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FOCK–BARGMANN SPACE AND $SU(3)$ MODEL

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The aim of this review is to present an approach to unitary models that has been developed during the last ten years. The approach uses wave functions and operators defined in Fock–Bargmann space. A solution for the $SU(3)$ shell model is introduced as well as some extensions that are related to a classical treatment of $SU(3)$ theory.

В обзоре представлен развитый в последние годы новый подход к унитарным моделям. Этот подход использует волновые функции и операторы унитарных моделей, определенные в пространстве Фока–Баргманна. Приводятся решения для $SU(3)$ модели оболочек и некоторые их обобщения, относящиеся к классической трактовке $SU(3)$ теории.

1. OUTLINE OF $SU(3)$ SHELL-MODEL DEVELOPMENTS

The $SU(3)$ model was introduced as a scheme for analysing the problem of valence nucleon dynamics within the framework of a harmonic oscillator shell-model basis. An important feature of the $SU(3)$ model is its reproduction of the dynamics of the rotor model. This feature is clearly demonstrated in the work of the LSU group [1-3], and independently in the work of the Kiev team [4-6]. The LSU and Kiev teams came to the same conclusion using different approaches: the LSU team used group-theoretical techniques and computational methods, whereas the Kiev team developed a theory that exploits well-known expressions for the wave mechanics of a rigid body.

The simplest version of the $SU(3)$ model takes only the so-called leading irreducible representation (irrep) into account. The Hamiltonian of the model is constructed either from invariants of $SU(3)$ and/or its SO(3) subgroup (for instance, QQ, QQQ, QQQQ) or from a many-particle Hamiltonian built with a Gaussian nucleon-nucleon interaction. The LSU work follows the traditional treatment which takes the leading irrep as the one for which the second-order Casimir operator of $SU(3)$ has the maximum value. The leading representation for the Kiev group is defined to be the irrep for which the interaction of even nucleon pairs is the most attractive.

Extended versions of both the LSU and Kiev approaches include more than just the leading irrep. A focus of this article is on the motivation for making such an extension. A further issue is the collective motion of the valence nucleons taken as a whole versus that of the valence neutrons and protons with respect to one another. To probe these and related matters, it is necessary to use many $SU(3)$ irreps. As will be seen, in the simplest case this leads to a two rotor problem, one for describing the neutron motion and the other for the protons. These rotors can move either together or relative to one another. The goal of this work is to provide a comprehensive theory of the valence nucleon dynamics, to appreciate the simplest modes in terms of a geometrical picture, and to understand what features of this dynamics are realized in nuclei.

2. SOME ELEMENTS OF RIGID ROTOR THEORY

Why can the microscopic $SU(3)$ model be presented as a rigid rotor model? To answer this question, it is first important to understand some elements of rigid rotor theory. The wave function Ψ_{LM} of the rigid non-axial rotor in a state with orbital momentum L and projection M has the form

$$\Psi^{\alpha}_{LM} = \sum_{K} \; C^{L,\alpha}_{K} \; D^{L}_{K,M}(\phi,\theta,\psi), \tag{1}$$

where $D^{L}_{K,M}$ are the spherical Wigner functions; K is the orbital momentum projection on the intrinsic axis of the rotor; ϕ,θ,ψ are Euler angles defining the orientation of the rotor with respect to the laboratory frame of reference; $C^{L,\alpha}_{K}$ are coefficients which show the contributions of states with different values of K; and finally, α is an additional quantum number used to distinguish distinct states with the same values of L, M.

The Hamiltonian $\hat{H}_{rot}$ of the rigid rotor [7] has the form:

$$\hat{H}_{rot} = \frac{1}{2}a \; \hat{L}^2_{\xi} + \frac{1}{2}b \; \hat{L}^2_{\eta} + \frac{1}{2}c \; \hat{L}^2_{\zeta}. \tag{2}$$

Now we would like to consider two central questions. First question: “How can one realize a transition from a microscopic many-particle wave function of the $SU(3)$ model to a typical rigid rotor wave function (1)?” Second question: “How can one derive from the many-particle microscopic Hamiltonian,

$$\hat{H} = \sum_{i=1}^{A} \; \hat{t}_i \; + \; \sum_{i<j} \; \hat{u}_{i,j}, \tag{3}$$

a Hamiltonian that reminds us of expression (2)?”

3. WAVE PACKET OF $SU(3)$ BASIS STATES

To address these questions, it is convenient to introduce an important construction; namely, the wave packet $\Phi(\lambda,\mu)$ for basis functions of an $SU(3)$ irrep (λ,μ). The wave packet of the highest weight irrep is most important. The state is a Slater determinant built from one-particle oscillator orbitals with specific cartesian quantum numbers. Whereas many such functions can be constructed, for a given nucleus there is only one or two such configurations with minimum value of the oscillator quanta number allowed by the Paul Principle.

$$\Phi(\lambda,\mu) = Det\ \left|\left|\varphi_{n_x,n_y,n_z}(\mathbf{r}_i,\sigma_i,\tau_i)\right|\right|, \tag{4}$$

$$\varphi_{n_x,n_y,n_z}(\mathbf{r}_i,\sigma_i,\tau_i) =$$

$$N_{n_x,n_y,n_z} H_{n_x}(\mathbf{u}_1\cdot\mathbf{r}_i) H_{n_y}(\mathbf{u}_2\cdot\mathbf{r}_i) H_{n_z}(\mathbf{u}_3\cdot\mathbf{r}_i)$$

$$\times \exp\{-r^2/2\}\xi(\sigma_i,\tau_i). \tag{5}$$

In (5), $H_n(x_i)$ are Hermitian polynomials and $\xi(\sigma_i,\tau_i)$ are the usual spin-isospin functions. Three mutually orthogonal unit vectors, $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$, that are identical for all particles, have also been introduced. If $\mathbf{u}_1$ is directed along the x axis, $\mathbf{u}_2$ along the y axis, and $\mathbf{u}_3$ along the z axis, then instead of the variables $x_i;\ y_i;\ z_i$ in the arguments of the Hermitian polynomials, one can write the scalar products $\mathbf{u}_1\cdot\mathbf{r}_i;\ \mathbf{u}_2\cdot\mathbf{r}_i;\ \mathbf{u}_3\cdot\mathbf{r}_i$. The vectors $\mathbf{u}_1,\ \mathbf{u}_2,\mathbf{u}_3$ are a convenient system for describing rotations of the Slater determinant. $\Phi(\lambda,\mu)$ is a superposition of states of different angular moment from which states of definite angular momentum can be obtained by means of a generalization of Peierls–Yoccoz projection method [8].

4. STANDARD PROJECTION METHOD

In this section we review the definition of the standard projection method. Let the wave function Ψ be a superposition of states with different L and M values.

$$\Psi = \sum_{L,M} C_{L,M}\Psi_{L,M}. \tag{6}$$

The same function has a new form after a rotation of the frame of reference, namely,

$$\Psi \to \tilde{\Psi} = \sum_{L,M,M'} C_{L,M}\Psi_{L,M} D^L_{M,M'}. \tag{7}$$

The overlap integral of Ψ and $\tilde{\Psi}$ (or, in other words, the integral of the product $\tilde{\Psi} \cdot \Psi$) can be expressed through the coefficients $C_{L,M}$ and a spherical Wigner function:

$$< \tilde{\Psi}|\Psi >= \sum_{L,M,M'} C_{L,M} C_{L,M'} D^{L}_{M,M'}. \tag{8}$$

As a result, we come to the following well-known relation:

$$C_{L,M} C_{L,M'} = \int d\Omega < \tilde{\Psi}|\Psi > D^{L}_{M,M'}. \tag{9}$$

The last formula gives us an algorithm for the selection of states with definite value of the angular momentum.

5. FOCK–BARGMANN MAP OF MANY-PARTICLE HARMONIC OSCILLATOR BASIS FUNCTIONS

The $SU(3)$ symmetry indices (λ, μ) are introduced to remove the degeneracy of the translationally invariant oscillator shell-model states. If K denotes the projections of the angular moment L on the principal intrinsic symmetry axis, the many-particle wave functions $\Psi^{\alpha}_{L,M}(\lambda, \mu)$ of the $SU(3)$ shell model have the following form:

$$\Psi^{\alpha}_{L,M}(\lambda, \mu) = \sum_{K} A^{L,a}_{K}(\lambda, \mu) \Psi^{L}_{K,M}(\{\mathbf{r}_i, \sigma_i, \tau_i\}), \tag{10}$$

where the quantum number α is analogous to the additional quantum number of the non-axial rigid rotor.

The starting point for the construction of wave function (10) can be the highest weight vector of the (λ, μ) irrep which has the form of the Slater determinant $\Phi(\lambda, \mu)$ or a linear combination of the Slater determinants and is characterized by definite numbers of quanta along the Cartesian axes of the system. The Peierls–Yoccoz projection method insures that this highest weight vector, after an arbitrary rotation of the coordinate system axes, is a superposition of all basis functions of the (λ, μ) irrep, namely,

$$\Phi(\lambda, \mu) = \sum_{L,M} \sum_{\alpha} B^{L,\alpha}_{M}(\lambda, \mu) \sum_{K} A^{L,a}_{K}(\lambda, \mu) \Psi^{L}_{K,M}(\{\mathbf{r}_i, \sigma_i, \tau_i\}). \tag{11}$$

The coefficients in this superposition (11) are not numbers, they are functions of the Euler angles ϕ, θ, ψ, which appear together with the three unit vectors, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, introduced (see Sec. 3) in the generating invariant $\Phi(\lambda, \mu)$ which is

also a generalized coherent state. Because $\Phi(\lambda,\mu)$ is an invariant form, the coefficients

$$B_M^{L,\alpha}(\lambda,\mu) = N_L^{\alpha}(\lambda,\mu)\sum_{K'} A_{K'}^{L,\alpha}(\lambda,\mu) D_{K',M}^{L}(\phi,\theta,\psi) \tag{12}$$

are also $SU(3)$ basis states, but in this case defined in Fock–Bargmann space [9,10].

We now give the wave packet expansion again (in other words, an expansion of the $SU(3)$ generating invariant):

$$\Phi(\lambda,\mu) = \sum_{L,M}\sum_{\alpha}\sum_{K} A_K^{L,a}(\lambda,\mu)\Psi_{K,M}^{L}(\{\mathbf{r}_i,\sigma_i,\tau_i\})\times$$

$$\times N_L^{\alpha}(\lambda,\mu)\sum_{K'} A_{K'}^{L,\alpha}(\lambda,\mu) D_{K',M}^{L}(\phi,\theta,\psi). \tag{13}$$

The last expansion plays the important role in our approach. It introduces the Fock–Bargmann map of the basis function and gives the following connection

$$\sum_K A_K^{L,\alpha}\Psi_{K,M}^{L} \to N_L^{\alpha}(\lambda,\mu)\sum_K A_K^{L,\alpha} D_{K,M}^{L}. \tag{14}$$

The left side of this connection is a complicated construction of the many-particle function, and the right side is its simple map in Fock–Bargmann space. This map reminds us of the rigid rotor wave function, but with a normalization factor $N_L^{\alpha}(\lambda,\mu)$. We shall work with the map because of its visual character and simplicity.

For the projection we need another Slater determinant $\tilde{\Phi}(\lambda,\mu)$ which differs from $\Phi(\lambda,\mu)$ in a simple way. To obtain $\tilde{\Phi}(\lambda,\mu)$ we should change the vectors $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$ to vectors $\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2,\tilde{\mathbf{u}}_3$. The next step is a calculation of the overlap integral

$$< \tilde{\Phi}(\lambda,\mu)|\Phi(\lambda,\mu) > .$$

This overlap has the following simple form because (λ,μ) is fixed:

$$< \tilde{\Phi}(\lambda,\mu)|\Phi(\lambda,\mu) >= (\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^{\lambda}(\tilde{\mathbf{u}}_3\cdot\mathbf{u}_3)^{\mu}. \tag{15}$$

Two conclusions follow from (15). First, this overlap doesn't depend on $\mathbf{u}_2$. Second, it can be presented in a form that is convenient for calculation:

$$< \tilde{\Phi}(\lambda,\mu)|\Phi(\lambda,\mu) >= d_{11}^{\lambda} d_{33}^{\mu}, \tag{16}$$

where d_{11} and d_{33} are the elements of the rotation matrix in the three-dimension space. Regarding the latter, it can be shown that

$$d_{11}^{\lambda} d_{33}^{\mu} = \sum_{L,M} \sum_{\alpha} N_L^{\alpha}(\lambda,\mu) \sum_{K} A_K^{L,a}(\lambda,\mu) D_{K,M}^{L}(\tilde{\phi},\tilde{\theta},\tilde{\psi}) \times$$

$$\times N_L^{\alpha}(\lambda,\mu) \sum_{K'} A_{K'}^{L,\alpha}(\lambda,\mu) D_{K',M}^{L}(\phi,\theta,\psi), \tag{17}$$

which defines the amplitudes

$$A_K^{L,a}(\lambda,\mu)$$

as well as the normalization coefficients

$$N_L^{\alpha}(\lambda,\mu).$$

Of course, the simplest situation is for μ equal to zero in which case we can realize the projection in the following way:

$$d_{11}^{\lambda} = \cos^{\lambda}\Theta = \sum_{L} C_L^{(\lambda)} \; P_L(\cos\Theta), \tag{18}$$

where $P_L(\cos\Theta)$ is the Legendre polynomial and

$$C_L^{\lambda} = \int_{-1}^{1} t^{\lambda} P_L(t) dt \tag{19}$$

are the expansion coefficients, which give the contribution of the state with angular momentum L.

6. $SU(3)$ GENERATORS IN FOCK–BARGMANN SPACE

Because our basis functions depend only on the three Euler angles, all the $SU(3)$ operators should be expressed through indices λ, μ and projections $\hat{M}_1, \hat{M}_2, \hat{M}_3$ of the angular momentum operator $\hat{\mathbf{M}}$ in the principal-axis system. We shall show two matrices. The first one is the matrix $||\hat{C}_{\alpha,\beta}||$ of the $SU(3)$ generators:

$$\left|\left| \begin{array}{ccc} 2\lambda/3+\mu/3 & \hat{M}_3 & -\hat{M}_2 \\ 0 & -\lambda/3+\mu/3 & \hat{M}_1 \\ 0 & 0 & -\lambda/3-2\mu/3 \end{array} \right|\right|. \tag{20}$$

The second is the matrix $||\hat{Q}_{\alpha,\beta}||$ of the mass quadrupole moment:

$$\left|\left| \begin{array}{ccc} 2\lambda/3+\mu/3 & \hat{M}_3/2 & -\hat{M}_2/2 \\ \hat{M}_3/2 & -\lambda/3+\mu/3 & \hat{M}_1/2 \\ -\hat{M}_2/2 & \hat{M}_1/2 & -\lambda/3-2\mu/3 \end{array} \right|\right|. \tag{21}$$

We can check these matrices by noting that they should give us the well-known expression for the $SU(3)$ second order Casimir g_2. By definition,

$$g_2 = \sum_{i,j} \hat{C}_{i,j}\hat{C}_{j,i}. \tag{22}$$

Here $\hat{C}_{i,j}$ are defined in the laboratory system, however, we have $\hat{C}_{\alpha,\beta}$ in the principal axes system. For this reason

$$g_2 = \sum_{i,j}\sum_{\alpha,\beta}\sum_{\delta,\gamma} d_{i\alpha}d_{j\beta}\hat{C}_{\alpha,\beta}d_{j\delta}d_{i\gamma}\hat{C}_{\delta,\gamma}. \tag{23}$$

The summation in (23) is not trivial because we must take into account the fact that the matrix elements $\hat{C}_{\alpha,\beta}$ contain the operators $\hat{M}_{\alpha,\beta}$, which act on the matrix elements $d_{j\delta}$ and $d_{i\gamma}$,

$$\hat{M}_1 d_{j,1} = 0, \quad \hat{M}_2 d_{j,1} = -d_{j,3}, \quad \hat{M}_3 d_{j,1} = d_{j,2}, \tag{24}$$

and so on,

$$d_{i,1}d_{j,2}\hat{M}_3\{d_{j,1}d_{i,1}\hat{Q}_{1,1} + d_{j,2}d_{i,2}\hat{Q}_{2,2} + d_{j,3}d_{i,3}\hat{Q}_{3,3}\} =$$

$$= d_{i,1}d_{j,2}\{(d_{j,2}d_{i,1} + d_{j,1}d_{i,2})\hat{Q}_{1,1} - (d_{j,1}d_{i,2} + d_{j,2}d_{i,1})\hat{Q}_{2,2}\}. \tag{25}$$

We have after summation over i and j of the last expression the simple result

$$\sum_{i,j} d_{i,1}d_{j,2}\hat{M}_3\{d_{j,1}d_{i,1}\hat{Q}_{1,1} + d_{j,2}d_{i,2}\hat{Q}_{2,2} + d_{j,3}d_{i,3}\hat{Q}_{3,3}\} =$$

$$= \hat{Q}_{1,1} - \hat{Q}_{2,2}. \tag{26}$$

By this method we obtain from (23)

$$g_2 = \frac{2}{3}(\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu). \tag{27}$$

Note that the linear terms in λ and μ are a direct result of the last calculation.

7. BARGMANN–MOSHINSKY OPERATORS $\hat{\Omega}$ AND $\hat{\Omega}_1$

Another important example is the operator $\hat{\Omega} = \hat{\mathbf{M}}\hat{\mathbf{Q}}\hat{\mathbf{M}}$, considered in the work of Bargmann and Moshinsky [11]. By definition,

$$\hat{\Omega} = \sum_{i,j}\sum_{\alpha,\beta}\sum_{\gamma,\delta} d_{i,\alpha}\hat{M}_\alpha d_{i,\beta} d_{j,\gamma}\hat{Q}_{\beta,\gamma} d_{j,\delta}\hat{M}_\delta =$$

$$= \frac{2\lambda+\mu}{3}\hat{M}_1^2 + \frac{-\lambda+\mu}{3}\hat{M}_2^2 + \frac{-\lambda-2\mu}{3}\hat{M}_3^2 +$$

$$+\hat{M}_2\hat{M}_3\hat{M}_1 + \frac{1}{2}\{\hat{M}_1^2 - \hat{M}_2^2 - \hat{M}_3^2\}. \tag{28}$$

We will now show, separately, the additional terms from the off-diagonal matrix elements of $\hat{Q}_{\beta,\gamma}$,

$$\hat{M}_2\hat{M}_3\hat{M}_1 + \frac{1}{2}\{-\hat{M}_1^2 - \hat{M}_2^2 + \hat{M}_3^2\}, \tag{29}$$

and from the action of operators $\hat{Q}_{\beta,\gamma}$ on the matrix elements $d_{j,\delta}$,

$$\hat{M}_1^2 - \hat{M}_3^2. \tag{30}$$

The action of $\hat{M}_\alpha$ gives a zero result.

It is convenient to rewrite the operator $\hat{\Omega}$ in the terms of the operators $\hat{L}_\alpha$, where $\hat{L}_\alpha = -i\hat{M}_\alpha$,

$$\hat{\Omega} = -\frac{2\lambda+\mu}{3}\hat{L}_1^2 + \frac{\lambda-\mu}{3}\hat{L}_2^2 + \frac{\lambda+2\mu}{3}\hat{L}_3^2 -$$

$$-i\hat{L}_2\hat{L}_3\hat{L}_1 + \frac{1}{2}\{\hat{L}_2^2 + \hat{L}_3^2 - \hat{L}_1^2\}. \tag{31}$$

As usual for the rigid rotor, we use the basis states

$$|L, K+> = \frac{1}{\sqrt{2}}\{|L, K> + |L, -K>\}, \tag{32}$$

if L are even, and

$$|L, K-> = \frac{1}{\sqrt{2}}\{|L, K> - |L, -K>\}, \tag{33}$$

if L are odd. Here K is the projection of the orbital momentum on the intrinsic axes. For simplicity we shall omit the $+$ or $-$ signs behind K in the notation of the basis functions.

The matrix of the operator $\hat{\Omega}$ is traceless and has a tridiagonal form. Its matrix elements are the following for fixed values of the orbital momentum L:

$$< L, K|\hat{\Omega}|L, K >= \frac{2\lambda + \mu + 3}{6}\{L(L+1) - 3K^2\}, \tag{34}$$

$$< L, K+2|\hat{\Omega}|L, K >=$$

$$= \frac{\mu - K}{4}\sqrt{(L-K)(L-K-1)(L+K+1)(L+K+2)}, \tag{35}$$

$$< L, K-2|\hat{\Omega}|L, K >=$$

$$= \frac{\mu + K}{4}\sqrt{(L+K)(L+K-1)(L-K+1)(L-K+2)}. \tag{36}$$

In the special case, if $K = 0$ or $K = 2$,

$$< L, 2|\hat{\Omega}|L, 0 >= \frac{\mu}{2\sqrt{2}}\sqrt{L(L-1)(L+1)(L+2)}, \tag{37}$$

$$< L, 0|\hat{\Omega}|L, 2 >= \frac{\mu + 2}{2\sqrt{2}}\sqrt{L(L-1)(L+1)(L+2)}. \tag{38}$$

The matrix of $\hat{\Omega}$ is not symmetric because the operator $\hat{\Omega}$ is non-Hermitian. This feature of the matrix appears as a result of the well-known fact that the quantum number K cannot be greater than μ. If $K = \mu$, the matrix element (35) disappears and the states for which $K \leq \mu$ are separated from the states with $K > \mu$. Because $\hat{\Omega} = -\hat{\mathbf{L}}\hat{\mathbf{Q}}\hat{\mathbf{L}}$, the eigenvalue for which $K = 0$ dominates and the angular moment $\mathbf{L}$ is directed orthogonal to the symmetry axes $\mathbf{Q}$, lying higher, for example, than those with dominant $K = \mu$ with $\mathbf{L}$ directed along $\mathbf{Q}$.

Some examples of characteristic equations will now be given. Consider the L = 2 case. Then we have the following determinant, which should be equal to zero,

$$\begin{vmatrix} 2\lambda + \mu + 3 - E & (\mu + 2)\sqrt{3} \\ \mu\sqrt{3} & -(2\lambda + \mu + 3) - E \end{vmatrix} = 0. \tag{39}$$

In another form the same equation is

$$E^2 - 6g_2 - 9 = 0, \qquad E_{1,2} = \pm\sqrt{6g_2 + 9}. \tag{40}$$

For $L = 4$ one finds

$$\begin{vmatrix} \frac{10}{3}(2\lambda+\mu+3)-E & 3(\mu+2)\sqrt{5} & 0 \\ 3\mu\sqrt{5} & \frac{4}{3}(2\lambda+\mu+3)-E & (\mu+4)\sqrt{7} \\ 0 & (\mu-2)\sqrt{7} & -\frac{14}{3}(2\lambda+\mu+3)-E \end{vmatrix} =$$

$$= E^3 - (104g_2 + 100)E + 448g_3 = 0, \tag{41}$$

where the third order Casimir is given by

$$g_3 = \frac{5}{27}(2\lambda+\mu+3)(\lambda+2\mu+3)(\lambda-\mu). \tag{42}$$

The result is trivial if $L = 3$:

$$E = 0. \tag{43}$$

(Note that if the operator $\hat{\Omega}$ is used to fit experimental spectrum, it should be taken with a negative sign. Remarks regarding the calculation of the normalization factor for eigenvectors can be found in the last section of this lecture, where, as an important example, the operator $\hat{\Omega}$ is given careful consideration.)

What are some simple conclusions one can draw regarding the operator $-\hat{\Omega}$?

1. It does not shift the 3^+ level.
2. It shifts the 2^+ levels symmetrical down and up.
3. Two of the three 4^+ levels go down and one 4^+ goes up.
4. If $\lambda \gg \mu$, the shift of the lowest 4^+ level is more than $10/3$ times (from initial position) the 2^+ shift.

The next most important operator is $\hat{\Omega}_1 = \hat{\mathbf{M}}\hat{\mathbf{Q}}\hat{\mathbf{Q}}\hat{\mathbf{M}}$, which has the following form:

$$\frac{1}{9}(2\lambda+\mu)^2\hat{M}_1^2 + \frac{1}{9}(\lambda-\mu)^2\hat{M}_2^2 + \frac{1}{9}(\lambda+2\mu)^2\hat{M}_3^2 +$$

$$+\frac{1}{3}(5\lambda+\mu)\hat{M}_1^2 + \frac{1}{3}(\lambda-\mu)\hat{M}_2^2 + \frac{1}{3}(\lambda+5\mu)\hat{M}_3^2 - \frac{2}{3}(\lambda-\mu)\hat{M}_2\hat{M}_3\hat{M}_1 +$$

$$+\hat{M}_3^2\hat{M}_1^2 - 2\hat{M}_2\hat{M}_3\hat{M}_1 - 2\hat{M}_2^2 - 2\hat{M}_3^2. \tag{44}$$

If $\mu = 0$, the last operator reduces to the axial rotor.

Operator (44) has diagonal matrix elements given by

$$< L, K|\hat{\Omega}_1|L, K >= -(g_2 - 2)L(L+1) +$$

$$+\frac{1}{18}(2\lambda^2+2\lambda\mu-\mu^2+6\lambda)\{L(L+1)-3K^2\}+$$

$$+\frac{1}{2}\{L(L+1)-K^2-2\}K^2; \tag{45}$$

and the off-diagonal ones given by

$$< L,K+2|\hat{\Omega}_1|L,K >= -\frac{\mu-K}{12}(2\lambda+\mu+3K+6)\times$$

$$\times\sqrt{(L-K)(L-K-1)(L+K+1)(L+K+2)}; \tag{46}$$

$$< L,K-2|\hat{\Omega}_1|L,K >= -\frac{\mu+K}{12}(2\lambda+\mu-3K+6)\times$$

$$\times\sqrt{(L+K)(L+K-1)(L-K+1)(L-K+2)}; \tag{47}$$

$$< L,2|\hat{\Omega}_1|L,0 >= -\frac{\mu}{6\sqrt{2}}(2\lambda+\mu+6)\sqrt{L(L-1)(L+1)(L+2)}; \tag{48}$$

$$< L,0|\hat{\Omega}_1|L,2 >= -\frac{\mu+2}{6\sqrt{2}}(2\lambda+\mu)\sqrt{L(L-1)(L+1)(L+2)}. \tag{49}$$

Again, we obtain a tridiagonal matrix. Its trace is equal to

$$L(L+1)\{-g_2+2+\frac{1}{30}(L-2)(2L+1)\}\frac{L+2}{2}, \tag{50}$$

if L is even, and

$$L(L+1)\{-g_2+2+\frac{1}{30}(L+3)(2L-1)\}\frac{L-1}{2}, \tag{51}$$

if L is odd.

If $\mu < L$, the characteristic equation becomes the product of two factors or two equations, one gives us the states with $K \leq \mu$; and the other, the non-physical states with $K > \mu$. As an example, we present the characteristic equation for $L = 2$

$$E^2+12(g_2-2)E+36(g_2-2)^2-g_2^2=0 \tag{52}$$

and its solutions

$$E_{1,2} = -6(g_2-2) \pm g_2. \tag{53}$$

A contribution of $K = 0$ is greater in the state with sign $+$, a result that can be predicted qualitatively *a priori* and that follows directly from the definition of the operator $\hat{\Omega}_1$.

Having matrices of the operators $\hat{\Omega}$ and $\hat{\Omega}_1$, means one can construct the matrix of an arbitrary linear superposition of such operators as well as an arbitrary power of those superpositions. In all cases it is sufficient to find the sum and product of the matrices obtained above.

8. OVERLAP INTEGRAL AND DENSITY MATRIX

The overlap integral can be considered to be a density matrix. What do we mean by a density matrix? Consider the wave function of a many-particle system:

$$\Psi = \Psi(\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_n; \mathbf{q}_1, \mathbf{q}_2, .., \mathbf{q}_N), \tag{54}$$

where $\mathbf{q}_1, \mathbf{q}_2, .., \mathbf{q}_N$ are overall control variables and $\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_n$ are the usual position variables of the subsystem. Then the density matrix ρ can be written as an integral

$$\rho(\mathbf{r}_1, \mathbf{r}_2, ..\mathbf{r}_n, \mathbf{r}'_1, \mathbf{r}'_2, .., \mathbf{r}'_n) =$$

$$= \int \Psi^*(\mathbf{r_1}, \mathbf{r}_2, .., \mathbf{r}_n; \mathbf{q}_1, \mathbf{q}_2, ..\mathbf{q}_N)\Psi(\mathbf{r}'_1, \mathbf{r}'_2, .., \mathbf{r}'_n; \mathbf{q}_1, \mathbf{q}_2, .., \mathbf{q}_N)d\mathbf{q}_1 d\mathbf{q}_2 ..d\mathbf{q}_N$$

over all the control variables. In the case of the overlap, we integrate $< \tilde{\Phi}|\Phi >$ over all variables because we do not know in explicit form the subsystem variables. However, we can introduced the three parameters (three Euler angles) and in this way separate the system into two subsystem through the integration procedure, one subsystem being the close shells and the other one representing the valence nucleons. The treatment of the overlap from this point of view gives an obvious picture — the motion of valence nucleons on a background of the fixed state of close shell nucleons.

9. SYMMETRY OF THE $SU(3)$ DYNAMICAL EQUATION ABOUT PERMUTATION OF THE PRINCIPAL AXES

All scalar $SU(3)$ operators should be symmetrical with respect to a permutation of the intrinsic principal axes. The S_3 group of this permutation has three irreps: symmetrical, antisymmetrical, and a two-dimensional one. In order to describe this symmetry, we introduce two expressions: $2\lambda+\mu+3$ and $\sqrt{3}(\mu+1)$.

These are elements of a two dimensional representation. It is convenient to introduce two other parameters β and γ, defined by equalities

$$2\lambda + \mu + 3 = \beta\cos\gamma, \quad \sqrt{3}(\mu+1) = \beta\sin\gamma. \tag{55}$$

It is obvious that g_2 and g_3 are the two simplest invariants of S_3 that one can construct

$$6g_2 = \beta^2 - 12, \quad g_3 = \frac{5}{27}\beta^3\cos 3\gamma. \tag{56}$$

Eigenvalues of higher order Casimir operators can be expressed through

$$\beta^2 \text{ and } \beta^3\cos 3\gamma.$$

The operator $\hat{\Omega}$ has an elegant invariant form in terms of $\beta,\ \gamma$:

$$-\hat{\Omega} = \frac{\beta}{3}\{\cos\gamma(\hat{L}_1^2 - \frac{\hat{L}_2^2 + \hat{L}_3^2}{2}) + \sin\gamma\frac{\sqrt{3}}{2}(\hat{L}_2^2 - \hat{L}_3^2)\} +$$

$$+\frac{i}{6}\{\hat{L}_1\hat{L}_2\hat{L}_3 + \hat{L}_2\hat{L}_3\hat{L}_1 + \hat{L}_3\hat{L}_1\hat{L}_2 + \hat{L}_1\hat{L}_3\hat{L}_2 + \hat{L}_3\hat{L}_2\hat{L}_1 + \hat{L}_2\hat{L}_1\hat{L}_3\}. \tag{57}$$

10. SOME EXAMPLES

We consider now some particular examples of eigenfunctions and eigenvalues of the $-\hat{\Omega}$ operator. The simplest case is if $\mu = 2$ since the problem then reduces to the second order algebraic equation for the eigenvalues

$$\begin{vmatrix} (2\lambda+5)L(L+1)/6 - E & \sqrt{2L(L+1)(L-1)(L+2)} \\ \sqrt{(L(L+1)(L-1)(L+2)/2} & (2\lambda+5)\{L(L+1)-12\}/6 - E \end{vmatrix} = 0.$$

There are two solutions for each even $L \leq \lambda$. However, there is only one physical solution, if $L = \lambda + 2$. In this case

$$E = -\frac{1}{6}(2\lambda+7)(\lambda+3)(\lambda-2), \tag{58}$$

with wave function

$$\Psi = N\{\sqrt{2(\lambda+2)(\lambda+1)}D^{\lambda+2}_{0,M} - \sqrt{(\lambda+4)(\lambda+3)}D^{\lambda+2}_{2+,M}\}, \tag{59}$$

where N is the normalization coefficient. The other solution is

$$E_s = -\frac{1}{6}(2\lambda+3)(\lambda+7)(\lambda+2), \tag{60}$$

$$\Psi_s = N_s\{\sqrt{2(\lambda+4)(\lambda+3)}D^{\lambda+2}_{0,M} + \sqrt{(\lambda+2)(\lambda+1)}D^{\lambda+2}_{2+,M}\}. \quad (61)$$

We obtain the same allowed function after expansion of the Elliott overlap integral

$$d^{\lambda}_{1,1}d^{2}_{3,3} = \frac{1}{8}\frac{\lambda!}{(2\lambda+3)!!}\{2(\lambda+2)(\lambda+1)D^{\lambda+2}_{0,0} -$$

$$-\sqrt{2(\lambda+4)(\lambda+3)(\lambda+2)(\lambda+1)}(D^{\lambda+2}_{2+,0} + D^{\lambda+2}_{0,2+}) +$$

$$+(\lambda+4)(\lambda+3)D^{\lambda+2}_{2+,2+}\} + ... \quad (62)$$

The overlap (62) gives the value of the normalization coefficient N of the function (59):

$$N = \sqrt{\frac{\lambda!}{8(2\lambda+3)!!}}. \quad (63)$$

We need this normalization factor for calculating transitions.

Because there is only one function in the $(\lambda, 2)$ irrep for $L = \lambda + 2$, it should also be an eigenfunction of the $-\Omega_1$ operator. The eigenvalue of the last one is

$$E = \frac{2}{3}(\lambda+3)(\lambda+2)(\lambda^2+5\lambda+10) - \frac{1}{9}(\lambda+3)(\lambda^3+13\lambda^2+38\lambda+20). \quad (64)$$

In the work of Draayer et al. [1,12] it is suggested that the $SU(3)$ form

$$\hat{\mathbf{H}} = a\hat{\mathbf{L}}^2 + b\hat{\mathbf{L}}\hat{\mathbf{Q}}\hat{\mathbf{L}} + c\hat{\mathbf{L}}\hat{\mathbf{Q}}\hat{\mathbf{Q}}\hat{\mathbf{L}} \quad (65)$$

is an approximate map of the rotor Hamiltonian,

$$\hat{\mathbf{H}}_{ASR} = A_1\hat{L}^2_1 + A_2\hat{L}^2_2 + A_3\hat{L}^2_3, \quad (66)$$

an approximation that improves with increasing λ, μ values. Now we can present the exact map $\hat{\mathcal{H}}$ of (65) which contains additional terms that are neglected in the large λ, μ and small L limit:

$$\hat{\mathcal{H}} = A_1\hat{L}^2_1 + A_2\hat{L}^2_2 + A_3\hat{L}^2_3 + iB\hat{L}_2\hat{L}_3\hat{L}_1 + C\hat{L}^2_3\hat{L}^2_1, \quad (67)$$

where

$$A_1 = a + b(\frac{2\lambda+\mu}{3} + \frac{1}{2}) + c\{\frac{(2\lambda+\mu)^2}{9} + \frac{5\lambda+\mu}{3}\}, \quad (68)$$

$$A_2 = a + b(\frac{-\lambda+\mu}{3} - \frac{1}{2}) + c\{\frac{(\lambda-\mu)^2}{9} + \frac{\lambda-\mu}{3} - 2\}, \quad (69)$$

$$A_3 = a + b(\frac{-\lambda - 2\mu}{3} - \frac{1}{2}) + c\{\frac{(\lambda + 2\mu)^2}{9} + \frac{\lambda + 5\mu}{3} - 2\}, \tag{70}$$

$$B = b + c(-2\frac{\lambda - \mu}{3} - 2), \quad D = -c. \tag{71}$$

It is necessary to use some selection rules in constructing eigenfunctions of the last Hamiltonian, specifically, in order to exclude non-physical spurious states and to add an algorithm for calculating the normalization coefficients of eigenvectors. Obviously, we can present immediately the normalized eigenvectors and eigenvalues of the operator $\hat{\mathcal{H}}$ for the case $L = \lambda + 2,\ \mu = 2$, using (58), (64), (59) and (63).

11. NORMALIZATION COEFFICIENTS

Because of the expansion,

$$d^{\lambda}_{1,1} d^{\mu}_{3,3} =$$

$$= \frac{1}{2\sqrt{\pi}} \frac{\Gamma((\lambda+1)/2)\Gamma((\lambda+\mu+2)/2)\Gamma((\mu+1)/2)}{\Gamma((\lambda+2)/2)\Gamma((\lambda+\mu+3)/2)\Gamma((\mu+2)/2)} D^0_{0,0} + ..., \tag{72}$$

where on the right side all except the first term is omitted, we can give the normalization coefficients $N(L = 0, (\lambda, \mu))$ for all states $\Psi_0(\lambda, \mu)$ with $L = 0$:

$$N^2(L = 0, (\lambda, \mu)) =$$

$$= \sqrt{\frac{1}{2\sqrt{\pi}} \frac{\Gamma((\lambda+1)/2)\Gamma((\lambda+\mu+2)/2)\Gamma((\mu+1)/2)}{\Gamma((\lambda+2)/2)\Gamma((\lambda+\mu+3)/2)\Gamma((\mu+2)/2)}}. \tag{73}$$

Then using the answer to the last exercise, we can show how to calculate normalization coefficients of functions defined in Fock–Bargmann space. These coefficients are needed for determining electomagnetic transition probabilities.

Consider two eigenvectors of the Bargmann–Moshinsky operator $\hat{\Omega}$ for the states $L = 2$ (see Sec. 7, equation (39)) and $\lambda = \mu = 2$. One eigenvector is

$$\psi(1) = N_1\{4\sqrt{3}D^2_{0,M} + (\sqrt{105} - 9)D^2_{2,M}\}, \tag{74}$$

with eigenvalue $E_1 = \sqrt{105}$. This function (74) is a superposition of the two orthonormalized functions. The latter are a complete basis for the Elliott states. The function $\psi(1)$ can be presented as the linear combination:

$$\psi(1) = a\Psi_2(1) + b\Psi_2(2). \tag{75}$$

Because the functions $\Psi_2(1),\ \Psi_2(2)$ are orthogonal and normalized, the function $\psi(1)$ should be also normalized and therefore

$$a^2 + b^2 = 1, \tag{76}$$

$$a = b = \frac{\sqrt{3}}{2}(21 - \sqrt{105})N_1 = \frac{1}{\sqrt{2}}, \quad N_1 = \sqrt{\frac{2}{3}}\frac{1}{21 - \sqrt{105}}. \tag{77}$$

In other words

$$\psi(1) = \frac{1}{\sqrt{2}}\{\Psi_2(1) + \Psi_2(2)\}. \tag{78}$$

The second eigenvector of the $\hat{\Omega}$ operator is

$$\psi(2) = N_2\{4\sqrt{3}D^2_{0,M} - (\sqrt{105} + 9)D^2_{2,M}\}, \tag{79}$$

with egenvalue $E_2 = -\sqrt{105}$. An analogous calculation gives the following result:

$$\psi(2) = \frac{1}{\sqrt{2}}\{\Psi_2(1) - \Psi_2(2)\}, \quad N_2 = \sqrt{\frac{2}{3}}\frac{1}{21 + \sqrt{105}}. \tag{80}$$

From this we see now that the eigenvectors $\psi(1),\ \psi(2)$ of the operator $\hat{\Omega}$ are really orthogonal, because they can be considered to result from an orthogonal transformation of two orthogonal Elliott functions.

General remark: The Elliott integral contains information on the normalization coefficients of the orthogonal basis function. One can use instead integrals with the Bargmann measure which are simpler as this involves a direct integration of the square of a basis function. The normalization of the operator $\hat{\Omega}$ eigenvectors is the result of the well-known fact of the nonorthogonality of the Wigner D-function in the Fock–Bargmann space. From this work one can see that it is possible to start with orthogonal combinations of D-functions that are normalized. However, in this case the theory becomes less transparent, more complicated, and loses its obvious connection with the problem of the rigid rotor.

12. HAMILTONIAN OVERLAP INTEGRAL — THE SPECIAL (λ,0) CASE

We shall consider a many-particle Hamiltonian. Matrix elements of the kinetic energy operator can be written down immediately:

$$< \tilde{\Phi}| \sum_i \hat{t}_i|\Phi >= \frac{1}{2}\{N_0 + \lambda + 2\mu + \frac{3}{2}(A-1)\}\hbar\omega < \tilde{\Phi}|\Phi >, \tag{81}$$

where A is the number of nucleons and N_0 is the number of the closed shell quanta. The overlap integral in this expression is a constant within irreps of $SU(3)$. We shall approximate the potential energy operator by a Gaussian nucleon-nucleon interaction:

$$\hat{U}_{i,j} = V_0 \exp\{-\frac{(\mathbf{r}_i - \mathbf{r}_j)^2}{b_0^2}\}. \tag{82}$$

Of course, this is only a schematic form. Furthermore, we have omitted some details in the definition (for instance, the spin and isospin dependence is not shown) which are not important for understanding the main picture. The structure of the matrix elements of such operators is quite simple. For the $\mu = 0$ case it has the form:

$$< \tilde{\Phi}| \sum_{i<j} \hat{U}_{i,j}|\Phi >= C_0(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^\lambda + C_1(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)^2(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda-2} + ... =$$

$$= \sum_k C_k(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)^{2k}(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda-2k}. \tag{83}$$

We now define the potential energy operator $\hat{\mathbf{U}}(\mathbf{u}_1)$ in Fock–Bargmann space. We put, by definition,

$$< \tilde{\Phi}| \sum_{i<j} \hat{U}_{i,j}|\Phi >=< \tilde{\Phi}|\hat{\mathbf{U}}(\mathbf{u}_1)|\Phi >= \hat{\mathbf{U}}(\mathbf{u}_1) < \tilde{\Phi}|\Phi > . \tag{84}$$

We shall use the following equality:

$$\hat{\mathbf{M}}(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^\lambda = \lambda(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda-1}, \tag{85}$$

where $\hat{\mathbf{M}}$ is the angular momentum operator introduced above (see the first lecture). This operator acts only on the vector $\mathbf{u}_1$ and can be expressed as:

$$\hat{\mathbf{M}} = (\mathbf{u}_1 \times \nabla_{\mathbf{u}_1}). \tag{86}$$

The expression

$$-i\frac{\hat{\mathbf{M}}(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^\lambda}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^\lambda} = \frac{\hat{\mathbf{L}}(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^\lambda}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^\lambda} = -i\lambda\frac{(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)} \tag{87}$$

is the average value of the angular momentum vector in the state Φ. Strictly speaking, the vector $\mathbf{u}_1$ is complex and $\tilde{\mathbf{u}}_1$ is its complex conjugate:

$$\mathbf{u}_1 = \vec{\xi}_1 + i\vec{\eta}_1, \quad \tilde{\mathbf{u}}_1 = \vec{\xi}_1 - i\vec{\eta}_1, \tag{88}$$

where $\vec{\xi}_1,\ \vec{\eta}_1$ are the real vectors. Then

$$\frac{\hat{\mathbf{L}}(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^\lambda}{(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^\lambda}=\lambda\frac{2(\vec{\xi}_1\times\vec{\eta}_1)}{(\xi_1^2+\eta_1^2)}, \tag{89}$$

$$\hat{\mathbf{M}}^2(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^\lambda=\lambda(\lambda-1)(\tilde{\mathbf{u}}_1\times\mathbf{u}_1)^2(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^{\lambda-2}-\lambda(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^\lambda. \tag{90}$$

Or, in another form,

$$(\tilde{\mathbf{u}}_1\times\mathbf{u}_1)^2(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^{\lambda-2}=\{\frac{1}{\lambda-1}-\frac{\hat{\mathbf{L}}^2}{\lambda(\lambda-1)}\}(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^\lambda. \tag{91}$$

Continuing this process for other terms in the potential energy (83), we come to the following form for the operator $\hat{\mathbf{U}}(\mathbf{u}_1)$ in Fock–Bargmann space:

$$\hat{\mathbf{U}}(\mathbf{u}_1)=\sum_k B_k\hat{\mathbf{L}}^{2k}, \tag{92}$$

where the coefficients B_k are expressed through λ and C_k. Because Hamiltonian $\hat{\mathbf{H}}$ of the system has the form

$$\hat{\mathbf{H}}=T_0+\hat{\mathbf{U}}(\mathbf{u}_1),\quad T_0=\frac{1}{2}\{N_0+\lambda+2\mu+\frac{3}{2}(A-1)\}\hbar\omega, \tag{93}$$

its eigenvalues E can be written down immediately

$$E=T_0+\sum_k B_k\{L(L+1)\}^k. \tag{94}$$

This result is well-known for p-shell nuclei, for which $k=1$ only.

13. HAMILTONIAN OVERLAP INTEGRAL — THE GENERAL (λ,μ) CASE

In the general case, when λ and μ are both non-zero,

$$<\tilde{\Phi}|\sum_{i<j}\hat{U}_{i,j}|\Phi>=C_{0,0}(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^\lambda(\tilde{\mathbf{u}}_3\cdot\mathbf{u}_3)^\mu+$$

$$+C_{1,0}(\tilde{\mathbf{u}}_1\times\mathbf{u}_1)^2(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^{\lambda-2}(\tilde{\mathbf{u}}_3\cdot\mathbf{u}_3)^\mu+$$

$$+C_{0,1}(\tilde{\mathbf{u}}_3\times\mathbf{u}_3)^2(\tilde{\mathbf{u}}_1\cdot\mathbf{u}_1)^\lambda(\tilde{\mathbf{u}}_3\cdot\mathbf{u}_3)^{\mu-2}+$$

$$+C_{1,1}((\tilde{\mathbf{u}}_1 \times \mathbf{u}_1) \cdot (\tilde{\mathbf{u}}_3 \times \mathbf{u}_3))(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda-1}(\tilde{\mathbf{u}}_3 \cdot \mathbf{u}_3)^{\mu-1} + ... \tag{95}$$

Again, the challenge is to find an operator which when acting on the overlap $< \tilde{\Phi}|\Phi >$ gives the right side of (95).

First, we define an operator $\hat{\mathbf{M}}$ in the terms of the vectors $\mathbf{u}_1,\ \mathbf{u}_3$:

$$\hat{\mathbf{M}} = \hat{\mathbf{M}}(1) + \hat{\mathbf{M}}(3), \quad \hat{\mathbf{M}}(1) = (\mathbf{u}_1 \times \nabla_{\mathbf{u}_1}), \quad \hat{\mathbf{M}}(3) = (\mathbf{u}_3 \times \nabla_{\mathbf{u}_3}). \tag{96}$$

This operator is simplest in a coordinate system with the first axis directed along the vector $\mathbf{u}_1$ and the third axis is directed along vector $\mathbf{u}_3$. Then

$$\hat{M}_1 = \hat{M}_1(3), \quad \hat{M}_3 = \hat{M}_3(1) \tag{97}$$

and

$$\hat{M}_2 = \hat{M}_2(1) + \hat{M}_2(3). \tag{98}$$

Our goal is to express $\hat{M}_2(1)$ and $\hat{M}_2(3)$ in terms of $\hat{M}_1,\ \hat{M}_2,\ \hat{M}_3$. The answer is the following:

$$\hat{M}_2(1) = \frac{\lambda\hat{M}_2 - \hat{M}_1\hat{M}_3}{\lambda+\mu+1}, \quad \hat{M}_2(3) = \frac{\mu\hat{M}_2 + \hat{M}_3\hat{M}_1}{\lambda+\mu+1}. \tag{99}$$

Of course,

$$(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)^2(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda-2}(\tilde{\mathbf{u}}_3 \cdot \mathbf{u}_3)^{\mu} =$$

$$= \frac{2\lambda + \hat{\mathbf{M}}^2(1)}{\lambda(\lambda-1)}(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda}(\tilde{\mathbf{u}}_3 \cdot \mathbf{u}_3)^{\mu}, \tag{100}$$

$$(\tilde{\mathbf{u}}_3 \times \mathbf{u}_3)^2(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda}(\tilde{\mathbf{u}}_3 \cdot \mathbf{u}_3)^{\mu-2} =$$

$$= \frac{2\mu + \hat{\mathbf{M}}^2(3)}{\mu(\mu-1)}(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda}(\tilde{\mathbf{u}}_3 \cdot \mathbf{u}_3)^{\mu}, \tag{101}$$

$$((\tilde{\mathbf{u}}_1 \times \mathbf{u}_1) \cdot (\tilde{\mathbf{u}}_3 \times \mathbf{u}_3))(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda-1}(\tilde{\mathbf{u}}_3 \cdot \mathbf{u}_3)^{\mu-1} =$$

$$= \frac{(\hat{\mathbf{M}}(1) \cdot \hat{\mathbf{M}}(3))}{\lambda\mu}(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda}(\tilde{\mathbf{u}}_3 \cdot \mathbf{u}_3)^{\mu}. \tag{102}$$

The last formulae can be used to give a realization of (95) in terms of the operators $\hat{L}_1,\ \hat{L}_2,\ \hat{L}_3$. As a consequence, the system Hamiltonian $\hat{\mathbf{H}}(\mathbf{u}_1, \mathbf{u}_3)$ will have the following form:

$$\hat{\mathbf{H}}(\mathbf{u}_1, \mathbf{u}_3) = B_0 + B_1\hat{\mathbf{L}}^2 + D_1\hat{\Omega} + D_2\hat{\Omega}_1 + ... \tag{103}$$

14. CLASSICAL EQUATIONS FOR THE $(\lambda, 0)$ CASE

We can consider $\mathbf{u}_1$ to be a vector function of time t, $\mathbf{u}_1(t)$. To find the optimal function $\mathbf{u}_1(t)$, we will construct a Lagrangian $\mathcal{L}$, define an action functional $\mathcal{S}$, and employ the minimum action principle.

The Lagrangian of our system has the general form:

$$\mathcal{L} = i\frac{<\tilde{\Phi}|\frac{\partial}{\partial t}|\Phi>}{<\tilde{\Phi}|\Phi>} - \frac{<\tilde{\Phi}|\hat{\mathbf{H}}|\Phi>}{<\tilde{\Phi}|\Phi>}. \tag{104}$$

By definition,

$$\mathcal{S} = \int \mathcal{L}dt. \tag{105}$$

For the $(\lambda, 0)$ case, $C_0 = C_2 = ... = 0$ with only $C_1 \neq 0$ (see (3)), and we omit T_0. Then

$$\mathcal{L} = i\lambda\frac{(\tilde{\mathbf{u}}_1 \cdot \dot{\mathbf{u}}_1)}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)} - C_1\frac{(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)^2}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^2}. \tag{106}$$

Variation of $\tilde{\mathbf{u}}_1$ gives

$$i\lambda\{\dot{\mathbf{u}}_1 - \frac{(\tilde{\mathbf{u}}_1 \cdot \dot{\mathbf{u}}_1)}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)}\mathbf{u}_1\} = 2C_1\{\frac{(\mathbf{u}_1 \times (\tilde{\mathbf{u}}_1 \times \mathbf{u}_1))}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)} - \frac{(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)^2}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^2}\mathbf{u}_1\}. \tag{107}$$

The vector $\mathbf{u}_1$ was defined as a unit vector. Its square, ${\mathbf{u}_1}^2 = 1$, does not change with time,

$${\mathbf{u}_1}^2 = \xi_1^2 - \eta_1^2 + i(\vec{\xi_1} \cdot \vec{\eta_1}) = 1, \tag{108}$$

and orthogonality requires that

$$\xi_1^2 - \eta_1^2 = 1. \tag{109}$$

Let $\xi_1^2 = \cosh^2 a,\ \eta_1^2 = \sinh^2 a$, then

$$(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1) = \cosh(2a), \quad (\tilde{\mathbf{u}}_1 \times \mathbf{u}_1) = -\sinh(2a)\mathbf{n}_3, \tag{110}$$

where $\mathbf{n}_3$ is a unit vector that is orthogonal to both $\vec{\xi_1}$ and $\vec{\eta_1}$ and is conserved,

$$E = C_1\frac{(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)^2}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^2} = C_1 \tanh^2(2a), \tag{111}$$

$$\mathbf{L} = -i\lambda\frac{(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)}{\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)} = \lambda \tanh(2a)\mathbf{n}_3. \tag{112}$$

It follows from this that a is constant and the vector $\mathbf{n}_3$ does not change its direction. As should be the case (understood from quantum mechanics), the absolute values of $\mathbf{L}$ are restricted by the conditions

$$0 \leq |\mathbf{L}| \leq \lambda \tag{113}$$

because $0 \leq a \leq \infty$. The value of the energy is expressed through $L^2 = \mathbf{L}^2$,

$$E = -\frac{C_1}{\lambda^2} L^2. \tag{114}$$

For transverse components of $\mathbf{u}_1$, equation (107) reduces to

$$i\lambda \dot{\mathbf{u}}_1 = 2C_1 \frac{(\mathbf{u}_1 \times (\tilde{\mathbf{u}}_1 \times \mathbf{u}_1))}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)}, \tag{115}$$

or in another form,

$$i\lambda \dot{\mathbf{u}}_1 = \frac{2C_1}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)} \{\tilde{\mathbf{u}}_1 - (\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)\mathbf{u}_1\}. \tag{116}$$

In the terms $\vec{\xi}_1,\ \vec{\eta}_1,\ a$

$$\lambda \dot{\vec{\xi}}_1 = -2C_1 \frac{1 + \cosh^2 2a}{\cosh 2a} \vec{\eta}_1, \tag{117}$$

$$\lambda \dot{\vec{\eta}}_1 = 2C_1 \frac{1 - \cosh^2 2a}{\cosh 2a} \vec{\xi}_1. \tag{118}$$

In addition,

$$\ddot{\vec{\xi}}_1 + \omega^2 \vec{\xi}_1, \quad \ddot{\vec{\eta}}_1 + \omega^2 \vec{\eta}_1, \quad \omega^2 = (2C_1/\lambda)^2 \tanh^2 2a. \tag{119}$$

A solution of two last equations follows:

$$\vec{\xi}_1(t) = \cosh a \{\mathbf{n}_1 \cos(\omega t) + \mathbf{n}_2 \sin(\omega t)\}, \tag{120}$$

$$\vec{\eta}_1(t) = \sinh a \{-\mathbf{n}_1 \sin(\omega t) + \mathbf{n}_2 \cos(\omega t)\}, \tag{121}$$

where $\mathbf{n}_1,\ \mathbf{n}_2$ are two unit, mutually orthogonal vectors. Of course,

$$(\mathbf{n}_1 \times \mathbf{n}_2) = \mathbf{n}_3. \tag{122}$$

What is the conclusion of this discussion? We obtained a very simple picture; namely, two vectors, $\mathbf{u}_1$ and $\tilde{\mathbf{u}}_1$ or $\vec{\xi}_1$ and $\vec{\eta}_1$, that move in the plane defined by $\mathbf{n}_1$ and $\mathbf{n}_2$ and that rotate about the $\mathbf{n}_3$ axis. The frequency of rotation is constant and proportional to the angular momentum L, and for fixed L inversely proportional λ^2. The energy of system is proportional to L^2.

15. TWO INTERACTING ROTORS, $(\lambda_\pi, 0)$ AND $(\lambda_\nu, 0)$

To describe two interacting rotors it is necessary to introduce two independent pairs of the generating vectors $\mathbf{u}_1, \tilde{\mathbf{u}}_1$ and $\mathbf{v}_1,\ \tilde{\mathbf{v}}_1$. The overlap is then

$$< \tilde{\Phi}|\Phi >= (\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda_\pi} (\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)^{\lambda_\nu}. \tag{123}$$

Now we give the potential energy overlap of such a system:

$$\begin{aligned} < \tilde{\Phi}| \sum_{i<j} U_{i,j} |\Phi >= C_{0,0} (\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda_\pi} (\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)^{\lambda_\nu} + \\ + C_{1,0} (\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)^2 (\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda_\pi - 2} (\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)^{\lambda_\nu} + \\ + C_{0,1} (\tilde{\mathbf{v}}_1 \times \mathbf{v}_1)^2 (\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda_\pi} (\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)^{\lambda_\nu - 2} + \\ + C_{1,1} ((\tilde{\mathbf{u}}_1 \times \mathbf{u}_1) \cdot (\tilde{\mathbf{v}}_1 \times \mathbf{v}_1)) (\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda_\pi - 1} (\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)^{\lambda_\nu - 1} - \\ - D (\tilde{\mathbf{u}}_1 \cdot \mathbf{v}_1)(\tilde{\mathbf{v}}_1 \cdot \mathbf{u}_1) (\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^{\lambda_\pi - 1} (\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)^{\lambda_\nu - 1} + ... \end{aligned} \tag{124}$$

The last term has the meaning of the second order Casimir operator of $SU(3)$. If D is positive, this operator favors the most symmetric $SU(3)$ irrep.

To understand the role of the second order Casimir, consider the $D = 0$ case. As for the genaral case, we then have two subsystems and for each we can introduce an angular momentum operator, $\hat{\mathbf{M}}(\mathbf{u}_1) = (\mathbf{u}_1 \times \nabla_{\mathbf{u}_1})$ and $\hat{\mathbf{M}}(\mathbf{v}_1) = (\mathbf{v}_1 \times \nabla_{\mathbf{v}_1})$. We shall consider also the total angular momentum

$$\hat{\mathbf{M}} = \hat{\mathbf{M}}(\mathbf{u}_1) + \hat{\mathbf{M}}(\mathbf{v}_1). \tag{125}$$

Of course, the integrals of motion are the energy

$$E = C_{1,0} \frac{(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)^2}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)^2} + C_{0,1} \frac{(\tilde{\mathbf{v}}_1 \times \mathbf{v}_1)^2}{(\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)^2} + C_{1,1} \frac{((\tilde{\mathbf{u}}_1 \times \mathbf{u}_1) \cdot (\tilde{\mathbf{v}}_1 \times \mathbf{v}_1))}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)(\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)} + ... \tag{126}$$

and the angular momentum

$$\mathbf{L} = -i\lambda_\pi \frac{(\tilde{\mathbf{u}}_1 \times \mathbf{u}_1)}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)} - i\lambda_\nu \frac{(\tilde{\mathbf{v}}_1 \times \mathbf{v}_1)}{(\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)}. \tag{127}$$

The classical dynamic equations of motion can be written directly in the terms of the subsystem angular momenta, $\mathbf{L}_\pi = \mathbf{L}(\mathbf{u}_1)$ and $\mathbf{L}_\nu = \mathbf{L}(\mathbf{v}_1)$,

$$\frac{d}{dt}\mathbf{L}_\pi = -\frac{C_{1,1}}{\lambda_\pi \lambda_\nu}(\mathbf{L}_\pi \times \mathbf{L}_\nu), \quad \frac{d}{dt}\mathbf{L}_\nu = \frac{C_{1,1}}{\lambda_\pi \lambda_\nu}(\mathbf{L}_\pi \times \mathbf{L}_\nu). \tag{128}$$

The two last equations can be reduced to

$$\frac{d^2}{dt^2}(\mathbf{L}_\pi \times \mathbf{L}_\nu) + \omega^2(\mathbf{L}_\pi \times \mathbf{L}_\nu) = 0, \quad \omega^2 = (C_{1,1}L/\lambda_\pi\lambda_\nu)^2, \tag{129}$$

where $L = |\mathbf{L}|$. The vector $(\mathbf{L}_\pi \times \mathbf{L}_\nu)$ rotates about $\mathbf{L}$ with rotation frequency ω. Its magnitude is directly proportional to L and inversely proportional to λ_π and λ_ν. The absolute values of $\mathbf{L}_\pi$ and $\mathbf{L}_\nu$ and the angle between them are preserved in time. The angles for which the energy is a minimum depends on the sign of $C_{1,1}$: zero for a positive $C_{1,1}$ and π for the negative $C_{1,1}$.

The Hamiltonian follows immediately from the expression (124):

$$\hat{\mathbf{H}} = T_0 + B_{1,0}\hat{\mathbf{L}}_\pi^2 + B_{0,1}\hat{\mathbf{L}}_\nu^2 + B_{1,1}(\hat{\mathbf{L}}_\pi \cdot \hat{\mathbf{L}}_\nu) + ... =$$

$$= T_0 + (B_{1,0} - \frac{1}{2}B_{1,1})\hat{\mathbf{L}}_\pi^2 + (B_{0,1} - \frac{1}{2}B_{1,1})\hat{\mathbf{L}}_\nu^2 + \frac{1}{2}B_{1,1}\hat{\mathbf{L}}^2 + ... \tag{130}$$

The eigenvalues of this Hamiltonian are

$$E = T_0 + (B_{1,0} - \frac{1}{2}B_{1,1})L_\pi(L_\pi + 1) +$$

$$+(B_{0,1} - \frac{1}{2}B_{1,1})L_\nu(L_\nu + 1) + \frac{1}{2}B_{1,1}L(L + 1). \tag{131}$$

Its eigenfunctions are

$$|L, M; L_\pi, L_\nu >= \sum_{M_\pi, M_\nu} C^{L,M}_{L_\pi, M_\pi; L_\nu, M_\nu} |L_\pi, M_\pi > |L_\nu, M_\nu > . \tag{132}$$

It follows that for the $D = 0$ case there are four integrals of the motion, the total angular momentum L and its projection M, the proton angular momentum L_π and the neutron angular momentum L_ν.

The general case ($D \neq 0$) is more complicated. The proton and neutron angular momenta are no longer preserved in time and two other quantum numbers are needed. As before, we start with a classical description, using the overlap function (124). We have two complex vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ and therefore what appears to be a total of twelve degrees of freedom. However, this is not true because we must take into account four additional conditions which follow from the equalities

$$\mathbf{u}_1^2 = 1, \quad \mathbf{v}_1^2 = 1. \tag{133}$$

We put

$$\mathbf{u}_1 = \vec{\xi}_1 + i\vec{\eta}_1, \quad \vec{\xi}_1 = \cosh a_1 \ \mathbf{n}_1, \quad \vec{\eta}_1 = \sinh a_1 \ \mathbf{n}_2, \tag{134}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are two unit, mutually orthogonal vectors, and

$$\mathbf{v}_1 = \vec{\xi}_2 + i\vec{\eta}_2, \quad \vec{\xi}_2 = \cosh a_2 \ \mathbf{m}_1, \quad \vec{\eta}_2 = \sinh a_2 \ \mathbf{m}_2, \tag{135}$$

where again $\mathbf{m}_1,\ \mathbf{m}_2$ are two other unit, mutually orthogonal vectors. Each pair of unit vectors is defined by three Euler angles (ϕ_1, θ_1, ψ_1 for the first pair and ϕ_2, θ_2, ψ_2 for the second pair). As a result, we find eight time-dependent degrees of freedom, including $a_1,\ a_2$.

First, we introduce the classical dynamics equations in the terms of the vectors $\mathbf{u}_1$ and $\mathbf{v}_1$

$$i\lambda_\pi \ \dot{\mathbf{u}}_1 = 2C_{1,0}\frac{(\mathbf{u}_1 \times (\tilde{\mathbf{u}}_1 \times \mathbf{u}_1))}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)} + C_{1,1}\frac{(\mathbf{u}_1 \times (\tilde{\mathbf{v}}_1 \times \mathbf{v}_1))}{(\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)} -$$

$$-D\frac{(\tilde{\mathbf{v}}_1 \cdot \mathbf{u}_1)}{(\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)}\{\mathbf{v}_1 - (\mathbf{v}_1 \cdot \mathbf{u}_1)\mathbf{u}_1\} \tag{136}$$

$$i\lambda_\nu \ \dot{\mathbf{v}}_1 = 2C_{0,1}\frac{(\mathbf{v}_1 \times (\tilde{\mathbf{v}}_1 \times \mathbf{v}_1))}{(\tilde{\mathbf{v}}_1 \cdot \mathbf{v}_1)} + C_{1,1}\frac{(\mathbf{v}_1 \times (\tilde{\mathbf{u}}_1 \times \mathbf{u}_1))}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)} -$$

$$-D\frac{(\tilde{\mathbf{u}}_1 \cdot \mathbf{v}_1)}{(\tilde{\mathbf{u}}_1 \cdot \mathbf{u}_1)}\{\mathbf{u}_1 - (\mathbf{u}_1 \cdot \mathbf{v}_1)\mathbf{v}_1\}. \tag{137}$$

These represent eight equations for eight functions. We can consider separately the five intrinsic variables and the three Euler angles of orientation of the system as a whole. There are the following intrinsic variables:

$$a_1(t), \quad a_2(t), \quad (\mathbf{n}_1 \cdot \mathbf{m}_1) = d_{1,1}(t), \quad (\mathbf{n}_1 \cdot \mathbf{m}_2) = d_{1,2}(t),$$

$$(\mathbf{n}_2 \cdot \mathbf{m}_1) = d_{2,1}(t), \quad (\mathbf{n}_2 \cdot \mathbf{m}_2) = d_{2,2}(t), \tag{138}$$

Of the four matrix elements $d_{1,1},\ d_{1,2},\ d_{2,1},\ d_{2,2}$, only three are independent because

$$d_{1,1}^2 + d_{1,2}^2 + d_{2,1}^2 + d_{2,2}^2 = 1 + \{d_{1,1}d_{2,2} - d_{1,2}d_{2,1}\}^2. \tag{139}$$

The same equations can be expressed in other terms.

16. CONCLUSION

We have shown that Fock–Bargmann space is a useful representation for gaining a deeper understanding of the $SU(3)$ model of atomic nuclei. In this space the wave equations have a differential form that is similar to those of a rigid body if Hamiltonian of the valence nucleons is constructed from the Bargmann–Moshinsky operators. This Hamiltonian also gives rise to new terms that strongly influenced the eigenvalues spectrum in the case of the large angular momentum L and its projection K. The Fock–Bargmann representation also gives us a simple way to derive classical equation analogs of the $SU(3)$ model and opens a new paradigm for visualizing $SU(3)$ model results.

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