+ + HF_0 + X^- F_-$ be a solution of (9.1) which is connected with f via the Discrete transformation. The explicit form of this transformation will be given below.

Let $F_- = \frac{1}{f_+}$. This suggestion is not accidental, but comes from the explicit form of the soliton solutions to the main chiral field problem (section 7). From (9.1) we have the equation for F_-:

$$(\theta_i - \theta_j)\frac{\partial^2 F_-}{\partial x_i \partial x_j} = -2\left\{\frac{\partial F_0}{\partial x_i}\frac{\partial F_-}{\partial x_j} - \frac{\partial F_0}{\partial x_j}\frac{\partial F_-}{\partial x_i}\right\}.$$

Substituting $F_- = \frac{1}{f_+}$ into the last equation and using the equation for f_+, which follows from (9.1), we find the equation for $F^* = F_0 f_+$:

$$\frac{\partial F^*}{\partial x_i} = f_0\frac{\partial f_+}{\partial x_i} - \frac{\partial f_0}{\partial x_i}f_+ + \theta_i\frac{\partial f_+}{\partial x_i}. \tag{9.2}$$

The second mixed derivatives calculated from (9.2) are equal due to (9.1). Thus for derivatives of F_0 we have:

$$\frac{\partial F_0}{\partial x_i} = (f_0 - F_0 + \theta_i)\frac{\partial}{\partial x_i}\ln f_+ - \frac{\partial f_0}{\partial x_i}. \tag{9.3}$$

Substituting (9.3) into null component of equation (9.1), we arrive at

$$\frac{\partial F_+}{\partial x_i} = (f_0 - F_0 + \theta_i)^2 \frac{\partial f_+}{\partial x_i} - 2f_+ (f_0 - F_0 + \theta_i) \frac{\partial f_0}{\partial x_i} - f_+^2 \frac{\partial f_-}{\partial x_i} . \quad (9.4)$$

Finally, substituting (9.3) and (9.4) into the «positive» component of the system (9.1) for F, we conclude that the corresponding equations are satisfied identically. Let us rewrite relations (9.3) and (9.4) in the matrix form,

$$\frac{\partial F}{\partial x_i} = \exp\,[-X^+ (f_0 - F_0 + \theta_i) f_+] \exp\,[H \ln f_+]\, r \frac{\partial f}{\partial x_i} r^{-1} =$$

$$= \exp\,[-H \ln f_+] \exp\,[X^+ (f_0 - F_0 + \theta_i) f_+], \quad (9.5)$$

where r is an automorphism of the algebra A_1 with the properties $rX^{\pm} r^{-1} = -X^{\mp}$, $rHr^{-1} = -H$. In what follows rfr^{-1} will be also denoted as f for brevity. Define the element S with values in the $SL(2, R)$ group,

$$S = \exp\,[-X^+ (f_0 - F_0) f_+] \exp H \ln f_+ .$$

By direct calculation one gets convinced that

$$S^{-1} \frac{\partial S}{\partial x_i} = \frac{1}{f_+} \left[\frac{\partial f}{\partial x_i}, X^+ \right] - \theta_i \frac{\partial}{\partial x_i} \frac{1}{f_+} X^+ . \quad (9.6)$$

In terms of S equation (9) can be rewritten in the form

$$\frac{\partial F}{\partial x_i} = S \frac{\partial f}{\partial x_i} S^{-1} + \theta_i \frac{\partial S}{\partial x_i} S^{-1} . \quad (9.7)$$

Equations (9.6) and (9.7), being equivalent of equations (9.3) and (9.4), realize the Discrete transformation for system (9.1) in the form which can be generalized for the case of an arbitrary semisimple Lie algebra.

In the case of any semisimple Lie algebra for element f, which takes values in it, and obeys the system (9.1), the following statement takes place:

There exists such an element S taking the values in the gauge group that

$$S^{-1} \frac{\partial S}{\partial x_i} = \frac{1}{f_-} \left[\frac{\partial f}{\partial x_i}, X_M^+ \right] - \theta_i \frac{\partial}{\partial x_i} \frac{1}{f_-} X_M^+ . \quad (9.8)$$

Here X_M^+ is the element of the algebra corresponding to its maximal root divided by its norm, i.e., $[X_M^+, X^-] = H$, $[H, X^{\pm}] = \pm 2X^{\pm}$; f_- is the coefficient function in the decomposition of f at the element corresponding to the minimal root of the algebra. To prove the statement it is necessary to convince oneself that the Cartan–Maurer identity is satisfied. After substitution (9.8) into this identity with the account of the definition of f_- and X^+_M given above, we obtain the following expression which should vanish:

$$\left[\frac{\partial f_-}{\partial x_j}\frac{\partial f}{\partial x_i} - \frac{\partial f_-}{\partial x_i}\frac{\partial f}{\partial x_j}, X^+_M \right] + (\theta_i - \theta_j)\frac{\partial^2 f_-}{\partial x_i \partial x_j} X^+_M - \left[\left[\frac{\partial f}{\partial x_j}, X^+_M \right], \left[\frac{\partial f}{\partial x_i}, X^+_M \right] \right].$$

This fact becomes obvious if one commutes twice equation (9.1) with the element X^+_M. Now define the element F taking values in the algebra by the following relations:

$$\frac{\partial F}{\partial x_i} = S \frac{\partial f}{\partial x_i} S^{-1} + \theta_i \frac{\partial S}{\partial x_i} S^{-1}. \tag{9.9}$$

To prove the compatibility of (9.9), compare the second derivatives of F:

$$S^{-1}\left(\frac{\partial^2 F}{\partial x_i \partial x_j} - \frac{\partial^2 F}{\partial x_j \partial x_i} \right) S =$$

$$= \left[S^{-1}\frac{\partial S}{\partial x_j}, \frac{\partial f}{\partial x_i} \right] - \left[S^{-1}\frac{\partial S}{\partial x_i}, \frac{\partial f}{\partial x_j} \right] + \theta_i \frac{\partial}{\partial x_j}\left(S^{-1}\frac{\partial S}{\partial x_j} \right) - \frac{\partial}{\partial x_j}\left(\theta_i S^{-1}\frac{\partial S}{\partial x_i} \right) =$$

$$= \frac{1}{f_-}\left[\left\{ (\theta_i - \theta_j)\frac{\partial^2 f}{\partial x_i \partial x_j} - \left[\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_i} \right] \right\}, X^+_M \right] = 0.$$

In the same way, with the use of (9.9) we come to the following equality:

$$(\theta_i - \theta_j)\frac{\partial^2 F}{\partial x_i \partial x_j} - \left[\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right] = S\left\{ (\theta_i - \theta_j)\frac{\partial^2 f}{\partial x_i \partial x_j} - \left[\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right] \right\} S^{-1}.$$

It means that (9.8), (9.9) realize the Discrete transformation for the main chiral field problem in the case of an arbitrary semisimple Lie algebra.

As a direct consequence of (9.8), (9.9) one gets the Discrete transformation for two dimensional main chiral problem with moving poles,

$$(\xi - \bar{\xi})\frac{\partial^2}{\partial \xi \, \partial \bar{\xi}} f = \left[\frac{\partial}{\partial \xi} f, \frac{\partial}{\partial \bar{\xi}} f \right]. \tag{9.10}$$

In this case relations (9.8), (9.9) realizing the Discrete transformation are changed as follows:

$$s^{-1}\frac{\partial}{\partial\xi}s=\frac{1}{f_-}\left[\frac{\partial}{\partial\xi}f,X_M^+\right]-\xi\frac{\partial}{\partial\xi}\frac{1}{f_-}X_M^+,$$

$$s^{-1}\frac{\partial}{\partial\bar\xi}s=\frac{1}{f_-}\left[\frac{\partial}{\partial\bar\xi}f,X_M^+\right]-\bar\xi\frac{\partial}{\partial\bar\xi}\frac{1}{f_-}X_M^+ \tag{9.11}$$

and discrete transformation by itself takes the form:

$$\frac{\partial}{\partial\xi}F=s\left(\frac{\partial}{\partial\xi}f\right)s^{-1}-\xi\frac{\partial s}{\partial\xi}s^{-1},$$

$$\frac{\partial}{\partial\bar\xi}F=s\left(\frac{\partial}{\partial\bar\xi}f\right)s^{-1}-\bar\xi\frac{\partial s}{\partial\bar\xi}s^{-1}. \tag{9.12}$$

10. THE LIST OF DISCRETE TRANSFORMATIONS FOR INTEGRABLE SYSTEMS [20]

Now we present the discrete transformation and its integration (in some manner) for the most known and applicable integrable systems. Here, discrete transformation plays the role of some nonlinear mapping which transfers any given solution into another one. However, we do not investigate the properties of the transformation, its geometrical interpretation (if any), etc. At present, we have not used the general method for construction of the transformation in question. General properties of discrete transformation together with the system of equations which determine it will be considered in one of the next sections. To prove the validity of all formulae of this section, one can make a direct check which uses only one operation — differentiation.

As a hint for obtaining discrete transformation it is possible to use a purely algebraic method for the construction of the soliton type solutions, which is given in previous sections (6—8).

The starting point of the construction below uses the following two facts. The integrable systems under consideration admit the transformation s:

$$\theta\Rightarrow\tilde\theta\equiv s\tilde\theta=F(\theta,\theta_i^{(1)}\ldots\theta_i^{n}),\quad s^N\neq 1.$$

Here θ and $\tilde\theta$ are unknown functions (variables) satisfying the corresponding PDEs, $\theta_i^s\equiv\dfrac{\partial^s\theta}{\partial x_i^s}$.

There is an obvious solution of the nonlinear system in question which depends on a set of arbitrary functions. The soliton type solutions, reductions related with the discrete groups, solutions with definite boundary conditions are defined by a special choice of arbitrary functions mentioned above. Let us note that θ_0 is a solution of a linear system of partial differential equations and it can be presented as a parametric integral on the plane of the complex variable λ.

In the case of integrable systems this circumstance is just the main reason for applying the methods of the theory of functions of complex variables, the technique of the Riemann problem, and, at last, the methods of the inverse scattering problem. The results of this section reduce the inverse scattering method to a simple technical rule.

Here we give a list of integrable systems together with discrete transformations for them and corresponding solutions.

10.1. Hirota Equation

$$\dot{v} + \alpha(v''' - 6uvv') - i\beta(v'' - 2v^2u) + \gamma v' + i\delta v = 0,$$

$$\dot{u} + \alpha(u''' - 6uvu') + i\beta(u' - 2u^2v) + \gamma u' - i\delta u = 0;$$

$$\dot{} \equiv \frac{\partial}{\partial t}, \quad {}' \equiv \frac{\partial}{\partial x};$$

$$\tilde{v} \equiv sv = \frac{1}{u}, \quad \tilde{u} \equiv su = u(uv - (\ln u)''), \quad v_0 = 0,$$

$$\dot{u}_0 + \alpha u_0''' + i\beta u_0'' + \gamma u_0' - i\delta u_0 = 0. \tag{10.1}$$

In this and in the other cases the main role will be played by the principal minors of the following matrix:

$$\begin{pmatrix} \phi^s & \phi^{s+1} & \phi^{s+2} & \dots \\ \phi^{s+1} & \phi^{s+2} & \phi^{s+3} & \dots \\ \phi^{s+2} & \phi^{s+3} & \phi^{s+4} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

The principal minors of these matrices will be denoted by the symbol D_r^n. Here n is the rank of the matrix and r is the symbol of its element at the left upper corner. In the case of Hirota equation the solution of the discrete transformation acquires the form

$$v_n = (-1)^n \frac{D_0^{n-1}}{D_0^n}, \qquad u_n = (-1)^{n+1} \frac{D_0^{n+1}}{D_0^{n+1}}. \tag{10.2}$$

The methods of the theory of functions of complex variables lead to the same expression, with D_0^n given by the nonlocal integral

$$D_0^n = \int d\lambda_1 \dots d\lambda_n c(\lambda_1) \dots c(\lambda_n)\, W_n^2(\lambda_1, \dots, \lambda_n), \tag{10.3}$$

where $W_n(\lambda)$ is the Vandermonde determinant and $c(\lambda)$ is the integral in the representation for u_0.

10.2. Nonlinear Schrödinger equations. *10.2.1. Nonlinear Schrödinger equation*

$$\dot{q} + q'' - 2rq^2 = 0, \quad \tilde{q} = \frac{1}{r}, \quad \tilde{r} = r(rq - (\ln r)''];$$
$$-\dot{r} + r'' - 2qr^2 = 0, \quad q_0 = 0, \quad \dot{r}_0 = r_0''. \tag{10.4}$$

The solution of the discrete transformation is the same as in the last subsection.

10.2.2. Modified Nonlinear Schrödinger Equation

$$\dot{q} + q'' + 2(rq)\, q' = 0, \quad \tilde{q} = \frac{1}{r}, \quad \tilde{r} = r\left[(rq) + \left(\ln \frac{r}{r'} \right)' \right];$$
$$-\dot{r} + r'' - 2(rq)\, r' = 0, \quad q_0 = 0, \quad \dot{r}_0 = r_0''. \tag{10.5}$$

The solution of the discrete transformation is as follows

$$q_n = (-1)^n \frac{D_1^{n-1}}{D_0^n}, \qquad r_n = (-1)^{n+1} \frac{D_0^{n+1}}{D_1^{n+1}}. \tag{10.6}$$

10.2.3. Nonlinear Schrödinger Equation with Derivative

$$\dot{q} + q'' - 2(rq^2)' = 0, \quad \tilde{q} = r, \quad \tilde{r} = q - \left(\frac{1}{r} \right)';$$
$$-\dot{r} + r'' + 2(r^2 q)' = 0, \quad q_0 = 0, \quad \dot{r}_0 = r_0''. \tag{10.7}$$

The solution of the discrete transformation is as follows

$$q_{2n} = \frac{D_1^{n-1} D_1^n}{(D_0^n)^2}, \qquad r_{2n} = \frac{D_0^{n-1} D_0^n}{(D_1^n)^2},$$

$$q_{2n+1} = \frac{D_0^{n-1} D_0^n}{(D_0^n)^2}, \qquad r_{2n+1} = \frac{D_1^{n-1} D_1^n}{(D_0^{n+1})^2}. \tag{10.8}$$

10.2.4. Nonlinear Schrödinger Equation with Cubic Nonlinearity

$$\dot{q} + q'' - 2q^2(r' + r^2 q) = 0, \quad -\dot{r} + r'' + 2r^2(q' - q^2 r) = 0.$$

Discrete transformation in this case is a bit more complicated:

$$\tilde{q} = (r' + qr^2)^{-1}, \quad \tilde{r} = -(r' + qr^2)' + r^{-1}(r' + qr^2)^2.$$

As in the last cases $q_0 = 0$, $-\dot{r}_0 + r_0'' = 0$ and the solution of discrete transformation under this boundary conditions has the form:

$$q_n = \frac{D^1_{n-1}}{D^1_n}, \quad r_n = \frac{D^0_{n+1}}{D^0_n}, \quad D^\alpha_{-1} = 0, \quad D^\alpha_0 = 0.$$

10.3. One-Dimensional Heisenberg Ferromagnet in Classical Region (XXX-Model)

$$\dot{S} = [S, S''], \quad S = (S_-, S_0, S_+), \quad S_0^2 + S_-S_+ = 1;$$

$$\tilde{S}_- = S_- + 2\left(\frac{1}{\left(\frac{s_+}{1+s_0}\right)'}\right)', \quad \tilde{S}_+ = S_+ + 2\left(\frac{1}{\left(\frac{s_-}{1-s_0}\right)'}\right)',$$

$$\tilde{S}_0 + 1 = -\tilde{S}_- \frac{S_+}{1+S_0}, \quad S_-^0 = 0, \quad S_0^0 = 1, \quad \dot{S}_+ = 2S_+'',$$

$$S_-^n = \frac{D_2^{n-1} D_2^n}{(D_1^n)^2}, \quad S_0^n + 1 = 2\frac{D_0^n D_2^n}{(D_1^n)^2},$$

$$S_0^n - 1 = 2\frac{D_2^{n-1} D_0^{n+1}}{(D_1^n)^2}, \quad S_+^n = -4\frac{D_0^{n+1} D_0^n}{(D_1^n)^2}. \tag{10.9}$$

10.4. XYZ-Model in Classical Region. The Landau–Lifshits Equation

$$(\dot{\mathbf{S}}) = \mathbf{S} \times \mathbf{S}'' + \mathbf{S} \times (\mathbf{J}\mathbf{S}),$$

$$\mathbf{S} = (S_1, S_2, S_3), \quad (\mathbf{S})^2 = 1, \quad J = \mathrm{diag}\,(J_1, J_2, J_3).$$

Under the stereographic projection

$$u = \frac{S_1 + iS_2}{1 + S_3}, \quad v = \frac{S_1 - iS_2}{1 + S_3}$$

and change $-it \to t$ it becomes a system of the following equations:

$$\dot{u} + u'' - 2v\frac{(u')^2 + R(u)}{1 + uv} + \frac{1}{2}\frac{\partial}{\partial u}R(u) = 0,$$

$$-\dot{v} + v'' - 2u\frac{(v')^2 + R(v)}{1 + uv} + \frac{1}{2}\frac{\partial}{\partial v}R(v) = 0, \tag{10.10}$$

where $R(x) = \alpha x^4 + \gamma x^2 + \alpha$, $\frac{\partial R}{\partial x} = 4\alpha x^3 + 2\gamma x = 2\frac{R + \alpha(x^4 - 1)}{x}$, $\alpha = \frac{J_2 - J_1}{4}$, $\gamma = \frac{J_1 + J_2}{2} - J_3$. The system (10.10) is invariant under transformation $u \to U$, $v \to V$:

$$U = \frac{1}{v}, \quad \frac{1}{1+VU} - \frac{1}{1+uv} = \frac{vv'' - (v')^2 + \alpha(v^4 - 1)}{(v')^2 + R(v)}, \tag{10.11}$$

which is the discrete transformation for this system. The reader can find the corresponding solution in [40,41].

10.5. Lund–Pohlmeyer–Regge Model

$$\dot{y}' - 4y + 2(xy)\,\dot{y} = 0, \quad \tilde{x} = (y' + xy^2)^{-1},$$

$$\dot{x}' - 4x - 2(xy)\,\dot{x} = 0, \quad \tilde{y} = -(y' + xy^2)y' + \frac{(y' + xy^2)^2}{y};$$

$$x_0 = 0, \quad \dot{y}_0' = 4y_0. \tag{10.12}$$

$$x_n = (-1)^{n+1}\frac{D_1^{n-1}}{D_1^n}, \quad y_n = (-1)^n \frac{D_0^{n+1}}{D_0^n}. \tag{10.13}$$

It is interesting to note that discrete transformation of L–P–R system coincides with the nonlinear Schrödinger equation (10.2.4). And indeed these two systems belong to the same hierarchy [42].

10.6. The Main Chiral Field Problem in a Space of n Dimensions (the Case of Algebra A_1). The main chiral field problem in n-dimensional space is described by the following system of equations:

$$(\theta_i - \theta_j)\frac{\partial^2 f}{\partial x_i\,\partial x_j} = \left[\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}\right], \tag{10.14}$$

where the function f takes values in A_1 algebra, θ_i are numerical parameters.

The last system is invariant under transformation [10]

$$F_- = \frac{1}{f_+}, \quad \frac{\partial F_0}{\partial x_i} = (f_0 - F_0 + \theta_i)\frac{\partial}{\partial x_i}\ln f_+ - \frac{\partial f_0}{\partial x_i},$$

$$\frac{\partial F_+}{\partial x_i} = (f_0 - F_0 + \theta_i)^2\frac{\partial f_+}{\partial x_i} - 2f_+(f_0 - F_0 + \theta_i)\frac{\partial f_0}{\partial x_i} - f_+^2\frac{\partial f_-}{\partial x_i}. \tag{10.15}$$

The last equations can be rewritten in the matrix form

$$\frac{\partial F}{\partial x_i} = \exp\,[-X^+(f_0 - F_0 + \theta_i)f_+]\exp\,[H\ln f_+]r\frac{\partial f}{\partial x_i}r^{-1} =$$

$$= \exp\,[-H\ln f_+]\exp\,[X^+(f_0 - F_0 + \theta_i)f_+], \tag{10.16}$$

where r is an automorphism of the algebra A_1 with the properties

$$rX^{\pm}r^{-1} = -X^{\mp}, \quad rHr^{-1} = -H;$$

$$f_-^0 = 0, \quad f_0^0 = \tau, \quad f_+^0 = \alpha^0,$$

where

$$\frac{\partial^2\tau}{\partial x_i\,\partial x_j} = 0, \quad (\theta_i - \theta_j)\frac{\partial^2\alpha^0}{\partial x_i\,\partial x_j} = 2\left[\frac{\partial\tau}{\partial x_i}\frac{\partial\alpha^0}{\partial x_j} - \frac{\partial\tau}{\partial x_j}\frac{\partial\alpha^0}{\partial x_i}\right]. \qquad (10.17)$$

To solve the discrete transformation, let us consider the linear system of equations:

$$\theta_i\frac{\partial\alpha^l}{\partial x_i} - 2\frac{\partial\tau}{\partial x_i}\alpha^l = \frac{\partial\alpha^{l+1}}{\partial x_i}. \qquad (10.18)$$

From the last equations it follows that each function α^l is a solution of the equation for α^0. We have an explicit expression for α^s:

$$\tau = \sum \phi_i(x_i), \quad \alpha^s = \int d\lambda(\lambda)^s\, c(\lambda)\exp\left(\sum\frac{\phi_i(x_i)}{\lambda - \theta_i}\right). \qquad (10.19)$$

In terms of the α^l the discrete transformation has the solution

$$f_0^n = \frac{D_0^{n-1}}{D_0^n}, \quad f_0^n = \tau - \frac{\dot{D}_0^n}{D_0^n}, \quad f_+^n = \frac{D_0^{n+1}}{D_0^n}. \qquad (10.20)$$

The number of the indices of the last row in the determinant $\dot{D}_0^n$ is enlarged by one. The application of the results of this subsection to the problem of construction of multisoliton solutions of Sigma-chiral model can be found in [43].

10.7. The Main Chiral Field Problem for an Arbitrary Semisimple Lie Algebra. For a semisimple Lie algebra and for an element f being a solution of (12), the following statement takes place [10]: *There exists such an element S taking values in a gauge group that*

$$S^{-1}\frac{\partial S}{\partial x_i} = \frac{1}{f_-}\left[\frac{\partial f}{\partial x_i}, X_M^+\right] - \theta_i\frac{\partial}{\partial x_i}\frac{1}{f_-}X_M^+. \qquad (10.21)$$

Here X_M^+ is the element of the algebra corresponding to its maximal root divided by its norm, i.e.,

$$[X_M^+, X^-] = H, \qquad [H, X^{\pm}] = \pm 2X^{\pm},$$

— f_- is the coefficient function in the decomposition of f of the element corresponding to the minimal root of the algebra. In this terms the discrete transformation reads as

$$\frac{\partial F}{\partial x_i} = S \frac{\partial f}{\partial x_i} S^{-1} + \theta_i \frac{\partial S}{\partial x_i} S^{-1}. \qquad (10.22)$$

The system of equations in the case under consideration may be written in the form of equality between the group g and algebra f valued functions as

$$g_{x_i} g^{-1} = \theta_i f_{x_i}.$$

Discrete transformation for group valued function takes the form

$$G = Sg,$$

where the group element is determined by the above relations. The explicit expression of the group element g_n after n-th application of discrete transformation can be found in [7].

10.8. The System of Self-Dual Equations on Four-Dimensional Space (the Case of Algebra A_1). The self-dual equations for an element f with values in a semisimple Lie algebra have the following form:

$$\frac{\partial^2 f}{\partial y\, \partial \bar{y}} + \frac{\partial^2 f}{\partial z\, \partial \bar{z}} = \left[\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]. \qquad (10.23)$$

The discrete transformation for this system is [44],[45]

$$F_- = \frac{1}{f_-},$$

$$\frac{\partial}{\partial y} F_0 = \frac{\partial}{\partial \bar{z}} \ln f_- - \frac{\partial}{\partial y} f_0 + (f_0 - F_0) \frac{\partial}{\partial y} \ln f_-,$$

$$\frac{\partial}{\partial z} F_0 = - \frac{\partial}{\partial \bar{y}} \ln f_- - \frac{\partial}{\partial z} f_0 + (f_0 - F_0) \frac{\partial}{\partial z} \ln f_-,$$

$$\frac{\partial}{\partial y} F_+ = -f_- \left\{ (f_0 - F_0) \frac{\partial}{\partial y} (f_0 - F_0) + \frac{\partial}{\partial \bar{z}} (f_0 - F_0) \right\} - f_-^2 \frac{\partial}{\partial y} f_+,$$

$$\frac{\partial}{\partial z} F_+ = -f_- \left\{ (f_0 - F_0) \frac{\partial}{\partial z} (f_0 - F_0) - \frac{\partial}{\partial \bar{y}} (f_0 - F_0) \right\} - f_-^2 \frac{\partial}{\partial z} f_+. \qquad (10.24)$$

The substitution of (12.20) into the density of the topological charge yields

$$Q_F = g_f + \square \quad \square \quad \ln f_-.$$

For the interaction of the discrete transformation we have the linear system of equations

$$\frac{\partial \alpha^l}{\partial \bar{y}} + 2 \frac{\partial \tau}{\partial z} \alpha^l = \frac{\partial \alpha^{l+1}}{\partial z}, \qquad \frac{\partial \alpha^l}{\partial \bar{z}} - 2 \frac{\partial \tau}{\partial y} \alpha = - \frac{\partial \alpha^{l+1}}{\partial y}. \qquad (10.25)$$

In these terms the solution of the self-dual system is given by

$$f_-^n = \frac{D_0^{n-1}}{D_0^n}, \quad f_0^n = \frac{\dot{D}_0^n}{D_0^n} + \tau, \quad f_+^0 = \frac{D_0^{n+1}}{D_0^n}. \tag{10.26}$$

10.9. The System of Self-Dual Equations for an Arbitrary Semisimple Algebra. The following statement takes place [44,] [45]:

There exists such an element S taking the values in the gauge group, that

$$S^{-1}\frac{\partial S}{\partial y} = \frac{1}{f_-}\left[\frac{\partial f}{\partial y}, X_M^+\right] - \frac{\partial}{\partial \bar{z}}\left(\frac{1}{f_-}\right)X_M^+,$$

$$S^{-1}\frac{\partial S}{\partial z} = \frac{1}{f_-}\left[\frac{\partial f}{\partial z}, X_M^+\right] + \frac{\partial}{\partial \bar{y}}\left(\frac{1}{f_-}\right)X_M^+. \tag{10.27}$$

Here X_M^+ is the element of the algebra corresponding to its maximal root, divided by its norm, i.e.,

$$[X_M^+, X^-] = H, \qquad [H, X^{\pm}] = \pm 2X^{\pm},$$

— f_- is the coefficient function in the decomposition of f on the element corresponding to the minimal root of the algebra. The discrete transformation has the form

$$\frac{\partial F}{\partial y} = S\frac{\partial f}{\partial y}S^{-1} + \frac{\partial S}{\partial \bar{z}}S^{-1}, \qquad \frac{\partial F}{\partial z} = S\frac{\partial f}{\partial z}S^{-1} - \frac{\partial S}{\partial \bar{y}}S^{-1}. \tag{10.28}$$

10.10. The Main Chiral Field Problem with the Moving Poles. Many integrable systems arise from the equations (10.23) by imposing symmetry requirements on their solutions. The cylindrically symmetric condition in four dimensional space restricts the form of the function f,

$$f = \frac{1}{\bar{y}}f(\xi, \bar{\xi}), \qquad \xi = \frac{z-\bar{z}}{2} + \left[\left(\frac{z+\bar{z}}{2}\right)^2 + y\bar{y}\right]^{1/2}, \qquad \bar{\xi} = -\xi^*.$$

$$(\xi - \bar{\xi})\frac{\partial^2}{\partial\xi\,\partial\bar{\xi}}f = \left[\frac{\partial}{\partial\xi}f, \frac{\partial}{\partial\bar{\xi}}f\right]. \tag{10.29}$$

This is the equation for the main chiral field with moving poles.

The result of integration of equation (10.9) is given by

$$S = S(\xi, \bar{\xi})\exp X_M^+\frac{z}{f_-};$$

$$S^{-1}\frac{\partial}{\partial\xi}S = \frac{1}{f_-}\left[\frac{\partial}{\partial\xi}f, X_M^+\right] - \xi\frac{\partial}{\partial\xi}\frac{1}{f_-}X_M^+,$$

$$S^{-1}\frac{\partial}{\partial\bar\xi}S=\frac{1}{f_-}\left[\frac{\partial}{\partial\bar\xi}f,X_M^+\right]-\bar\xi\frac{\partial}{\partial\bar\xi}\frac{1}{f_-}X_M^+, \tag{10.30}$$

and the discrete transformation has the following form:

$$\frac{\partial}{\partial\xi}F=S\left(\frac{\partial}{\partial\xi}f\right)S^{-1}-\xi\frac{\partial S}{\partial\xi}S^{-1},$$

$$\frac{\partial}{\partial\bar\xi}F=S\left(\frac{\partial}{\partial\bar\xi}f\right)S^{-1}-\bar\xi\frac{\partial S}{\partial\bar\xi}S^{-1}. \tag{10.31}$$

The relations (10.10), (10.31) describe the discrete transformation for the main chiral field with moving poles [20].

10.11. The Self-Dual Equation under Condition of Cylindric Symmetry in Three-Dimensional Space. The condition of cylindrical symmetry in three-dimensional space leads to the following form of the solution:

$$f=\frac{1}{\bar y}f(\xi,\bar\xi),\qquad \xi=\frac{z+\bar z}{2}+i(y\bar y)^{1/2},\qquad \bar\xi=-\xi^*;$$

$$(\xi-\bar\xi)\frac{\partial^2 f}{\partial\xi\,\partial\bar\xi}=\frac{1}{2}\left(\frac{\partial f}{\partial\bar\xi}-\frac{\partial f}{\partial\xi}\right)+\left[\frac{\partial f}{\partial\bar\xi}\frac{\partial f}{\partial\xi}\right]. \tag{10.32}$$

The discrete transformation for equation (10.32) arising from (10.9), (10.28) has the form [20]

$$S^{-1}\frac{\partial}{\partial\xi}S=\frac{1}{f_-}\left[\frac{\partial}{\partial\xi}f,X_M^+\right]+\left(\frac{1}{f_-}-\frac{\xi-\bar\xi}{2}\frac{\partial}{\partial\xi}\frac{1}{f_-}\right)X_M^+,$$

$$S^{-1}\frac{\partial}{\partial\bar\xi}S=\frac{1}{f_-}\left[\frac{\partial}{\partial\bar\xi}f,X_M^+\right]+\left(\frac{1}{f_-}-\frac{\xi-\bar\xi}{2}\frac{\partial}{\partial\bar\xi}\frac{1}{f_-}\right)X_M^+,$$

$$\frac{\partial}{\partial\xi}F=S\left(\frac{\partial}{\partial\xi}f\right)S^{-1}+\frac{\bar\xi-\xi}{2}\frac{\partial S}{\partial\xi}S^{-1},$$

$$\frac{\partial}{\partial\bar\xi}F=S\left(\frac{\partial}{\partial\bar\xi}f\right)S^{-1}-\frac{\bar\xi-\xi}{2}\frac{\partial S}{\partial\bar\xi}S^{-1}.$$

The system (10.32) in the case of algebra A_1 arises in the integration problem of the general relativity with two commuting Killing vectors [13].

10.12. The Cylindric Symmetric Solution Invariant under Two Orthogonal Four-Dimensional Axes

$$x_1\frac{\partial^2 F}{\partial x_1^2}+x_2\frac{\partial^2 F}{\partial x_2^2}=\left[\frac{\partial F}{\partial x_1},\frac{\partial F}{\partial x_2}\right]. \tag{10.33}$$

The explicit form of the discrete transformation in the case of the algebra A_1 is as follows [20]

$$F_- = \frac{1}{f_-},$$

$$\frac{\partial F_0}{\partial x_1} = -1 + (f_0 - F_0)\frac{\partial \ln f_-}{\partial x_1} + x_2 \frac{\partial \ln f_-}{\partial x_2} - \frac{\partial f_0}{\partial x_1},$$

$$\frac{\partial F_0}{\partial x_2} = 1 + (f_0 - F_0)\frac{\partial \ln f_-}{\partial x_2} - x_1 \frac{\partial \ln f_-}{\partial x_1} - \frac{\partial f_0}{\partial x_2},$$

$$\frac{\partial F_+}{\partial x_1} = f_-\left[(f_0 - F_0)\frac{\partial (f_0 - F_0)}{\partial x_1} + x_2 \frac{\partial (f_0 - F_0)}{\partial x_2}\right] - f_-^2 \frac{\partial f_+}{\partial x_1},$$

$$\frac{\partial F_+}{\partial x_2} = -f_-\left[(f_0 - F_0)\frac{\partial (f_0 - F_0)}{\partial x_2} - x_1 \frac{\partial (f_0 - F_0)}{\partial x_1}\right] - f_-^2 \frac{\partial f_+}{\partial x_2}. \qquad (10.34)$$

The linear system of equations has the form

$$x_1 \frac{\partial \alpha^l}{\partial x_1} - \alpha^l + 2\alpha^l \frac{\partial \tau^l}{\partial x_2} = \frac{\partial \alpha^{l+1}}{\partial x_2},$$

$$x_2 \frac{\partial \alpha^l}{\partial x_2} - \alpha^l - 2\alpha^l \frac{\partial \tau^l}{\partial x_1} = -\frac{\partial \alpha^{l+1}}{\partial x_1},$$

$$\tau^{l+1} = \tau^l + \frac{x_1 - x_2}{2}. \qquad (10.35)$$

The solution of the discrete transformation coincides with the self-dual case (see (10.25) and (10.26)).

10.13. (2 + 1)-Matrix Davey–Stewartson System. The matrix Davey–Stewartson equation is the system of two equations for two unknown $s \times s$-matrix functions u, v:

$$u_t + au_{xx} + bu_{yy} - 2au \int dy(vu)_x - 2b \int dx(uv)_y u = 0,$$

$$-v_t + av_{xx} + bv_{yy} - 2a \int dy(vu)_x v - 2bv \int dx(uv)_y = 0, \qquad (10.36)$$

where a, b are arbitrary numerical parameters; x, y are the coordinates of two-dimensional space. As $s = 1$, the order of multipliers is not essential, and (1.1) is the usual Davey–Stewartson equation in its original form [15].

By direct but not very simple computations one finds that the system (10.36) is invariant under the following change of unknown functions,

$$\tilde{u} = \frac{1}{v}, \quad \tilde{v} = [vu - (v_x v^{-1})_y]\, v \equiv v[uv - (v^{-1} v_y)_x]. \tag{10.37}$$

The substitution (10.37) is the discrete transformation under which all equations of matrix Davey–Stewartson hierarchy are invariant [46]. In the case of one-dimensional space this substitution was mentioned in [47].

The substitution (10.37) may be also rewritten in the form of infinite chain

$$((v_n)_x\, v_n^{-1})_y = v_n v_{n-1}^{-1} - v_{n+1} v_n^{-1}, \qquad \left(u_{n+1} = \frac{1}{v_n} \right), \tag{10.38}$$

where (v_{n-1}, u_{n-1}) means the result of the n-th application of (10.37) to some given matrix functions (v_0, u_0).

Under the boundary condition $v_{-1}^{-1} = v_N = 0$ (the so-called matrix Toda chain with fixed ends) the general solution of (10.38) takes the form [16]

$$v_0 = \sum_{t=0}^{N} \phi_t(x)\, \bar{\phi}_t(y),$$

where $\phi_t(x)$, $\bar{\phi}_t(y)$ are arbitrary $s \times s$ matrix functions of the corresponding arguments.

In scalar case $s = 1$ the general solution of the Toda chain with fixed ends was found in [48] for all series of semisimple algebras except of E_7, E_8. In [49] this result was reproduced in terms of invariant root technique applicable to all semisimple series.

11. THE THEORY OF INTEGRABLE SYSTEMS FROM THE POINT OF VIEW OF REPRESENTATION THEORY OF THE DISCRETE GROUP OF INTEGRABLE MAPPINGS [50–53]

Here we want to consider the results of the previous section from some more general point of view. We will start from the short historical discussion. Our aim is to find place for including discrete transformation in usual, more traditional ways of investigation of the theory of integrable systems.

Liouville has introduced the term «integrability» for dymanical systems. He proved that if a dynamical system possesses a sufficiently large number of integrals of motion in involution then such a system is integrable. But neither general methods for the construction of a solution in explicit form nor any mention of the symmetry of the system under consideration are contained within Liouville's criterion.

In the case of Lie symmetries the theorem by E.Noether fills this gap. It teaches us that the number of conservation laws coincides with the dimensions of the Lie group and gives the possibility (in the case of Lagrange theory) of obtaining explicit expressions for the integrals of motion.

Roughly speaking the modern theory of integrable systems up to now has maintained the Liouville definition (the integrable system must contain an infinite number of integrals of motion in involution) and many people have found various consequences which follow from this fact.

The aim of this section is to show in a deductive way that the theory of integrable systems may be understood as a theory of the linear representation of the discrete group of integrable mappings. This does not mean that we can propose at this moment a theory of this connection which would be complete from mathematical point of view. We only demonstrate that all known results of the theory of integrable systems do not contradict this hypothesis.

11.1. Discrete Transformation of Integrable Systems. Let us consider a local invertible transformation described by the substitution

$$\tilde{u} = \phi(u, u', u'', \ldots, u^{(r)}) \equiv \phi(u), \tag{11.1}$$

where u is an s-dimensional vector function and u', u'',... are its derivatives of the corresponding order with respect to «space» coordinates (the dimension of the space may be arbitrary).

The property of invertibility means that equality (11.1) may be resolved and «odd» function u may be locally expressed in terms of new functions $\tilde{u}$ and their derivatives. The connection (11.1) with integrable systems is illustrated by examples of the previous section.

The Frechet derivative $\phi'(u)$ corresponding to the substitution (11.1) is the $s \times s$ matrix operator defined as

$$\phi'(u) = \phi_u + \phi_{u'} D + \phi_{u''} D^2 + \ldots \tag{11.2}$$

where D^m is an operator of m-th differentiation with respect to corresponding space coordinates u, u', u''... d. More detailed information about this construction can be found in [47], [50].

Let us consider the equation which appeared first in another notation in [51]

$$F_n(\phi(u)) = \phi'(u)\, F_n(u), \tag{11.3}$$

where $F_n(u)$ is an unknown s-component vector function, each component of which depends on u and its derivatives up to a maximal order n.

For each substitution the equation (11.3) possesses one obvious trivial solution $F_n(u) = u'$. To prove this it is sufficient to differentiate the equation (11.1) once with respect to one of its space coordinates.

If the equation (11.2) possesses some other solution (for a given $\phi(u)$) except of trivial one, then we will call such a substitution an integrable substitution or mapping.

We specially emphasize that equation (11.3) contains two unknown functions $\phi(u)$ and $F_n(u)$, and it possesses nontrivial solution for the function $F_n(u)$ only for a narrow class of integrable substitutions.

Each nontrivial solution of (11.2) generates the equation (system) of evolution type

$$u_t = F_n(u), \tag{11.4}$$

which is obviously invariant under the substitution $u \to \phi(u)$. (In this connection let us emphasize that the equation $u_t = u'$ is indeed invariant under an arbitrary substitution).

Let us now compare the equation (11.3) with the definition of a linear representation $T(g)$ of some group (for definiteness one can keep in mind a Lie group)

$$\Phi(gx) = T(g)\,\Phi(x), \tag{11.5}$$

where g is the group element, $T(g)$ is the group operator for some representation, $\Phi(x)$ is the basis of the corresponding representation space.

The obvious correspondence occurs whenever the definition relationship (11.5) is compared with equation (11.3):

$$\Phi(x) \to F_n(u), \qquad T(g) \to \phi'(u).$$

Using this correspondence, let us interpret equation (11.3) on the group theoretical level. We have some discrete group of transformations, the group element of which is exactly the substitution $u \to \phi(u)$; $\phi'(u)$ (the Frechet derivative) is a linear representation of the group element; and finally $F_n(u)$ (the equations of the hierarchy) is a basis in representation space. If this representation is irreducible (it is necessary to verify that by independent methods), then all possible bases of this representation (solutions of equation (11.3) with different n) must be connected by some operator $W_{n,n'}$

$$F_n(u) = W_{n,n'} F_{n'}. \tag{11.6}$$

Certainly the same situation occurs in the theory of (1 + 1) integrable systems. All equations of the same hierarchy are connected by the «raising» operators constructed from the skew symmetrical (nonlocal) Hamiltonian operators $J_n = -J_n^T$

$$W_{n,n'} = J_n J_{n'}^{-1}. \tag{11.7}$$

11.2. Equations for «Raising» and Hamiltonian Operators. Two equations will be important for further considerations:

$$\phi'(u)\, W(u)\, \phi'(u)^{-1} = W(\phi(u)), \qquad \phi'(u)\, J(u)\, \phi'(u)^T = J(\phi(u)), \qquad (11.8)$$

where $\phi'(u)^T = \phi^T_u - D\phi^T_{u'} + D^2\phi^T_{u''} - \ldots$, and $W(u)$, $J(u)$ are unknown $s \times s$ matrix operators, with entries in the form of polynomials of some finite order in operator of differentiation D (of both positive and negative degrees).

From (11.8) and (11.3) it follows immediately that if $F_n(u)$ is some solution of the principal equation (11.3) then $W^p(u)\, F_n(u)$ (p is an arbitrary natural number) will be another solution of the same equation.

The solution of the second equation (11.8) under the additional restriction of skew symmetry may be connected (interpreted) as a Poisson structure which is invariant under the transformation of discrete symmetry. Skew symmetrical operators $J(u)$ are known as Hamiltonian ones. Two different solutions of the second equation (11.8) (if it is possible to find them), say $J_1(u)$ and $J_2(u)$ in the combination $J_1 J_2^{-1}$ obey the first equation (11.8). The operator $J_1 J_2^{-1} J_1(u)$ is again the solution of the second equation (11.8) and so on. This is the way in which Hamiltonian operators usually arise in the theory of integrable systems. It is necessary to find two different Poisson structures from some independent assumptions. All other objects may be constructed by the above scheme.

In the problem of the construction of Hamiltonian operators for integrable systems, the equations (11.8) were used first in [52].

11.3. Conditions under Which the Evolution Equation May Be Rewritten in Hamiltonian Form. Let us consider some scalar function $H(u)$ locally dependent on u and its derivatives and obeying the equality (equation)

$$H(\phi(u)) - H(u) = Ker \in \frac{\delta}{\partial u}\,. \qquad (11.9)$$

In other words, the difference between the function after one application of the discrete transformation and its original value is a divergence with respect to space coordinates. Let us compare the variational derivatives $H(u)$ before and after the discrete transformation. We have

$$\frac{\delta H(u)}{\delta u} = \phi'^T(u)\, \frac{\delta H(\phi(u))}{\delta \phi(u)}\,. \qquad (11.10)$$

The last equality is a direct corollary of (11.9) and the obvious fact that the variational derivative of divergence vanishes identically.

Let $J(u)$ be any solution of (11.8). Consequently we have

$$\phi'(u)\, J(u)\, \frac{\delta H(u)}{\delta u} = \phi'(u)\, J(u)\, \phi'^T(u)\, \frac{\delta H\,(\phi(u))}{\delta \phi(u)} = J(\phi(u))\, \frac{\delta H(\phi(u))}{\delta \phi(u)}\,. \qquad (11.11)$$

So we see that function $F(u) = J(u)\,\frac{\delta H(u)}{\delta u}$ is just a solution of our main equation (11.3) and corresponding evolution equation (11.4) takes a Hamiltonian form. Compare with [5].

11.4. Conservation Laws. All known integrable substitutions in one-dimensional space ((1 + 1) integrable systems) obey all conditions of the previous section. This means that it is possible to find an infinite number of Hamiltonian functions $H_n(u)$ and an infinite number of Hamiltonian operators $J_n(u)$ in explicit form. And so in the (1 + 1) dimensional case all integrable systems of evolutional type (11.4) may be written in Hamiltonian form. As a consequence, it is possible to determine the Poisson brackets between two local functions by the rule

$$\{N(u), M(u)\} \equiv \left(\frac{\delta N(u)}{\delta u}\, J(u)\, \frac{\delta M(u)}{\delta u} \right) \qquad (11.12)$$

and to prove with the use of some technical manipulations that all conserved integrals are in involution

$$\{H_n(u), H_{n'}(u)\} = Ker \in \frac{\delta}{\delta u}\,. \qquad (11.13)$$

This result is interpreted usually as fulfilling of the Liouville criterion of integrability.

In the case of (1 + 2)-dimensional integrable systems it is impossible to write down the systems investigated in Hamiltonian form (except for some trivial cases). But whenever in the (1 + 2)-dimensional case the functions obeying equation (11.9) can be found, they are in general nonlocal, the number of them is infinite and they are invariant under time evolution in the sense:

$$(H_n^0(u))_t = \sum_s (H_n^s(u))_{x_s},$$

where x_s are independent space coordinates of the problem.

It is true that we can present infinite number of concrete examples of validity of the last propositions but also it is true that at this moment we have no idea how to prove it on group-theoretical level in general case.

11.5. The General Hypothesis. As a conclusion of the previous consideration we are able to formulate the following general hypothesis about the structure of a future theory of integrable systems:

The problem of integrable systems is equivalent to the theory of representations of the discrete group of integrable mappings.

Indeed if from independent considerations it turns out to be possible to obtain a solution of our main equation (11.3), then we automatically produce an integrable equation of evolution type (11.4) and each space of irreducible representation of (11.5) will give us the exact solution of it. We are well aware that the form of our main equation (11.3) is not very suitable for obtaining direct conclusions from it. In this connection we can notice by analogy with the distance between the original definition of semisimple algebras (in the sense of absence of nontrivial ideals) and the Cartan classification into A,B,C,D,E,F,G and E, it may be of comparable magnitude to the problem of declassification of the solutions of our main equation.

We hope that something of this kind will be achieved in the case of representation theory of discrete groups of integrable mappings.

11.6. Conclusion. The main result of the present section is contained in the new equation (11.3). Its solution will provide the answers to two most important questions of the theory of integrable systems. The first question can be regarded as the «quantization» of substitutions, i.e., substitutions the choice of which would be integrable in the above sense among the infinite number of invertible ones. Except for the obvious remark that this will depend essentially upon the dimensions of the spaces involved, the author knows almost nothing about how to solve this problem and concludes that it is not going to be resolved quickly.

The second, more tractable problem from our point of view is the question of the solution of the main equation (11.3) for a given (ad hoc) integrable substitution $\phi(u)$ (in this connection see the next section). It is possible to suppose that the solution to this problem is closely connected with the theory of representations of the discrete group of integrable mappings. From known examples of integrable systems it follows that the discrete group of integrable mapping possesses a rich storage of different irreducible representations. A definite class of exact solutions of corresponding integrable system may be connected with each of these representations. In some sence the soliton-like solutions (which will be discussed below) correspond to finite-dimensional representations of such groups.

12. TWO-DIMENSIONAL INTEGRABLE MAPPINGS AND EXPLICIT FORM OF EQUATIONS OF (1 + 2)-DIMENSIONAL HIERARCHIES OF INTEGRABLE SYSTEMS [54–57]

Here we shall complete the second part of the general programme of the last section: we shall find the explicit form of solution of our main equation (11.3) for an ad hoc given integrable mapping. The equations of (1 + 2) integrable systems belonging to Darboux–Toda, Heisenberg and Lotka–Volterra hierarchies which are invariant with respect to discrete transformations of corresponding integrable mappings will be presented in an explicit form.

12.1. Two-Dimensional Integrable Mappings. Below we will discuss three concrete examples of two-dimensional integrable mappings which can be considered by similar methods.

12.1.1. Darboux–Toda Substitutions. The explicit form of the direct and inverse *D–T* integrable substitution is as follows,

$$\overleftarrow{u} = \frac{1}{v}, \quad \overleftarrow{v} = v(uv - (\ln v)_{xy}),$$
$$\overrightarrow{v} = \frac{1}{u}, \quad \overrightarrow{u} = u(vu - (\ln u)_{xy}). \tag{12.1}$$

Let $\overset{\leftarrow s}{f}$ and $\overset{s \rightarrow}{f}$ be the results of the s-th application of direct and inverse transformations to the function $f(u, v)$, with the following agreement $\overset{\leftarrow(-m)}{f} \equiv$ $\equiv \overset{m\rightarrow}{f}$, $m \geq 0$.

As a direct corollary of (12.1) one finds the Toda-like recurrence relation for function $T_0 = uv$. It will be of importance for our further considerations.

$$(\ln T_0)_{xy} = -\overleftarrow{T_0} + 2T_0 - \overrightarrow{T_0}. \tag{12.2}$$

The corresponding to (12.1) Frechet derivative has the form

$$\phi'(u) = \begin{pmatrix} 0 & -\dfrac{1}{v^2} \\ v^2 & 2(uv) - \dfrac{v_x v_y}{v^2} + \dfrac{v_x}{v} D_y + \dfrac{v_y}{v} D_x - D_{xy} \end{pmatrix}, \tag{12.3}$$

where $D_y \equiv \dfrac{\partial}{\partial y}$, $D_x \equiv \dfrac{\partial}{\partial x}$.

The system (11.3) in the concrete case *D–T* of substitution may be rewritten as

$$\overleftarrow{F_1} = -\frac{1}{v^2}, F_2,$$
$$\overleftarrow{F_2} = v^2 F_1 + \left(2(uv) - \frac{v_x v_y}{v^2} + \frac{v_x}{v} D_y + \frac{v_y}{v} D_x - D_{xy} \right) F_2. \tag{12.4}$$

It is not difficult to verify by direct computations that $F_0 = (u, -v)$ is the solution of the last equation and so substitution (12.1) is integrable in the sense of [51].

After introduction of the new functions $F_1 = uf_1$, $F_2 = vf_2$, the system (12.4) takes the form of a single equation for only one unknown function f_2

$$(\overleftarrow{uv})\,(\overleftarrow{f_2} - f_2) - (uv)\,(f_2 - \overrightarrow{f_2}) = -D_{xy} f_2, \quad f_1 = -\overrightarrow{f_2}. \tag{12.5}$$

The meaning of notations in the last equation is explained after formula (12.1).

In further transformations of (12.5) we will use the fact that condition of invariance of some function with respect to the discrete transformation $\overleftarrow{F} = F$ is equivalent to the $F \equiv \text{const}$. This is in some sense analogous to Liouville theorem in the theory of analytic functions. Using this fact for function $T\,(f_2 = \int dy\,(\overleftarrow{T} - T\,))$ we obtain the Toda chain-like equation:

$$-T_x = T_0 \int dy\,(\overleftarrow{T} - 2T + \overrightarrow{T}), \qquad T_0 = uv. \tag{12.6}$$

In terms of solution of (12.6) the evolution type equation (11.4) (invariant with respect to D–T substitution (12.1)) takes the form:

$$v_t = v \int dy\,(\overleftarrow{T} - T), \qquad u_t = u \int dy\,(\overrightarrow{T} - T). \tag{12.7}$$

12.1.2. Two-Dimensional Heisenberg Substitution. Under this term we will mean the direct and inverse transformations of two functions (u, v) of the form:

$$\overleftarrow{u} = v^{-1}, \qquad \frac{1}{1 + \overleftarrow{uv}} = \frac{1}{1 + uv} + \frac{\phi_{xy}}{\phi_x\,\phi_y}, \qquad \phi = \ln v,$$

$$\overrightarrow{v} = u^{-1}, \qquad \frac{1}{1 + \overrightarrow{uv}} = \frac{1}{1 + uv} + \frac{\psi_{xy}}{\psi_x\,\psi_y}, \qquad \psi = \ln u. \tag{12.8}$$

One can check that functions $t_m \left(t_1 = \frac{v_y v_x}{(1 + uv)^2} = -\frac{(\overrightarrow{v})_y u_x}{(\overrightarrow{v} + v)^2},\ t_2 = \frac{v_y u_x}{(1 + uv)^2} = -\frac{(\overrightarrow{v})_x v_y}{(\overrightarrow{v} + v)^2} \right)$ obey the Toda-like recurrence relations

$$(t_m)_x = t_m \int dy \Delta_m, \qquad (m = 1, 2), \tag{12.9}$$

where $\Delta_m = \overleftarrow{t_m} - 2t_m + \overrightarrow{t_m}$.

The explicit form of the Frechet derivative operator reads:

$$\phi'(u) = \begin{pmatrix} 0 & -v^{-2} \\ \left(\frac{v \overleftarrow{R}^{1}}{R} \right)^2 & -\left(1 + \left(\frac{vu \overleftarrow{R}^{1}}{R} \right)^2 + (\overleftarrow{R}^{1})^2\, \delta \left(\phi_x^{-1} D_x + \phi_y^{-1} D_y - \frac{v}{v_{xy}} D_{xy} \right) \right) \end{pmatrix},$$

$$\delta = \frac{v v_{xy}}{v_x v_y}, \qquad R = 1 + uv, \qquad \overleftarrow{R} = 1 + \overleftarrow{u}v. \tag{12.10}$$

By a short calculation it is possible to show that equation (11.3) possesses nontrivial solution $F_1 = u$, $F_2 = -v$ and so the Heisenberg substitution by definition is integrable.

Now we can rewrite equation (11.3) in more transparent form. Let us denote $F_1 = uB$, $F_2 = vA$. From the first equation (11.3) we obtain immediately $B = -\overrightarrow{A}$. The second equation after some transformations may be rewritten in the form of a single equation for function A:

$$\left(\frac{\overleftarrow{uv}}{(1+uv)^2}\right)(\overleftarrow{A} - A) - \frac{uv}{(1+uv)^2}(A - \overrightarrow{A}) =$$

$$= (\phi_x \phi_y)^{-1}\left(\frac{\phi_{xy}}{\phi_x} A_x + \frac{\phi_{xy}}{\phi_y} A_y - A_{xy}\right). \tag{12.11}$$

As we know, the main equation (11.3) always possesses the trivial solution $F_1 = u_x, (u_y)$; $F_2 = v_x, (v_y)$ or $A = \phi_x, (\phi_y)$. Let us look for solution of (12.10) in the form $A = \phi_x \alpha$. Instead of (12.10) we obtain the equation for α:

$$\left(\frac{\overleftarrow{u_x v_x}}{(1+uv)^2}\right)(\overleftarrow{\alpha} - \alpha) - \frac{u_x v_x}{(1+uv)^2}(\alpha - \overset{1\rightarrow}{\alpha}) = \left(\frac{\alpha_y}{\theta}\right)_x, \qquad \theta = \frac{\phi_y}{\phi_x}. \tag{12.12}$$

Resolving (12.12) by the substitution

$$\left(\frac{\alpha_y}{\theta}\right)_x = \overleftarrow{T} - T,$$

we arrive at the equation for determining T:

$$T_x = T_0 \int dy\, [\theta(\overleftarrow{T} - T) - \overrightarrow{\theta}(T - \overrightarrow{T})], \tag{12.13}$$

where

$$T_0 = \frac{u_x v_x}{(1+uv)^2}.$$

12.1.3. Lotka–Volterra Substitution. In this case direct and inverse transformations have the form

$$\overleftarrow{u} = u + (\ln v)_x, \qquad \overleftarrow{v} = v + (\ln \overleftarrow{u})_y,$$

$$\overrightarrow{u} = u - (\ln \overrightarrow{v})_x, \qquad \overrightarrow{v} = v - (\ln u)_y. \tag{12.14}$$

As in the previous case, the functions $t_1 = uv$, $t_2 = \overleftarrow{u}v$ obey the Toda-like recurrence relations (12.9).

The Frechet operator in this case has the form:

$$\phi'(u) = \begin{pmatrix} 1 & D_x v^{-1} \\ D_y(\overleftarrow{u})^{-1} & 1 + D_y(\overleftarrow{u})^{-1} D_x v^{-1} \end{pmatrix}. \qquad (12.15)$$

By the same technique as in the previous subsections we obtain the single equation for unknown function T and expressions for the equations of hierarchy via this solution

$$T_y = v \int dx \, [\overleftarrow{u}^{-1}(\overleftarrow{T} - T) + u(T - \overrightarrow{T})], \qquad (12.16)$$

and finally

$$u_t = u(T - \overrightarrow{T}), \qquad v_t = D_y T.$$

12.2. Solution of the Main Equation. In spite of essential differences in the form of the Frechet operators in three above cases, the main equations of the problems (12.6), (12.14) and (12.16) have the same structure and may be solved by the similar methods. We shall demonstrate these methods on the more complicated example of Heisenberg substitution and represent the results of calculations for other cases.

First of all let us note that equation (12.14) has the partial solution

$$T = T_0$$

in what one can be convinced with the help of equality below, which is the direct corollary of (12.8) and (12.9)

$$\overleftarrow{T}_0 - T_0 = 2\phi_x \left(\frac{1}{1+uv} \right)_x + 2\phi_{xy} \frac{\phi_x}{\phi_y} \frac{1}{1+uv} + \phi_x \left(\frac{\phi_{xy}}{\phi_x \phi_y} \right)_x - \phi_{xy} \frac{\phi_x}{\phi_y} + \frac{\phi_{xy}^2}{\phi_y^2}.$$

Let us now seek the solution of (12.14) as $T = T_0 \int dy \, \alpha_0$. Instead of (12.14) we obtain the equation for determining the function α_0

$$(\alpha_0)_x + \alpha_0 \int dy \, [\overleftarrow{t}_1 - t_1 + \overrightarrow{t}_2 - t_2] = \overleftarrow{t}_1 \int dy \, (\overleftarrow{\alpha}_0 - \alpha_0) + \overrightarrow{t}_2 \int dy \, (\overrightarrow{\alpha}_0 - \alpha_0). \quad (12.17)$$

As it will be shown later, this equation will arise many times. Two possible ways of its further evolution will be important. Let us use the following ansatz

$$\alpha_0 = \overleftarrow{t}_1 \alpha_1 + \overrightarrow{t}_2 \beta_1.$$

After substitution of this expression into (12.17) and equating to zero coefficients in front of the terms $\overleftarrow{t}_1, \overrightarrow{t}_2$ (this is some additional assumption), we come to equations for unknown functions α_1, β_1:

$$(\alpha_1)_x + \alpha_1 \int dy\,[\overset{\leftarrow 2}{t}_1 - \overleftarrow{t}_1 + \overrightarrow{t}_2 - t_2] = \int dy\,(\overleftarrow{\alpha}_0 - \alpha_0),$$

$$(\beta_1)_x + \beta_1 \int dy\,[\overleftarrow{t}_1 - t_1 + \overset{2\rightarrow}{t}_2 - \overrightarrow{t}_2] = \int dy\,(\overleftarrow{\alpha}_0 - \alpha_0). \qquad (12.18)$$

Setting the second equation (12.18) by direct transformation and adding the result to the first one, we get

$$(\alpha_1 + \overleftarrow{\beta}_1)_x + (\alpha_1 + \overleftarrow{\beta}_1) \int dy\,[\overset{\leftarrow 2}{t}_1 - \overleftarrow{t}_1 + \overrightarrow{t}_2 - t_2] = 0.$$

It means that the system (12.18) has partial solution $\alpha_1 + \overleftarrow{\beta}_1 = 0$. We will use it in what follows.

For this solution the system (12.18) is equivalent to single equation for unknown function α_1:

$$(\alpha_1)_x + \alpha_1 \int dy\,[\overset{\leftarrow 2}{t}_1 - \overleftarrow{t}_1 + \overrightarrow{t}_2 - t_2] = \int dy\,[(\overset{\leftarrow 2}{t}_1 \overleftarrow{\alpha}_1 - t_2\alpha_1) - (\overset{\leftarrow 1}{t}_1 \alpha_1 - \overrightarrow{t}_2 \overrightarrow{\alpha}_1)].$$

It has obvious solution $\alpha_1 = 1$. As a corollary we obtain the second partial solution of our main equation:

$$T_1 = T_0 \int dy\,(\overleftarrow{t}_1 - \overrightarrow{t}_2).$$

Further evolution of equation for α_1 is connected with representation of unknown function in integral form $\alpha_1 \to \int dy\alpha_1$ (we keep the same symbol for unknown function because it can't lead to misunderstanding),

$$(\alpha_1)_x + \alpha_1 \int dy\,[\overset{\leftarrow 2}{t}_1 - \overleftarrow{t}_1 + \overrightarrow{t}_2 - t_2] =$$

$$= \overset{\leftarrow 2}{t}_1 \int dy\,(\overleftarrow{\alpha}_1 - \alpha_1) + \overrightarrow{t}_2 \int dy\,(\overrightarrow{\alpha}_1 - \alpha_1). \qquad (12.19)$$

Up to obvious replacement $\overleftarrow{t}_1 \to \overset{\leftarrow 2}{t}_1$ it coincides with the equation (12.17) for α_0.

We can repeat the same trick with this equation as with equation for α_0 and after k steps will come to substitution

$$\alpha_k = \overset{\leftarrow(k+1)}{t_1} \alpha_{k+1} - \overset{\to\to}{t_2} \alpha_{k+1}$$

and equation for α_{k+1}

$$(\alpha_{k+1})_x + \alpha_{k+1} \int dy [\overset{\leftarrow k+2}{t_1} - \overset{\leftarrow k+1}{t_1} + \overset{\to}{t_2} - t_2] =$$

$$= \int dy \, [(\overset{\leftarrow k+2\leftarrow}{t_1} \alpha_{k+1} - t_2 \alpha_1) - (\overset{\leftarrow k+1}{t_1} \alpha_1 - \overset{\to\to}{t_2 \alpha_1})]$$

with obvious solution $\alpha_{k+1} = 1$.

Collecting all results together we obtain partial solution of the main equation in the following formula

$$T_n = T_0 \prod_{i=1}^{n} (1 - L_i \exp [-(i+1) d_i - \sum_{k=i+1}^{n} d_k]) \int dy \overset{\leftarrow 1}{t_1} \int dy \overset{\leftarrow 2}{t_1} \ldots \int dy \overset{\leftarrow n}{t_1}, \tag{12.20}$$

where symbol $\exp d_s$ means the shift of the agreement of s-repeated integral $(\ldots \int dy \overset{h\to}{t_1} \ldots \to \ldots \int dy \overset{h+1\to}{t_1} \ldots)$ in (12.13) by 1 and symbol L_p means exchange $\overset{r\to}{t_1}$ on the $\overset{r\to}{t_2}$ in the p-repeated integral $\ldots \int dy \overset{r\to}{t_1} \ldots \to \ldots \int dy t_2 r \ldots$.

The expression (3.20) is directly applicable to Heisenberg and Lotka–Volterra integrable hierarchies. In the case of D–T hierarchy it is necessary to put all operators $L_i = 1$ and keep in mind the equality $t_1 = t_2 = T_0$.

12.3. Examples. In this subsection we present the simplest integrable systems for usual unknown functions u, v corresponding to the lowest solutions T_n of the main equation for D–T, Heisenberg and L–V substitutions.

12.3.1. Darboux–Toda Substitution

$n = 0$

$$T_0 = uv, \quad u_t = au_x + bu_y, \quad v_t = av_x + bv_y.$$

In examples below we shall choose $a = 1$, $b = 0$ keeping in mind that it is always possible to add the term (with arbitrary numerical coefficient) in which x is changed by y and vice versa.

$n = 1$

$$T_1 = vu_x - v_x u,$$

$$u_t = u_{xx} - u \int dy (uv)_x, \quad -v_t = v_{xx} - v \int dy (uv)_x.$$

This is the Davey–Stewartson equation in its original form.

$n=2$

$$T_2 = (uv)_{xx} - 3u_x v_x - 3uv \int dy(uv)_x,$$

$$u_t = u_{xxx} - 3u_x \int dy(uv)_x - 3u \int dy(u_x v)_x,$$

$$v_t = v_{xxx} - 3v_x \int dy(uv)_x - 3v \int dy(v_x u)_x.$$

This is the Veselov–Novikov equation.

$n=3$

$$T_3 = -(T_1)_{xx} - 2(u_x v_{xx} - v_x u_{xx}) + 2uv \int dy(T_1)_x + 4T_1 \int dy(uv)_x,$$

$$v_t = -v_{xxxx} + 4v_{xx} \int dy(uv)_x - 2v_x (\int dy(T_1)_x - 2 \int dy(uv)_{xx}) +$$

$$+ 2v(\int dy(uv)_{xxx} - \int dy(u_x v_x)_x + \int (uv_{xx})_x - ([\int dy(uv)]^2)_{xx} - [\int dy(uv)_x]^2).$$

Equation for u may be obtained from the equation for u under the change $u \to v$, $v \to u$, $t \to -t$.

12.3.2. Heisenberg Substitution

$n=0$

$$v_t = -v_{xx} + 2v_x \int dy \left(\frac{uv_y}{1+uv} \right)_x, \qquad -u_t = -u_{xx} + 2u_x \int dy \left(\frac{vu_y}{1+uv} \right)_x.$$

$n=1$

$$v_t + v_{xxx} - 3v_{xx} \int dy \left(\frac{uv_y}{1+uv} \right)_x + 3v_x \left[\int dy \left(\frac{uv_y}{1+uv} \right)_x \right]^2 +$$

$$+ 3v_x \int dy \left(\frac{u_x v_y}{(1+uv)^2} \right)_x - 3v_x \int dy \left(\frac{uv_y}{1+uv} \right)_{xx},$$

$$u_t + u_{xxx} - 3u_{xx} \int dy \left(\frac{vu_y}{1+uv} \right)_x + 3u_x \left[\int dy \left(\frac{vu_y}{1+uv} \right)_x \right]^2 +$$

$$+ 3u_x \int dy \left(\frac{v_x u_y}{(1+uv)^2} \right)_x - 3u_x \int dy \left(\frac{uv_y}{1+uv} \right)_{xx}.$$

12.3.3. Lotka–Volterra Substitution

$n=0$

In the case $T_0 = v$ we obtain the trivial system with the help of (12.2)

$$u_t = u_y, \quad v_t = v_y.$$

$n=1$

In this case

$$S_1 = v \int dx\,(\overleftarrow{t_1} - \overrightarrow{t_2}) = v_y + v^2 + 2v \int dx(u_y).$$

The corresponding integrable system has the form

$$u_t = -u_{yy} + 2(uv)_y + 2u_y \int dx(u_y), \qquad v_t = (v^2 + v_y + 2v \int dx(u_y))_y.$$

In one-dimensional case $D_x = D_y$ this system is a partial case of more general integrable system described in [25].

$n=2$

In this case

$$S_2 = v^3 + 3vv_y - v_{yy} + 3vD_x^{-1}(uv)_y + 3(v_y + v^2)\, D_x^{-1}(u_y) + 3v(D_x^{-1}(u_y))^2.$$

The corresponding integrable system can be written as

$$u_t = D_y(u_{yy} - 3(vu_y) + 3v^2u - 3(u_y - uv)\, D_x^{-1}(u)_y) +$$

$$+ D_x(3D_x^{-1}(u_y)\, D_x^{-1}(uv)_y + (D_x^{-1}(u_y))^3),$$

$$v_t = D_y(v^3 + 3vv_y + v_{yy} + 3vD_x^{-1}(uv)_y + 3(v_y + v^2)D_x^{-1}(u_y) + 3v(D_x^{-1}(u_y))^2).$$

13. FORMALISM OF SCALAR L-A PAIR APPLIED TO PERIODIC TODA LATTICES [11,12,35]

Now we consider concrete realizations of the general results of section 4 and apply them to the case of the system of equations of periodic Toda lattice related to classical A_n-series. The case of algebra A_1 (the sin-Gordon equation) has been considered in detail in the former paragraph.

We use the following formulation of the equations of the generalized Toda lattice in two-dimensional space:

$$\text{(a)}\ \frac{\partial^2 x_i}{\partial z\,\partial \bar{z}} = \exp\,(\tilde{K}x)_i, \quad \text{(b)}\ \frac{\partial^2 \rho_\alpha}{\partial z\,\partial \bar{z}} = \exp\,\delta_\alpha - t_\alpha W_\alpha^{-1} \exp - \sum \delta_\nu t_\nu,$$

$$\delta_\alpha = (k\rho)_\alpha, \tag{13.1}$$

where in the case (a) the index i takes the values 1, 2,...$(r + 1)$; $(r + 1)$ is the rank of the simple infinite-dimensional algebra of finite growth with the generalized Cartan matrix $\tilde{K}$. In the case (b) the number of values that index α takes is less than in (a); k is the Cartan matrix (corresponding to $\tilde{K}$) of the finite-dimensional semisimple algebra and t_ν are the coefficients of the expansion of the maximal root of algebra over the set of its simple roots. The system (a) admits the transformation $x_i \to x_i + (\Theta(z) + \overline{\Theta}(\bar{z}))\, N_i$, where $\vec{N}$ is the null vector of the generalized Cartan matrix: $(\tilde{K}N) = 0$. Equation (b) is equivalent to (a) after excluding the trivial solution of the homogeneous Laplace equation with the help of conformal transformation. Equation (b) is a direct consequence of the Lax representation

$$A_z = (h\rho_z) + \lambda\Big(\sum_{\alpha=1}^{r} X_\alpha^+ + X_M^-\Big),$$

$$A_{\bar{z}} = \lambda^{-1}\Big(\sum_{\alpha=1}^{r} \exp - \delta_\alpha X_\alpha^- + \exp\,(M\rho)\, X_M^+\Big),$$

$$[\partial_z - A_z, \partial_{\bar{z}} - A_{\bar{z}}] = 0, \qquad (13.2)$$

where $X_\alpha^\pm$ and $X_M^\pm$ are the root vectors of the simple and maximal roots of the algebra; $(M\rho) \equiv \sum t_\nu \delta_\nu$ and λ is the spectral parameter. For $X_M^\pm$ we take the normalization:

$$[X_M^+, X_M^-] = \sum t_\nu W_\nu^{-1} h_\nu, \qquad Wk = (kW)^T.$$

The algebra, whose local part consists of the subspaces

$$g_{-1} = (\lambda^{-1}X_\alpha^-, \lambda^{-1}X_M^+), \qquad g_1 = (\lambda X_\alpha^+, \lambda X_M^-), \qquad g_0 = (h_\alpha)$$

is an infinite-dimensional semisimple algebra of finite growth. The degree of the parameter λ distinguishes the identical elements of the finite-dimensional algebra, relating them to the subspaces with whose grading index they are compared. Thus they eliminate the degeneracy of the representations of algebras realized by finite-dimensional matrices. The Cartan elements of the finite-dimensional algebra appear in the subspaces whose grading index is the product of some integral number on the height of the maximal root $m = \sum t_\nu$ increased by 1. The only one element that is not distinguished by degree of λ is the element H of the null subspace $H = [X_M^+, X_M^-] = \sum t_\nu W_\nu^{-1} h_\nu$. This circumstance explains why the number of unknown functions in (b) is less than in (a). Thus the operators of the L–A pair (13.2) should be treated as operator-valued functions of the finite-dimensional representation of an infinite-dimensional algebra of the finite growth. The spectral parameter λ plays here the role of a grading parameter. The algebra of internal symmetry of the equations of the generalized Toda lattice is infinite-dimensional and coincides with solvable part of such infinite-dimensional semi-simple algebra.

Equations (13.2) mean a «gradientness» of the L–A pair operators:

$$g_z g^{-1} = (h\rho_z) - \sum_{\alpha=1}^{r} X_\alpha^+ + \lambda^{n+1} X_M^-,$$

$$g_{\bar z} g^{-1} = \sum_{\alpha=1}^{r} \exp - \delta_\alpha X_\alpha^- + \lambda^{-n-1} \exp(M\rho)\, X_M^+ \qquad (13.3)$$

((13.3) differs from (13.2) by the gauge transformation $g \to \exp \frac{1}{2} \ln H$, which results in the change $\lambda \to \lambda - 1$ at the generators of the simple roots; g is an element of the complex hull of the group, spanned over the elements of the semisimple algebra. Equation (13.3) may be considered as a system of equations for the parameters of the element g. This system is naturally invariant under the choice of a concrete representation of the algebraic elements $X_\alpha^\pm$ in (13.3). Parametrize the element g by the Gauss decomposition $g = Z^- \exp(h\tau) Z^+$. Then (13.3) has as its consequence the essentially nonlinear system of equations relating the parameters of the elements Z^+, Z^-, τ. It seems quite remarkable that the equations for the parameters τ split from the general system, remaining essentially nonlinear. However, they split, in their turn, to a linear differential equation for the functions $\Psi_l = \exp \sum \tau_\alpha l_\alpha$, which are equal to the matrix elements of the group element g between the highest states of the representation $(l_1, l_2, \ldots, l_r)$. However, in order to obtain the scalar L–A pair equations of the given representation, one does not need to write the complete system of equations for the parameters of element g and then single out the linear system for Ψ_l from it. It would be enough to calculate the derivatives of the Ψ_l up to the order $N_l - 1$ using (13.3), and then to express them in terms of linear combinations of the matrix elements $\langle \alpha \| g \| l \rangle$ as follows (see section 4):

$$\dot\Psi_l = \langle l \| \dot g \| l \rangle = \langle l \| [(h\rho_z) - \sum_{\alpha=1}^{r} X_\alpha^+ + \lambda^{n+1} X_M^-] \| l \rangle =$$

$$= (l\rho_z)\, \Psi_l + \sum f_\alpha^1 \langle \alpha \| g \| l \rangle.$$

Analogously, for an s-order derivative, we obtain

$$\Psi_l^{[.s]} = \sum_{\alpha=1}^{N_l} f_\alpha^s \langle \alpha \| g \| l \rangle.$$

Inverting the latter equation we arrive at equality

$$\langle \alpha \| g \| l \rangle = \sum_0^{N_l - 1} F_s^\alpha \Psi_l^{[.s]} = \sum_0^{N_l - 1} (\bar{F})_t^\alpha \Psi_l^{['t]}.$$

The matrix elements $\langle \alpha \| g \| l \rangle$ may be calculated with the help of (13.3) in two forms, i.e., they may be expressed as through the derivatives with respect to the argument z or with respect to the argument $\bar{z}$. As a result, one obtains two forms of the matrix element $\langle \alpha \| g \| l \rangle$, which are equations $N_l - 1$ of the scalar L–A pair in the representation l. Two missing equations appear if someone excludes the N_l matrix elements $\langle \alpha \| g \| l \rangle$ from $N_l + 1$ linear relations connecting them with the derivatives Ψ_l up to the order N_l with respect to both arguments. These are two spectral equations of the representation l. In the general case, the structure of the spectral equations is as follows:

$$\Theta_{N_l}(D) \Psi_l = \lambda^{m+1} \Theta_{N_l - m - 1}(D) \Psi_l,$$

where $\Theta_n(D)$ denotes the differential operator of the n-th order, whose coefficient functions are homogeneous (with respect to the differentiation) polynomials in ρ_α (13.3). As $N_l = m + 1$, the right-hand side of the spectral equation does not contain the differentiation at all. Such a situation occurs only in the case of the simplest representations (of the lowest dimensions) of algebras A_k, C_k, $(AB)_k$. For the classical series B_k and D_k the degrees of the differential operator in the right-hand side of the spectral equation are one and two, respectively.

14. SOLUTION OF sin-GORDON EQUATION IN THE FORM INVARIANT UNDER THE CHOICE OF THE REPRESENTATION [10]

Now, on the example of the sin-Gordon equation we demonstrate the method of constructing the solutions without using a concrete realization of the algebra. This is the partial case of the general construction of the previous section.

The sin-Gordon equation results from the compatibility of the linear system (the Lax pair)

$$\dot{g}g^{-1} = h\dot{\rho} + \lambda(X^+ + X^-), \qquad g'g^{-1} = \lambda^{-1}(X^+ \exp 2\rho + X^- \exp - 2\rho)$$

with g being the element of the $SL(2, C)$ group, X^+, X^-, h-elements of its algebra $[X^+, X^-] = h$, $[h, X^{+-}] = 2X^{\pm}$. The internal symmetry algebra of the sin-Gordon equation is connected with the graded algebra of finite growth $SL(2, C) \times Z_2 = \tilde{A}_1$, that has the background elements $X^{\pm}_{1,2}$, $h_{1,2}$ from which the whole algebra is constructed. The commutation relations are

$$[X^+_\alpha, X^-_\beta] = \delta_{\alpha,\beta}\, h_\alpha, \qquad [h_\alpha, X^{+-}_\beta] = k_{\alpha\beta}\, X^{+-}_\beta,$$
$$\alpha, \beta = 1, 2, \qquad k = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

By direct check we confirm that the algebra $\tilde{A}_1$ has a multidimensional representation with the spectral parameter and generators:

$$X^+_1 = \lambda X^+, \quad X^+_2 = \lambda X^-, \quad X^-_1 = \lambda^{-1}X^-, \quad X^-_2 = \lambda^{-1}X^+, \quad h_1 = - h_2 = h,$$

where $X^{\pm}$, h are the elements of the algebra $SL(2, R)$ introduced above. Here the algebra as a whole consists of the sets of elements:

$$R^- = (\lambda^{-2s-1}X^{\pm}, \lambda^{-2s}h), \qquad R^0 = (h),$$
$$R^+ = (\lambda^{2s+1}X^{\pm}, \lambda^{2s}h), \qquad s > 0,$$

i.e., the positive, negative and zero subalgebras of the initial algebra. Going back to the L–A pair representation, we see that the element g may be considered as belonging not to the group $SL(2, C)$ but to some degenerate representation of the $\tilde{A}_1$ group. The arbitrary element of the $SL(2, C)$ group can be represented as the Gauss decomposition $g = \exp X^+\alpha \exp h\tau \exp X^-\beta$. Consequently, from the Lax representation, there appears a system of equations connecting the functions α, β, τ, which is obviously invariant under the choice of a certain representation of the algebra $SL(2, C)$. Thus we have

$$\dot{g}g^{-1} = h\dot{\rho} + \lambda(X^+ + X^-) =$$
$$= (\dot{\alpha} - 2\alpha\dot{\tau} - \alpha^2\dot{\beta} \exp - 2\tau)X^+ + (\dot{\tau} + \alpha\dot{\beta} \exp - 2\tau)h + \dot{\beta} \exp - 2\tau X^-,$$

whereof

$$\dot{\beta} \exp - 2\tau = \lambda, \qquad \alpha = \frac{\dot{\rho} - \dot{\tau}}{\lambda}, \qquad - \ddot{\tau} + (\dot{\tau})^2 = \lambda^2 - \ddot{\rho} + (\dot{\rho})^2.$$

Similarly,

$$\beta' \exp - 2\tau = \lambda^{-1} \exp 2\rho, \quad \alpha = - \tau' \exp - 2\rho, \quad - \ddot{\tau} + 2\dot{\tau}\dot{\rho} + (\dot{\rho})^2 = \lambda^{-2}.$$

As a result of the last equalities we have a system of three equations

$$\ddot{\Psi} = (\lambda^2 - \ddot{\rho} + (\dot{\rho})^2)\,\Psi, \qquad (\Psi \exp \rho)'' = (\lambda^{-2} - \rho'' + (\rho')^2)\,\Psi,$$
$$\Psi' \exp - 2\rho = \lambda^{-2}(\dot{\Psi} + \dot{\rho}\Psi), \tag{14.1}$$

where $\Psi = \exp - \tau$, $\alpha \exp - \tau$. Now we search for a solution of this system in the form

$$\Psi = \exp(\lambda z + \lambda^{-1}\bar{z}) \prod_{k=1}^{n} (\lambda - a_k),$$

where a_k are some unknown functions, that are defined from the system of equations, which arose after substitution Ψ into previous equations and comparing the terms at the same powers of λ

$$\ddot{a}_i + 2a_i\dot{a}_i + 2\sum \frac{\dot{a}_i\,\dot{a}_k}{a_k - a_i} = 0, \qquad - \ddot{\rho} + (\dot{\rho})^2 = 2\sum \dot{a}_k. \tag{14.2}$$

The last equality may be satisfied by the substitution $\exp - \rho = \dfrac{\prod a_k}{\prod \lambda_k}$ (see recurrent equalities, which follow from the system (7.3)). System (14.2) contains n first integrals, which give expressions for the first derivatives a_k:

$$\dot{a}_k = \frac{P_n(a_k^2)}{\prod_{l \neq k} (a_k^2 - a_l^2)}, \qquad P_n(a_k^2) = \prod_{l=1}^{n} (a_k^2 - a_l^2), \qquad \dot{x}_l = 0.$$

The set of parameters x_l independent of z, in fact, represents the first integrals of system (14.2). Calculating $\ddot{a}_k$ from the latter expression for $\dot{a}_k$, we confirm that (14.2) is valid. Further integration of the system is connected with the following identity from the theory of the symmetric function of n arguments. Namely

$$\Theta^s(x_j) = \sum_{i=1}^{n} \frac{x_i^s}{\prod_{i \neq j} (x_i - x_j)} = 0,$$

if $1 < s < n - 2$ and $\Theta^{n-1}(x_j) = 1$. In fact, being reduced to a common denominator, $\Theta^s(x_j)$ represents a ratio of two homogeneous symmetric functions. Here the denominator is the Vandermond determinant, which is antisymmetric under permutation of all its arguments. Hence, the numerator should be characterized by the same property, which is possible only when s is not less than $n-1$. Due to this, we may fulfill the further integration with the result

$$\sum_{k=1}^{n} \int \frac{da_k}{a_k^2 - x_s^2} = z + \frac{y_s}{x_s}, \qquad \prod_{k=1}^{n} \frac{a_k - x_s}{a_k + x_s} = \exp 2(x_s z + y_s).$$

Thus, the last n relations mainly determine a general solution of system (14.2) depending exactly on $2n$ parameters. By now only the first equation of the system (14.1) has been integrated, and to obtain an explicit dependence of the parameters x_s, y_s on argument $\bar{z}$, one needs to use the two remaining equations. As far as $\exp - \rho = \prod a_k / \prod \lambda_k$, the function $\Psi \exp \rho$ has the form:

$$\Psi \exp \rho = \lambda^n \exp(\lambda z + \lambda^{-1}\bar{z}) \prod_{k=1}^{n} (\lambda^{-1} - a_k^{-1}),$$

i.e., with respect to the argument $\bar{z}$, the function $\Psi \exp \rho$ has the same structure as Ψ with respect to z with evident substitution $a_k \to a_k^{-1}$, $\lambda \to \lambda^{-1}$. Due to this, the second equation of (14.1) entails the system of equations with respect to differentiation over the argument $\bar{z}$

$$(a_k^{-1})' = \frac{\tilde{P}_n(a_k^{-2})}{\prod_{l \neq k} (a_k^{-2} - a_l^{-2})}, \qquad \tilde{P}_n(a_k^{-2}) = \prod_{l=1}^{n} (a_k^{-2} - a_l^{-2}), \qquad (x_l)' = 0,$$

or

$$a_k' = (-1)^n a_k^{2n} \frac{\tilde{P}_n(a_k^{-2})}{\prod_{l \neq k} (a_k^2 - a_l^2)} \prod_{l \neq k} a_l^2 .$$

We have not used yet the third equation of system (14.1). After simple transformations it can be written as

$$\dot{a}_k = \exp 2\rho a_k^2 a_k', \qquad \exp 2\rho(1 + \sum a_k') = 1, \qquad \dot{\rho} = \exp 2\rho \sum a_k' a_k,$$

which in their turn, after corresponding substitution, lead to

$$P_n(a_k^2) = (-1)^n a_k^{2n} \tilde{P}_n(a_k^{-2}) \exp 2\rho \prod a_l^2 .$$

This means that $\dot{a}_k$ and a_k' are defined by the same polynomial, whose roots x_l^2 depend neither on z nor on $\bar{z}$, i.e.,

$$\tilde{x}_l = x_l, \qquad \exp - \rho = \frac{\prod a_l}{\prod x_l}.$$

It is easy to see that the system of equations is invariant under the substitution $z \to \bar{z}$, $a_k \to a_k^{-1}$, $x_k \to x_k^{-1}$. Then, from our previous results we find

$$\prod_{k=1}^{n} \frac{a_k - x_s}{a_k + x_s} = \exp 2(x_s z + x_s^{-1}\bar{z} + y_s) \equiv \exp 2z_s,$$

where parameters x_k, y_k are independent both of z and of $\bar{z}$. From the last equality we obtain the linear algebraic system for homogeneous symmetric functions $s_r = \sum\limits_{i \neq j \neq \ldots \neq k} a_i a_j \ldots a_k$ in the form

$$\sinh z_l s_n - x_k \cosh z_k s_{n-1} + x_k^2 \sinh z_k s_{n-2} - \ldots, = 0, \quad s = 1, 2 \ldots n.$$

Solution of the latter system with respect to $s_n = \prod a_l$ yields the well-known n-soliton solution of sin-Gordon equation in the form of the ratio of two determinants of n-th order.

15. GENERALIZED BARGMANN POTENTIALS [13]

In this section we establish a condition for the ordinary differential equation of the $(k + 1)$-th order

$$\Psi^{[k+1]} + \sum_{i=0}^{k-1} u_i \Psi^{[i]} = \lambda^{k+1}\Psi \tag{15.1}$$

under which it has a solution with the following analytic dependence on λ:

$$\Psi = \exp \lambda z \prod_{k=1}^{n} (a_k - \lambda).$$

The problem of this type, applied to quantum-mechanical one-dimensional Schrödinger equation, was first considered by Bargmann. For this reason, the coefficient functions of the last equation u_i will be called generalized Bargmann potentials. To solve Bargmann problem, we need the expression for the coefficient functions of an ordinary differential equation through the full set of its linearly independent solutions. The following statement generalizing the Wiett and Gauss theorems for the case of polynomials takes place. The equation

$$\Psi^{[.k+1]} + \sum_{i=0}^{k-1} u_i \Psi^{[.i]} = 0$$

may be represented in the form:

$$V_k^{-1}(V_k^2 V_{k-1}^{-1} (V_k^{-1} V_{k-1}^2 V_{k-2}^{-1}(\ldots(V_1^2 V_2^{-1}(V_1^{-1}\Psi)^{\cdot})^{\cdot}\ldots)^{\cdot} = 0,$$

where V_i are the principal minors of the matrix of the Wronskian $V_\alpha^\beta = \Psi_\beta^{[.\alpha-1]}$, $(1 < \alpha, \beta < k+1)$, and generate the full set of $k+1$ linearly independent solutions of the equation. This is the Frobenius theorem. The condition that Wronskian is a constant $V_{k+1} = 1$ is solved as follows:

$$\Psi_1 = \varphi_1, \quad \Psi_1 = \varphi_1 \int^z dz_1 \varphi_2,$$

$$\Psi_s = \varphi_1 \int^z dz_1 \varphi_2 \int^{z_1} dz_2 \varphi_3 \ldots \int^{z_s - 1} dz_{s-1} \varphi_s, \tag{15.2}$$

where the functions φ_l obey the only condition $\prod_{l=1}^{k+1} \varphi_l^{k+2-l} = V_{k+1}$.

1. The set of $(k+1)$ functions Ψ_l manifestly obeys the equation

$$(\varphi_{k+1}^{-1} (\varphi_k^{-1}(\ldots(\varphi_2^{-1}(\varphi_1^{-1}\Psi)^{\cdot})^{\cdot})\ldots)^{\cdot} = 0.$$

All that remains is to express φ_l through Ψ_k. As a consequence of the definition of the matrix V and Ψ we find

$$V_s = \prod_{l=1}^{s} \varphi_l^{s-l+1}, \quad \varphi_1 = V_1, \quad \varphi_2 = V_1^{-2} V_2, \ldots \; \varphi_{l+1} = V_{l-1} V_l^{-2} V_{l+1}.$$

The substitution of the expressions obtained for φ into the previous equation completes the proof of the theorem.

The problem concerning the generalized Bargmann potentials is solved according to the following theorem.

The solution of equation (15.1) has an analytic dependence on the parameter λ of the form $\Psi = \exp \lambda z \prod_{c=1}^{n} (a_c - \lambda)$, if the functions a_c are defined by the condition of vanishing of the function

$$\tilde{\Psi} = \sum_{\alpha=1}^{k+1} c(\lambda_\alpha) \exp \lambda_\alpha z \prod_{c=1}^{n} (a_c - \lambda_\alpha), \quad \lambda_\alpha^{k+1} = \lambda^{k+1}$$

at n different points of the λ^{k+1} plane $\lambda_b^{k+1}(1 < b < n)$. The generalized Bargmann potentials u_i are expressed through symmetric combinations constructed from a_c and their derivatives via quantities $B_s^{\,i}$, which are defined from the expressions for derivative of the s-th order of the function Ψ

$$\Psi^{[s]} = \left(\lambda^s + \sum_{i=0}^{s-2} B_s^{\,i} \lambda^i + \sum_{c=1}^{n} \frac{A_c^{\,s}}{a_c - \lambda} \right) \Psi$$

as follows:

$$u_i = -\tilde{B}_{k+1}^{\,i} \equiv -(B_{k+1}^{\,i} - \sum_{l=0}^{i-2} B_{k+1}^{\,l} \tilde{B}_l^{\,i}),$$

$$B_j^{\,i} = 0, \quad j < i + 2. \tag{15.3}$$

Equation (15.1) may be represented in the form

$$\exp \delta_k (\exp \delta_{k-1} (\ldots (\exp \delta_1 (\exp - \rho_1 \Psi)^{\cdot})^{\cdot} \ldots)^{\cdot} = \lambda^{k+1} \exp \rho_k \Psi, \tag{15.4}$$

where $\exp \rho_s = J_{s-1}^{\,0} \prod_{c=1}^{n} a_c^s$ and $J_b^{\,0}$ are the principal minors of the matrix of the conserved integrals and $\delta_s = -\rho_{s-1} + 2\rho_s - \rho_{s+1}$, $\rho_0 = \rho_{k+1} = 0$,

$$J_{i,j} = \delta_{i,j} + B_i^{\,j} + \sum_{c=1}^{n} \frac{A_c^{\,i} a_c^{k-j}}{a_c^{k+1} - \lambda^{k+1}} \tag{15.5}$$

under the null value of the parameter λ.

After calculating the logarithmic derivative of Ψ, we obtain

$$\dot{\Psi} = \left(\lambda + \sum_{c=1}^{n} \frac{\dot{a}_c}{a_c - \lambda} \right) \Psi \equiv \varphi^1 \Psi.$$

For the s-th derivative we have by induction

$$\Psi^{[s]} = \varphi^s \Psi, \qquad \varphi^{s+1} = \dot{\varphi}^s + \varphi^s \varphi^1.$$

With the help of the last equalities we find the recurrence relations for $A_c^{\,s}$ and $B_i^{\,s}$.

Substituting the proposed form of Ψ into general equation (15.1) and equating the quantities at different powers of λ, we obtain the expressions for the Bargmann potentials according to the conditions of the theorem.

From the condition of vanishing of the residues in the poles at the points $\lambda = a_c$ we obtain the nonlinear system of differential equations for the functions a_c:

$$\tilde{A}_c^{k+1} \equiv A_c^{k+1} - \sum_{i=1}^{k-1} \tilde{B}_{k+1} A_c^i = A_c^{k+1} - \sum_{i=1}^{k-1} B_{k+1}^i \tilde{A}_c^i = 0. \qquad (15.6)$$

Let us show that a_c, as defined by the conditions of the theorem, obey the relations (15.6). For this purpose, consider the Wronskian constructed by the functions Ψ_α. In the notations of the previous sections, we get

$$V_{k+1} = \| \Psi, \dot{\Psi}, \ldots \Psi^{[k]} \| = \prod_{c=1}^{n} (a_c^{k+1} - \lambda^{k+1}),$$

$$\| 1, \lambda + \sum_{c=1}^{n} \frac{A_c^s}{a_c - \lambda} \ldots\ldots \lambda^k + \sum_{i=0}^{k-2} B_k^i \lambda^i + \sum_{c=1}^{n} \frac{A_c^k}{a_c - \lambda} \| =$$

$$= \prod_{c=1}^{n} (a_c^{k+1} - \lambda^{k+1})\, W(\lambda_1, \lambda_1, \ldots \lambda_{k+1}) \det_k J, \qquad (15.7)$$

where W is Vandermond determinant. The calculations in (15.7) were performed by the standard procedure, i.e., subtracting the first column from the remaining one and removing the factor $\prod_{\alpha=1}^{k+1} (\lambda_\alpha - \lambda_1)$, etc. It follows from (15.7) that, up to W_{k+1}, V_{k+1} is a polynomial of the n-th order of the argument λ^{k+1} that vanishes due to the linear dependence of Ψ_α, in accordance with the conditions of the theorem, at n points λ_b^{k+1}, i.e.,

$$V_{k+1} = W_{k+1} \prod_{b=1}^{n} (\lambda_b^{k+1} - \lambda^{k+1}).$$

Consequently, $\dot{V}_{k+1} = \| \Psi, \dot{\Psi}, \ldots, \Psi^{[k-1]} \Psi^{[k+1]} \| = 0$. Calculating the latter determinant in the same way as (15.7), we verify that it has (15.4) as its consequence. To prove (15.5), we make use of the fact that both the Bargmann potentials u_i and the equations (15.6) for A_c^i do not depend on the parameter λ. According to the Frobenius theorem, we have

$$\varphi_1 = \Psi_1(\lambda = 0) = \prod_{c=1}^{n} a_c, \quad \varphi_1^2 \varphi_2 = \lim_{\lambda \to 0} (\lambda_2 - \lambda_1)^{-1} \| \Psi, \dot{\Psi} \| =$$

$$= \lim_{\lambda \to 0} (\lambda_2 - \lambda_1) \parallel 1, \lambda + \sum_{c=1}^{n} \frac{A_c^1}{a_c - \lambda} \parallel \prod_{c=1}^{n} (a_c - \lambda_1)(a_c - \lambda_2) =$$

$$= \lim_{\lambda \to 0} \prod_{c=1}^{n} (a_c - \lambda_1)(a_c - \lambda_2) \left(1 + \sum_{c=1}^{n} \frac{A_c^1}{(a_c - \lambda_1)(a_c - \lambda_2)} \right) =$$

$$= \prod_{c=1}^{n} a_c^2 \left(1 + \sum_{c=1}^{n} \frac{A_c^1}{a_c^2} \right) = \prod_{c=1}^{n} a_c^2 J_1^0.$$

Continuing the reduction procedure, we find

$$\prod_{\alpha=1}^{s} \varphi_\alpha^{s-\alpha+1} = \lim_{\lambda \to 0} W_s^{-1}(\parallel \Psi, \dot{\Psi}, \dots \Psi^{[s]} \parallel) = \prod_{c=1}^{n} a_c^s J_{s-1}^0.$$

From here we see that all the statements of the theorem are fulfilled and the other form of the Bargmann potentials may be found after performing differentiation in equation (15.4).

16. SOLUTION OF PERIODIC TODA LATTICE FOR A_k-SERIES [11,12,35]

In this section a system of equations is constructed for a scalar L-A pair of the first fundamental representation ($k + 1$-th dimension) of the algebra A_k. With the help of the results of the previous section, its «wave function» and the solutions of the periodic Toda lattice are obtained.

The highest vector of the first fundamental representation $\parallel l \rangle (\langle l \parallel)$ obeys the conditions

$$X_\alpha^+ \parallel l \rangle = 0, \quad \langle l \parallel X_\alpha^- = 0 h_\alpha \parallel l \rangle = \delta_{\alpha,1}, \quad \langle l \parallel h_\alpha = \delta_{\alpha,1}.$$

The set of its basic vectors is as follows,

$$\parallel l \rangle, \quad X_1^- \parallel l \rangle, \quad X_2^- X_1^- \parallel l \rangle, \quad X_k^- \dots X_2^- X_1^- \parallel l \rangle,$$

$$\langle l \parallel, \quad \langle l \parallel X_1^+, \quad \langle l \parallel X_1^+ X_2^+, \quad \langle l \parallel X_1^+ X_2^+, \dots X_k^+.$$

We introduce the wave function $\langle l \parallel g \parallel l \rangle$ and, using (13.3) calculate its derivatives, with respect to z

$$\dot{\Psi} = \langle l \parallel \dot{g} \parallel l \rangle = \langle l \parallel (h\dot{\rho} + \sum_{\alpha=1}^{k} X_\alpha^+ + \lambda^{k+1} X_M^-) g \parallel l \rangle =$$

$$= \dot{\rho}_1 \Psi + \langle l \parallel X_1^+ g \parallel l \rangle,$$

or

$$\exp \rho_1 (\exp - \rho_1 \Psi) = \langle l \| X_1^+ g \| l \rangle.$$

Next,

$$(\exp \rho_1 (\exp - \rho_1 \Psi)) = \langle l \| X_1^+ g \| l \rangle =$$

$$= \langle l \| X_1^+ (h\dot{\rho}) g \| 1 \rangle + \langle l \| X_1^+ X_2^+ g \| l \rangle$$

has its consequence

$$\exp \rho_2 - \rho_1 (\exp \delta_1 (\exp - \rho_1 \Psi)) = \langle l \| X_1^+ X_2^+ g \| l \rangle.$$

Continuing the reduction procedure up to the s-th step, we obtain

$$\exp \rho_s - \rho_{s-1} (\exp \delta_{s-1} (\exp \delta_{s-2} \ldots (\exp \delta_1 (\exp - \rho_1 \Psi)^{\cdot})^{\cdot} \ldots)^{\cdot})^{\cdot}) =$$

$$= \langle l \| X_1^+ X_2^+ \ldots X_s^+ g \| l \rangle. \tag{16.1}$$

Finally the $(k + 1)$-th step

$$\langle 1 \| X_1^+ X_2^+ \ldots X_k^+ (h\dot{\rho} + \lambda^{k+1} X_M^-)\, g \| l \rangle =$$

$$= - \dot{\rho}_k \langle 1 \| X_1^+ X_2^+ \ldots X_k^+ g \| l \rangle + \lambda^{k+1} \Psi$$

leads to the spectral equation

$$(\exp \delta_k (\exp \delta_{k-1} \ldots (\exp \delta_1 (\exp - \rho_1 \Psi)^{\cdot})^{\cdot} \ldots)^{\cdot})^{\cdot} = \lambda^{k+1} \exp \rho_k \Psi.$$

Quite similarly, by using the differentiation with respect to $\bar{z}$ we obtain

$$\exp \delta_{s+1} (\exp \delta_{s+2} \ldots (\exp \delta_k (\exp - (\rho_1 + \rho_k)\, \Psi)')' \ldots)' =$$

$$= \lambda^{-(k+1)} \langle 1 \| X_1^+ X_2^+, \ldots X_s^+ g \| l \rangle. \tag{16.2}$$

Excluding the matrix elements of the element g from the (16.1) and (16.2), we obtain

$$\exp \rho_s - \rho_{s-1} (\exp \delta_{s-1} (\exp \delta_{s-2} \ldots (\exp \delta_1 (\exp - \rho_1 \Psi)^{\cdot})^{\cdot} \ldots)^{\cdot})^{\cdot} =$$

$$= \lambda^{k+1} \exp \delta_{s+1} (\exp \delta_{s+2} \ldots (\exp \delta_k (\exp - (\rho_1 + \rho_k)\, \Psi)')' \ldots)',$$

$$(\exp \delta_k (\exp \delta_{k-1} \ldots (\exp \delta_1 (\exp - \rho_1 \Psi)^{\cdot})^{\cdot} \ldots)^{\cdot})^{\cdot} = \lambda^{k+1} \exp \rho_k \Psi,$$

$$(\exp \delta_2 (\exp \delta_3 \ldots (\exp \delta_k (\exp - (\rho_1 + \rho_k)\, \Psi)')' \ldots)')' =$$

$$= \lambda^{-(k+1)} \exp - \delta_1 \Psi. \tag{16.3}$$

The system of $(k + 2)$ equations (16.3) is, in fact, a scalar L–A pair of the first (vector) fundamental representation of the algebra A_k. System (16.3) is invariant under the substitution

$$z \to \bar{z},\ \Psi \to \exp - \rho_1 \Psi,\ \lambda \to \lambda_{-1},\ \rho_1 \to - \rho_1,\ \rho_{k+2-s} \to \rho_s - \rho_1$$

$(1 < s < k + 1,\ \rho_{k+1} = 0)$, i.e., under the Weyl reflection of the first simple root of the algebra A_k.

We shall look for the wave function of the system in the «soliton» form

$$\Psi = \exp (\lambda z + \lambda^{-1} \bar{z}) \prod_{c=1}^{n} (a_c - \lambda)\, \lambda_\alpha^{k+1}.$$

Excluding $\Psi^{[.k]}$ from the Wronskian $\| \Psi, \dot{\Psi},.. \Psi^{[.k]} \|$, using the equation connecting $\Psi, \dot{\Psi},\ldots \Psi^{[.k]}$ and Ψ' (the equation with $s = k$ in (16.3)), we arrive at the equality

$$\| \Psi, \dot{\Psi},.. \Psi^{[.k]} \| = \lambda^{k+1} \exp - (\rho_1 + \rho_k) \| \Psi, \dot{\Psi},.. \dot{\Psi}^{[.k-1]}, \Psi^{[,]} \| .$$

Continuing the procedure of the further exclusion of the derivatives with the help of (16.3), we get the following chain of equations for the determinants

$$\| \Psi, \dot{\Psi},.. \Psi^{[.k]} \| = \lambda^{k+1} \exp - (\rho_1 + \rho_k) \| \Psi, \dot{\Psi},.. \Psi^{[.k-1]}, \Psi^{[']} \| =$$
$$= \lambda^{2(k+1)} \exp - (2\rho_1 + \rho_{k-1}) \| \Psi, \dot{\Psi},.. \Psi^{[.k-2]}, \Psi^{['']}, \Psi^{[']} \| = \ldots$$
$$= \lambda^{s(k+1)} \exp - (s\rho_1 + \rho_{k+1-s}) \| \Psi, \dot{\Psi},.. \Psi^{[.k-s]}, \Psi^{['s]} \ldots \Psi' \| =$$
$$= \lambda^{k(k+1)} \exp - (k + 1) \rho_1 \| \Psi, \Psi^{['k]} \ldots \Psi' \| ,$$
$$\lambda^{k(k+1)} (-1)^{k(k+1)} \| \bar{\Psi}, \bar{\Psi}' \ldots \bar{\Psi}^{['k]} \| , \tag{16.4}$$

where $\bar{\Psi} \equiv \exp (- \rho_1 \Psi)$. The chain of equations (16.4), completed with two spectral equations, is completely equivalent to the system of equations of the scalar L–A pair (16.3). It follows from the explicit form of the spectral equation with respect to the argument z that the first term in the equality chain (16.4) does not depend on z; the last one does not depend on $\bar{z}$, and, therefore, each term of the chain is equal to some constant.

As for the equations of the scalar L–A pair in form (16.3), the following theorem, which generalizes the results of the previous section in the natural way, is valid.

The solution of the system of equations for the scalar L–A pair (16.4) is the wave function $\Psi = \exp(\lambda z + \lambda^{-1}\bar{z}) \prod_{c=1}^{n} (a_c - \lambda)$, where $a_c(z, \bar{z})$ are defined by the condition of vanishing of the function

$$\tilde{\Psi} = \sum_{\alpha=1}^{k+1} c(\lambda_\alpha) \exp(\lambda_\alpha z + \lambda_\alpha^{-1}) \prod_{c=1}^{n} (a_c - \lambda_\alpha) \equiv$$

$$\equiv \sum_{\alpha=1}^{k+1} c_\alpha \Psi_\alpha, \quad \lambda_\alpha^{k+1} = \lambda^{k+1}$$

at n different points λ_b^{k+1} of the λ^{k+1} plane $(1 < b < n)$. The solutions of the equations of the periodic Toda lattice are given by the relations

$$\exp \rho_s = \prod_{c=1}^{n} \left(\frac{a_c}{\lambda_c} \right)^s J_{s-1}^0 = \prod_{c=1}^{n} \left(\frac{a_c}{\lambda_c} \right)^{s-k-1} \tilde{J}_{s-k-1}^\infty,$$

where J_s^0 are principal minors of the conserved integral matrix (see previous section), when $\lambda = 0$ and $\tilde{J}_{s-k-1}^\infty$ are the principal minors of the matrix $\tilde{J} = \sigma J(a^{-1}, \lambda^{-1}\sigma)$ the infinite value of λ (σ is the constant $(k+1)\,x\,(k+1)$ matrix with 1 on its antidiagonal and 0 on the other places).

As follows from the results of the previous section, the first spectral equation (with respect to the argument z) (16.3) is satisfied when the first expression for $\exp \rho_s$ from formulation of the theorem is used; the second spectral equation, which is obtained from the first one by the Weyl transformation, is satisfied if the second expression for ρ is used. Thus, all that remains to do is to prove the consistency of these identifications. For this purpose we calculate the determinants in the equality chain. The first determinant was calculated in the previous section with the result

$$\| \Psi, \dot{\Psi}, \dots \Psi^{[k]} \| = W_{k+1} \prod_{b=1}^{n} (\lambda_b^{k+1} - \lambda^{k+1}).$$

For the simplification of all the following formulae we propose $\prod_{c=1}^{n} \lambda_c^{k+1} = 1$. For the function $\tilde{\Psi} = \exp - \rho_1 \Psi$, we have

$$\tilde{\Psi} = \prod_{c=1}^{n} a_c^{-1} \exp(\lambda z + \lambda^{-1}\bar{z}) \prod_{c=1}^{n} (a_c - \lambda) =$$

$$= (-1)^n \lambda^n \exp(\lambda z + \lambda^{-1}\bar{z}) \prod_{c=1}^{n} (a_c^{-1} - \lambda^{-1}) =$$

$$= (-1)^n \lambda^{-1}\Psi(z \to \bar{z}, \lambda \to \lambda^{-1}, a \to a^{-1}).$$

A common term in (16.4), rewritten in the adopted notations, is calculated by the general scheme and leads to the result

$$\exp - \rho_{k+1-s}(-1)^{ns} \lambda^{s(k+1)} \, \| \Psi, \dot{\Psi}, \ldots \Psi^{[k-s]}, \Psi^{[,s]} \ldots \Psi' \| =$$

$$= W_{k+1} \exp - \rho_{k+1-s} \prod_{b=1}^{n} (\lambda_b^{k+1} - \lambda^{k+1}) \det J^{k,s},$$

where the first $k + 1 - s$ rows of the matrix $J^{k,s}$ coincide with those of the integral of the motion matrix J, with elements

$$J_{i,j} = \delta_{i,j} + B_i^j + \sum_{c=1}^{n} \frac{A_c^i a_c^{k-j}}{a_c^{k+1} - \lambda^{k+1}},$$

$$\tilde{J}_{i,j} = \delta_{i,j} + \tilde{B}_{k+1-i}^{k+1-j} + \sum_{c=1}^{n} \frac{\tilde{A}_c^{k+1-i} a_c^{-k-1+j}}{a_c^{k+1} - \lambda^{k+1}}.$$

The quantities $\tilde{A}$, $\tilde{B}$ are obtained from the corresponding quantities A, B by the Weyl transformation. By virtue of the condition of the theorem, the determinant $\| \Psi, \dot{\Psi}, \ldots \Psi^{[k-s]}, \Psi^{[,s]} \ldots \Psi' \|$ becomes zero at n points of the λ^{k+1} plan. Thus the concerned determinant (which is a polinom of the n-th power of the λ^{-1}) may differ from the Wandermond determinant only in some factor. Finally, we have

$$\exp \rho_{k+1-s} = \prod_{c=1}^{n} \frac{a_c^{k+1} - \lambda^{k+1}}{\lambda_c^{k+1} - \lambda^{k+1}} \det J^{k,s}. \tag{16.5}$$

The last expression does not depend on λ and it is convenient to calculate it when $\lambda = 0$, $\lambda = \infty$. In the first case, the matrix $\tilde{J}$ transforms into upper triangular matrix with unities on the principal diagonal; for this reason, from the last equality we obtain

$$\exp \rho_{k+1-s} = \prod_{c=1}^{n} a_c^{k+1-s} J_{k-s}^{0}.$$

In the second case, J transforms into the lower triangular matrix and so (16.5) results in

$$\exp \rho_{k+1-s} = \prod_{c=1}^{n} a_c^{-s} \tilde{J}_s^{\infty}.$$

Thus the theorem is proved and one more expression (16.5) is obtained for the solution of the periodic Toda lattice equations of the series A_k.

Let us now write some relations, that are useful for concrete calculations. Introducing the function $F = \sum_{\alpha=1}^{k+1} c(\lambda_\alpha) \exp (\lambda_\alpha z + \lambda_\alpha^{-1})$ and the notation s_l for the elementary symmetric functions constructed from a_c, $s_l = \sum_{c \neq b \neq \ldots \neq d} a_c a_b \ldots a_d$, we rewrite the expression for the function $\tilde{\Psi}$, which appeared in the formulation of the theorem, in the form

$$\tilde{\Psi} = \sum_{c=0}^{n} (-1)^c s_{n-s} F^{[.c]}, \quad s_0 = 1, \quad F^{.(k+1)} = \lambda^{k+1} F, \quad F^{,(k+1)} = \lambda^{-k-1} F.$$

The system of equations for determining s_l is written out in the form

$$\sum_{c=0}^{n} (-1)^c s_{n-s} F_b^{[.c]} = 0, \qquad 1 < b < n,$$

where each of n functions F_b satisfy the equations

$$F_b^{.(k+1)} = \lambda_b^{k+1} F_b, \qquad F_b^{,(k+1)} = \lambda_b^{-k-1} F_b.$$

For the matrix $J_{i,j}^0$, we have a recurrence relation, that relates every row with the previous ones and thus allows one to reconstruct the matrix as a whole, using only the elements of its first row. To do this, we take into consideration the fact, in accordance with the definition of J^0 (see (15.5) and the following formulae), that the matrix elements J_{si}^0 appear in the expansions of the functions φ^s in the powers of λ. That is,

$$\varphi^s = \lambda^s + \sum_{i=0}^{s-2} B_s^i \lambda^i + \sum_{c=1}^{n} \frac{A_c^s}{a_c - \lambda} =$$

$$= \lambda^s + \sum_{i=0}^{s-2} B_s^i \lambda^i + \sum_{c=1}^{n} \frac{A_c^s}{a_c} + \sum_{j=1}^{\infty} \lambda^j \sum_{c=1}^{n} \frac{A_c^s}{a_c^{j+1}} =$$

$$= B_s^0 + \sum_{c=1}^{n} \frac{A_c^s}{a_c^{-1}} + \sum_{1}^{\infty} J_{si}^0 \lambda^i = \varphi_0^s + \sum_{1}^{\infty} J_{si}^0 \lambda^i.$$

The recurrence relations connecting the functions φ^s make it possible to establish the dependence of interest,

$$\varphi^{s+1} = \varphi_0^s + \sum_1^\infty J_{si}^0 \lambda^i = \dot\varphi^s + \varphi^s\varphi^1 = \dot\varphi_0^s + \varphi_0^s\varphi_0^1 +$$

$$+ \varphi_0^1 \sum_1^\infty J_{si}^0 \lambda^i + \varphi_0^s \sum_1^\infty J_{1i}^0 \lambda^i + \sum_1^\infty \lambda^i \sum_{k=1}^{i-1} J_{sk}^0 J_{k,i-k}^0 + \sum_1^\infty \dot J_{si}^0 \lambda^i,$$

$$\dot J_{si}^0 = \varphi_0^s J_{1,i}^0 + \dot\varphi_0^1 J_{s,i}^0 + \dot J_{si}^0 + \sum_{k=1}^{i-1} J_{sk}^0 J_{k,i-k}^0 .$$

In the latter equation, the first two terms, being proportional to the elements of the first and the s-th rows, do not contribute to the principle minors (one can show that they may be omitted in the recurrence procedure as well). Finally, we arrive at

$$J_{s+1,i}^0 = \dot J_{si}^0 + \sum_{k=1}^{i-1} J_{sk}^0 J_{k,i-k}^0 .$$

As in the case of Toda lattice with fixed end-points, it is possible to construct solutions for some other series from solutions of the periodic Toda chain for the A_k series. The system of equations of the periodic Toda lattice is invariant under substitution $\rho_\alpha \to \rho_{k+1-\alpha}$ and consequently among its solutions are those that $\rho_\alpha = \rho_{k+1-\alpha}$. The direct check shows that in the case $k = 2n + 1$ the system of equations of the periodic Toda lattice series A_{2n+1} goes to the system of equations of the periodic Toda lattice series C_n; in the case $k = 2n$, to series $(AB)_k$.

17. THE GENERAL SOLUTION OF THE PERIODIC TODA LATTICE [36,37]

Here we will consider the problem of constructing the general solution of the systems under consideration; the soluton which possesses the sufficient set of arbitrary functions for the solution of the Caushy or Goursat problems. We use the methods of construction of the general solution of the Toda chain with the fixed end-points. As is known, the algebra of the inner symmetry of Toda lattice with fixed ends is finite and we thus have the finite number of terms in the expression for exp $(-\rho)$ in its solution; in the periodic case, the algebra of the inner symmetry is infinite-dimensional and the number of terms in the corresponding expression if infinite. But it may be possible to prove that these series converge absolutely due to the properties of the semisimple infinite-dimensional algebras of the finite growth.

From the beginning, for convenience we restrict ourselves by the case of one-dimensional equations, which arise from the general system of Toda lattice:

$$\frac{\partial^2 \rho_\alpha}{\partial z\, \partial \bar{z}} = \sum_{\beta=1}^{r} K_{\alpha,\beta} \exp \rho_\beta,$$

$$\frac{\partial^2 x_\alpha}{\partial z\, \partial \bar{z}} = \exp \sum_{\beta=1}^{r} K_{\alpha,\beta}\, x_\beta,$$

where $K_{\alpha,\beta}$ coincides with the generalized Cartan matrix of the semisimple infinite-dimensional algebra of the restricted growth. Generalized Cartan matrix for the graded algebras of the second rank brings the latter system of equations to the form

$$\frac{\partial^2 x_1}{\partial z\, \partial \bar{z}} = \exp(2x_1 - 2x_2), \qquad \frac{\partial^2 x_2}{\partial z\, \partial \bar{z}} = \exp(-2x_1 + 2x_2)$$

$$\frac{\partial^2 x_1}{\partial z\, \partial \bar{z}} = \exp(2x_1 - x_2), \qquad \frac{\partial^2 x_2}{\partial z\, \partial \bar{z}} = \exp(-4x_1 + 2x_2) \tag{17.1}$$

which, if variables $x_1 - x_2$, $2x_1 - x_2$ are introduced, yields the sin-Gordon $\frac{\partial^2 x}{\partial z\, \partial \bar{z}} = \exp(2x) - \exp(-2x)$ and Dodd–Boullow-Jeber–Schabat $\frac{\partial^2 x}{\partial z\, \partial \bar{z}} = \exp x - \exp(-2x)$, respectively in the first and in the second cases. Note that these equations together with the Liouville equation $\frac{\partial^2 x}{\partial z\, \partial \bar{z}} = \exp 2x$ are exceptional among all the equations of the form $\frac{\partial^2 x}{\partial z\, \partial \bar{z}} = f(x)$ due to the presence of the nontrivial group of internal symmetry.

We know that in the case of Toda chain with the fixed endpoints a solution for the $\exp(-x_\alpha)$ is expressed up to mulptipliers dependent only on z and $\bar{z}$, through the powers of the repeated integrals of arbitrary functions. Let us assume that such a structure is also valid for the solutions in the contragradient case and rewrite system (17.1) in the form (further on we will use only the first system):

$$\frac{\partial^2 x_1}{\partial z\, \partial \bar{z}} = \varphi_1 \bar{\varphi}_1 \exp(2x_1 - 2x_2), \qquad \frac{\partial^2 x_2}{\partial z\, \partial \bar{z}} = \varphi_2 \bar{\varphi}_2 \exp(-2x_1 + 2x_2),$$

where φ_1, φ_2 are arbitrary functions of the argument z; $\bar{\varphi}_1$, $\bar{\varphi}_2$ are the same of the argument $\bar{z}$. We have introduced arbitrary functions into (17.1) (it may be done by substitution $x_\alpha \to x_\alpha + \ln \varphi_\alpha + \ln \bar{\varphi}_\alpha$ and further conformal transformation) which play the role of unhomogeneities. After replacing $\exp(-x_j) \to X_j$, the previous system becomes

$$X_1 \frac{\partial^2 X_1}{\partial z\, \partial \bar{z}} - \frac{\partial X_1}{\partial z} \frac{\partial X_1}{\partial \bar{z}} = \varphi_1 \bar{\varphi}_1 X_2^2 \quad (X_2),$$

$$X_2 \frac{\partial^2 X_2}{\partial z\, \partial \bar{z}} - \frac{\partial X_2}{\partial z} \frac{\partial X_2}{\partial \bar{z}} = \varphi_2 \bar{\varphi}_2 X_1^2 \quad (X_1^4).$$

In the brackets in the latter equation there are given the r.h.s. for the second system (17.1). In the finite case, X_j are the polynomials in the repeated integrals; the first term being equal to unity. Therefore, we assume that in the «zeroth» approximation in φ_l $X_1 = X_2 = 1$. Then the equations may be solved through «iterations», where the small quantities are the corresponding powers of φ_α, $\bar{\varphi}_\alpha$. The first order approximation gives

$$X_1^1 = -\int dz\, \varphi_1 \int d\bar{z}\, \bar{\varphi}_1 \equiv -(1)\,(\bar{1}),$$

$$X_2^1 = -\int dz\, \varphi_2 \int d\bar{z}\, \bar{\varphi}_2 \equiv -(2)\,(\bar{2}).$$

The results of the calculations up to the eight order are listed below. It is worth noting that the proposed procedure for solving the systems with the exponential interaction is also applicable in the case of finite semisimple algebras. The only difference from the case of infinite graded algebras is the finiteness of the series in powers of the repeated integrals.

$$X_1 = \sum_{k=0}^{\infty} (-1)^k \sum_{s=1} X_s^k \bar{X}_s^k = 1 - X_1^1\bar{X}_1^1 + 2X_1^2\bar{X}_1^2 - 4X_1^3\bar{X}_1^3 - 2X_2^3\bar{X}_2^3 + 4X_1^4\bar{X}_1^4 +$$

$$+ 8X_2^4\bar{X}_2^4 - 8X_1^5\bar{X}_1^5 - 8X_2^5\bar{X}_2^5 - 16X_3^5\bar{X}_3^5 + 8X_1^6\bar{X}_1^6 + 16X_2^6\bar{X}_2^6 + 16X_3^6\bar{X}_3^6 +$$

$$+ 32X_4^6\bar{X}_4^6 - 16X_1^7\bar{X}_1^7 - 16X_2^7\bar{X}_2^7 - 16X_3^7\bar{X}_3^7 - 32X_4^7\bar{X}_4^7 - 64X_5^7\bar{X}_5^7 +$$

$$+ 32X_1^8\bar{X}_1^8 + 32X_2^8\bar{X}_2^8 + 64X_3^8\bar{X}_3^8 + 64X_4^8\bar{X}_4^8 + 64X_5^8\bar{X}_5^8 + 128X_6^8\bar{X}_6^8.$$

The upper indices of $X_s^k(\bar{X}_s^k)$ mark the number of approximation, while the lower ones stand for the order number in it. Here are the values for X_s^k,

$$X_1^1 = (1), \quad X_1^2 = (12), \quad X_1^3 = (122), \quad X_2^3 = (121),$$
$$X_1^4 = (1212) + (1221), \quad X_2^4 = (1221), \quad X_1^5 = (12122) + (12212),$$
$$X_2^5 = (12121) + 2(12211), \quad X_3^5 = (12211),$$
$$X_1^6 = (122121) + (121221), \quad X_2^6 = (121211) + 3(122111),$$
$$X_3^6 = (121212) + (122121) + (21221) + 2(122112), \quad X_4^6 = (122112),$$

where $(ij \ldots k) = \int^{z} dz_1 \varphi_i \int^{z_1} dz_2 \varphi_j \ldots \int^{z_{n-1}} dz_n \varphi_k$ and $\bar{X}_j^i$ arose from X_j^i by the substitution $\varphi_s \to \varphi$, $z \to \bar{z}$. The X_2 is obtained from X_1 by replacing indices $1 \to 2$ in the expressions for X_j^i. The number of terms in X_j^i will be called a length of $X_j^i - L(X_j^i)$. Thus $L(X_1^2) = 1$, $L(X_2^5) = 3$, $L(X_3^6) = 5$, etc. Taking into account that $(i, i, \ldots i) = \frac{(i)^s}{s!}$ (where s is the number of the repeated integrals) and the presence of an evident solution $X_1 = X_2 = \exp - (1)(\bar{1})$, when $\varphi_1 = \varphi_2$, we will find from its definition that

$$\sum_s c_s (L(X_s^k))^2 = k!$$

whereof, it follows that for arbitrary functions $\varphi_{1,2}$ and $\bar{\varphi}_{1,2}$ bounded on the intervals (z_0, z) and $(\bar{z}_0, \bar{z})$, there is the estimation of the term of the k-th approximation

$$\sum_s = c_s X_s^k \bar{X}_s^k \le \frac{M^k \bar{M}^k}{k!} (z - z_0)^k (\bar{z} - \bar{z}_0)^k,$$

where M is the supremum of the functions $\varphi_{1,2}$ on the interval (z, z_0), $\bar{M}$ — the same for the functions $\bar{\varphi}_{1,2}$ on the corresponding interval $\bar{z}, \bar{z}_0$. The series, which gives the solutions $X_{1,2}$, converges absolutely. For this estimation it is essential that all the terms of the k-th approximation, as well as all the terms in X_j^i enter with the same sign. This is a direct consequence of the properties of the contragraded algebras of the restricted growth. To obtain closed expressions for $X_{1,2}$, which would allow one, in particular, to calculate any term in series, it is necessary to have some information about the representation theory of such algebras.

The set of simple roots of the graded algebras of finite growth $X_\alpha^{\pm}$ and its Cartan elements h_α obeys the system of commutation relations

$$[X_\alpha^+, X_\beta^-] = \delta_{\alpha,\beta} h_\beta, \qquad [h_\beta, X_\alpha^{+-}] = \pm K_{\alpha,\beta} X_\alpha^\pm, \tag{17.2}$$

where K is the generalized Cartan matrix. Classification theorems and explicit form of the matrix K for the algebras under consideration are well known. The Cartan matrix for the considered equations (a, b) has the form

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

In the whole analogue to the finite case, the graded semisimple algebras possess the set of the fundamental representations. Each of these representations is determined by its highest vector $\alpha\rangle$ with the properties

$$X_\beta^+ \alpha\rangle = 0, \qquad h_\beta \alpha\rangle = \delta_{\alpha,\beta} \alpha\rangle$$

and all other basic vectors of representation, which is infinite-dimensional in this case, are constructed by consequent applications of the generators of the negative simple roots to the highest vector. The properties of the graded semisimple algebras allow one to construct the invariant bilinear form in the representation space. To calculate the scalar products of basis vectors of the representations with the highest vector it is sufficient to know only the background commutation relations (17.2).

Now we describe the way of constructing the solutions for the case of arbitrary semisimple graded algebras of the finite growth. First of all two equations of the S-matrix type should be solved

$$\frac{\partial M_+}{\partial z} = M_+ L^+(z), \qquad \frac{\partial M_-}{\partial \bar{z}} = M_- L^-(\bar{z}),$$

where

$$L^+ = \sum_{\alpha=1}^{r} \varphi_\alpha(z)\, X_\alpha^+, \qquad L^- = \sum_{\alpha=1}^{r} \bar{\varphi}_\alpha(\bar{z})\, X_\alpha^-.$$

The functions $\varphi_\alpha(z)$, $\bar{\varphi}_\alpha(\bar{z})$ contained in the definition of Lagrangians $L^\pm$ are the arbitrary functions of its arguments. The solutions of S-matrix equations may be represented in the form of ordered integrals, but the number of terms in this expansion will be infinite. In the above notations and definitions for the X_α for the arbitrary semisimple algebras of finite growth we have

$$X_\alpha = \langle \alpha \| M_+^{-1} M_- \| \alpha \rangle. \tag{17.3}$$

The results at the beginning of this section, obtained by the methods of the perturbation theory, are in fact the special case of the general formulae (17.3).

Coming back to the beginning of this section, we have for the solution of the sin-Gordon equation

$$\exp x = \varphi_1^{\frac{1}{2}} \bar{\varphi}_1^{\frac{1}{2}} X_2 X_1^{-1},$$

where in expressions for $X_{1,2}$, which follow from (17.3), one has to put $\varphi_2 = \varphi_1^{-1}$, $\bar{\varphi}_2 = \bar{\varphi}_1^{-1}$. For the solution of second equation (b) we correspondingly get

$$\exp x = \varphi_1 \bar{\varphi}_1 X_2 X_1^{-2}, \quad \varphi_2 = \varphi_1^{-2}, \bar{\varphi}_2 = \bar{\varphi}_1^{-2}.$$

It should be noticed that, at present, we have no proof of (17.1) except of the series expansion in powers of the repeated integrals and direct check of the validity of (17.3) in each order.

Thus, in the considered case the form of general solution of the periodic Toda lattice is the same as for Toda chain with fixed end points. The main difference is that in the case of finite-dimensional algebras the series (given by the perturbation theory) are finite and in the case of the infinite-dimensional algebras they are infinite. But the demand of restricted growth guarantees their absolute convergence.

Now it is not known how to choose the «creating» functions φ, $\bar{\varphi}$ for constructing the soliton solution of the previous section and what is the criterium of the summation of the series corresponding to this situation. It is an interesting unsolved problem.

18. THE SOLUTION OF THE MAIN CHIRAL PROBLEM WITH MOVING POLES BY THE METHODS OF RIEMANN PROBLEM [38,39]

In this section, by the special choice of the coefficient function of the homogeneous Riemann problem we show that its solution is connected with the main chiral field problem with moving poles. This approach is by no means unique; the method of the Backlund transformation leads to the same results.

The main chiral field problem with moving poles is described by the equation

$$(\xi - \bar{\xi}) \frac{\partial^2 F}{\partial \xi \, \partial \bar{\xi}} + \left[\frac{\partial F}{\partial \xi}, \frac{\partial F}{\partial \bar{\xi}} \right] = 0.$$

We illustrate the general scheme of its integration by the example of the simplest case of the $SL(2, C)$ algebra. Let the homogeneous Riemann problem on some contour has its usual form $\Omega_0 \Omega_+ = \Omega_-$, where $\Omega_\pm$ are the boundary values of two functions analytic out and within the contour respectively. Element Ω_0 is chosen in the form

$$\begin{pmatrix} a & b\left(\dfrac{\lambda-\xi}{\lambda-\bar{\xi}}\right)^{n} \\ c\left(\dfrac{\lambda-\xi}{\lambda-\bar{\xi}}\right)^{-n} & d \end{pmatrix},$$

where a, b, c, d $(ad - cd = 1)$ are arbitrary functions of argument λ without any peculiarities under analytic continuation within the contour; the points $\lambda = \xi$, $\lambda = \bar{\xi}$, $\lambda = 0$ dispose there. The condition in the neighbourhood of the infinity point of the λ plane is $\Omega_+ \to 1 + \frac{f}{\lambda}$.

Let us rewrite the Riemann problem in the form more useful for our aim

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tilde{\Omega}_+ = \exp - n \left(\frac{\lambda-\bar{\xi}}{\lambda-\xi}\right) \frac{H}{2}\, \Omega^-, \tag{18.1}$$

where $\tilde{\Omega}_+ = \exp - n \ln \frac{\lambda-\bar{\xi}}{\lambda-\xi} \frac{H}{2}\, \Omega_+$ with new asymptotic condition $\tilde{\Omega}_+ \to 1 + \dfrac{f + \frac{n}{2}(\xi - \bar{\xi})}{\lambda} = 1 + \dfrac{F}{\lambda}$. The problem (18.1), being rewritten in the differential form, reads

$$(\lambda - \xi)\, (\tilde{\Omega}_+)^{-1}\, (\tilde{\Omega}_+)_\xi = (\lambda - \xi)\, \Omega_-^{-1}(\Omega_-)_\xi + \Omega_-^{-1} \frac{nH}{2}\, \Omega_-,$$

$$(\lambda - \bar{\xi})\, (\tilde{\Omega}_+)^{-1}\, (\tilde{\Omega}_+)_{\bar{\xi}} = (\lambda - \bar{\xi})\, \Omega_-^{-1}(\Omega_-)_{\bar{\xi}} - \Omega_-^{-1} \frac{nH}{2}\, \Omega_-.$$

Taking into account the Liouville theorem and the properties of the Riemann problem we conclude that both the latter expressions are the polynomials in the whole complex plane. The asymptotic condition added is that the degrees of the polynomial are the zeroes. Calculating these polynomials in the neighbourhood of the infinite point and at the point $\lambda = 0$, which lies within the contour, we have

$$F_\xi = -\xi G^{-1} G_\xi, \quad F_{\bar{\xi}} = -\bar{\xi} G^{-1} G_{\bar{\xi}}, \quad G = \exp - \frac{n}{2} \ln \frac{\bar{\xi}}{\xi}\, H\Omega(0).$$

With the help of the Maurer–Cartan identity, we conclude now that F obeys the equation of the main chiral field with moving poles.

Let us find the solution of the Riemann problem in the form

$$\Omega_+ = \begin{pmatrix} 1 + \sum_1^n \frac{e_s}{(\lambda - \xi)^s} & \sum_1^n \frac{f_s}{(\lambda - \xi)^s} \\ \sum_1^n \frac{g_s}{(\lambda - \xi)^s} & 1 + \sum_1^n \frac{h_s}{(\lambda - \xi)^s} \end{pmatrix}.$$

Here e_s, f_s, g_s and h_s are $4n$ parameters. They must be chosen so as the element Ω_- would be analytic at the point $\lambda = \xi$, $\lambda = \bar{\xi}$. Let us denote by $\varphi_s(y)$ the first s terms of its expansion in Taylor series near the point $\lambda = y$, i.e.

$$\varphi_s(y) = \varphi(y) + \frac{\lambda - y}{1!} \varphi(y)^{(1)} + \ldots + \frac{(\lambda - y)^s}{s!} \varphi(y)^{(s)}.$$

For the matrix element $(\Omega_-)_{1,1}$ we have

$$(\Omega_-)_{1,1} = a(\lambda) + \sum_1^n \frac{a(\lambda)\, e_s}{(\lambda - \xi)^s} + \sum_1^n \frac{b(\lambda)\,(\lambda - \xi)^{n-s} g_s}{(\lambda - \xi)^n} =$$

$$= a(\lambda) + \sum_1^n \frac{a(\lambda) - a_{s-1}(\xi) e_s}{(\lambda - \xi)^s} + \sum_1^n \frac{b(\lambda)(\lambda - \bar{\xi})^{n-s} - (b(\lambda)(\lambda - \bar{\xi})^{n-s})_{n-1}(\xi) g_s}{(\lambda - \xi)^n} +$$

$$+ \sum_1^n \frac{a_{s-1}(\xi) e_s}{(\lambda - \xi)^s} + \sum_1^n \frac{(b(\lambda)\,(\lambda - \bar{\xi})^{n-s})_{n-1}(\xi) g_s}{(\lambda - \xi)^n}.$$

Within the contour C, the singularities may have only the terms of the last line of the previous equality. The absence of them in $(\Omega_-)_{1,1}$ is equivalent to the zero values of the residues up to the n-th order in the mentioned expression. The same conditions on the matrix elements $(\Omega_-)_{1,2}$, $(\Omega_-)_{2,2}$, $(\Omega_-)_{2,1}$ lead to the linear systems of the algebraic equations, which determine the unknowns e_s, g_s, f_s, h_s. In what follows we shall write them in the form of the n-ordered columns. We have

$$\begin{pmatrix} \Gamma_+(\varphi^{-1})\, e + \Gamma_-(\xi, \bar{\xi})\, g = 0 & \Gamma_+(\varphi^{-1}) f + \Gamma_-(\xi, \bar{\xi})\, h = T \\ (\Gamma_-(\xi, \bar{\xi}))^{-1} e + \Gamma_+(\bar{\varphi})\, g = -T & (\Gamma_-(\xi, \bar{\xi}))^{-1} f + \Gamma_+(\bar{\varphi})\, h = 0 \end{pmatrix},$$

where $\Gamma_+(\varphi)$ $(\Gamma_+(\bar{\varphi}))$ are upper triangle matrices all elements of which parallel to the main diagonal are the same and equal to $\frac{1}{s!} \varphi^{(s)}(\xi)$, $\left(\frac{1}{s!} \bar{\varphi}^{(s)}(\bar{\xi})\right)$, where s is the distance from the main diagonal. On the main diagonal there is the function $\varphi(\xi)$, $(\overline{\varphi}(\overline{\xi}))$ by itself, on the next place — its first derivative and so on. The functions $\varphi(\xi)$, $\overline{\varphi}(\overline{\xi})$ are connected with the matrix elements of the homogeneous Riemann problem by expressions

$$\varphi(\xi) = \frac{a}{b}\,(\lambda = \xi), \quad \overline{\varphi}(\overline{\xi}) = \frac{d}{c}\,(\lambda = \overline{\xi}), \quad \Gamma_+(\varphi_1)\,\Gamma_-(\varphi_2) = \Gamma_+(\varphi_1\varphi_2).$$

The s-th line of the lower triangle matrix $= \Gamma_-(\xi, \overline{\xi}) = \Gamma_-(\xi - \overline{\xi}) \equiv \Gamma_-(x)$, consist of terms of the binomial expansion $(1 + x)^s$ $(\Gamma_-(-x) = (\Gamma_-(x))^{-1})$

$$\Gamma_-(x) = \begin{pmatrix} 1 & 0 & \dots & \dots\dots.0 \\ x & 1 & 0 & \dots\dots.0 \\ x^2 & 2x & 1 & \dots\dots.0 \\ \dots & \dots & \dots & \dots\dots.0 \\ \dots & \dots & \dots & \dots\dots.0 \\ x^n & x^{n-1}C_n^1 & x^{n-2}C_n^2 & \dots\dots.1 \end{pmatrix}, \quad T = \begin{pmatrix} (-x)C_n^{n-1} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ (-x)^{n-2}C_n^2 \\ (-x)^{n-1}C_n^1 \\ (-x)^n \end{pmatrix}.$$

In accordance with the previous results, the solution of the main chiral field problem with moving poles is determined by the asymptotics of the homogeneous Riemann problem and its explicit form is

$$F = g_1X_- + \left(e_1 + (\xi - \overline{\xi})\,\frac{n}{2}\right)H + f_1X_+.$$

The explicit expressions for $e_1 = -h_1, g_1, f_1$ as a solution of the linear system of the algebraic equations () may be represented in the form of the sum of the entries of $n \times n$ matrix, which we denote by the title corresponding to e, g, h, f letters

$$G = (\varphi_- - \varphi_+)^{-1}, \quad E = -\frac{1}{2}\,(\varphi_- + \varphi_+)\,(\varphi_- - \varphi_+)^{-1}, \quad F = -\,\varphi_-(\varphi_- - \varphi_+)^{-1}\varphi_+,$$

where φ_-, φ_+ are the lower and upper triangular matrices with equal entries, represented on the equal distances from the main diagonal. They can be written as

$$\varphi_-^s = -\frac{1}{(s-1)!}\,\frac{\partial^{s-1}}{\partial\xi^{s-1}}\,(\xi - \overline{\xi})^s\,\frac{\partial\varphi}{\partial\xi}, \quad \varphi_+^s = \frac{1}{(s-1)!}\,\frac{\partial^{s-1}}{\partial\overline{\xi}^{s-1}}\,(\xi - \overline{\xi})^s\,\frac{\partial\overline{\varphi}}{\partial\overline{\xi}},$$

$$s = 1, 2, \dots \quad \varphi_-^0 = \varphi(\xi), \quad \overline{\varphi}_+^0 = \overline{\varphi}(\overline{\xi}).$$

Now consider some more simple examples. Let $n = 1$. In this case, all matrices G, E, F are one-dimensional and the solution of the main chiral field problem takes the form

$$F = \frac{1}{\varphi - \overline{\varphi}} X_- - \frac{1}{2} \frac{\varphi + \overline{\varphi}}{\varphi - \overline{\varphi}} H - \frac{\varphi\overline{\varphi}}{\varphi - \overline{\varphi}} X_+ .$$

This is no more than (up to the gauge transformation) the 'Hooft solution in the spherical symmetric case.

Let $n = 2$. All matrices in the problem are two-dimensional. The matrices $\varphi_\pm$ have the form

$$\varphi_+ = \begin{pmatrix} \overline{\varphi} & x\dot{\overline{\varphi}} \\ 0 & \overline{\varphi} \end{pmatrix}, \quad \varphi_- = \begin{pmatrix} \varphi & 0 \\ -x\varphi' & \varphi \end{pmatrix},$$

where ˙' mean the differentiation with respect to the independent arguments ξ and $\overline{\xi}$, respectively, as before $x = (\varphi - \overline{\varphi})$. For the solution F we get

$$\frac{x}{D} [(2\delta + x(\varphi' + \dot{\overline{\varphi}}))\, X_+ + (\delta(\overline{\varphi} + \varphi) + x(\overline{\varphi}\varphi' + \varphi\dot{\overline{\varphi}}))\, H -$$

$$- (2\overline{\varphi}\varphi\delta + x(\overline{\varphi}^2\varphi' + \varphi^2\dot{\overline{\varphi}}))\, X_+],$$

where $D = \delta^2 - x^2\dot{\overline{\varphi}}\varphi'$ and $\delta = \varphi - \overline{\varphi}$. The expression for the «instanton» charge density for the main chiral field problem with moving poles was established in the first chapter. For arbitrary n, the solution of the present section reads

$$q \infty \ln \frac{\mathrm{Det}\,(\varphi_- - \varphi_+)}{(\xi - \overline{\xi})^{n^2}} =$$

$$= \frac{1}{(\xi - \overline{\xi})^2} \left[\frac{\partial^2}{\partial\xi\, \partial\overline{\xi}} \frac{(\xi - \overline{\xi})^2}{2} + 1 \right] \frac{\partial^2}{\partial\xi\, \partial\overline{\xi}} \ln \frac{\mathrm{Det}\,(\varphi_- - \varphi_+)}{(\xi - \overline{\xi})^{n^2}} .$$

If the functions $\overline{\varphi}$ and φ are chosen in the «pole» form

$$\varphi = \sum_1^N \frac{c_s}{\xi + ia_s}, \quad \overline{\varphi} = \sum_1^N \frac{c_s}{\overline{\xi} + ia_s},$$

where c_s, a_s are the real parameters, then after substitution into the charge density and integration over invariant measure we come to the whole charge equal to $N (N \geq n)$.

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