«ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА»
1998, ТОМ 29, ВЫП. 4

УДК 537.8

SOME REMARKABLE CHARGE-CURRENT CONFIGURATIONS

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We investigate how different magnetization distributions interact with an external electromagnetic field. Strong selectivity to the time dependence of the external electromagnetic field arising for particular magnetizations suggests that it can be used for the practical applications.

We review the properties of the known charge-current radiationless configurations. The radiation field of toroidal-like time-dependent current configurations is investigated. The infinitesimal time-dependent configurations are found outside which the electromagnetic strengths disappear but the potentials survive. For a number of time dependences, their finite radiationless counterparts can be found. In these cases topologically nontrivial (unremovable by a gauge transformation) electromagnetic potentials exist outside sources. The well-defined rule obtained for constructing of time-dependent infinitesimal sources suggests the existence of finite nontrivial radiationless sources with a rather arbitrary time dependence. The latter can be used to carry out time-dependent Aharonov–Bohm-like experiments.

Examples are given of nonstatic current configurations generating the static electric field and adequately described by the electric vector potential rather than by the scalar one.

Показано, как различные намагниченности взаимодействуют с внешним электромагнитным полем. Для некоторых намагниченностей характерна сильная избирательность к временной зависимости поля. Это может быть использовано для практических приложений.

Обсуждаются свойства известных неизлучающих распределений зарядов и токов. Дан рецепт построения бесконечно малых распределений, вне которых напряженности электромагнитного поля, но не потенциалы, равны нулю. Для ряда временных зависимостей найдены распределения зарядов и токов конечных размеров. В этих случаях вне источников существуют топологически-нетривиальные (то есть неустранимые калибровочными преобразованиями), зависящие от времени электромагнитные потенциалы. Это может быть использовано для постановки зависящего от времени эффекта Ааронова–Бома.

Приведены примеры нестационарных чисто токовых распределений, генерирующих статическое электрическое поле и адекватно описываемых электрическим векторным (не скалярным) потенциалом.

1. INTRODUCTION

Probably, it should be at first explained what words "remarkable charge-current configurations" in the title of this paper mean. Under them we understand charge-current distributions with unusual paradoxical properties. For example, it is well known that a point charge radiates the electromagnetic energy when it moves with acceleration. However, there are known specific finite-extension configurations of charges which do not radiate when they exhibit acceleration [1—9].

Further, everybody knows that time-dependent currents emit electromagnetic energy into the surrounding space. However, there are known time-dependent current configurations which do not radiate the electromagnetic energy [6,9-12]. Up to now only those nonradiating time-dependent configurations of charges and currents were known for which the electromagnetic field (EMF) strengths $\vec{E}, \vec{H}$ as well as electromagnetic potentials $\vec{A}, \Phi$ have disappeared outside the finite space region S. It turns out that finite time-dependent configurations of charges and currents exist outside which electromagnetic strengths $\vec{E}, \vec{H}$ vanish, but the nontrivial electromagnetic potentials $\vec{A}, \Phi$ differ from zero [13]. Under the term "nontrivial" we mean that the treated physical situation is described adequately by electromagnetic potentials rather than electromagnetic strengths.

Further, it is known that electric and magnetic dipoles interact with electric and magnetic field, respectively. However, there are known finite configurations of magnetic (electric) dipoles whose interaction with an external EMF is proportional to the time derivative (of the definite order) of the electric (magnetic) field [14-19].

It is the goal of present consideration to study the properties of these remarkable charge-current configurations.

The plan of our exposition is as follows.

In Sec. 2 we study how different configurations of electric and magnetic dipoles interact with external EMF. It turns out that the selectivity of the interaction to the time dependence of an external EMF can be used for the storage and ciphering of information.

In the same section the classification of current sources according to their interaction with external field is given.

Consider the metallic ring embracing the cylindrical solenoid. When the metallic ring becomes superconductive, the supercurrent arises on its surface. This in turn leads to the appearance of magnetic field in the surrounding space. These quantities are evaluated in Sec. 3.

The review of known radiationless time-dependent sources is given in Sec. 4. The exposition is illustrated by the concrete examples of accelerated nonradiating charge distributions.

In Sec. 5 we construct the toroidal charge-current configuration having the property that the time-dependent magnetic field differs from zero only inside the impenetrable torus while the time-dependent magnetic vector potential (VP) and time-independent electric scalar potential differ from zero everywhere. In the accessible region (i.e., outside the impenetrable torus) the static electric field differs from zero inside the torus hole. Although charged particles may scatter on this electric field, the latter contributes only to the static background. It is just the time variation of the magnetic flux confined to the excluded region that leads to the time dependence of the interference picture. This may be viewed as a new channel for the information transfer and can be used for the performance of the time-dependent Aharonov–Bohm effect. The classification of more general radiationless sources is given in the same section. Examples are given of current configurations generating the static electric field adequately described by the electric vector potential rather than by the scalar one.

2. INTERACTION OF MAGNETIZATIONS WITH EXTERNAL ELECTROMAGNETIC FIELD

The plan of our exposition is as follows. In Sec. 2.1, we study how the choice of magnetization inside the sample affects its interaction with external EMF. The generalization of Ampere hypothesis is discussed in the same section. The physical meaning of the scalar functions entering into the Debye parametrization of the current density is clarified in Sec. 2.2. It turns out that the selectivity of the interaction to the time dependence of an external EMF arises for a specific choice of these functions. Probably,this can be used for the storage and ciphering of information. In the same section, we give the classification of the point-like and extended current sources according to their interaction with an external EMF.

2.1. Magnetization, Toroidization and Generalization of Ampere Hypothesis. Consider the circular current in the $Z = 0$ plane (the upper part of Fig.1):

$$\vec{j} = \vec{n}_\phi I\delta(\rho - d)\delta(z) = \\ = \frac{1}{d}\vec{n}_\phi I\delta(\rho - d)\delta(\theta - \frac{\pi}{2}). \tag{2.1}$$

As div $\vec{j} = 0$, the equivalent magnetization can be used instead of $\vec{j}$ (see, e.g.,[20]):

$$\vec{j} = \text{rot } \vec{M} \tag{2.2}$$

$$\vec{M} = I\vec{n}_z\delta(z)\Theta(d - \rho) = \\ = -I\vec{n}_\theta\frac{1}{d}\Theta(d - r)\delta(\theta - \frac{\pi}{2}), \quad \text{div } \vec{M} \neq 0 \tag{2.3}$$

($\Theta(x)$ is the step function).This relation is a mathematical expression of the Ampere hypothesis according to which the closed circular current is equivalent to the magnetized sheet. The magnetic field can be evaluated from either (2.1) or (2.3). For example, the magnetic vector potential is given by

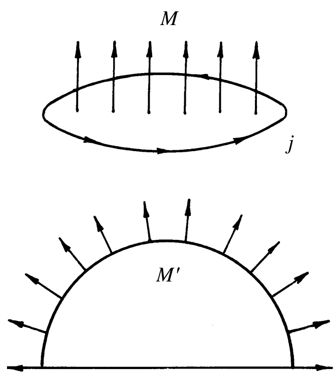

Fig. 1. Two magnetizations $\vec{M}$ and $\vec{M}'$ corresponding to the same current density $\vec{j}$

$$\vec{A} = \frac{I}{c}\int \frac{1}{|\vec{r} - \vec{r}'|}\vec{n}'_\phi\delta(\rho' - d)\delta(z')dV' = -\frac{1}{c}\int \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \times \vec{M}(\vec{r}')dV'. \tag{2.4}$$

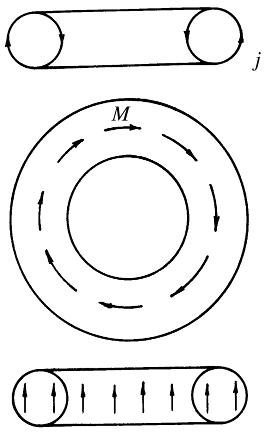

Fig. 2. The poloidal current $\vec{j}$ flowing on the torus surface is equivalent to the magnetization $\vec{M}$ which in turn is equivalent to the toroidization $\vec{T}$

For an infinitely small d, the current $\vec{j}$ in Eq. (2.1) is not well defined (the vector $\vec{n}_\phi$ loses its sense at the origin). On the other hand, the magnetization $\vec{M}$ in Eq. (2.3) is still well defined. In the limit $d \to 0$, Eqs. (2.1)—(2.3) mean that the circular current of an infinitely small radius is equivalent to the magnetic dipole.

A more complicated case is the poloidal current flowing on the torus surface $(\rho - d)^2 + z^2 = R^2$ (see Fig.2). As far as we know, the term "poloidal" originates to the Elsasser paper [21]. To parametrize $\vec{j}$, it is convenient to introduce the coordinates $\tilde{R}, \Psi$ (see Fig.3)

$$x = (d + \tilde{R}\cos\Psi)\cos\phi, \quad y = (d + \tilde{R}\cos\Psi)\sin\phi,$$

$$z = \tilde{R}\sin\Psi.$$

In these coordinates,

$$\vec{j} = \vec{n}_\psi \frac{\delta(R - \tilde{R})}{d + \tilde{R}\cos\Psi}\frac{j_0}{R^2 d}. \tag{2.5}$$

Here $\vec{n}_\psi$ is the unit vector tangent to the torus surface

$$\vec{n}_\psi = \vec{n}_z \cos\Psi - \vec{n}_\rho \sin\Psi.$$

It lies in the $\phi = \text{const}$. plane and defines the direction of j. The factor $R^2 d$ in the denominator of $\vec{j}$ is introduced for convenience and may be absorbed into j_0 . The constant j_0 may be expressed in terms of either magnetic flux Φ penetrating solenoid or the number of coils N and a current I in each of them:

$$j_0 = \frac{R^2 dc\Phi}{8\pi^2(d - \sqrt{d^2 - R^2})} = \frac{NIR^2 d}{2\pi}.$$

As div $\vec{j} = 0$, the current $\vec{j}$ may be expressed through the magnetization: $\vec{j} =$ $=$ rot $\vec{M}$. It turns out that $\vec{M}$ is enclosed inside the torus T and has only the ϕ component:

$$\vec{M} = -\vec{n}_\phi \frac{\Theta(R - \tilde{R})}{d + \tilde{R}\cos\Psi}\frac{j_0}{R^2 d}. \tag{2.6}$$

As div $\vec{M} = 0$, it can be represented as $\vec{M} =$ rot $\vec{T}$, div $\vec{T} \neq 0$, where $\vec{T}$ is given by

$$\vec{T} = \vec{n}_z j_0 T / Rd. \tag{2.7}$$

Here

$$T = \ln \frac{d - \sqrt{R^2 - z^2}}{d + \sqrt{R^2 - z^2}} \qquad (2.8)$$

inside the torus hole ($0 \le \rho \le d - \sqrt{R^2 - z^2}, \quad -R \le z \le R$),

$$T = \ln \frac{\rho}{d + \sqrt{R^2 - z^2}} \qquad (2.9)$$

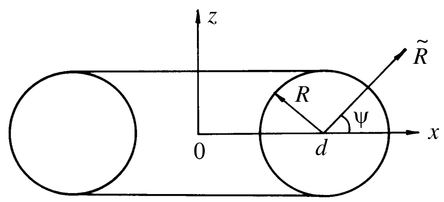

Fig. 3. Geometrical depiction of $\tilde{R}$, Ψ coordinates used in the text

inside the torus itself ($d - \sqrt{R^2 - z^2} \le \rho \le d + \sqrt{R^2 - z^2}, \quad -R \le z \le R$) and $T = 0$ in other space regions. Similarly to the magnetization $\vec{M}$, the distribution $\vec{T}$ may be called the toroidization (as far as we know, this term has been introduced by M.A. Miller [22]). It follows from Eqs. (2.5)—(2.9) that

$$\vec{j} = (\mathrm{rot}\ )^2 \vec{T}, \quad \mathrm{div}\ \vec{T} \neq 0 \qquad (2.10)$$

while the vector potential is given by

$$\vec{A} = \frac{4\pi}{c}\vec{T}(\vec{r}) + \frac{1}{c}\nabla \int \frac{1}{|\vec{r} - \vec{r}'|} \mathrm{div}\ \vec{T}(\vec{r}')dV'. \qquad (2.11)$$

The magnetic field strength differs from zero only inside the torus:

$$H_\phi = -\frac{4\pi}{c}\frac{j_0}{dR^2}\frac{1}{\rho}.$$

Physically, Eqs. (2.5)—(2.11) mean that the poloidal current $\vec{j}$ given by Eq. (2.5) is equivalent (i.e., produces the same magnetic field) to the toroidal tube with the magnetization $\vec{M}$ defined by (2.6) and to the toroidization $\vec{T}$ given by (2.7). This is illustrated in Fig. 2.

We consider now the case when the torus dimensions d, R tend to zero. Since R is always less than d, we let R tend to zero first and d later. In the limit $R \to 0$ the current $\vec{j}$ (see Fig. 2) becomes ill-defined. On the other hand, $\vec{M}$ and $\vec{T}$ remain well-defined:

$$\vec{M} \to -\vec{n}_\phi \frac{\pi}{d^2} j_0 \delta(\rho - d)\delta(z) \quad (\mathrm{div}\ \vec{M} = 0),$$

$$\vec{T} \to -\vec{n}_z \frac{\pi}{d^2} j_0 \Theta(d - \rho)\delta(z) \quad (\mathrm{div}\ \vec{T} \neq 0) \quad \text{for} \quad R \to 0. \qquad (2.12)$$

After performing the $R \to 0$ limit we let d go to zero. Now it is the turn of the magnetization $\vec{M}$ to be ill-defined, but the vector $\vec{T}$ is still well-defined:

$$\vec{T} \to -\vec{n}_z j_0 \pi^2 \delta^3(\vec{r}) \quad (\mathrm{div}\ \vec{T} \neq 0) \quad \text{for} \quad d \to 0 \qquad (2.13)$$

($\delta^3(\vec{r}) = \delta(\rho)\delta(z)/2\pi\rho$). The VP corresponding to this toroidization is given by [15,23]:

$$A_x = -\frac{3\pi^2 j_0}{c}\frac{xz}{r^5}, \quad A_y = -\frac{3\pi^2 j_0}{c}\frac{yz}{r^5}, \quad A_z = \frac{\pi^2 j_0}{c}\frac{r^2 - 3z^2}{r^5} - \frac{8\pi^3}{3c} j_0 \delta^3(\vec{r}). \tag{2.14}$$

We consider now the sequence of toroidal solenoids each turn of which is again a toroidal solenoid. The simplest configuration is obtained if we take the usual toroidal solenoid (upper part of Fig. 2) and install new toroidal solenoid into each of its turns. As a result, we arrive at the current configuration shown in Fig. 4. For this case

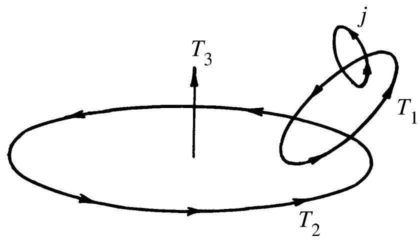

Fig. 4. The family of toroidal solenoids each turn of which is again toroidal solenoid (only the particular turns are shown)

$$\vec{j} \sim (\text{rot })^3\vec{T}(\vec{r}), \quad \text{div }\vec{T} \neq 0,$$

$$\vec{A} \sim \frac{4\pi}{c}\text{rot }\vec{T}(\vec{r}). \tag{2.15}$$

We see that for this current both the vector potential and the magnetic field differ from zero only in those space regions, where $\vec{T} \neq 0$. When the space region in which $\vec{T} \neq 0$ shrinks to a point, the vector potential and magnetic field differ from zero only at that point [18,19].

2.1.1. Currents, Magnetic Dipoles and Monopoles. We rewrite Eq. (2.14) in a condensed form:

$$\vec{A} = \frac{1}{r^3}[\frac{3}{r^2}\vec{r}(\vec{m}\vec{r}) - \vec{m}] + \frac{8\pi}{3}\vec{m}\delta^3(\vec{r}), \quad m_i = -\delta_{iz}\frac{\pi^2 j_0}{c}. \tag{2.16}$$

This equation is an analog of the well-known expression [24] for the magnetic field created by the magnetic dipole $\vec{m}$:

$$\vec{B} = \frac{1}{r^3}[\frac{3}{r^2}\vec{r}(\vec{m}\vec{r}) - \vec{m}] + \frac{8\pi}{3}\vec{m}\delta^3(\vec{r}). \tag{2.17}$$

Sometimes in a physical literature another representation of $\vec{B}$ is used [25]:

$$\vec{B} = \frac{1}{r^3}[\frac{3}{r^2}\vec{r}(\vec{m}\vec{r}) - \vec{m}] - \frac{4\pi}{3}\vec{m}\delta^3(\vec{r}). \tag{2.18}$$

This difference is due to the following reason [25]. If we identify the magnetic dipole with the electric current flowing in the infinitely small circular current loop, then VP is given by

$$\vec{A} = \frac{1}{cr^3}(\vec{m}_e \times \vec{r}), \quad \vec{m}_e = \frac{1}{2}\int(\vec{r} \times \vec{j})dV. \tag{2.19}$$

Applying to $\vec{A}$ the rot operator and using the identity (see, e.g., [26])

$$\frac{\partial}{\partial x_j}(\frac{x_i}{r^3}) = \frac{1}{r^3}(\delta_{ij} - 3\frac{x_i x_j}{r^2}) + \frac{4\pi}{3}\delta_{ij}\delta^3(\vec{r}), \tag{2.20}$$

we get (2.17). On the other hand, if we suggest that magnetic dipoles consist of the magnetic monopoles

$$\vec{m}_m = \int \rho_m \vec{r} dV, \quad \int \rho_m dV = 0, \tag{2.21}$$

then the magnetic induction is obtained from the scalar magnetic potential:

$$\vec{B} = -\vec{\nabla}\Phi_m, \quad \Phi_m = \frac{\vec{m}_m \vec{r}}{r^3}. \tag{2.22}$$

Again, using the differentiation rule (2.20) we arrive at (2.18). This means that different coefficients at $\delta^3(\vec{r})$ terms in (2.17) and (2.18) are due to different definitions of magnetic dipoles.

The expression (2.17) leads to the so-called hyperfine contact interaction derived by Fermi. It was observed experimentally by measuring the splitting of hydrogen atomic s levels. Above, we have used the fact that $\vec{B} = \vec{H}$ in the absence of medium.

Consider a semi-infinite cylindrical solenoid of the radius R formed either by the circular currents or the magnetic current dipoles (Fig. 5). The magnetic VP of a particular current j lying in the $z = z_0$ plane is given

$$\vec{A} = \frac{1}{c} j \int \frac{d\phi'}{|\vec{r} - \vec{r}'|}\vec{n}_{\phi'},$$

where $\vec{n}_\phi = \vec{n}_y \cos\phi - \vec{n}_x \sin\phi$ is the vector defining the current direction. The sole non-vanishing component of VP is

$$A_\phi(\rho, z) = $$

$$= \frac{2j}{c\sqrt{\rho R}} Q_{1/2}(\frac{\rho^2 + R^2 + (z - z_0)^2}{2\rho R}).$$

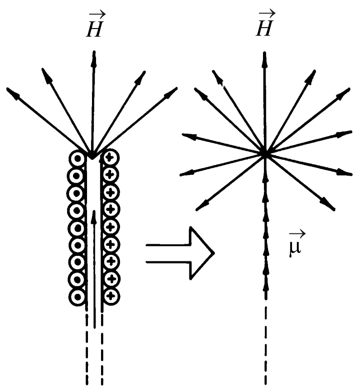

Fig. 5. The magnetic fields of the semi-infinite solenoid and the magnetized filament coincide with the field of magnetic monopole everywhere except for the position of solenoid or filament

Here $Q_\nu(x)$ is the Legendre function of the second kind. Using its asymptotic behaviour

$$Q_\nu(x) \to \sqrt{\pi}\Gamma(\nu + 1)/2^{\nu+1}\Gamma(\nu + 3/2)x^{\nu+1} \quad x \to \infty,$$

one obtains for the infinitely small radius R (or large distances):

$$\vec{A}=A_\phi\vec{n}_\phi,\quad A_\phi\approx\frac{\pi Rj\rho}{c\tilde{r}^3},\quad \tilde{r}=[x^2+y^2+(z-z_0)^2]^{1/2},\quad \text{div }\vec{A}=0. \quad (2.23)$$

The nonvanishing components of magnetic strength of a particular current coil are given by

$$H_x=\frac{\pi Rj}{c}\frac{\partial^2}{\partial x\partial z}\frac{1}{\tilde{r}},\quad H_y=\frac{\pi Rj}{c}\frac{\partial^2}{\partial y\partial z}\frac{1}{\tilde{r}},$$

$$H_z=\frac{\pi Rj}{c}[\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}}+4\pi\delta(x)\delta(y)\delta(z-z_0)],\quad \text{div }\vec{H}=0. \quad (2.24)$$

The magnetic field of semi-infinite solenoid is obtained by integrating $\vec{H}$ from $z_0=-\infty$ to $z_0=0$. This results in

$$\vec{H}=\frac{\pi Rj}{c}[\frac{\vec{r}}{r^3}+4\pi\vec{n}_z\delta(x)\delta(y)\Theta(-z)],\quad \text{div }\vec{H}=0. \quad (2.25)$$

Thus, an infinitely thin semi-infinite magnetized filament generates the field of a magnetic monopole everywhere except for the position of the filament itself. The equalities

$$\text{div }\vec{B}=0,\quad \int\int B_n dS=0$$

guarantee the absence of free magnetic charges. Due to the presence of the δ function term in (2.25) thus obtained monopoles are not true ones.

Earlier, in a qualitative manner these results were obtained in [22].

2.1.2. Interaction with External Electromagnetic Field. Now we explain how the current distributions just obtained interact with an external electromagnetic field ($\vec{E}_{\text{ext}},\vec{H}_{\text{ext}}$). In the absence of medium, $\vec{E}_{\text{ext}}$ and $\vec{H}_{\text{ext}}$ satisfy the Maxwell equations:

$$\text{div }\vec{B}_{\text{ext}}=0,\quad \text{div }\vec{D}_{\text{ext}}=4\pi\rho_{\text{ext}},\quad \text{rot }\vec{E}_{\text{ext}}=-\frac{1}{c}\frac{\partial\vec{B}_{\text{ext}}}{\partial t},$$

$$\text{rot }\vec{H}_{\text{ext}}=\frac{1}{c}\frac{\partial\vec{D}_{\text{ext}}}{\partial t}+\frac{4\pi}{c}\vec{j}_{\text{ext}},\quad \vec{D}=\vec{E},\quad \vec{B}=\vec{H}.$$

Let $\vec{j}$ be of the form

$$\vec{j}\sim(\text{rot })^n\vec{T}(\vec{r}),\quad \text{div }\vec{T}\neq 0, \quad (2.26)$$

where $\vec{T}$ is either confined to the finite region of space or decreases sufficiently fast at infinity. Then the interaction energy of this configuration with an external electromagnetic field is given by

$$U=-\frac{1}{c}\int\vec{j}\vec{A}_{\text{ext}}\sim\int\vec{T}(\text{rot })^{n-1}\vec{H}_{\text{ext}}dV. \quad (2.27)$$

The final answer is different for n even and odd. If $n = 2k+1$, then

$$U \sim (-1)^k \int \vec{T}(\frac{1}{c}\frac{\partial}{\partial t})^{2k}\vec{H}_{\rm ext}dV. \quad (2.28)$$

For $n = 2k+2$ one has

$$U \sim (-1)^k \int \vec{T}(\frac{1}{c}\frac{\partial}{\partial t})^{2k+1}\vec{E}_{\rm ext}dV. \quad (2.29)$$

For the distances large as compared to the dimensions of the particular current configuration, the interaction energy has the form:

$$U \sim \vec{t}\vec{E}^{(2k+1)}_{\rm ext} \quad \text{and} \quad U \sim \vec{t}\vec{H}^{(2k)}_{\rm ext}, \quad (2.30)$$

where the superscripts mean the corresponding time derivative and $\vec{t} = \int \vec{T}(\vec{r})dV$ is the vector depending on the geometrical dimensions of the treated current, magnetized or toroidized configuration, resp. The explicit form of $\vec{t}$ for the particular toroidal current configuration may be found in Ref. 27. In particular, for the toroidization given by (2.12) the vector $\vec{t}$ is: $\vec{t} = -2\pi^2 j_0 \vec{n}_z$.

The current configuration corresponding to $k = 0$ in (2.29) (poloidal current on the torus surface) was considered by Ya.B. Zeldovich [14] who referred to it as to the anapole. In the modern physical literature, the anapoles are associated with the radiationless charge-current sources (see, e.g., [15,28,29]), while the charge-current configurations corresponding to Eqs. (2.15) and (2.16) are referred to as toroidal moments [15,30,31]. The next terms ($k = 1$ in Eqs (2.28) and (2.29)) in the development of the interaction energy were written out in Ref. 19. The general form of interactions (2.28), (2.29) was given in [18].

Thus, we obtain the sequence of current configurations (or, magnetizations corresponding to them) which interact with the time-dependent magnetic or electric field. For example, the usual current loop interacts with an external magnetic field in the same way as the magnetic dipole orthogonal to it. The poloidal current shown in the upper part of Fig. 2, the magnetized ring in its middle part and the toroidal distribution in its lower part, all of them interact with the first derivative of the electric field.

We turn now to Fig. 4. The current distribution $\vec{J}$ shown in it, is obtained if instead of each turn of TS shown in the upper part of Fig. 2 we insert new TS. The current configuration $\vec{j}$, $\quad$ div $\vec{j} = 0$ of Fig. 4, the magnetization $\vec{T}_1$, $\quad$ div $\vec{T}_1 = 0$ distributed over the torus surface (in the same way as the current $\vec{j}$ in Fig. 2), the toroidization $\vec{T}_2$, $\quad$ div $\vec{T}_2 = 0$ confined to the interior of the torus (similarly to the magnetization $\vec{M}$ shown in Fig. 2) and the toroidization $\vec{T}_3$, $\quad$ div $\vec{T}_3 \neq 0$, all of them interact with the second derivative of the magnetic field. The words "interact with time derivative..." mean that the interaction energy has the form

(2.30), i.e., it is proportional to the time derivative (of the definite order) of the electric or magnetic field.

Obviously, the equivalence between the current distributions and magnetizations (toroidizations) established in this section is the straightforward generalization of the original Ampere hypothesis.

One may ask why Eqs. (2.28)—(2.30) do not contain the even time derivatives of the electric field and the odd derivatives of the magnetic one. It turns out [15,18,19,32] that missing terms describe the interaction of the closed configurations composed of electric dipoles. To see this, consider the electric dipoles distributed inside the space region S with the vector density $\vec{d}(\vec{r})$. Their interaction with an external EMF is given by

$$U \sim \int \vec{d}(\vec{r})\vec{E}_{\rm ext}(\vec{r})dV. \tag{2.31}$$

Let $\vec{d}(\vec{r})$ be distributed over the torus surface in the same way as the magnetization $\vec{M}$ shown in the middle part of Fig. 2. As in the treated case div $\vec{d} = 0$, the vector function can be represented in the form $\vec{d} = \text{rot } \vec{T}, \quad \text{div } \vec{T} \neq 0$, where $\vec{T} \sim \vec{n}_z T$ and T is defined by Eqs. (2.8), (2.9) and shown at the bottom of Fig. 2. Substituting $\vec{d}$ into (2.31) and integrating by parts one gets for the distances large as compared to the dimensions of the torus:

$$U \sim \frac{\partial \vec{H}_{\rm ext}(\vec{r}_0)}{\partial t}\vec{t}, \tag{2.32}$$

where $\vec{t} = \int \vec{T} dV$ and $\vec{r}_0$ is some point inside the torus. Further, let the electric dipoles be distributed over the torus surface like the current $\vec{J}$ in the upper part of Fig. 2. Then

$$\vec{d} = ({\rm rot})^2 \vec{T}(\vec{r}), \quad {\rm div}\ \vec{T} \neq 0, \tag{2.33}$$

where T is the same as before (see Eqs. (2.8), (2.9)). Substituting this $\vec{d}$ into (2.18) one easily obtains

$$U \sim \frac{\partial^2 \vec{E}_{\rm ext}(\vec{r}_0)}{\partial t^2}\vec{t}. \tag{2.34}$$

The continuation of this procedure ensures us that the interaction of electric dipoles with the external EMF is indeed the missing link in Eqs. (2.28)—(2.30). In particular, the term $\vec{t}(\partial \vec{H}/\partial t)$ describes the interaction of the closed electric dipole ring (see the middle part of Fig. 2 where the distribution M of the magnetic dipoles should be changed by the distribution of electric ones) with the time derivative of an external magnetic field. The corresponding experiments were performed by Tolstoy and Spartakov [16], their interpretation was given in [17].

2.2. Magnetizations and the Debye Potential Representation. According to the Helmholtz-Neumann theorem (see, e.g., [33]) an arbitrary vector function and, in particular, the current density can be presented as the sum of the longitudinal and transversal parts:

$$\vec{j} = \vec{j}_l + \vec{j}_t, \quad \text{rot } \vec{j}_l = 0, \quad \text{div } \vec{j}_t = 0,$$

$\vec{j}_l$ and $\vec{j}_t$ can be presented in the form

$$\vec{j}_l = \vec{\nabla}\Psi_1, \quad \vec{j}_t = \text{rot } (\vec{r}\Psi_2) + (\text{rot })^2(\vec{r}\Psi_3).$$

As a result, one arrives at

$$\vec{j} = \nabla\Psi_1 + \text{rot } (\vec{r}\Psi_2) + (\text{rot })^2(\vec{r}\Psi_3). \tag{2.35}$$

The functions Ψ_1, Ψ_2 and Ψ_3 are known as the Debye potentials. They were introduced by Debye [34] when evaluating the light pressure on a sphere of arbitrary material. Various other authors (Thomson, Mie Whittaker, Bromwich, Sommerfeld) applied these potentials to the electromagnetic problems. Earlier, Lamb used representation (2.35) when he studied fluid mechanics and electromagnetic problems [35-37].

Comparing (2.35) with (2.1), (2.2) we get

$$\Psi_1 = \Psi_3 = 0, \quad \Psi_2 = \frac{1}{2}\delta(\rho - d)\Theta(\frac{\pi}{2} - \theta). \tag{2.36}$$

The corresponding magnetization is given by

$$\vec{M}' = \vec{n}_r\delta(r - d)\Theta(\frac{\pi}{2} - \theta). \tag{2.37}$$

This magnetization covers the upper semisphere of the radius d and is directed along its radius (see Fig. 1). It certainly differs from the magnetization (2.3). The magnetizations $\vec{M}$ and $\vec{M}'$ are connected by the gradient transformation

$$\vec{M}' = \vec{M} + \nabla\chi, \quad \chi = -\Theta(d - z)\Theta(\frac{\pi}{2} - \theta),$$

i.e., the function χ differs from zero inside the upper semisphere. This equation means that the magnetizations $\vec{M}$ and $\vec{M}'$, despite their different functional forms, lead to the same observable effects. The reason for the appearance of different magnetizations is that the equation rot $\vec{M} = \vec{J}$ does not fix $\vec{M}$ uniquely. We note that the magnetic strength $\vec{H}$ satisfies almost the same equation rot $\vec{H} = \vec{J}$ but with the auxiliary condition div $\vec{H} = 0$. These two equations are sufficient for fixing $\vec{H}$. In general, the condition div $\vec{M} = 0$ is not imposed on $\vec{M}$. It turns out that the requirement for $\vec{M}$ to disappear in the nearest vicinity of $\vec{J}$ does not fix $\vec{M}$ unambiguously. On the other hand, if both rot $\vec{M}=\vec{j}$ and div $\vec{M}$ are known, then (see, e.g., [33])

$$4\pi\vec{M} = \text{rot} \int \text{rot } \vec{M}(\vec{r}\,')\frac{dV'}{|\vec{r}-\vec{r}\,'|} - \vec{\nabla}\int \frac{\text{div } \vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}. \qquad (2.38)$$

Obviously, rot $\vec{M}$ and div $\vec{M}$ define $\vec{M}$ up to a constant vector which is chosen to be zero in Eq. (2.38).

2.2.1. On the Inversion of the Debye Parametrization. An interesting question is the inversion of the Debye parametrization (2.35), i.e., the expression of Ψ_1, Ψ_2 and Ψ_3 functions in terms of the current density $\vec{j}$. Rearranging the terms in (2.35) one gets

$$\vec{j} = \vec{\nabla}\Psi_1' + (\vec{r}\times\vec{\nabla})\Psi_2' + \vec{r}\Psi_3'. \qquad (2.39)$$

This parametrization is used on the same footing as (2.35) (see, e.g., [38,39]). To find Ψ_i' one applies to $\vec{j}$ the div and rot operators

$$(\vec{r}\cdot\vec{j}) = r\frac{d\Psi_1'}{dr} + r^2\Psi_3', \quad r^2\text{div } \vec{j} = (\vec{r}\times\vec{\nabla})^2\Psi_1' + \frac{d}{dr}[r(\vec{r}\cdot\vec{j})],$$

$$\vec{r}\cdot\text{rot } \vec{j} = (\vec{r}\times\vec{\nabla})^2\Psi_2', \quad \vec{r}\cdot\text{rot rot } \vec{j} = -(\vec{r}\times\vec{\nabla})^2\Psi_3'.$$

As a result, the following equations are obtained for ψ_i'

$$(\vec{r}\times\vec{\nabla})^2\Psi_1' = r^2\text{div } \vec{j} - \frac{d}{dr}[r(\vec{r}\cdot\vec{j})],$$

$$(\vec{r}\times\vec{\nabla})^2\Psi_2' = \vec{r}\cdot\text{rot } \vec{j}, \quad (\vec{r}\times\vec{\nabla})^2\Psi_3' = -\vec{r}\cdot\text{rot rot } \vec{j}. \qquad (2.40)$$

Consider the equation

$$(\vec{r}\times\vec{\nabla})^2\Psi = f. \qquad (2.41)$$

Its Green function

$$G(\vec{n},\vec{n}\,') = -\sum_{l,m}\frac{1}{l(l+1)}Y_l^m(\vec{n})Y_l^{m*}(\vec{n}\,'), \quad \vec{n}=(\theta,\phi), \quad l\geq 1$$

satisfies the equation

$$(\vec{r}\times\vec{\nabla})^2 G = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi') - \frac{1}{4\pi}.$$

Thus,

$$\Psi = \int G(\vec{n},\vec{n}\,')f(\vec{r}\,')d\Omega'. \qquad (2.42)$$

The functions Ψ_i' are obtained if one substitutes the right-hand sides of (2.40) instead of f.

We still need the relations between Ψ_i and Ψ'_i. They are

$$\Delta\Psi_3 = \Psi'_3, \quad \Psi_2 = -\Psi'_2, \quad \Psi_1 = \Psi'_1 + (1 + \vec{r}\vec{\nabla})\Psi'_3. \tag{2.43}$$

These equations are easily resolved:

$$\Psi_3 = -\frac{1}{4\pi}\int \frac{1}{|\vec{r}-\vec{r}'|}\Psi'_3(\vec{r}')dV', \quad \Psi_1 = \Psi'_1 - \frac{1}{4\pi}(1+\vec{r}\vec{\nabla})\int \frac{1}{|\vec{r}-\vec{r}'|}\Psi'_3(\vec{r}')dV'. \tag{2.44}$$

Equations similar to (2.42)—(2.44) were proved with different degree of rigour many times (see, e.g., [40—45]).

2.2.2. Physical meaning of the Ψ functions. Now we are able to clarify the physical meaning of the functions Ψ_i defining the current density $\vec{j}$. For this purpose, we consider the interaction of the pure current density $\vec{j}$ (which corresponds to $\Psi_1 = 0$) with an external electromagnetic field defined by the vector potential $\vec{A}_{\rm ext}$

$$U = -\frac{1}{c}\int \vec{j}\vec{A}_{\rm ext}dV. \tag{2.45}$$

Substituting here $\vec{j}$, integrating by parts and assuming that $\vec{j}$ does not overlap with the space region S, where $J_{\rm ext} \neq 0$, we get

$$U = U_d + U_t, \quad U_d = -\frac{1}{c}\int \vec{r}\vec{H}\Psi_2 dV, \quad U_t = -\frac{1}{c^2}\int \vec{r}\dot{\vec{E}}\Psi_3 dV. \tag{2.46}$$

Here $\vec{H}$ and $\vec{E}$ are the electromagnetic strengths of the external field. The dot above the letter means the time derivative. Let the dimensions of S be small as compared with the distance from the sources of the external field. Then, the external fields varying rather slowly over S can be approximated by their values taken at some point $\vec{r}_0$ inside S:

$$U_d^{(1)} = -\frac{1}{c}\vec{H}(0)\int \vec{r}\Psi_2 dV, \quad U_t^{(1)} = -\frac{1}{c^2}\dot{\vec{E}}(0)\int \vec{r}\Psi_3 dV. \tag{2.47}$$

Here $\vec{H}(0) = \vec{H}(r_0)$, $\quad \vec{E}(0) = \vec{E}(r_0)$. It then follows that $\vec{\mu}_d = \int \vec{r}\Psi_2 dV$ and $\vec{\mu}_t = \int \vec{r}\Psi_3 dV$ are the magnetic dipole and toroidal moments (as they interact with external magnetic field and with time derivative of the external electric field, resp.) The next terms in the development of U_d are

$$U_d^{(2)} = -\frac{1}{c}\frac{\partial H_i(0)}{\partial x_k}\mu_{ik}, \quad \mu_{ik} = \int q_{ik}^{(2)}\Psi_2 dV, \quad q_{ik}^{(2)} = (x_i x_k - \frac{1}{3}\delta_{ik}r^2)$$

$$U_d^{(3)} = \frac{1}{2c}\frac{\partial^2 H_i(0)}{\partial x_k \partial x_j}\mu_{ijk} - \frac{1}{10c^3}\frac{\partial^2 \vec{H}(0)}{\partial t^2}\vec{\mu}_d^{(2)}, \quad \vec{\mu}_d^{(2)} = \int \vec{r}r^2\Psi_2 dV$$

$$\mu_{ijk} = \int q^{(3)}_{ijk}\Psi_2 dV, \quad q^{(3)}_{ijk} = [x_i x_j x_k - \frac{1}{5}(\delta_{ij}x_k + \delta_{ik}x_j + \delta_{jk}x_i)r^2]. \quad (2.48)$$

Obviously, μ_{ij} and μ_{ijk} are the quadrupole and octupole magnetic moments, resp. Thus, the function Ψ_2 describes the set of magnetic moments of different multipolarities. Similarly, one obtains the next terms in the expansion of U_t:

$$U^{(2)}_t = -\frac{1}{c^2}\frac{\partial \dot{E}_i(0)}{\partial x_k}t_{ik}, \quad t_{ik} = \int q^{(2)}_{ik}\Psi_3 dV$$

$$U^{(3)}_t = -\frac{1}{2c^2}\frac{\partial^2 \dot{E}_i(0)}{\partial x_k \partial x_j}t_{ijk} - \frac{1}{10c}(\frac{\partial}{c\partial t})^3 \vec{E}(0)\vec{\mu}^{(2)}_t, \quad \vec{\mu}^{(2)}_t = \int \vec{r}r^2\Psi_3 dV$$

$$t_{ijk} = \int q^{(3)}_{ijk}\Psi_3 dV. \quad (2.49)$$

This means that the function Ψ_3 describes the toroidal moments of higher multipolarities [15]. Their physical realization via the toroidal solenoids embedded into each other has been given in Ref. 46.

Let Ψ_2 be of the form

$$\Psi_2 = \Delta\Psi^{(1)}_2. \quad (2.50)$$

Then,

$$U_d = -\frac{1}{c^3}\frac{\partial^2 \vec{H}(0)}{\partial t^2}\int \vec{r}\Psi^{(1)}_2 dV. \quad (2.51)$$

It follows from this that such a current configuration interacts neither with the stationary nor with the linearly growing with time external magnetic field. It interacts with the magnetic field whose polynomial growth is not slower than t^2. Further, if Ψ_2 is presented in the form

$$\Psi_2 = (\Delta)^n\Psi^{(n)}_2, \quad n \geq 1, \quad (2.52)$$

then

$$U_d = -\frac{1}{c}(\frac{\partial}{c\partial t})^{2n}\vec{H}(0)\int \vec{r}\Psi^{(n)}_2 dV. \quad (2.53)$$

Such a current distribution interacts with the magnetic field whose polynomial growth is not slower than t^{2n}. If the external magnetic field grows as t^α (where α is not integer), then the interaction energy decreases as a function of time for $\alpha < 2n$ and increases for $\alpha > 2n$.

Now we turn to the toroidal moments. Taking into account the Maxwell equations and the fact that at large distances $j_{\rm ext}$ is not overlapping with S, we rewrite U_t as

$$U_t = -\frac{1}{c^2}\int \vec{r}\dot{\vec{E}}\Psi_3 dV = -\frac{1}{c^2}\dot{\vec{E}}(0)\int \vec{r}\Psi_3 dV. \quad (2.54)$$

Now let Ψ_3 be of the form

$$\Psi_3 = (\Delta)^n \Psi_3^{(n)}, \quad n \geq 1. \tag{2.55}$$

Then

$$U_t = -\frac{1}{c}(\frac{\partial}{c\partial t})^{2n+1}\vec{E}(0)\int \vec{r}\Psi_3{}^{(n)}dV. \tag{2.56}$$

This means that this current configuration interacts with the polynomial electric field which grows not slower than t^{2n+1}.

It then follows that the magnetized sample consisting of magnetic dipoles, all of which are united into the ring-like structures (thus, realizing toroidal magnetic moments), does not interact with a spatially uniform magnetic field $\vec{H}_0$ (although each of magnetic dipoles does interact with $\vec{H}_0$). This sample interacts with the rot of $\vec{H}_0$ (or, that is the same, with the time derivative of the electric field). The magnetized sample, all magnetic moments of which are organized into the toroidal moments of higher multipolarities, interacts with higher derivatives of the electric and magnetic fields. Thus, we obtain a one-to-one correspondence between the hierarchy of magnetic structures and the electromagnetic fields interacting with them. Probably, this selectivity of interaction can be used for the storage and ciphering of information. There are known first practical attempts in this direction (see,e.g., [47]).

When representing Ψ_2 or Ψ_3 in the form (2.52) or (2.55), we have implicitly assumed that $\Psi_2{}^{(n)}$ or $\Psi_3{}^{(n)}$ are confined to a finite space region or that they decrease sufficiently fast for large distances. This is required for the disappearance of surface integrals arising when the transition from (2.50) to (2.51), or from (2.55) to (2.56) is performed. In fact, every function Ψ can be represented in the form

$$\Psi = \Delta f, \quad \text{where} \quad f = -\frac{1}{4\pi}\int |\vec{r}-\vec{r}'|^{-1}\Psi(\vec{r}')dV',$$

but there is no guarantee that f decreases sufficiently fast (which is needed for its physical meaning). As a result, Eqs. (2.51), (2.53), (2.54) and (2.56) are valid for very specific current configurations.

We elucidate now which magnetic field corresponds to the choice of functions in the form (2.50) and (2.55). The convenient parametrization of VP corresponding to the stationary current density has been found in Ref. 45 (see Eqs. (2.10) and (2.14) therein). Substituting the current parametrization (2.35) into it, we get, outside the space region S to which the current density is confined,

$$\vec{A} = \frac{4\pi}{c}\sum \frac{1}{2l+1}r^{-l-1}(\vec{r}\times\vec{\nabla})Y_l^m \int r^l Y_l^{m*}\Psi_2 dV +$$
$$+\frac{4\pi}{c}\vec{\nabla}\sum \frac{l}{2l+1}r^{-l-1}Y_l^m \int r^l Y_l^{m*}\Psi_3 dV. \tag{2.57}$$

The magnetic field H disappears if

$$\int r^l Y_l^{m*} \Psi_2 dV = 0. \tag{2.58}$$

This relation is automatically satisfied if Ψ_2 has the form (2.52). The condition for the vector potential to vanish is (2.58) and

$$\int r^l Y_l^{m*} \Psi_3 dV = 0. \tag{2.59}$$

Obviously, it is satisfied if Ψ_3 has the form (2.55). Thus, the simultaneous fulfillment of Eqs. (2.52) and (2.55) leads to the disappearance both of the VP and magnetic field outside of the space region S to which the current configuration J is confined.

The representation (2.57) of VP, valid only outside S, disappears for specific current distributions defined by Eqs. (2.52) and (2.55). This does not mean that VP vanishes everywhere. Inside S one should use either the general formula

$$\vec{A} = \frac{1}{c} \int \frac{1}{|\vec{r} - \vec{r}'|} \vec{j}(\vec{r}')dV'$$

(as it was done in Sec. 3.1) or its development over the vector spherical harmonics. The latter certainly differs from (2.16) inside S. It follows from this that none of experiments performed outside S (including the Aharonov–Bohm-like experiments) can give information on the just mentioned current distribution inside S.

Obviously, Eqs. (2.53) and (2.56) generalize Eqs. (2.28) and (2.29) obtained earlier. In fact, Eqs. (2.53) and (2.56) contain two arbitrary functions Ψ_2 and Ψ_3 while only one function T enters into (2.28) and (2.29).

The inspection of Eqs. (2.46)—(2.56) ensures us that there are two degrees of freedom. First of them is due to the appearance of definite multipoles in the expansions of $\vec{E}$ and $\vec{H}$ (see Eqs. (2.47)—(2.49)). Let Ψ_2 (or Ψ_3) be transformed according to the particular representation of the rotation group with the fixed value l_2 (l_3) of an angular momentum. Then, only terms with these angular momenta survive in the expansion of U_d (or U_t). In particular, for $l_2 = l_3 = 1$ one gets:

$$U_d(l_2 = 1) = -\frac{1}{c}\vec{H}(0) \int \vec{r}\Psi_2 dV - \frac{1}{10c^3}\ddot{\vec{H}}(0) \int \vec{r}r^2 \Psi_2 dV -$$

$$-\frac{1}{280c^5}\vec{H}^{(4)}(0) \int \vec{r}r^4 \Psi_2 dV - \ldots .$$

$$U_t(l_3 = 1) = -\frac{1}{c^2}\dot{\vec{E}}(0) \int \vec{r}\Psi_3 dV - \frac{1}{10c^4}\vec{E}^{(3)}(0) \int \vec{r}r^2 \Psi_3 dV -$$

$$-\frac{1}{280c^6}\vec{E}^{(5)}(0)\int \vec{r}r^4\Psi_3 dV - ....$$

Let $l_2 = l_3 = 2$. Then,

$$U_d(l=2) = -\frac{1}{c}\frac{\partial H_i(0)}{\partial x_k}\mu_{ik} - \frac{1}{42c^3}\frac{\partial \ddot{H}_i(0)}{\partial x_k}\int q_{ik}^{(2)} r^2 \Psi_2 dV...$$

$$U_t(l=2) = -\frac{1}{c^2}\frac{\partial \dot{E}_i(0)}{\partial x_k}t_{ik} - \frac{1}{42c^4}\frac{\partial E_i^{(3)}(0)}{\partial x_k}\int q_{ik}^{(2)} r^2 \Psi_3 dV...$$

The second degree of freedom is due to the fact that for the given multipole it is possible to change the interaction with an external electromagnetic field by choosing Ψ_2 and Ψ_3 in the form (2.52) and (2.55), resp. To this end, we have a wonderful electromagnetic object with a number of interesting properties. It does not act on the test charge or magnetic needle. On the other hand, it interacts with a time-dependent external electromagnetic field. The difficult question on the equality of action and counter-action is beyond the scope of the present consideration. The question arises on the practical realizations of this object. One of them is the family of toroidal solenoids considered in Sec. 2.1 (when each turn of a solenoid is changed by a toroidal solenoid). The ambiguity in the magnetization choice implies that this realization is not unique.

2.2.3. Transition to the Point-Like Sources. For the point-like current source carrying the magnetic moment of the multipolarity l_2 and the toroidal moment of the multipolarity l_3 we have [13,18]

$$\Psi_2^{(k_2,l_2)} = f_2(t)\Delta^{k_2}(Q^{(l_2)}\vec{\nabla})\delta^3(\vec{r}), \quad \Psi_3^{(k_3 l_3)} = f_3(t)\Delta^{k_3}(Q^{(l_3)}\vec{\nabla})\delta^3(\vec{r}). \quad (2.60)$$

Here $f_2(t)$ and $f_3(t)$ are functions of time only,

$$(Q^{(l)}\vec{\nabla}) = Q^{(l)}_{i_1 i_2 \ldots i_l}\vec{\nabla}_{i_1}\vec{\nabla}_{i_2}...\vec{\nabla}_{i_l}$$

(the summation over the repeated indices is implied), $\vec{\nabla}_i = \frac{\partial}{\partial x_i}$. Further, $Q^{(l)}_{i_1 i_2 \ldots i_l}(n_k)$ is the traceless symmetric form of the order l of the unit vectors $n_i, \quad i = 1, ..., 3$ defining the orientation of the current configuration (e.g., $Q_i^1 = n_i, \quad Q_{i,j}^{(2)} = n_i n_j - \delta_{ij}/3$ (i=1,..,3), etc.). Then, the point-like analogues of the interaction energies defined by Eqs. (2.46) are given by

$$U_d = (-1)^{l_2+1} f_2(t) c^{-2k_2-1} l_2(\vec{v}_{l_2}\vec{H}^{(2k_2)}),$$

$$U_t = (-1)^{l_3+1} f_3(t) c^{-2k_3-1} l_3(\vec{v}_{l_3}\vec{E}^{(2k_3+1)}).$$

Here $\vec{v}$ is a vector with Cartesian components $(\vec{v}_l)_i = Q^{(l)}_{i i_2 \ldots i_l}\vec{\nabla}_{i_2}...\vec{\nabla}_{i_l}$. Superscripts of $\vec{E}$ and $\vec{H}$ denote the time derivatives. We write out a few particular terms. The choices $l_2 = 1, k_2 = 0$ and $l_3 = 1, k_3 = 0$ correspond to the dipole magnetic and toroidal moments, resp. Their interaction with the external EMF is given by

$$U_d^{(1)} = f_2(\vec{n}\vec{H})/c \quad \text{and} \quad U_t^{(1)} = f_3(\vec{n}\dot{\vec{E}})/c^2.$$

The quadrupole magnetic and toroidal moments correspond to $l_2 = 2, k_2 = 0$ and $l_3 = 2, k_3 = 0$, resp. The interaction energies are

$$U_d^{(2,0)} = -\frac{2}{c} f_2(\vec{n}\vec{\nabla})(\vec{n}\vec{H}), \quad U_t^{(2,0)} = -\frac{2}{c^2} f_3(\vec{n}\vec{\nabla})(\vec{n}\dot{\vec{E}}).$$

Further, for $l_2 = 1, k_2 = 1$ and $l_3 = 1, k_3 = 1$ one gets

$$U_d^{(1,1)} = \frac{1}{c^3} f_2(\vec{n}\ddot{\vec{H}}), \quad U_t^{(1,1)} = \frac{1}{c^3} f_3(\vec{n}\vec{E}^{(3)}).$$

Again we see that indices l and k describe different degrees of freedom. The index l defines the particular multipole, while k shows how much is "toroidized" the treated magnetic distribution.

2.2.4. Interaction of Charge Densities with an External Field. One may wonder why we have limited ourselves to the consideration of pure current configurations imbedded into the external electromagnetic field. The obvious generalization including charge density is (see, e.g., [24, 48])

$$U = \int \rho\phi_{\text{ext}} dV - \frac{1}{c}\int \vec{j}\vec{A}_{\text{ext}} dV. \qquad (2.61).$$

We rewrite this equation as

$$U = U_q + U_d + U_t. \qquad (2.62)$$

Here U_t and U_d were defined earlier (see Eqs. (2.46)) and

$$U_q = \int \rho\phi_{\text{ext}} dV - \frac{1}{c}\int \vec{j}_l\vec{A}_{\text{ext}} dV. \qquad (2.63)$$

Here $\vec{j}_l$ is the longitudinal part of $\vec{j}$ ($\vec{j}_l = \vec{\nabla}\Psi_1, \quad \text{div}\,\vec{j} = -\dot{\rho}$). Developing U_q we get [18, 31]:

$$U_q = e\phi_{\text{ext}}(0) + (\vec{d}\vec{\nabla})\phi_{\text{ext}}(0) + \frac{1}{2} q_{ik}\frac{\partial^2\phi_{\text{ext}}(0)}{\partial x_i \partial x_k} -$$

$$-\frac{1}{c}\dot{\vec{d}}\vec{A}_{\text{ext}}(0) - \frac{1}{2c}\dot{q}_{ik}\frac{\partial (A_{\text{ext}})_i(0}{\partial x_k} - \frac{1}{2}\vec{\mu}_l\vec{H}_{\text{ext}}(0) + \cdots \qquad (2.64)$$

where $\vec{d} = \int \rho\vec{r} dV, \quad q_{ik} = \int x_i x_k \rho dV$ and $\vec{\mu}_l = \frac{1}{2c}\int(\vec{r}\times\vec{j}_l)dV$ are electric dipole, electric quadrupole and longitudinal magnetic dipole moments, resp. Let the function Ψ_1 entering into the Debye parametrization (2.35) of $\vec{j}_l$ ($\vec{j}_l = \vec{\nabla}\Psi_1$) decrease sufficiently fast outside the region S to which the current j is confined. Then, $\vec{\mu}_l$ disappears. If, in addition, the external field is a pure induction (i.e., it is generated by the pure current density), then:

$$\phi_{\rm ext} = 0, \quad \vec{A}_{\rm ext} = \frac{1}{c}\int R^{-1}\vec{j}_{\rm ext}(\vec{r}', t - R/c)dV', \quad \vec{E}_{\rm ext} = -\dot{\vec{A}}_{\rm ext}/c,$$

$$\vec{H}_{\rm ext} = {\rm rot}\ \vec{A}_{\rm ext}, \quad R = |\vec{r} - \vec{r}'|.$$

It follows from this that

$$U_q = -\frac{1}{c}\dot{\vec{d}}\vec{A}_{\rm ext} - \frac{1}{2c}\frac{\partial A_i}{\partial x_k}\dot{q}_{ik}. \tag{2.65}$$

On the other hand, for the charge configuration carrying the electric dipole moment $\vec{d}$ and the quadrupole moment q_{ik} the appearance of the following interaction term

$$-\vec{d}\vec{E}_{\rm ext} - \frac{1}{2}\frac{\partial(E_{\rm ext})_i}{\partial x_k}q_{ik} \tag{2.66}$$

is intuitively expected. But these terms are absent in (2.65). According to K.H.Yang and D.H.Kobe (see, e.g., [49] and Refs. therein) this is due to the gauge noninvariance of the interaction energy:

$$\phi_{\rm ext} \to \phi'_{\rm ext} = \phi_{\rm ext} - \dot{\chi}/c, \quad \vec{A}_{\rm ext} \to \vec{A}'_{\rm ext} = \vec{A}_{\rm ext} + \vec{\nabla}\chi,$$

$$U \to U' = U - \frac{1}{c}\frac{d}{dt}\int \rho\chi dV.$$

It follows from this that the interaction energy is a gauge-invariant quantity for a pure current density ($\rho = 0$). There is no ambiguity mentioned above.

We note also that the transformed interaction energy differs from the original one by the total time derivative. This means that both of them should lead to the same equations of motion. In particular, after the insertion of (2.65) into the Lagrangian and subsequent variation relative to $\dot{d}_i$ and $\dot{q}_{ik}$ we obtain the usual expression for the Lorentz force acting on the dipole and quadrupole moments.

3. ON THE SUPERCURRENT ARISING IN A SUPERCONDUCTING RING

Consider the closed circular metallic ring C encircling the infinite cylindrical solenoid with a constant flux Φ_0 in it (Fig. 6). Suppose that initially there is no current in C. Let the ring C be cooled. At some temperature T_c its transition to the superconductive state occurs. The following two properties were observed experimentally [50-52] and explained theoretically [53-55]:

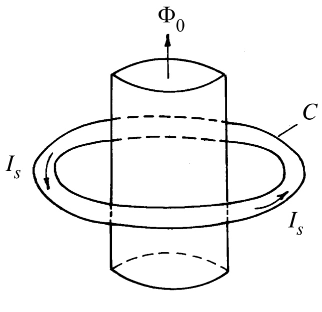

Fig. 6. The cylindrical solenoid with magnetic flux Φ_0 is encircled by the metallic ring C. When C becomes superconductive, the supercurrent I_s arises on its surface (although C is in the region where electromagnetic strengths are zero)

1) Magnetic field $\vec{H}$ vanishes inside C (it is, therefore, assumed that penetration depth is zero);

2) The total magnetic flux trapped by C turns out to be integer (in units $hc/2e$).

The appearance of the supercurrent flowing on the surface of C (despite its location in a field-free region, where $\vec{E} = \vec{H} = 0$) for $T < T_c$ was predicted in Refs. 56, 57. Indeed, as the flux inside the cylindrical solenoid is not in general integer, the supercurrent in C arises making the total flux to be integer. This supercurrent was, in fact, observed in Tonomura experiments (see Refs. 59, 62, where this fact was clearly stated). It is our aim to explicitly evaluate the distribution of supercurrent on the surface of C and the arising magnetic field. The density of the current J_s flowing on the torus C surface and providing $\vec{H} = 0$ inside C was obtained in [58]. Let the surface of C be given by

$$(\rho - d)^2 + z^2 = R^2.$$

It is convenient to introduce toroidal coordinates

$$\rho = a\frac{\sinh\mu}{\cosh\mu - \cos\theta}, \quad z = a\frac{\sin\theta}{\cosh\mu - \cos\theta}, \quad \phi = \phi. \tag{3.1}$$

For a given value of μ the points ρ, z, ϕ (where ρ, z, ϕ are defined in (3.1)) fill the surface of the torus with the parameters $d = a\coth\mu$, $R = a/\sinh\mu$, ($a = \sqrt{d^2 - R^2}$) (see Fig. 7). Let $\mu = \mu_0$ corresponds to the surface of C. Then, the surface current providing the vanishing of $\vec{H}$ inside C is given by [58]:

$$\vec{J}_s = \delta(\mu - \mu_0) j(\theta) \vec{n}_\phi$$

$$j(\theta) = -\frac{C_0}{2\sqrt{2}\pi^2 a^2} \frac{(\cosh\mu_0 - \cos\theta)^{5/2}}{\sinh\mu_0} \sum \frac{\cos n\theta}{1 + \delta_{n0}} [P^1_{n-1/2}(\cosh\mu_0)]^{-1}.$$

This current gives the following VP

$$A_\phi = C_0 \frac{\cosh\mu - \cos\theta}{\sinh\mu}$$

inside $C(\mu > \mu_0)$ and

$$A_\phi = C_0 \frac{\sqrt{2}}{\pi} (\cosh\mu - \cos\theta)^{1/2} \times$$

$$\times \sum \frac{\cos n\theta}{1+\delta_{n0}} \frac{1}{n^2-1/4} \frac{Q^1_{n-1/2}(\cosh \mu_0)}{P^1_{n-1/2}(\cosh \mu_0)} P^1_{n-1/2}(\cosh \mu)$$

outside $C(\mu < \mu_0)$. In particular, on the circle $z = 0, \rho = d - R$ (that is, for $\mu = \mu_0, \theta = \pi$) one gets

$$A_\phi = C_0 \frac{1+\cosh \mu_0}{\sinh \mu_0}.$$

The integral

$$\oint A_\phi dl = 2\pi C_0 a$$

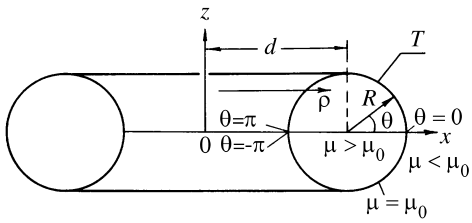

Fig. 7. Geometrical depiction of toroidal coordinates

taken along the same circle coincides with the flux Φ_s of the magnetic field produced by the supercurrent J_s. The total magnetic flux trapped by the superconducting ring is the sum of the cylinder solenoid flux Φ_0 and the supercurrent flux Φ_s:

$$2\pi C_0 a + \Phi_0 = \frac{hcn}{2e},$$

where n is the integer nearest to $2e\Phi_0/hc$. From this we find C_0

$$C_0 = -(\Phi_0 - \frac{hcn}{2e})/2\pi a.$$

The corresponding magnetic field is given by

$$H_\mu = \frac{(\cosh \mu - \cos \theta)^2}{a \sinh \mu} \frac{\partial}{\partial \theta} (\frac{\sinh \mu A_\phi}{\cosh \mu - \cos \theta}),$$

$$H_\theta = -\frac{(\cosh \mu - \cos \theta)^2}{a \sinh \mu} \frac{\partial}{\partial \mu} (\frac{\sinh \mu A_\phi}{\cosh \mu - \cos \theta}).$$

At large distances VP and field strengths fall like r^{-2} and r^{-3}, resp.:

$$A_\phi \sim \frac{2a^2}{\pi r^2} \sin \theta_s \text{const.}, \quad \text{const.} = C_0 \sum \frac{1}{1+\delta_{n0}} \frac{Q_{n-1/2}(\cosh \mu_0)}{P_{n-1/2}(\cosh \mu_0)}$$

$$H_r \sim \frac{4a^2}{\pi r^3} \cos \theta_s \cdot \text{ const.}, \quad H_\theta \sim \frac{2a^2}{\pi r^3} \sin \theta_s \cdot \text{ const.}$$

Here r and θ_s are usual spherical coordinates.

It turns out that the cooling of the ring C below the critical temperature T_c inevitably leads to the appearance of the magnetic field in a space surrounding

C. Obviously, the appearance of supercurrent in C is a pure quantum effect as the ring C is located in the region where $\vec{E} = \vec{H} = 0$. But for the creation of supercurrent in C the energy is needed. Where it comes from? Theory says [59] that for $T > T_c$ the electrons in C are in chaotic motion and the average current is zero. For $T < T_c$ the external vector potential correlates the phases of the electrons wave functions. As a result, the macroscopic flow of electrons arises in C.

It would be interesting to observe this supercurrent experimentally. This is not an easy task as the quantity $\Phi_0 - hcn/2e$ entering into the definition of the vector potential and field strengths is rather small (it is of the order $hc/2e$).

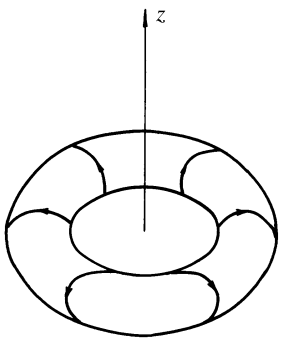

Fig. 8. The lines with arrows mean the poloidal current flowing on the torus surface

Theoretically, in Tonomura experiments the reason for the quantization of the total magnetic flux penetrated by the toroidal solenoid is the appearance (for $T < T_c$) of the poloidal supercurrent on the torus surface. But the poloidal supercurrent (Fig. 8) uniformly distributed over the torus suface produces no magnetic field outside the toroidal solenoid. Thus, the magnetic flux quantization observed in Tonomura experiments is only indirect evidence of the supercurrent existence. On the other hand, the supercurrent arising in a particular circular turn embracing either cylindrical or toroidal solenoids may be observed by the detection of the magnetic field created by this supercurrent. There are many experiments in which the dependence of the physical parameters (e.g., resistivity) of the multi-connected sample embracing the magnetic flux (but lying outside the region where $H = 0$) was studied as a function of magnetic flux value (see, e.g., Resource Letter [60]). As in Tonomura experiments, the arising supercurrent is not measured directly, but its existence is needed for the explanation of experimental data.

4. RADIATIONLESS TIME-DEPENDENT CHARGE-CURRENT SOURCES

It is usually believed that a charged body radiates, when it exhibits acceleration. We demonstrate now that this intuition is not always correct. We follow closely Ref. 5.

At first we clarify under which conditions the accelerated configuration of charge $\rho(\vec{r}, t)$ and current $\vec{j}(\vec{r}, t)$ densities does not radiate. The corresponding

VP is given by

$$\vec{A} = \frac{1}{c}\int \frac{1}{R}\vec{j}(\vec{r}', t')\delta(t' - t + \frac{R}{c})dV'dt'.$$

Here $R = |\vec{r} - \vec{r}'|$. Obviously, only the terms of the order not higher than r^{-1} contribute to the radiation field. Developing VP over the powers of r'/r and neglecting the terms of the order r'^2/r^2 and higher ones we get

$$\vec{A} = \frac{1}{cr}\int \vec{j}(\vec{r}', t')\delta(t' - t + \frac{r}{c} - \frac{1}{c}\vec{n_r\vec{r}'})dV'dt'. \tag{4.1}$$

Here $\vec{n}_r = \vec{r}/r$. Now, making the Fourier transformation of $\vec{j}$

$$\vec{j}(\vec{r}, t) = \int \vec{j}(\vec{k}, \omega)\mathrm{e}^{i(\vec{k}\vec{r} - \omega t)}d^3k d\omega$$

and inserting this into (4.1) one gets

$$\vec{A} = \frac{(2\pi)^3}{cr}\int \vec{j}(\vec{n}_r\frac{\omega}{c}, \omega)\mathrm{e}^{-i\omega(t - r/c)}d\omega.$$

Obviously, $\vec{A}$ disappears if

$$\vec{j}(\vec{n}_r\frac{\omega}{c}, \omega) = 0. \tag{4.2}$$

Now, let $\vec{j}(\vec{r}, t)$ be a periodical function of time with a period T. Most easily this may be achieved if one chooses

$$\vec{j}(\vec{k}, \omega) = \sum \vec{j}_n(\vec{k})\delta(\omega - \omega_n), \quad \rho(\vec{k}, \omega) = \sum \frac{1}{\omega_n}(\vec{k}\vec{j}_n(\vec{k}))\delta(\omega - \omega_n),$$

$$\omega_n = \omega_1 n, \quad \omega_1 = 2\pi/T.$$

Then,

$$\vec{j}(\vec{r}, t) = \sum_n \int d^3k \mathrm{e}^{i(\vec{k}\vec{r} - \omega_n t)}\vec{j}_n(\vec{k}).$$

From this we find $\vec{j}_n(\vec{k})$

$$\vec{j}_n(\vec{k}) = \frac{1}{(2\pi)^3}\frac{1}{T}\int \vec{j}(\vec{r}, t)\mathrm{e}^{-i(\vec{k}\vec{r} - \omega_n t)}d^3x dt.$$

It turns out that the condition (4.2) reduces to

$$\vec{j}_n(\vec{n}_r\frac{\omega}{c},\ \omega) = 0. \tag{4.3}$$

Let $\rho(\vec{r},t)$ be centered around the time-dependent position $\vec{a}(t)$ which is the periodic function of time. That is, we suppose $\rho(\vec{r},t)$ and $\vec{j}(\vec{r},t)$ to be of the form

$$\rho(\vec{r},t) = ef(\vec{r}-\vec{a}(t)), \quad \vec{j}(\vec{r},t) = e\dot{\vec{a}}f(\vec{r}-\vec{a}(t)). \tag{4.4}$$

The Fourier components are given by

$$\vec{j}(\vec{k},t) = \frac{1}{(2\pi)^3}\int \vec{j}(\vec{r},t)\mathrm{e}^{-i\vec{k}\vec{r}}d^3x = \frac{1}{(2\pi)^3}e\dot{\vec{a}}\mathrm{e}^{i\vec{k}\vec{a}(t)}\int f(\vec{z})\mathrm{e}^{-i\vec{k}\vec{z}}d^3z.$$

Here $\vec{z} = \vec{r} - \vec{a}(t)$. Let $f(\vec{z})$ be spherically-symmetric: $f(\vec{z}) = f(|\vec{z}|) = f(z)$. Then,

$$I(k) = \int f(\vec{z})\mathrm{e}^{-i\vec{k}\vec{z}}d^3z = \int f(z)z^2dz\frac{\sin kz}{kz}.$$

Obviously, this expression should vanish for $k = \omega_n/c$. Consider the particular choices of $f(z)$. Let

$$f(z) = \frac{1}{4\pi r^2}\delta(z-R). \tag{4.5}$$

Then,

$$I_n = I(\omega_n) = \frac{c}{\omega_n R}\sin\frac{\omega_n R}{c}.$$

It is seen that I_n disappears if

$$\omega_1 R = l\pi c, \tag{4.6}$$

where l is integer. This means that charge-current distributions (4.4) and (4.5), where $\vec{a}_n(t)$ is an arbitrary vector periodical function of time, do not radiate if the condition (4.6) is fulfilled.

The charge-current configuration (4.5) corresponds to the surface distribution. The nonradiating volume distributions are also easily found. Let

$$\rho(y) = A\Theta(b-y)z^{-1}\cos\omega_m y, \tag{4.7}$$

here A is a constant, $\Theta(x)$ is a step function, m is an integer. Then,

$$I_{n,m} = 4\pi A\int_0^b dy\sin\omega_n y\cos\omega_m y =$$

$$= 2\pi Ac[\frac{1-\cos(\omega_n-\omega_m)\frac{b}{c}}{\omega_n-\omega_m} + \frac{1-\cos(\omega_n+\omega_m)\frac{b}{c}}{\omega_n+\omega_m}].$$

Clearly, $I_{n,m}$ disappears and, correspondingly, accelerated volume distribution (4.7) does not radiate if the condition $\omega_1 b = 2\pi c$ is satisfied.

In the same way, the spherically-symmetric configuration

$$\rho(y) = \sum_m A_m \Theta(b-y) y^{-1} \cos\omega_m y$$

does not radiate provided $\omega_1 b = 2\pi c$.

The examples of nonradiating spherically-nonsymmetric distributions may be also presented. Let

$$\rho(\vec{z}) = R_{l,m}(z) Y_{lm}(\theta_z, \phi_z), \tag{4.8}$$

where $R_{l,m}(z) = A_{lm}\Theta(b-z), \quad A_{lm} = const$. Then, condition for the absence of radiation is

$$I_{lm}(\omega_n) = A_{lm} \int_0^b dz z^2 j_l(\frac{\omega_n z}{c}) = 0$$

($j_l(x)$ is the spherical Bessel function). For $l = 1$ one gets

$$I_{1m}(\omega_n) \sim A_{1m}[2(1-\cos\frac{\omega_n b}{c}) - \frac{\omega_n b}{c}\sin\frac{\omega_n b}{c}]\frac{c^3}{\omega_n^3}.$$

It is easy to see that $I_{1m} = 0$ if $\omega_1 b = 2\pi c$.

Another example [9] of nonradiating charge distribution is the uniformly charged sphere

$$\rho = \sigma\delta(r-R), \quad \sigma = e/4\pi R^2,$$

which oscillates around a fixed axis with the angular velocity

$$\vec{\Omega} = U(\omega)\cos\omega t\, \vec{n}_\omega.$$

Here $\vec{n}_\omega$ is the constant unit vector. The current is given by

$$\vec{j} = \sigma\delta(r-R)(\vec{\Omega}(t) \times \vec{r})\cos\omega t.$$

The corresponding VP is

$$\vec{A}_\omega = \frac{ekR}{cr}(\vec{\Omega}(t) \times \vec{r}) j_1(kr_<) h_1(kr_>)\cos\omega t,$$

where $r_< = \min\,(r,R), \quad r_> = \max\,(r,R)$; $j_1(x)$ and $h_1(x)$ are the spherical Bessel and Hankel functions of the first order. Thus, outside the charged sphere one gets

$$\vec{A}_\omega = \frac{ekR}{cr}(\vec{\Omega}(t) \times \vec{r}) j_1(kR) h_1(kr)\cos\omega t.$$

We observe that the considered oscillating charge distribution does not radiate when $x = kR$ coincides with the zero of $j_1(x)$, i.e., when x satisfies the equation $\tan x = x$.

We summarize: There are charge distributions of the finite extension which do not radiate when they exhibit an arbitrary periodic accelerated motion described by the time-dependent vector $\vec{a}(t)$.

As we have seen the condition for the nonradiation of the treated charge-current configuration is

$$\vec{j}(\vec{k}, \omega)|_{k=\omega/c} = 0.$$

Now we apply this condition to the uniformly moving charge. In this case $\vec{j}(\vec{r}, t) = \vec{v}\rho(\vec{r} - \vec{v}t)$ and

$$\vec{j}(\vec{k}, \omega) = 2\pi\vec{v}\delta(\omega - \vec{k}\vec{v})\rho(\vec{k}).$$

Consider at first the motion in a vacuum. For $\omega = ck$ one gets

$$\delta(\omega - \vec{k}\vec{v}) = \omega^{-1}\delta(1 - \beta\cos\theta), \quad \beta = v/c.$$

As in a vacuum $\beta < 1$, the argument of the δ-function is always greater than 1 and the nonradiation condition is satisfied.

Let the charged particle move uniformly in the medium. Then, conditions for the absence of radiation are: $\omega = kc_n, \quad c_n = c/n$ (c_n is the light velocity in the medium, n is the refraction index) and

$$\vec{j}(\vec{k}, kc_n) = 2\pi\vec{v}\rho(\vec{k})\delta(1 - \beta_n\cos\theta) = 0.$$

It is seen that nonradiation condition is satisfied everywhere except for the angle $\cos\theta_n = 1/\beta_n$. For the arbitrary density the quantity

$$\rho(\vec{k})|_{\omega=kc_n, \cos\theta=1/\beta_n}$$

differs from zero and this is just the reason for the appearance of the Vavilov-Cherenkov radiation. This takes place, e.g., for the point-like charge and for an arbitrary spherically-symmetric charge distribution confined to the finite region of space. Now we prove the existence of the nonradiating finite charge distributions moving with the superluminal velocity in the medium. We choose ρ to be in the form

$$\rho(\vec{r}) = \rho(r)P_l(\cos\theta_{rv}),$$

where θ_{rv} is the angle between the charge velocity $\vec{v}$ and the radius-vector $\vec{r}$, P_l is the Legendre polynomial. The Fourier transform of this density is

$$\rho(\vec{k}) = \frac{1}{2\pi^2}(-i)^l P_l(\cos\theta_{kv}) \int j_l(kr)\rho(r)r^2\,dr.$$

As the Cherenkov radiation differs from zero only for the definite angle $\cos\theta_{kv} = 1/\beta_n$ the nonradiation condition is

$$P_l(1/\beta_n) = 0.$$

Let $l = 2$. $P_2(x)$ has zero at $x = 1/\sqrt{3}$ that corresponds to $\beta_n = \sqrt{3}$. This means that the charge distribution $\rho_2(\vec{r}) = \rho(r)P_2(\cos\theta_{rv})$ does not radiate if it moves with the velocity $\beta_n = \sqrt{3}$. Similarly, the charge distribution $\rho(r)P_3(\cos_{rv})$ does not radiate when its velocity in a medium is equal to $\beta_n = \sqrt{5/3}$. Further, there are two velocities for which the charge distribution $\rho(r)P_4(\cos\theta_{rv})$ does not radiate. These interesting results were obtained in Ref. 7.

Consider the current $\vec{j}$ flowing on the cylinder surface:

$$\vec{j} = \vec{n}_\phi j \delta(\rho - R).$$

Let j be a periodical function of time: $j = j_0 \cos\omega t$. Then, outside the cylinder the VP and field strengths disappear for a discrete set of frequencies satisfying equation [6, 61]:

$$J_1(kR) = 0$$

(J_1 is the Bessel function). The same is true for the sphere. Let on its surface (of the radius R) flows the current

$$\vec{j} = \vec{n}_\phi j\ P_l^1(\cos\theta)\ \delta(\rho - R)$$

which is a periodical function of time ($j = j_0 \cos\omega t$). Then, the VP and field strengths disappear outside the sphere for the infinite set of frequencies satisfying the equation

$$j_1(kR) = 0 \quad j_l(x) = (\pi/2x)^{1/2} J_{l+1/2}(x).$$

In Ref. 22 a point-like electric solenoid was considered by using the following nonstatic point charge and current densities

$$\rho = D\exp(-i\omega t)\Delta\delta^3(\vec{r}), \quad \vec{j} = i\omega D\exp(-i\omega t)\vec{\nabla}\delta^3(\vec{r}). \tag{4.9}$$

Here D is a constant. Under the electric solenoid we mean a charge-current configuration generating magnetic field equal to zero everywhere and the electric field confined to the finite region of space. The corresponding electromagnetic potentials are:

$$\Phi = -\exp(-i\omega t)D[4\pi\delta^3(\vec{r}) + \frac{k^2}{r}\exp(ikr)], \quad \vec{A} = ikD\exp(-i\omega t)\vec{\nabla}\frac{\exp(ikr)}{r}. \tag{4.10}$$

Only the electric field is nonzero:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = 4\pi D\exp(-i\omega t)\vec{\nabla}\delta^3(\vec{r}). \tag{4.11}$$

These relations are easily generalized to the case of charge and current distributions of finite size [12,32]. We choose ρ and $\vec{j}$ in the form

$$\rho = \exp(-i\omega t)\Delta f, \quad \vec{j} = i\omega\exp(-i\omega t)\vec{\nabla} f. \tag{4.12}$$

The following potentials and field strengths correspond to these sources:

$$\Phi = -\exp(-i\omega t)[4\pi f + k^2 \int G(\vec{r},\vec{r}')f dV'], \quad \vec{A} = ik\exp(-i\omega t)\vec{\nabla}\int G f dV',$$

$$\vec{E} = 4\pi\exp(-i\omega t)\vec{\nabla}f, \quad \vec{H} = 0, \quad G = \frac{\exp(ik|\vec{r}-\vec{r}'|}{|\vec{r}-\vec{r}'|}). \tag{4.13}$$

The factor $\exp(-i\omega t)$ will be omitted below when it is obvious.

Equations (4.9)—(4.11) are obtained for the choice $f = D\delta^3(\vec{r})$. It follows from (4.13) that if the function f is nonzero inside some region of space, $\vec{H} = 0$ everywhere, while $\vec{E} \neq 0$ only in the region where $f \neq 0$. On the other hand, the electromagnetic potentials differ from zero everywhere. Thus, Eqs. (4.12), (4.13) realize a nonstatic electric solenoid. In particular, f can be chosen to be nonzero inside the torus $(\rho - d)^2 + z^2 = R^2$. For this it is enough to take $f = D\Theta(R - \sqrt{(\rho-d)^2 + z^2})$, where D is a constant. As an example, consider a spherical capacitor which is obtained for a special choice of the function f. We have

$$\rho = \frac{e}{4\pi r^2}[\delta(r-r_1) - \delta(r-r_2)], \quad \vec{j} = \frac{i\omega e}{4\pi r^3}\vec{r}\Theta(r-r_1)\Theta(r_2-r), \quad r_1 < r_2. \tag{4.14}$$

This spherical capacitor consists of two oppositely charged spheres and a radial current between them. Using the general expressions

$$\Phi = \int G\rho(\vec{r}')dV', \quad \vec{A} = \frac{1}{c}\int G\vec{j}(\vec{r}')dV',$$

we easily find the scalar and vector potentials (only the radial component of the vector potential is nonzero):

$$\Phi = ikeh_0^{(1)}(kr)[j_0(1) - j_0(2)], \; A_r = -keh_1^{(1)}(kr)[j_0(1) - j_0(2)] \text{ for } r > r_2,$$

$$\Phi = ikej_0(kr)[h_0^{(1)}(1) - h_0^{(1)}(2)], \; A_r = -kej_1(kr)[h_0^{(1)}(1) - h_0^{(1)}(2)] \text{ for } r < r_1$$

and

$$\Phi = ike[h_0^{(1)}(kr)j_0(1) - j_0(kr)h_0^{(1)}(2)],$$

$$A_r = ek[j_1(kr)h_0^{(1)}(2) - h_1^{(1)}(kr)j_0(1)] - \frac{ie}{kr^2}$$

for $r_1 < r < r_2$. Here we put

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+1/2}(x), \quad h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}}H_{l+1/2}^{(1)}(x), \quad j_l(1) = j_l(kr_1), \quad \text{etc.}$$

The magnetic field is zero everywhere, while the electric field $\vec{E} = e\vec{r}/r^3$ differs from zero only inside the spherical capacitor (i.e., for $r_1 < r < r_2$).

It is seen that the waves of electromagnetic potentials appear outside a non-static electric solenoid. The question arises on the physical meaning of such waves and the possibility to detect them experimentally. Let the region S in which $\vec{E}$ and $\vec{H}$ are nonzero be inaccessible to observation. Can the observer located outside S verify the existence of electromagnetic potential waves? Since $\vec{E} = \vec{H} = 0$ in these waves, they do not carry an energy. Therefore, they can be detected only at the quantum level. This is the case because the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \quad H = -\frac{\hbar^2}{2m}(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A})^2 + e\Phi$$

describing the scattering of charged particles on the waves of electromagnetic potentials involves the potentials Φ and $\vec{A}$ rather than the fields $\vec{E}$ and $\vec{H}$. The gauge transformation

$$\Psi \to \Psi' = \Psi\exp(-ie\chi/\hbar c), \quad \chi = ik\exp(-i\omega t)\int GfdV' \tag{4.15}$$

eliminates the electromagnetic potentials outside S. If χ is a single-valued function outside S, Eq. (4.15) is a unitary transformation between the single-valued wave functions in the presence and absence of electromagnetic potentials outside S. In this case the presence of electromagnetic potential waves outside S does not lead to observable consequences. On the other hand, if χ is discontinuous outside S (which, in turn, depends on the choice of the source function f), the possibility in principle arises of observing electromagnetic potential waves, e.g., by observing a phase difference acquired by the wave function of a charged particle as the particle travels around a closed contour. A necessary condition is that the region of space accessible to charged test particles be multiply connected (as nontrivial electromagnetic potentials corresponding to $\vec{E} = \vec{H} = 0$ are allowed only in non-simply connected spaces).

Up to now we considered only those nonradiating charge-current sources outside which electromagnetic strengths $\vec{E}, \vec{H}$ disappeared. No attention was payed to the existence of electromagnetic potentials in the surrounded space. In the next section we will be interested in studying those charge-current distributions outside which $\vec{E} = \vec{H} = 0$, but $\vec{A}, \Phi \neq 0$. To be observable the nonvanishing electromagnetic potentials should be nontrivial, i.e., unremovable by a gauge transformation. The static analog of such distributions is TS with a constant current in its winding. Outside such TS $\vec{E} = \vec{H} = \Phi = 0$, but $\vec{A} \neq 0$. This static VP was observed in Tonomura experiments [62]. The existence of the nontrivial nonstatic electromagnetic potentials with mentioned above properties makes the observation of the time-dependent Aharonov–Bohm effect to be possible.

5. ELEMENTARY TIME-DEPENDENT TOROIDAL SOURCES

Interest in the time-dependent currents flowing in the toroidal coils is due to the following remark made by James Clerk Maxwell in his memoir "On Physical Lines of Force" [63]:

"Let B, Fig. 3, be a circular ring of uniform section, lapped uniformly with covered wire. It may be shown that if an electric current is passed through this wire, a magnet placed within the coil of wire will be strongly affected, but no magnetic effect will be produced on any external point. The effect will be that of magnet bent round till its two poles are in contact.

If the coil is properly made, no effect on a magnet placed outside it can be discovered, whether the current is kept constant or made to vary in strength; but if a conducting wire C be made to embrace the ring any number of times, an electromotive force will act on this wire whenever the current in the coil is made to vary; and if the circuit be closed, there will be an actual current in the wire C."

Figure 3 mentioned in this passage shows the torus with a poloidal winding on its surface (see our Fig. 7). At the present time, it is known that in general this Maxwell assertion is not correct. It turns out that for the time-dependent current in the toroidal coil the electromagnetic field strengths appear outside it. Qualitatively this was shown by Mitkevich [64] and Page [65]. The corresponding experiments were performed by Mitkevich [64], Ryazanov [66], Bartlett and Ward [67] and many others. The quantitative results were obtained in Ref. 58 where the electromagnetic fields were evaluated for a number of time dependences of the current flowing in the toroidal coil. After all, experimentalists widely use the toroidal transformers for their own purposes without philosophizing on this subject. The sole exception for which Maxwell's claim holds is the current linearly rising in time which flows in the toroidal coil. In this case $H = 0$ and E is independent of time outside the torus (see, e.g., Miller [22, 68]). The question of the energy transfer into the wire C embracing the torus was considered by Heald [69] (the difficulty is that the Poynting vector equals zero for the linear growing current).

In Sec. 3, we have studied the electromagnetic field of the static toroidal-like configurations, their interactions with the external electromagnetic field and possible physical applications. It is our nearest goal to study nonstatic current configurations. Probably, it would be appropriate to explain the meaning of the words "elementary toroidal sources" in the title of this section. The words "toroidal source" mean the poloidal current flowing in the winding of the toroidal solenoid (TS), which in turn may be an element of a more complex configuration. When the dimensions of this configuration tend to zero, we obtain an "elementary toroidal source". The reason for the treatment of an elementary toroidal source is due to the considerable simplification of the theoretical consideration. The TS with finite dimensions has a number of nontrivial topological properties (see, e.g., reviews [12, 61, 70]). Suppose that these properties survive when the TS dimensions tend to zero. Thus, if we find some interesting property for the elementary toroidal source, there is a chance for it to be survived for the finite toroidal configuration. This is confirmed for the simplest toroidal configurations for which the analytical solutions can be found. As an example, mention the configuration consisting of the TS with a linearly growing current flowing in its winding and the double charged layer lying at the hole of TS (see Sec. 5.3). Outside this configuration, there is time-dependent vector potential. The electromagnetic strengths everywhere disappear except for the static electric field filling the torus hole. Thus, the possibility arises to perform a time-dependent Aharonov–Bohm-like experiment. However, the linear time-dependence of the current is unrealistic. It is the aim of this consideration to find elementary charge-current configurations possessing radiationless properties mentioned above but with a rather arbitrary time dependence.

The plan of our exposition is as follows. The radiation of elementary time-dependent toroidal-like configurations, in the winding of which the time-dependent current flows, is studied in Sec. 5.1. It turns out that two different branches of these configurations generate essentially different electromagnetic fields. On the other hand, the current sources of the same branch generate the same electromagnetic field if their time dependencies are properly adjusted. In Secs. 5.2, 5.3 we give the examples of the elementary radiationless charge-current source having the property that electromagnetic field strengths disappear outside it, but the time-dependent potentials survive there. In Sec. 5.4 examples are given of current configurations generating the static electric field adequately described by the electric vector potential rather than by the scalar one. In Sec. 5.5 these results are applied to the consideration of the time-dependent Aharonov–Bohm effect. The extended toroidal-like current sources are considered in Sec. 5.6. By using the Neumann-Helmholtz parametrization for the current density the convenient formulas for the time-dependent electromagnetic fields are obtained. Basing on them, the more general elementary radiationless charge-current sources of different multipolarities are constructed in Sec. 5.7. These elementary configurations have their finite counterparts. Those of them which can be treated analytically are radiationless and have nontrivial electromagnetic potentials outside them. Although the electromagnetic field of more complicated finite configurations cannot be obtained in a closed form, the electromagnetic field of their infinitesimal analogues can. The well prescribed rule for the construction of the elementary radiationless configurations found in Sec. 5.7 suggests that their finite radiationless counterparts will also possess nontrivial electromagnetic potentials. Short discussion of the results obtained and their summary are given in Sec. 5.8.

5.1. The Radiation of the Elementary Toroidal Sources. *5.1.1. A Pedagogical Example: Time-Dependent Circular Current.* According to the Ampere hypothesis the distribution of the magnetic dipoles $\vec{M}(\vec{r})$ is equivalent to the current distribution $\vec{J}(\vec{r}) = \text{rot } \vec{M}(\vec{r})$. For example, the circular current flowing in the $Z = 0$ plane (upper part of Fig. 1)

$$\vec{J} = I\vec{n}_\phi \delta(\rho - d)\delta(z) \tag{5.1}$$

is equivalent to the magnetization

$$\vec{M} = I\vec{n}\Theta(d - \rho)\delta(z) \tag{5.2}$$

different from zero in the same plane and directed along its normal $\vec{n}$ ($\Theta(x)$ is a step function). When the radius d of the circumference along which the current flows tends to zero, the current $\vec{J}$ becomes ill-defined (it is not clear what does the vector $\vec{n}_\phi$ mean at the origin). On the other hand, the vector $\vec{M}$ is still well-defined. In this limit the elementary current (5.1) turns out to be equivalent to the magnetic dipole oriented normally to the plane of this current. It is convenient to introduce $I/\pi d^2$ instead of I in Eqs. (5.1), (5.2). Then, in the limit $d \to 0$ one gets

$$\vec{M} = I\vec{n}\delta^3(\vec{r}), \quad (\delta^3(\vec{r}) = \delta(\rho)\delta(z)/2\pi\rho) \tag{5.3}$$

and

$$\vec{J} = I\text{rot } (\vec{n}\delta^3(\vec{r})). \tag{5.4}$$

Eqs. (5.3) and (5.4) define the magnetization and current density corresponding to the elementary magnetic dipole. These questions were considered in detail in Sec. 3. Now let the intensity of the elementary current change with time

$$\vec{J}_0 = f_0(t)\text{rot } (\vec{n}\delta^3(\vec{r})) \tag{5.5}$$

(the factor I is absorbed into f_0). The VP corresponding to this current is elementarily obtained:

$$\vec{A}_0 = -\frac{D_0}{c^2 r^2}(\vec{r} \times \vec{n}), \quad D_0 = D(f_0) = \dot{f}_0 + \frac{c}{r} f_0. \tag{5.6}$$

From now the time derivative will be denoted either by the point above the letter or (especially for higher derivatives) by the superscripts. For example, $f^{(2)} = \ddot{f} = d^2 f/dt^2$. The argument of the f functions, if not indicated, means $t - r/c$ everywhere in this section. The electromagnetic field strengths are

$$\vec{E}_0 = \frac{1}{c^3 r^2}(\vec{r} \times \vec{n})\dot{D}_0, \quad \vec{H}_0 = \frac{\vec{r}\vec{n}}{c^3 r^3}\vec{r} F_0 - \frac{1}{c^3 r}\vec{n} G_0, \tag{5.7}$$

where for brevity we put

$$F_0 = F(f_0) = \ddot{f}_0 + 3\frac{c}{r}\dot{f}_0 + 3\frac{c^2}{r^2} f_0, \quad G_0 = G(f_0) = f_0^{(2)} + \frac{c}{r}\dot{f}_0 + \frac{c^2}{r^2} f_0.$$

The flux of the electromagnetic energy through the sphere of the radius r is

$$S = \int P_r r^2 d\Omega = \frac{2}{3c^5}\dot{D}_0 G_0, \quad \vec{P} = \frac{c}{4\pi}(E_0 \times H_0). \tag{5.8}$$

This flux is positive for large distances and determined by the second derivative of f_0, $\quad S_0 = \frac{2}{3c^5}|\ddot{f}_k|^2$. These results are well-known and may be found in many text-books (see, e.g., Stratton [71]).

In the subsequent consideration the following notation will be used

$$D_k = D(f_k) = \dot{f}_k + \frac{c}{r}f_k, \quad F_k = F(f_k) = \ddot{f}_k + 3\frac{c}{r}\dot{f}_k + 3\frac{c^2}{r^2}f_k,$$

$$G_k = G(f_k) = f_k^{(2)} + \frac{c}{r}\dot{f}_k + \frac{c^2}{r^2}f_k.$$

From the classical electrodynamics it is known ([24, 48]) that there are two types of multipole radiation. For the multipole radiation of magnetic type $\vec{r}\vec{E} = 0, \vec{r}\vec{H} \neq 0$, while for the radiation of electric type should be $\vec{r}\vec{H} = 0, \vec{r}\vec{E} \neq 0$ (it is, therefore, assumed that the origin lies within the region where $\rho, \vec{j} \neq 0$). It follows from (5.7) that $\vec{r}\vec{E}_0 = 0$, $\vec{r}\vec{H}_0 \neq 0$. Thus, radiation field of the time-dependent current flowing in a circular turn is of magnetic type.

5.1.2. The Elementary Radiating Toroidal Solenoid. The next in complexity case is the radiation of the current flowing in the winding of elementary (i.e., infinitely small) toroidal solenoid. As stated in Sec. 3 (see upper part of Fig. 2) this elementary current is given by

$$j_1 = f_1(t)\text{rot}^{(2)}(\vec{n}\delta^3(\vec{r})), \tag{5.9}$$

where $\text{rot}^{(2)} = \text{rot rot}$ and $\vec{n}$ means the normal to the equatorial plane of TS. The electromagnetic potentials and field strengths are equal to

$$\phi_1 = 0, \quad \vec{A}_1 = -\vec{n}\frac{1}{c^3 r}G_1 + \frac{1}{c^3 r^3}\vec{r}(\vec{r}\vec{n})F_1,$$

$$\vec{E}_1 = \vec{n}\frac{1}{c^4 r}\dot{G}_1 - \frac{1}{c^4 r^3}\vec{r}(\vec{r}\vec{n})\dot{F}_1, \quad H_1 = \frac{1}{c^4 r^2}\ddot{D}_1(\vec{r} \times \vec{n}). \tag{5.10}$$

In this and the following equations of this section we omit the δ-function terms giving the field values at the origin (to which the current is confined). Thus, Eqs. (5.10) are valid everywhere except for the origin. Since $\vec{r}\vec{H}_1 = 0, \vec{r}\vec{E}_1 \neq 0$, the electromagnetic field radiated by the time-dependent current flowing in the winding of TS is of electric type.

5.1.3. More Complicated Elementary Toroidal Sources. We consider now the hierarchy of TS each turn of which is again TS. The simplest example is the usual TS (which is obtained by an installing of the infinitely thin TS in a single turn with the current (5.5) in it). We denote this TS by T_1 (the initial current source (5.5) will be denoted by T_0). The next-in-complexity case is obtained when each turn of T_1 is replaced by the infinitely thin TS with alternating current in its winding. Thus obtained current configuration is denoted by T_2 (Fig. 4). When its dimensions tend to zero, we get (see Sec. 2):

$$\vec{j}_2 = f_2(t)\mathrm{rot}\ ^{(3)}\vec{n}\delta^3(\vec{r}). \tag{5.11}$$

The corresponding VP and field strengths are given by

$$\vec{A}_2 = \frac{1}{c^4 r^2} D_2^{(2)}(\vec{r}\times n),$$
$$\vec{E}_2 = -\frac{1}{c^5 r^2}\ D_2^{(3)}(\vec{r}\times n), \tag{5.12}$$
$$\vec{H}_2 = \vec{n}\frac{1}{c^5 r} G_2^{(2)} - \frac{1}{c^5 r^3}\vec{r}(\vec{r}\vec{n})F_2^{(2)}.$$

By comparing Eqs. (5.6), (5.7) with (5.12) we conclude that the electromagnetic fields coincide for the current configurations T_0 and T_2 everywhere except for the origin if the following relation between time-dependent intensities is fulfilled $f_2^{(2)} = -f_0/c^2$. This means, in particular, that the electromagnetic field of the static magnetic dipole ($f_0 = \mathrm{const}$) coincides with that of the current configuration T_2 if the current in it quadratically varies with time ($f_2 = -f_0 c^2 t^2/2$). It follows from this that the magnetic field of the usual magnetic dipole can be compensated everywhere (except for the origin) by the time-dependent current flowing in T_2.

Compare now the periodical currents flowing in T_0 and T_2: $f_0 = f_{00}\cos\omega t$ and $f_2 = f_{20}\cos\omega t$. It turns out that electromagnetic fields of T_0 and T_2 coincide if $f_{20} = f_{00}c^2/\omega^2$.

Obviously, the radiation emitted by T_2 is of magnetic type. Now we are able to write out the electromagnetic field for the point-like toroidal configuration of the arbitrary order. Let

$$\vec{j}_m = f_m(t)\mathrm{rot}\ ^{(m+1)}(\vec{n}\delta^3(\vec{r})). \tag{5.13}$$

Then, for m even ($m = 2k, k \geq 0$)

$$\vec{A}_{2k} = (-1)^{k+1}\frac{1}{c^{2k+2}r^2}\ D_{2k}^{(2k)}(\vec{r}\times n), \quad \vec{E}_{2k} = (-)^k \frac{1}{c^{2k+3}r^2} D_{2k}^{(2k+1)}(\vec{r}\times n),$$
$$\vec{H}_{2k} = (-1)^k \frac{1}{c^{2k+3}}[\frac{1}{r^3}\vec{r}(\vec{r}\vec{n})F_{2k}^{(2k)} - \vec{n}\frac{1}{r}G_{2k}^{(2k)}]. \tag{5.14}$$

From the facts that: (i) $\vec{A}$ transforms like a vector under space rotations, (ii) VP changes its sign under space reflections and (iii) $\vec{r}\vec{E}_{2k} = 0, \vec{r}\vec{H}_{2k} \neq 0$ it follows

[24, 48] that toroidal configuration of the even order emits the radiation of the magnetic type. The flux of the electromagnetic energy through the sphere of the radius r is equal to

$$S = \frac{2}{3c^{4k+5}} G_{2k}^{(2k)} D_{2k}^{(2k+1)}.$$

On the other hand, for m odd $(m = 2k+1, k \geq 0)$

$$\vec{A}_{2k+1} = (-1)^k \frac{1}{c^{2k+3}} [\frac{1}{r^3} \vec{r}(\vec{r}\vec{n}) F_{2k+1}^{(2k)} - \vec{n}\frac{1}{r} G_{2k+1}^{(2k)}],$$

$$\vec{E}_{2k+1} = (-1)^{k+1} \frac{1}{c^{2k+4}} [\frac{1}{r^3} \vec{r}(\vec{r}\vec{n}) F_{2k+1}^{(2k+1)} - \vec{n}\frac{1}{r} G_{2k+1}^{(2k+1)}],$$

$$\vec{H}_{2k+1} = (-1)^k \frac{1}{c^{2k+4} r^2} D_{2k+1}^{(2k+2)} (\vec{r} \times n) \quad S = \frac{2}{3c^{4k+7}} G_{2k+1}^{(2k+1)} D_{2k+1}^{(2k+2)}. \quad (5.15)$$

From the facts that: (i) VP $\vec{A}$ in (5.15) transforms like a vector under the rotations, (ii) VP does not change its sign under space reflections and (iii) $\vec{r}\vec{H}_{2k} = 0$, $\vec{r}\vec{E}_{2k} \neq 0$, it follows that electromagnetic field (5.15) is of electric type.

We see that there are two branches of toroidal point-like currents generating essentially different electromagnetic fields. A representative of the first branch is the usual magnetic dipole. The electromagnetic field of the k-th member of this family reduces to that of the circular current if the time dependences of these currents are properly adjusted:

$$f_{2k}^{(2k)} = (-1)^k f_0(t)/c^{2k}, \quad (k \geq 0). \quad (5.16)$$

We remember that the lower index of the f functions selects a particular member of the first branch, while the upper one means the time derivative.

The representative of the second branch is the elementary TS. Again, the electromagnetic fields of this family are the same if the time dependences of currents are properly adjusted:

$$f_{2k+1}^{(2k)} = (-1)^k f_1(t)/c^{2k}, \quad (k \geq 0). \quad (5.17)$$

From the equations defining the energy flux it follows that, for high frequencies, the toroidal emitters of the higher order are more effective (as the time derivatives of higher orders contribute to the energy flux).

Earlier, the electromagnetic fields of T_0, T_1 and T_2 current configurations were considered by Nevessky [11]. Further, the radiation field originating from the instantaneous change of dipole moments (i.e., the radiation emitted by the current configuration T_1 for the very particular choice of f_1) was given by Dubovik and Shabanov [72].

5.1.4. Toroidal Solenoids of Higher Multipolarities. So far we have used the usual TS as a corner-stone for constructing more complicated current configurations. Under the term "usual" we mean the torus $(\rho - d)^2 + z^2 = R^2$ with the poloidal current flowing on its surface. The VP corresponding to this current falls as r^{-3} at large distances:

$$\vec{A} \sim \frac{3\vec{r}(\vec{r}\vec{n}) - \vec{n}r^2}{r^5} \quad \text{for} \quad r \to \infty. \tag{5.18}$$

Here $\vec{n}$ is the unit vector normal to TS' equatorial plane. This VP can be represented in a slightly different form:

$$A_i \sim r^{-5} \sum Q_{ik}(x) n_k,$$

where $Q_{ik}(x) = x_i x_k - \delta_{ik} r^2/3$ is the symmetric traceless tensor of the second rank.

It has been shown in Ref. 46] that it is possible to distribute the currents inside the torus in such a way (for the same magnetic flux) as to cancel the leading term ($\sim r^{-3}$) in the expansion of the VP. It turns out that the first nonvanishing term in the expansion of the VP has the form

$$A_i \sim r^{-9} \sum n_j n_k n_l Q^{(4)}_{ijkl}(x), \tag{5.19}$$

where $Q^{(4)}_{ijkl}(x)$ is the symmetric traceless tensor of the fourth rank:

$$Q^{(4)}_{ijkl}(x) = x_i x_j x_k x_l -$$

$$-\frac{1}{7}(\delta_{ij}x_k x_l + \delta_{ik}x_j x_l + \delta_{il}x_j x_k + \delta_{jk}x_i x_l + \delta_{jl}x_i x_k + \delta_{kl}x_i x_j)r^2 +$$

$$+\frac{1}{35}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})r^4.$$

This VP falls like r^{-5} for $r \to \infty$ and carries the same magnetic flux as the initial solenoid with asymptotic behaviour r^{-3} of the VP. With this TS taken as a corner-stone and using the procedure described above we can construct a new hierarchy of TS's. This game may be continued further. More complicated current configuration may be found inside the torus for which the VP falls like r^{-7} [46]. This current configuration may be in turn used as a corner-stone for the construction of the TS installed in each other. These corner-stone current configurations correspond to higher order toroidal multipoles [15]. At large distances these VP have the following asymptotic behaviour

$$A_i^{(l)} \sim r^{-2l-1} \sum Q^l_{i_1, i_2, \ldots, i_l}(x) n_{i_1} n_{i_2} \ldots n_{i_l}. \tag{5.20}$$

Here $Q^l_{i_1,i_2,\ldots,i_l}$ is the symmetric traceless form of the order l. Correspondingly the VP $\vec{A}^{(l)}$ falls as r^{-l-1} for $r \to \infty$. Only even values of l correspond to the finite configurations of poloidal currents found in [46]. As asymptotic form (5.20) satisfy conditions div $\vec{A} = 0, \mathrm{rot}\ \ \vec{A} = 0$ for any l , the question arises on the possible existence of finite current toroidal-like configurations (i.e., ones outside which $\vec{E} = \vec{H} = 0$) corresponding to odd l. So far we did not identify them.

5.2. On the Radiationless Topologically Nontrivial Sources of Electromagnetic Fields. Consider the electric dipole oriented in the $\vec{n}$ direction. Its charge density is

$$\rho_d = e[\delta^3(\vec{r} + a\vec{n}) - \delta^3(\vec{r} - a\vec{n})].$$

For small separation a this reduces to

$$\rho_d = 2ea(\vec{n}\nabla)\delta^3(\vec{r}).$$

Let the intensity of this dipole change with time

$$\rho_d = f_d(t)(\vec{n}\nabla)\delta^3(\vec{r})$$

(the factor $2ea$ is absorbed into f_d). The corresponding current density is given by

$$\vec{j}_d = -\dot{f}_d(t)\vec{n}\delta^3(\vec{r}).$$

These densities generate the following potentials and field strengths (see, e.g., Weinstein [73]):

$$\phi_d = -\frac{1}{cr^2}(\vec{n}\vec{r})D_d, \quad \vec{A}_d = -\vec{n}\dot{f}_d/rc$$

$$\vec{H}_d = \frac{1}{c^2r^2}(\vec{r} \times \vec{n})\dot{D}_d, \quad E_d = \frac{1}{c^4r}\vec{n}G_d - \frac{1}{c^2r^3}(\vec{n}\vec{r})\vec{r}F_d. \tag{5.21}$$

Evidently, the radiation emitted by the oscillating electric dipole is of electric type.

From a comparison of Eqs. (5.10) and (5.21) we conclude that the field strengths of the time-dependent current flowing in the winding of the infinitely small TS can be compensated by that of the electric dipole if their time dependences are properly adjusted: $f_d = -\dot{f}_1/c^2$. Then, the total charge-current densities are

$$\rho = -\frac{1}{c^2}\dot{f}_1 \cdot (\vec{n}\nabla)\delta^3(\vec{r}), \quad \vec{j} = f_1(t)\mathrm{rot}\ ^{(2)}\vec{n}\delta^3(\vec{r}) + \frac{1}{c^2}\ddot{f}_1\vec{n}\delta^3(\vec{r}). \tag{5.22}$$

In the surrounding space $\vec{E} = \vec{H} = 0$, but the potentials differ from zero

$$\phi = \frac{1}{c^3r^2}(\vec{n}\vec{r})\dot{D}_1, \quad \vec{A} = -\frac{1}{c^2r^2}\vec{n}D_1 + \frac{1}{c^3r^3}\vec{r}(\vec{r}\vec{n})F_1. \tag{5.23}$$

Thus, outside this composite object (electric dipole and TS placed at the same point) there are nonvanishing time-dependent electric and vector potentials despite disappearance of the field strengths. The simplest example corresponds to $f_1 = const$. Then,

$$\phi = 0, \quad \vec{A} = f_1[3\vec{r}(\vec{r}\vec{n}) - \vec{n}r^2)/cr^5$$

which coincides with VP of the elementary (i.e., infinitely small) static TS. The next-in-complexity case is the composite object consisting of the static electric dipole ($f_d = f = const$) and the current which linearly changes with time in the winding of TS

$$\rho = f(\vec{n}\nabla)\delta^3(\vec{r}), \quad \vec{j} = -c^2 f t \text{rot}\ ^{(2)}\vec{n}\delta^3(\vec{r})$$

$$\vec{E} = \vec{H} = 0, \quad \phi = -f(\vec{n}\vec{r})/r^3, \quad \vec{A} = -ctf[3\vec{r}(\vec{r}\vec{n}) - \vec{n}r^2)/r^5. \qquad (5.24)$$

A counterpart of (5.24) with finite dimensions is linearly rising with time current flowing in the winding of TS and the double charged layer filling the hole of the same TS. Outside this configuration electromagnetic strengths vanish, but the nontrivial (that is, unremovable by a gauge transformation) VP exists.

Another interesting case is the compensation of the electromagnetic field generated by the oscillating electric dipole by that of the periodical current flowing in the winding of the TS:

$$\rho = \rho_d = f\cos\omega t(\vec{n}\nabla)\delta^3(\vec{r}),\ \vec{j} = \vec{j}_d + \vec{j}_1 = f\omega\sin\omega t[\vec{n}\delta^3(\vec{r}) - \frac{c^2}{\omega^2}\text{rot}\ ^{(2)}\vec{n}\delta^3(\vec{r})],$$

$$\vec{E} = \vec{H} = 0, \quad \phi = \frac{f}{cr^2}(\vec{n}\vec{r})(\omega\sin\Omega - \frac{c}{r}\cos\Omega), \quad \Omega = \omega(t - r/c),$$

$$\vec{A} = \frac{f}{r^2}\vec{n}(\cos\Omega + \frac{c}{\omega r}\sin\omega t) + \frac{\omega f}{cr^3}(\vec{r}\vec{n})\vec{r}(\sin\Omega - 3\frac{c}{\omega r}\cos\Omega - 3\frac{c^2}{\omega^2 r^2}\sin\omega t).$$

It turns out that the field strengths are compensated if the phase of the charge density of the electric dipole is shifted by $\pi/2$ relative to the phase of the current flowing in the winding of toroidal solenoid.

In the wave zone the equivalence of EMF radiated by the oscillating electric dipole to that of produced by the periodical current flowing in the winding of TS was established earlier in Ref. 30. There is no equivalence in the whole space if the finite-dimensional counterparts of the afore-mentioned charge-current configurations are nontrivial. In this case there is no global gauge transformation between the corresponding potentials and this could in principle be observable. The following sections illustrate this. There are references [6,10,12,22,32,45,74,75] in which the nonradiating sources were treated. Outside these sources both electromagnetic strengths and potentials were zero and, thus, they are of no interest for us. Up to now it was not known whether the nontrivial nonradiating time-dependent sources can exist in principle. As far as we know, the first such example has been given in Ref. 13. Nontrivial time-dependent electromagnetic potentials can be used as a new channel for the information transfer (by modulating the phase of the charged-particle wave function) and for the performance of time-dependent Aharonov–Bohm-like experiments.

5.3. On the Current Configurations Generating the Static Electric Field. Consider the poloidal current (Fig. 8) on the torus surface ($(\rho - d)^2 + z^2 = R^2$) which increases linearly with time: $\vec{j} = \vec{j}_0 t$. To parametrize j_0 it is convenient to introduce the coordinates $\tilde{R}, \Psi$ (Fig. 3):

$$x = (d + \tilde{R}\cos\Psi)\cos\phi, \quad y = (d + \tilde{R}\cos\Psi)\sin\phi, \quad z = \tilde{R}\sin\Psi.$$

In these coordinates,

$$\vec{j}_0 = \vec{n}_\psi \frac{j_0 t}{R^2} \frac{\delta(R - \tilde{R})}{d + R\cos\Psi}.$$

Here $\vec{n}_\psi$ is the unit vector tangential to the torus surface: $\vec{n}_\psi = \vec{n}_z \cos\psi - \vec{n}_\rho \sin\psi$. It lies in the $\phi = \text{const}$ plane on the torus surface ($\tilde{R} = R$) and defines the direction of $\vec{j}$. It turns out [58, 68] that for this current only the electric strength $\vec{E}$ differs from zero outside the torus. For simplicity we consider the infinitely thin torus ($R << d$). The following representation for the VP is valid [76, 77]:

$$A_x = \frac{\Phi_0 t}{4\pi} \frac{\partial^2 \alpha}{\partial x \partial z}, \quad A_y = \frac{\Phi_0 t}{4\pi} \frac{\partial^2 \alpha}{\partial y \partial z}, \quad A_z = -\frac{\Phi_0 t}{4\pi} (\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2}), \quad \text{div } \vec{A} = 0.$$

Here

$$\Phi_0 = -\frac{4\pi^2 j_0}{cd}, \quad \alpha = \int\int \frac{dx' dy'}{|\vec{r} - \vec{r}'|}. \tag{5.25}$$

The integration in α is performed over the circle $z = 0, \rho \le d$ coinciding with the hole of the infinitely thin torus ($R << d$). It was shown in [77] that VP has nowhere singularities except for the line $z = 0, \rho = d$ into which torus T degenerates itself. The electromagnetic strengths are

$$H_\rho = H_z = 0, \quad H_\phi = \Phi_0 t \delta(z) \delta(d - \rho),$$
$$E_x = -\frac{\Phi_0}{4\pi c} \frac{\partial^2 \alpha}{\partial x \partial z}, \quad E_y = -\frac{\Phi_0}{4\pi c} \frac{\partial^2 \alpha}{\partial y \partial z}, \quad E_z = \frac{\Phi_0}{4\pi c} (\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2}). \tag{5.26}$$

On the other hand, the electric field produced by two oppositely charged layers ($\rho \le d, \quad z = \pm\epsilon$) filling the torus hole is given by

$$E_x^d = \frac{2e\epsilon}{\pi d^2} \frac{\partial^2 \alpha}{\partial x \partial z}, \quad E_y^d = \frac{2e\epsilon}{\pi d^2} \frac{\partial^2 \alpha}{\partial y \partial z},$$

$$E_z^d = \frac{2e\epsilon}{\pi d^2} \frac{\partial^2 \alpha}{\partial z^2} = \frac{2e\epsilon}{\pi d^2} (\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2}) - \frac{8e\epsilon}{d^2} \delta(z) \Theta(d - \rho). \tag{5.27}$$

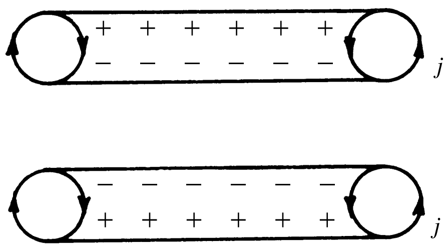

Fig. 9. The poloidal current $\vec{j}$ linearly growing with time is equivalent to the doubly charged layer (the upper part of the figure). The lower part of this figure illustrates that the electric field of the current may be compensated by that of the doubly charged layer

We see that E_z^d has a singularity on the circle $z = 0, \rho \le d$. It follows from Eqs. (5.26) and (5.27) that if

$$\frac{\Phi_0}{4\pi c} = \frac{2e\epsilon}{\pi d^2},$$

then the electric field of the linearly growing poloidal current is compensated by that of the double layer everywhere except for the position of the layer itself (see Fig. 9). The electromagnetic potentials and field strengths of this combined configuration are given by

$$\phi = -\frac{\Phi_0}{4\pi c}\frac{\partial\alpha}{\partial z},$$

$$A_x = \frac{\Phi_0 t}{4\pi}\frac{\partial^2\alpha}{\partial x\partial z}, \quad A_y = \frac{\Phi_0 t}{4\pi}\frac{\partial^2\alpha}{\partial y\partial z}, \quad A_z = -\frac{\Phi_0 t}{4\pi}(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}),$$

$$E_x = E_y = 0, \quad E_z = -\frac{1}{c}\Phi_0\delta(z)\Theta(d-\rho), \tag{5.28}$$

$$H_\rho = H_z = 0, \quad H_\phi = \Phi_0 t\delta(z)\delta(d-\rho).$$

We observe that the time-independent electric field $\vec{E}$ differs from zero only inside the torus hole ($\rho \le d, \quad z = 0$), while the magnetic field $\vec{H} \ne 0$ only on the filament $\rho = d, \quad z = 0$ coinciding with an infinitely thin torus.

The situation remains essentially the same for the TS with a finite value of R. Let the linearly rising current flow in its winding. The corresponding VP is $\vec{A}_{TS} = t\vec{A}_0$, where A_0 is independent of time and up to nonessential constant coincides with the VP of the static TS. The corresponding electric field strength is $\vec{E}_{TS} = -\vec{A}_0/c$. It is known [58, 77] that A_0 is everywhere continuous function of coordinates. Further, outside the solenoid $\vec{A}_0$ can be written as a gradient of some function χ: $\vec{A}_0 = grad\chi$. This representation is valid everywhere except for the circle $\rho \le d - R, z = 0$ filling TS hole. Function χ suffers a finite jump from the value $\chi = \Phi_0$ on the lower side ($z = 0-$) of this circle up to value $\chi = -\Phi_0$ on its upper side ($z = 0+$). Here $\Phi_0 = d\Phi/dt$ is the magnetic flux change per unit of time. Obviously, it does not depend on time. Now we identify $-\chi/c$ with the scalar potential of some electric field. The corresponding electric strength is:

$$\vec{E}_d = -\text{grad}\,(-\chi/c) = \frac{1}{c}\text{grad}\,\chi = \frac{1}{c}\vec{A}_0 - \frac{1}{c}\Theta(d-\rho)\delta(z)\Phi_0\vec{n}_z.$$

The associated charge density

$$\rho_d = (1/4\pi)\text{div}\ \vec{E}_d = -(1/4\pi c)\Theta(d-\rho)\dot{\delta}(z)\Phi_0$$

describes the electric dipoles layer filling the TS hole. The total electric field is:

$$\vec{E} = \vec{E}_{TS} + \vec{E}_d = -\frac{1}{c}\Theta(d-\rho)\delta(z)\Phi_0\vec{n}_z.$$

This means that EMF of the TS with a linearly rising current can be compensated by the EMF of the static electric dipoles layer filling the TS hole everywhere except for the TS hole itself. *5.3.1. On the Current Electrostatics.* Although the toroidal solenoid with a linearly growing current and the double charged layer produce the same electric field in the space surrounding them, they in fact represent quite different systems. The following example illustrates this. Consider an arbitrary closed curve C at each point of which we install (perpendicular to this curve) an infinitely thin toroidal solenoid with a current linearly growing with time. The whole set of these solenoids forms a toroidal-like surface S. The magnetic strength is everywhere zero except on the surface S. The electric strength and time-dependent magnetic VP will be different from zero only inside the tube T, surrounded by the surface S. It seems at first that this contradicts the vanishing of VP outside S (VP should be everywhere continuous). The reason is the same as the discontinuity of the usual electric scalar potential on the surface of a double charged layer: it turns out that the surface S is an example of the double current layer. This construction (Fig. 10) realises a pure current capacitor (the static electric field produced by the time-dependent current is confined to the interior of the tube T). If the set of charged layers (instead of TS) were installed along on the same curve C (perpendicular to it), the elec-

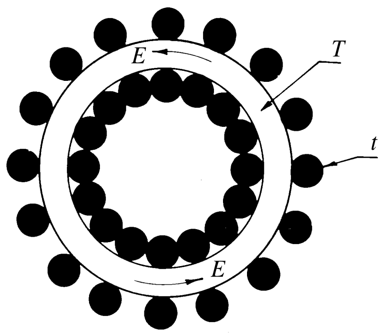

Fig. 10. The torus T is densely covered by the infinitely thin toroidal solenoids t (only few of them are shown) in the windings of which flows the current linearly rising with time. The magnetic field H differs from zero only inside t (that is, at the surface of T in the limit of infinitely thin t), while independent of time electric field E differs from zero inside T. A scalar electric potential is zero everywhere. The vector magnetic potential equals zero outside T and t. Although electromagnetic potentials and strengths are zero outside T and t, there is nonzero electric vector potential ($\vec{E}$ = rot $\vec{\alpha}$) there. The Stokes theorem (see the text) ensures us that $\vec{\alpha}$ cannot be removed by the gauge transformation

tric strength would vanish inside the tube T. However, the nontrivial electric induction will be different from zero there [32].

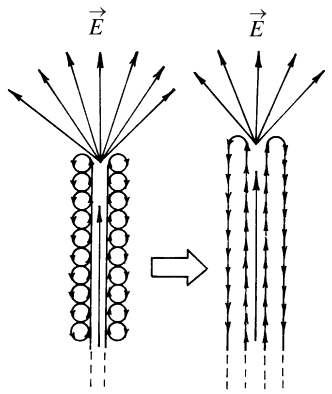

Fig. 11. A semi-infinite set of infinitely thin TS with linear rising currents in their windings (left part of the figure) and linear rising currents flowing along the semi-infinite parallel cylindrical surfaces (right part) generate the field of an electric charge everywhere except for the position of the cylinder

Consider a semi-infinite cylinder C densely covered by the infinitely thin toroidal solenoids (Fig. 11). For simplicity, consider the case when the radius of the cylinder tends to zero. In the limit one obtains a semi-infinite filament composed of the toroidal moments μ_t. The VP of a particular toroidal moment lying at $z = z_0$ is

$$A_x = \mu_t \frac{\partial^2}{\partial x \partial z}\frac{1}{\tilde{r}}, \quad A_y = \mu_t \frac{\partial^2}{\partial y \partial z}\frac{1}{\tilde{r}},$$

$$A_z = \mu_t [\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}} + 4\pi\delta(x)\delta(x)\delta(z - z_0)],$$

$$\tilde{r} = \sqrt{x^2 + y^2 + (z - z_0)^2}, \quad \text{div } \vec{A} = 0.$$

To obtain the VP of the semi-infinite filament composed of the toroidal moments, we integrate these equations from $z_0 = -\infty$ to $z_0 = 0$:

$$A_x = \mu_t \frac{x}{r^3}, \quad A_y = \mu_t \frac{y}{r^3},$$

$$A_z = \mu_t [\frac{z}{r^3} + 4\pi\delta(x)\delta(y)\Theta(-z)], \quad \text{div } \vec{A} = 0.$$

Let in the windings of toroidal solenoids covering the surface of C flows the current linearly rising with time. The VP of a particular infinitely small solenoid located at $z = z_0$ was obtained in Ref. 58. It is given by

$$A_x = t\dot{\mu}_t \frac{\partial^2}{\partial x \partial z}\frac{1}{\tilde{r}}, \quad A_y = t\dot{\mu}_t \frac{\partial^2}{\partial y \partial z}\frac{1}{\tilde{r}},$$

$$A_z = t\dot{\mu}_t [\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}} + 4\pi\delta(x)\delta(x)\delta(z - z_0)], \quad \text{div } \vec{A} = 0.$$

Here $\dot{\mu}_t$ is the independent of time constant characterizing the rate of the current change. The total VP of the semi-infinite filament densely covered by the infinitely small toroidal solenoids with time-dependent currents in their windings is obtained by integrating these equations from $z_0 = -\infty$ to $z_0 = 0$:

$$A_x = t\dot{\mu}_t \frac{x}{r^3}, \quad A_y = t\dot{\mu}_t \frac{y}{r^3}, \quad A_z = t\dot{\mu}_t [\frac{z}{r^3} + 4\pi\delta(x)\delta(y)\Theta(-z)], \quad \text{div } \vec{A} = 0.$$

To this semi-infinite filament corresponds the static electric field

$$\vec{D} = \vec{E}, \quad E_x = -\dot{\mu}_t \frac{x}{cr^3}, \quad E_y = -\dot{\mu}_t \frac{y}{cr^3},$$

$$E_z = -\frac{\dot{\mu}_t}{c}[\frac{z}{r^3} + 4\pi\delta(x)\delta(y)\Theta(-z)], \quad \text{div } \vec{E} = 0,$$

and the singular magnetic field confined to the negative z semiaxis

$$\vec{B} = \vec{H} = H_\phi \vec{n}_\phi, \quad H_\phi = -4\pi t \dot{\mu}_t \frac{d}{d\rho}\frac{\delta(\rho)}{2\pi\rho}, \quad \text{div } \vec{H} = 0.$$

The resulting EMF coincides with that of the point electric charge $e = -\dot{\mu}_t/c$ everywhere except for the semi-infinite filament (left part of Fig. 11). Above, we have used the fact that $\vec{D} = \vec{E}, \quad \vec{B} = \vec{H}$ in the absence of medium. The same electric field may be also realized via two linearly-rising currents flowing in opposite directions along the cylindrical surfaces parallel to the z axis (right part of Fig. 11). The equalities

$$\text{div } \vec{D} = 0, \quad \int D_n d\Omega = 0$$

guarantee the absence of free charges. Obviously, thus obtained electric charges are not true ones (due to the presence of δ function term).

In a qualitative manner these results were obtained earlier by M.A. Miller [68] who pointed out on the possibility to simulate the charge distributions by the time-dependent currents. He referred to it as to "current electrostatics". The present investigation may be viewed as a concrete realization of these ideas. Excellent measurements of the static electric fields produced by the time-dependent currents have been reported in [66].

There are known attempts (see, e.g., [78] and references therein) to measure the electric field arising from the stationary currents. Maxwell's theory negates the existence of this field. On the other hand, we have seen that there exist nonstatic current configurations generating the static electric field.

5.4. On the Electric Vector Potentials. As we have learned from the previous section, it is possible to find current configurations producing a static electric field $\vec{E}$ inside the tube T. As $\vec{E}$ is due to the currents, so div $\vec{E} = 0$, and it can be represented in the form $\vec{E} = \text{rot } \vec{A}_e$. The possibility of such representation for a free electromagnetic field was pointed out earlier by Stratton [71]. The integral $\int \vec{E} dS$ taken over the tube T cross section differs from zero. Then, the Stokes theorem $\int \vec{E} dS = \oint \vec{A}_e d\vec{l}$ (the linear integral is taken along the contour embracing the tube T but lying outside it) tells us that $\vec{A}_e$ differs from zero outside T. Or, in other words, there is a nontrivial electric VP outside T (Fig. 10). The same is valid for a closed chain of electric dipoles [32]. The drawback of the present consideration is that we have not taken into account singular fields in the infinitely thin layer on the surface of T (where the currents flow). It may happen that they exactly compensate the flux of $\vec{E}$ inside T. Then the total flux of the electric strength will be zero and there will be no need to introduce the electric vector potential. To clarify this point, we turn again to the closed chain of TS, installed along the closed curve C perpendicular to it. The total VP and electric field strength are given by

$$\vec{A}(\vec{r}) = \int \vec{A}_{TS}(\vec{r}, \vec{r}_0(s))ds, \quad \vec{E} = -\dot{\vec{A}}/c. \tag{5.29}$$

Here $\vec{A}_{TS}$ is the VP of the particular infinitely thin TS with its center at the point $\vec{r}_0(s)$. The integration in (5.29) is performed along the curve C defined as $\vec{r} = \vec{r}_0(s)$. For the treated case the time-dependent VP is given by [58]: $\vec{A}_{TS}(t) = t\vec{A}^0_{TS}$, where $\vec{A}^0_{TS}$ is the VP of TS with a static current. However, we are unable to evaluate the integral (5.29) along an arbitrary closed curve. Instead, we integrate along the infinite straight line parallel to the TS' symmetry axis. In the special gauge the VP of TS with its axis parallel to the Z one is [76, 77]: $\vec{A}^0_{TS} = -g\vec{n}_z T$, where T is given by Eqs. (2.8), (2.9) in which one should use $z - z_0$ instead of z (z_0 is the position of the TS' center); $g = \Phi_0[2\pi(d - \sqrt{d^2 - R^2}]^{-1}$ and Φ_0 is the magnetic flux inside TS, d and R are its geometrical parameters. Now we integrate this VP along the Z axis

$$A_z = \int (\vec{A}^0_{TS})_z dz_0.$$

It turns out that

$$A_z = \Phi_0 \quad \text{for} \quad \rho < d - R,$$

$$A_z = \Phi_0 - 2g\xi \ln \rho + 2g \int_0^\xi dz \ln(d - \sqrt{R^2 - z^2})$$

for $d - R < \rho < d, \quad \xi = \sqrt{R^2 - (\rho - d)^2}$,

$$A_z = -2g\xi \ln \rho + 2g \int_0^\xi dz \ln(d + \sqrt{R^2 - z^2}) \quad \text{for} \quad d < \rho < d + R$$

and $A_z = 0$ for $\rho > d + R$. The flux of the VP $\vec{A}$ is obtained by integration over the cylinder C cross section

$$\int A_z \rho d\rho d\phi = \pi^2 g d R^2.$$

In the limit $R \to 0$ this expression goes into $\pi d^2 \Phi_0$ that coincides with the integral of VP taken over the interior of the cylindrical tube without taking into account the singular magnetic field concentrated on the surface of the cylinder. This means that the surface magnetic field contributes nothing in the limit $R \to 0$.

Thus, we have proved that for the treated current configuration (TS continuously distributed over the cylinder C surface) the VP equals zero outside C, but its flux over the cross section of C differs from zero. As div $\vec{A} = 0$, we may put $\vec{A} = \text{rot}\ \vec{\alpha}$. Using the Stokes theorem one sees that there is the nontrivial vector function $\vec{\alpha}$ outside C although $A = 0$ there. The main problem is that α does not enter into the Schroedinger or the Dirac equation. Nevertheless, such a current configuration interacts with an external electromagnetic field (see Sec. 2.2) and, in particular, with that of the incoming charged particle.

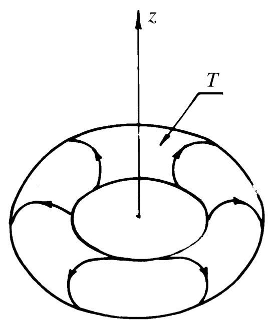

Fig. 12. The torus T is densely covered by the magnetized rings (only few of them are shown). The magnetic strength $\vec{H}$ differs from zero at the surface of T, while the magnetic vector potential $\vec{A}$ differs from zero only inside T and at its surface. Outside T there is nontrivial (that is, unremovable by the gauge transformation) vector $\vec{\alpha}$ the curl of which is vector potential $\vec{A}$

The existence of the nontrivial (that is, unremovable by the gauge transformation) vector $\vec{\alpha}$, the rot of which is just VP may be proved without recourse to the just considered rather complicated nonstatic current configurations. Consider the set of the closed magnetized filaments uniformly distributed over the surface of the torus T (see Fig. 12 where the lines on the torus surface mean the magnetized filaments). This configuration can be assembled from the ferromagnetic rings used in Tonomura experiments [62] testing the existence of the Aharonov–Bohm effect. The VP differs from zero only inside the torus T although the magnetic strength $\vec{H}$ vanishes there (it differs from zero on the surface of T). Then, reasonings similar to the previous ones prove the existence of the vector $\vec{\alpha}$ ($\vec{A} = \text{rot}\ \vec{\alpha}$) in the space external to T. It seems that $\vec{\alpha}$ cannot be eliminated by a gauge transformation.

5.5. Time-Dependent Aharonov–Bohm Effect. Consider the scattering of charged particles on a charge-current configuration shown in the lower part of Fig. 9. It consists of the impenetrable toroidal solenoid with a layer of electric dipoles filling torus hole. The corresponding Schroedinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} = [-\frac{\hbar^2}{2m}(\nabla - \frac{ie}{\hbar c}\vec{A})^2 + e\phi]\psi. \tag{5.30}$$

To prevent the particle penetration into the torus interior, it can be made impenetrable. Outside it the magnetic field $\vec{H} = 0$ everywhere, the electric field is also everywhere zero except for the torus hole where it has the δ-type singularity. The static scalar and linearly growing with time vector potentials differ from zero everywhere. The integral $\oint A_l dl$ taken along the closed path passing through the torus hole also grows linearly with time. The question arises to what extent the electromagnetic potentials can be removed from the Schroedinger equation (5.30).

But at first we remember the situation for the usual infinitely thin static magnetic toroidal solenoid without the double charged layer [76, 77]. In this case

$$\Phi = 0, \quad A_x = \frac{\Phi_0}{4\pi}\frac{\partial^2\alpha}{\partial x\partial z}, \quad A_y = \frac{\Phi_0}{4\pi}\frac{\partial^2\alpha}{\partial y\partial z}, \quad A_z = -\frac{\Phi_0}{4\pi}(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}),$$

$$\vec{E} = 0, \quad \vec{H} = \vec{n}_\phi \Phi_0 \delta(\rho - d)\delta(z)$$

(Φ_0 is the magnetic flux inside the TS and α is defined in Eq. (5.25)). The following gauge transformation

$$\vec{A} \to \vec{A}' = \vec{A} - \nabla\chi, \quad \psi \to \psi' = \psi \exp(ie\chi/\hbar c), \quad \chi = \frac{1}{4\pi}\Phi_0\frac{\partial\alpha}{\partial z}$$

leads to VP filling the torus hole:

$$A'_x = A'_y = 0, \quad A'_z = \Phi_0\delta(z)\Theta(d-\rho), \tag{5.31}$$

$$i\hbar\frac{\partial\psi'}{\partial t} = -\frac{\hbar^2}{2m}[\nabla_x^2 + \nabla_y^2 + (\nabla_z - \frac{ie}{\hbar c}\Phi_0\delta(z)\Theta(d-\rho))^2]\Psi'.$$

The VP cannot be expelled from this equation by the gauge transformation and this leads to the shift of interference picture on the screen installed behind TS. The corresponding experiments have been performed by Tonomura [62], their theoretical description is given in [79]. For the treated time-dependent case the gauge transformation which partially eliminates the electromagnetic potentials is

$$\vec{A} \to \vec{A}' = \vec{A} - \nabla\chi, \quad \phi \to \phi' = \phi + \dot{\chi}/c,$$

$$\psi \to \psi' = \psi \exp(ie\chi/\hbar c), \quad \chi = \frac{1}{4\pi}\Phi_0 t\frac{\partial\alpha}{\partial z}.$$

After this transformation

$$\phi' = A'_x = A'_y = E'_x = E'_y = 0, \quad A'_z = \Phi_0 t\delta(z)\Theta(d-\rho),$$

$$E'_z = E_z = -\frac{1}{c}\Phi_0\delta(z)\Theta(d-\rho), \quad H'_\phi = H_\phi = \Phi_0 t\delta(z)\delta(d-\rho).$$

$$i\hbar\frac{\partial\psi'}{\partial t} = -\frac{\hbar^2}{2m}[\nabla_x^2 + \nabla_y^2 + (\nabla_z - \frac{ie}{\hbar c}\Phi_0 t\delta(z)\Theta(d-\rho))^2]\Psi'. \tag{5.32}$$

Eqs. (5.31) and (5.32) have essentially the same form. Likewise the static VP cannot be expelled from Eq. (5.31), the time-dependent VP cannot be removed from Eq. (5.32). This means that changing with time interference picture inevitably arises on the screen installed behind the impenetrable toroidal solenoid (Fig. 13). The static electric field E filling the torus hole certainly deflects the incoming charged particles (via the Lorentz force). The charged particles scattering cross section evaluated according to the laws of classical mechanics does not depend upon the time. The time dependence of the interference picture is a pure quantum effect. It is due to the time-dependent magnetic flux enclosed into the impenetrable torus. We observe that effects of excluded fields (time-dependent magnetic field confined to the impenetrable torus) are observed against a background of accessible ones (i.e., the static electric field filling torus hole).This agrees with a standard definition of the Aharonov–Bohm effect as observable effects of enclosed (or, inaccessible) fields (see, e.g., [62]). For the cylindrical geometry the magnetic time-dependent AB effect was considered recently in Refs. 80, 81.

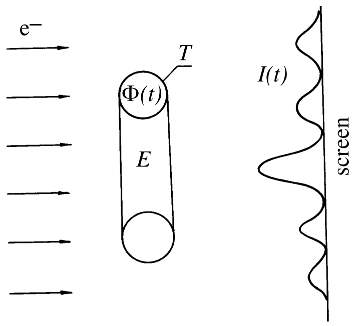

Fig. 13. The magnetic time-dependent AB effect. For the charge-current configuration discussed in the text the time-dependent magnetic flux differs from zero only inside the impenetrable torus T. Outside T the independent of time electric strength E differs from zero only inside the torus hole. It is the time-dependent magnetic flux inside T that leads to the time variation of the intensity of the scattered charged-particles

5.6. Finite Toroidal-Like Configurations. *5.6.1. The Debye Parametrization for the Electromagnetic Potentials and Strengths.* Consider now the time-dependent current distribution confined to a finite region of space

$$\vec{j}(\vec{r}, t) = f(t)\vec{j}(\vec{r}). \tag{5.33}$$

An arbitrary vector function and, in particular, the current distribution can be presented in the form (Debye parametrization)

$$\vec{j}(\vec{r}) = \nabla\Psi_1 + \text{rot }(\vec{r}\Psi_2) + \text{rot }^{(2)}(\vec{r}\Psi_3). \tag{5.34}$$

It turns out that the VP corresponding to the current density (5.33) in the Lorentz gauge (div $\vec{A} + \dot{\Phi}/c = 0$) is given by

$$\vec{A} = \vec{\nabla} a_1 + \text{rot }(\vec{r}a_2) + \text{rot }^2(\vec{r}a_3). \tag{5.35}$$

Clearly, Eq. (5.35) is the Debye parametrization of VP. The functions entering into it are

$$a_k = I_k/c, \quad I_k = \int \frac{1}{R} f(t - R/c)\Psi_k(\vec{r}')dV'. \tag{5.36}$$

Here $R = |\vec{r} - \vec{r}'|$. To be complete, we write out the corresponding scalar electric potential

$$\phi = -\dot{I}_1/c + 4\pi F(t)\Psi_1(\vec{r}) + \phi_{\rm stat}. \tag{5.37}$$

Here the point above I_k means the time derivative, $F(t) = \int^t f(t)dt$ and $\phi_{\rm stat}$ is the scalar potential arising from time-independent part of the charge density (if it exists): $\phi_{\rm stat} = \int R^{-1}\rho_{\rm stat}(\vec{r}')dV'$. It is convenient to represent the field strengths in the same form as j and A:

$$\vec{E} = \vec{\nabla}e_1 + {\rm rot}\ (\vec{r}e_2) + {\rm rot}\ ^{(2)}(\vec{r}e_3),$$

$$\vec{H} = \vec{\nabla}h_1 + {\rm rot}\ (\vec{r}h_2) + {\rm rot}\ ^{(2)}(\vec{r}h_3). \tag{5.38}$$

It turns out that

$$e_1 = -\phi_{\rm stat} - 4\pi F(t)\Psi_1(\vec{r}), \quad e_2 = -\dot{I}_2/c^2, \quad e_3 = -\dot{I}_3/c^2,$$

$$h_1 = 0, \quad h_2 = -\ddot{I}_3/c^3 + 4\pi f(t)\Psi_3(\vec{r})/c, \quad h_3 = I_2/c. \tag{5.39}$$

These representations are convenient because the potentials and strengths are obtained from the relatively simple integrals, their time and space derivatives.

We know from Sec. 3 that the functions Ψ_2 and Ψ_3 carry information on the magnetic and toroidal (electric) moments, resp. Thus, putting $\Psi_2(\vec{r}) = \psi_2(r)Y_{lm}(\theta,\phi)$ and $\Psi_3(\vec{r}) = \psi_3(r)Y_{lm}(\theta,\phi)$ we obtain the formulae describing the radiation of particular magnetic and toroidal (electric) multipoles. The functions ψ_2 and ψ_3 define the radial distribution of the current sources. Developing the function $g = f(t - R/c)/R$ over the spherical harmonics:

$$g = 4\pi \sum \frac{1}{2l+1} g_l(r, r', t)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \tag{5.40}$$

we obtain for the particular lm multipole

$$I_{lm} = \frac{4\pi}{c}\frac{1}{2l+1}Y_{lm}(\theta,\phi)\int g_l(r, r', t)\psi_k(r')r'^2dr' \tag{5.41}$$

(no sum over l, m here).

5.6.2. Transition to the Point-Like Limit. Eqs. (5.41) define the integrals for the finite spatial current distribution. To obtain the point current limit we follow the method used by E.G.P.Rowe [82] for the evaluation of the integral I_1 entering into the definition of ϕ (see Eq. (5.37)). One simply puts

$$\Psi_k(\vec{r}) \sim Y_{lm}(-\nabla)\delta^3(\vec{r}). \tag{5.42}$$

It should be clarified what does $Y_{lm}(-\nabla)$ mean in the r.h.s. of this equation. We write

$$Y_{lm}(x) = r^l Y_{lm}(\theta, \phi), \tag{5.43}$$

where $Y_{lm}(\theta, \phi)$ is the usual spherical harmonic. Clearly, $Y_{lm}(x)$ is the homogeneous function (of the order l) in Cartesian variables x, y, z. For example,

$$Y_{20}(x) \sim 2z^2 - x^2 - y^2. \tag{5.44}$$

To obtain $Y_{lm}(-\nabla)$ we change x_i by $-\partial/\partial x_i$ in Eq. (5.43). In particular,

$$Y_{20}(-\nabla) \sim 2\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}. \tag{5.45}$$

Many of the properties of the functions $Y_{lm}(x)$ and their physical applications are collected in Ref. 83. Now we substitute (5.42) into (5.36) and integrate by parts:

$$I_k \sim Y_{lm}(\nabla) f(t - r/c)/r. \tag{5.46}$$

Inserting this expression into Eqs. (5.35), (5.38) we obtain the electromagnetic potentials and strengths describing the elementary source.

5.7. More General Radiationless Sources. Having obtained the explicit expressions for the extended and point-like sources, we now try to construct the radiationless sources of higher multipolarities. Consider charge and current densities corresponding to the oscillating quadrupole moment:

$$\rho_q = f_q(t)[(\vec{n}\vec{\nabla})^2 - \frac{1}{3}\Delta]\delta^3(\vec{r}), \quad \vec{j}_q = -\dot{f}_q[\vec{n}(\vec{n}\vec{\nabla}) - \frac{1}{3}\nabla]\delta^3(\vec{r}). \tag{5.47}$$

On the other hand, consider the pure current density (5.34) with

$$\Psi_1 = \Psi_2 = 0, \quad \Psi_3 = [(\vec{n}\vec{\nabla})^2 - \frac{1}{3}\Delta]\delta^3(\vec{r}), \quad \vec{j}_c = f_c(t)\text{rot}\,^2(\vec{r}\Psi_3). \tag{5.48}$$

It turns out that the oscillating quadrupole charge-current configuration (5.47) and a pure current configuration (5.48) being placed at the same point generate the total field strengths equal to zero everywhere except for the origin if the following relation is fulfilled: $f_q = 2\dot{f}_c/c^2$. The total charge-current densities are equal to

$$\rho = \frac{2}{c^2}\dot{f}_c(t)[(\vec{n}\vec{\nabla})^2 - \frac{1}{3}\Delta]\delta^3(\vec{r}), \quad \vec{j} = f_c(t)\text{rot}\,^2(\vec{r}\Psi_3) - \frac{2}{c^2}\ddot{f}_c[\vec{n}(\vec{n}\vec{\nabla}) - \frac{1}{3}\nabla]\delta^3(\vec{r}). \tag{5.49}$$

Nevertheless, the electromagnetic potentials are not zero:

$$\phi = \phi_q = \frac{2}{c^4 r^3}[(\vec{n}\vec{r})^2 - \frac{1}{3}r^2]\dot{F}_c,$$

$$\vec{A}=\vec{A}_q+\vec{A}_c=-\frac{4}{c^3r^3}[(\vec{n}\vec{r})\vec{n}-\frac{1}{3}\vec{r}]F_c+\frac{2}{c^4r^4}\vec{r}[(\vec{n}\vec{r})^2-\frac{1}{3}r^2]$$

$$\times(f_c^{(3)}+6\frac{c}{r}f_c^{(2)}+15\frac{c^2}{r^2}\dot{f}_c+15\frac{c^3}{r^3}f_c),\quad (F_c=F(f_c)=\ddot{f}_c+3\frac{c}{r}\dot{f}_c+3\frac{c^2}{r^2}f_c). \tag{5.50}$$

For $f_c=\text{const.},\quad f_q=0$ we get the following static configuration:

$$\vec{j}=f_c(t)\text{rot}\,^2(\vec{r}\Psi_3),\quad \vec{A}=-\frac{12}{cr^5}f_c[(\vec{n}\vec{r})\vec{n}-\frac{1}{3}\vec{r}]+\frac{30}{cr^7}f_c\vec{r}[(\vec{n}\vec{r})^2-\frac{1}{3}r^2]. \tag{5.51}$$

This VP falling at large distances as r^{-4} corresponds to $l=3$ in Eq. (5.20). As we have mentioned at the end of Sec. 6.1.3, we did not succeeded in identifying the finite static current configuration whose infinitesimal limit coincides with (5.51) and corresponds to odd l in (5.20). The next-in-complexity case corresponds to octupole oscillations of the charge density

$$\rho_q=f_q(t)(\vec{n}\nabla)[(\vec{n}\vec{\nabla})^2-\frac{3}{5}\Delta]\delta^3(\vec{r}),\quad \vec{j}_q=-\dot{f}_q\vec{n}[(\vec{n}\vec{\nabla})^2-\frac{3}{5}\Delta]\delta^3(\vec{r}). \tag{5.52}$$

The elementary toroidal current distribution giving the same field strengths corresponds to

$$\Psi_1=\Psi_2=0,\quad \Psi_3=f_c(t)(\vec{n}\nabla)[(\vec{n}\vec{\nabla})^2-\frac{3}{5}\Delta]\delta^3(\vec{r}),\quad f_q=-3\dot{f}_c/c^2. \tag{5.53}$$

The finite poloidal current distribution whose infinitesimal limit coincides with Eq. (5.53) was obtained in Ref. 46. The asymptotic behaviour of the corresponding VP is determined by Eq. (5.19).

Now we are able to write out more general radiationless charge-current configurations. The extension of Eqs. (5.47) and (5.52) to an arbitrary multipolarity l is given by

$$\rho_q=f_q(t)(\vec{v}\nabla)\delta^3(\vec{r}),\quad \vec{j}_q=-\dot{f}_q(t)\vec{v}\delta^3(\vec{r}). \tag{5.54}$$

Here $\nabla_i=\partial/\partial x_i$, while $\vec{v}$ is the vector whose cartesian components are

$$v_i=\sum_{i_2\ldots i_l}Q^{(l)}{}_{i,i_2,\ldots,i_l}\nabla_{i_2}\ldots\nabla_{i_l};$$

$Q^{(l)}{}_{i,i_2,\ldots,i_l}$ is the symmetric traceless tensor (see Sec. 3.2.2) of the rank l of the variables n_x,n_y,n_z defining the direction of the fixed 3-vector (this vector can be identified with the direction of TS' axis). The electromagnetic potentials and field strengths corresponding to these densities are

$$\phi_q=(\vec{v}\vec{\nabla})\frac{f_q}{r},\quad \vec{A}_q=-\vec{v}\frac{1}{c}\frac{\dot{f}_q}{r},$$

$$E_q = -\nabla(\vec{v}\vec{\nabla})\frac{f_q}{r} + \frac{1}{c^2}\vec{v}\frac{\ddot{f}_q}{r}, \quad \vec{H}_q = -\frac{1}{c}(\nabla \times \vec{v})\frac{\dot{f}_q}{r} \tag{5.55}$$

(remember that argument of the f functions, if not indicated, means $t - r/c$). On the other hand, a pure current configuration generalizing Eqs. (5.48) and (5.53) is given by

$$\rho_c = 0, \quad \vec{j}_c = f_c(t)\text{rot}\,^{(2)}(\vec{r}\Psi_3), \quad \Psi_3 = (\vec{v}\nabla)\delta^3(\vec{r}). \tag{5.56}$$

The corresponding electromagnetic potentials and field strengths are

$$\phi_c = 0, \quad \vec{A}_c = -\frac{l}{c}\nabla(\vec{v}\vec{\nabla})\frac{f_c}{r} + \frac{l}{c^3}\vec{v}\frac{\ddot{f}_c}{r} + \frac{4\pi}{c}f_c(t)\vec{r}(\vec{v}\nabla)\delta^3(\vec{r}),$$

$$\vec{E}_c = \frac{l}{c^2}\nabla(\vec{v}\vec{\nabla})\frac{f_c}{r} - \frac{l}{c^4}\vec{v}\frac{f_c^3}{r} - \frac{4\pi}{c^2}\dot{f}_c(t)\vec{r}(\vec{v}\nabla)\delta^3(\vec{r}),$$

$$\vec{H}_c = \frac{l}{c^3}(\nabla \times v)\frac{\ddot{f}_c}{r} - \frac{4\pi l}{c}f_c(t)(\vec{r} \times \vec{\nabla})(\vec{v}\nabla)\delta^3(\vec{r}). \tag{5.57}$$

Now we place charge-current densities (5.54) and (5.56) at the same point. It turns out that if $f_q = \dot{f}_c/c^2$, then the total electromagnetic field strengths are everywhere zero except for the origin:

$$\vec{H} = -\frac{4\pi l}{c}f_c(t)(\nabla \times \vec{v})\delta^3(\vec{r}), \quad \vec{E} = \frac{4\pi l}{c^2}\dot{f}_c(t)\vec{v}\delta^3(\vec{r}). \tag{5.58}$$

Nevertheless, the electromagnetic potentials differ from zero in the whole space:

$$\phi = -\dot{\chi}/c, \quad \vec{A} = \nabla\chi - \frac{4\pi l}{c}f_c(t)vecv\delta^3(\vec{r}), \quad \chi = -\frac{l}{c}(\vec{v}\nabla)\frac{f_c}{r}. \tag{5.59}$$

Evidently, these equations generalize particular cases considered earlier.

5.8. Concluding Remarks on the Toroidal Radiationless Sources. In the previous section we have found elementary charge-current configurations with the property that electromagnetic strengths, not potentials, disappear outside them. Turning to Eq. (5.59) we observe that outside the source one gets $\vec{A} = \nabla\chi$ and $\phi = -\dot{\chi}/c$, that is, electromagnetic potential can be presented there as a 4-gradient of a singular function χ. Does this mean that electromagnetic potentials can be eliminated by a gauge transformation? One cannot comment on the topological nontriviality of electromagnetic potentials without going beyond the framework of the elementary source. This is due to the fact that it is not clear what is the topologically nontrivial point-like source. As an illustration consider the vector potential (5.18) of the usual static elementary toroidal solenoid. It turns out that outside the origin (where the TS is placed) the VP may be presented as a gradient of the singular function $\chi = -f_1(\vec{n}\vec{r})/r^3$. On the other hand, outside the finite

TS (whose infinitesimal counterpart is elementary source (5.18)) the VP cannot be eliminated by the gauge transformation (despite the fact that $E = H = 0$ there). This leads to numerous experimental consequences and, in particular, to the static magnetic Aharonov–Bohm effect. The experiments in which the electrons were scattered on the impenetrable magnetized ring were performed by Tonomura et al. [62].

Now we turn again to Eqs. (5.54), (5.56). We know [46] how to find finite counterparts of the elementary sources (5.52). For time dependences for which VP can be found in a closed form, the rules (6.50), (5.57) lead to the topologically nontrivial electromagnetic potentials outside the radiationless sources. The uniformity of these prescriptions suggests that nontrivial potentials should exist for an arbitrary time-dependence. To the best of our knowledge the nontrivial radiationless sources considered in [13] are their first concrete realizations.

Further, it turns out that the field strengths vanish in the space surrounding radiationless sources. Since the electromagnetic strengths generated by the oscillating charge densities and the elementary toroidal sources are the same (if their time dependences are properly adjusted), particular terms of the multipole expansions defining these strengths coincide and have the double names known in a physical literature as electric (see, e.g., [24, 48]) or toroidal [15, 30, 31] multipoles. Despite the coincidence of the electromagnetic strengths, the corresponding potentials may be physically different. In these cases the multipole expansion of the field strengths does not describe the whole physical situation (since the same multipole expansion of the field strengths corresponds to physically different electromagnetic potentials which can be discriminated experimentally).

We briefly summarize the main results obtained in this section:

1. The radiation fields of toroidal-like current configurations are investigated. There are two different representatives which generate essentially different electromagnetic fields. These representatives are the circular turn and toroidal solenoid with time-dependent currents flowing in them.

2. There are found elementary time-dependent charge-current configurations outside which the electromagnetic field strengths disappear but the potentials survive. In the solvable cases their finite-dimensions counterparts have nontrivial (i.e., unremovable by the gauge transformation) electromagnetic potentials outside them. This can be used for performing time-dependent Aharonov–Bohm-like experiments and the information transfer (modulating the phase of the charge particle wave function).

3. Using the Debye parametrization of the current density we present the electromagnetic field of the arbitrary time-dependent charge-current density in a form convenient for applications. The contributions of different multipoles in it are explicitly separated.

6. ACKNOWLEDGEMENTS

The authors are grateful to Prof. Lyuboshitz V.L. for the careful reading of the manuscript. His precise remarks made this paper to be a better one.

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