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# $m$-ADIC DYNAMICAL MODELS FOR FUNCTIONING OF COMPLEX INFORMATION SYSTEMS

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We develop a model of functioning of complex information systems in that information states are coded by $m$-adic integers. An information state evolves by iterations of a discrete $m$-adic dynamical system. The $m$-adic ultrametric on the space of information states describes the ability of an information system to operate with associations. The main attention is paid to the collective dynamics of families of associations.

The system of $p$-adic numbers $\mathbf{Q}_p$, constructed by K. Hensel in the 1890s, was the first example of an infinite number field (i.e., a system of numbers where the operations of addition, subtraction, multiplication and division are well defined) which was different from a subfield of the fields of real and complex numbers. During much of the last 100 years $p$-adic numbers were considered only in pure mathematics, but in recent years they have been extensively used in theoretical physics (see, for example, the books [1] and [2] and the pioneer paper [3]), the theory of probability [2] and investigations of chaos in dynamical systems [4]. In [4,5] $p$-adic dynamical systems were applied to the simulation of functioning of complex information systems (in particular, cognitive systems). In this paper we continue investigations started in [4,5]. There are no physical reasons to use only prime numbers $p$ as the basis for the description of a physical or information model. Therefore, we use systems of the so-called $m$-adic numbers, where $m > 1$ is an arbitrary natural number, see, for example, [2].

**1. $m$-Adic Hierarchic Chains for Coding of Information.** The abbreviation «$I$» will be used for information. The symbol $\tau$ will be used to denote an $I$-system.

Let $\tau$ be an $I$-system. We shall use neurophysiologic terminology: elementary units for $I$-processing are called neurons, a «thinking device» of $\tau$ is called brain. In our model it is supposed that each neuron $n$ has $m > 1$ levels of excitement, $\alpha = 0, 1, ..., m-1$. Individual neuron has no $I$-meaning in this model. Information is represented by chains of neurons, $\mathcal{N} = (n_0, n_1, ..., n_M)$. Each chain of neurons $\mathcal{N}$ can (in principle) perform $m^M$ different $I$-states

$$x = (\alpha_0, \alpha_1, ..., \alpha_{M-1}), \quad \alpha \in \{0, 1, ..., m-1\}, \tag{1}$$

corresponding to different levels of excitement for neurons in $\mathcal{N}$. Denote the set of all possible $I$-states by the symbol $X_I$.

In our model each chain of neurons $\mathcal{N}$ has a *hierarchic structure*: neuron $n_0$ is the most important, neuron $n_1$ is less important neuron than $n_0$,..., neuron $n_j$ is less important neuron than $n_0, ..., n_{j-1}$. This hierarchy is based on the possibility of a neuron to ignite other neurons in this chain: $n_0$ can ignite all neurons $n_1, ..., n_k, ..., n_M$, $n_1$ can ignite all neurons $n_2, ..., n_k, ..., n_M$, and so on; but the neuron $n_j$ cannot ignite any of the previous neurons $n_0, ..., n_{j-1}$. Moreover, the process of igniting has the following structure. If $n_j$ has the highest level of excitement, $\alpha_j = m-1$, then increasing of $\alpha_j$ to one unit induces the complete relaxation of the neuron $n_j, \alpha_j \to \alpha'_j = 0$, and increasing to one unit of the level of excitement $\alpha_{j+1}$ of the next neuron in the chain,

$$\alpha_{j+1} \to \alpha'_{j+1} = \alpha_{j+1} + 1 \ . \tag{2}$$

If neuron $n_{j+1}$ already was maximally excited, $\alpha_{j+1} = m-1$, then transformation (2) will automatically imply the change to one unit of the state of neuron $n_{j+2}$ (and the complete relaxation of the neuron $n_{j+1}$) and so on.*

We shall use the abbreviation HCN for *hierarchic chain of neurons.* This hierarchy is called a *horizontal hierarchy* in the $I$-performance in brain.

HCNs provide the basis for forming *associations.* Of course, a single HCN is not able to form associations. Such an ability is a feature of an ensemble $B^\tau$ of HCNs of $\tau$. Let $s \in \{0, 1, ..., m-1\}$. A set

$$A_s = \{x = (\alpha_0, ..., \alpha_M) \in X_I : \alpha_0 = s\} \subset X_I$$

is called an association of the order 1. This association is represented by a collection $B_s^\tau$ of all HCNs $\mathcal{N} = (n_0, n_1, ..., n_M)$ which have the state $\alpha_0 = s$ for neuron $n_0$. Thus any association $A_s$ of the order 1 is represented in the brain of $\tau$ by some set $B_s^\tau$ of HCNs. Of course, if the set $B_s^\tau$ is empty the association $A_s$ does not present in the brain (at this instance of time). Associations of higher orders are defined in the same way. Let $s_0, ..., s_{l-1} \in \{0, 1, ..., m-1\}, l \leq M$. The set

$$A_{s_0 \dots s_l} = \{x = (\alpha_0, ..., \alpha_M) \in X_I : \alpha_0 = s_0, ..., \alpha_{l-1} = s_{l-1}\}$$

is called an association of the order $l$. Such an association is represented by a set $B^\tau_{s_0 \dots s_l} \subset B^\tau$ of HCN. We remark that associations of the order $M$ coincide with $I$-states for HCN. We shall demonstrate that an $I$-system $\tau$ obtains large advantages by working with associations of orders $l \ll M$.

*In fact, transformation (2) is the addition with respect to $\mathrm{mod}\ m$.

Denote the set of all associations of order $l$ by the symbol $X_{A,l}$. We set $X_A = \cup_l X_{A,l}$. This is the set of all possible associations which can be formed on the basis of the $I$-space $X_I$.

Sets of associations $J \subset X_A$ also have an $I$-meaning. Such sets of associations will be called *ideas* of $\tau$ (of the order 1). Denote the set of all ideas by the symbol $X_{ID}$*.

The hierarchy $I$-state $\to$ association $\to$ idea is called a *vertical hierarchy* in the $I$-performance in the brain.

In our model «hardware» of the brain of $\tau$ is given by an ensemble $B^\tau$ of HCNs. For an HCN $\mathcal{N} \in B^\tau$, we set $i(\mathcal{N}) = x$, where $x$ is the $I$-state of $\mathcal{N}$. The map $i : B^\tau \to X_I$ gives the correspondence between states of brain and $I$-states. In general it may be that $i(\mathcal{N}_1) = i(\mathcal{N}_2)$ for $\mathcal{N}_1 \neq \mathcal{N}_2$. It is natural to assume that in general the map $i$ depends on the time parameter $t: \ i = i_t$. In particular, if $t$ is discrete, we obtain a sequence of maps $i_t : t = 0, 1, 2, ...$

**2. Dynamical Evolution of Information.** In this section shall we study the simplest dynamics of $I$-states, associations and ideas. Such $I$-dynamics is «ruled» by a function $f\colon X_I \to X_I$ which does not depend on time and random fluctuations. This «process of thinking» has no memory: the previous $I$-state $x$ determines a new $I$-state $y$ via the transformation $y = f(x)$. In this model time is discrete, $t = 0, 1, 2, \ldots, n, \ldots, K$. Set

$$U_0^\tau = i_o(B^\tau), U_1^\tau = i_1(B^\tau), \ldots, U_n^\tau(B^\tau), \ldots . \tag{3}$$

A set $U_n^\tau$ of $I$-states is called an $I$-universe of $\tau$. This is the set of all $I$-states which are generated by the ensemble $B^\tau$ of HCNs of $\tau$ at the instant of the time $t = n$. We suppose that sets $\{ U_n^\tau \}_{n=0}^\infty$ of $I$-states can be obtained by iterations of one fixed map $f\colon X_I \to X_I$. Thus dynamics (3) of $I$-universe of $\tau$ is induced by pointwise iterations:

$$x_{n+1} = f(x_n). \tag{4}$$

If $x \in U_n^\tau$, then $y = f(x) \in U_{n+1}^\tau$. Each $x_0 \in U_0^\tau$ evolves via in $I$-trajectory: $x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \ldots, x_{n+1} = f(x_n) = f^n(x_0), \ldots$ . Here the symbol $f^n$ denotes $n$th iteration of $f$.

Suppose that, for each association $A$, its image $B = f(A) = \{ y = f(x) : x \in A \}$ is again an association. Denote the class of all such maps $f$ by the symbol $\mathcal{A}(X_I)$. If $f \in \mathcal{A}(X_I)$, then dynamics (4) of $I$-states of $\tau$ induces dynamics of associations

$$A_{n+1} = f(A_n). \tag{5}$$

*In principle, it is possible to consider sets of ideas of the order 1 as new $I$-objects (ideas of the order 2) and so on. However, we restrict our attention to dynamics of ideas of order 1.

Starting with an association $A_0$ (which is a subset of $I$-universe $U_0^\tau$) $\tau$ obtains a sequence of associations: $A_0, A_1 = f(A_0), \ldots , A_{n+1} = f(A_n), \ldots$ . Dynamics of associations (5) induces dynamics of ideas: $J'{=}f(J){=}\{B^\tau{=}f(A)\ \ A{\in}J\}$. Thus each idea evolves by iterations:

$$J_{n+1} = f(J_n). \tag{6}$$

Starting with an idea $J_0$ $\tau$ obtains a sequence of ideas: $J_0, J_1 = f(J_0), \ldots , J_{n+1} = f(J_n), \ldots$ . In particular, by choosing $J_0 = U_0^\tau$ we obtain dynamics of $I$-universe (which is also an idea of $\tau$).

**3.** $m$**-Adic Representation for Information States.** It is surprising that number systems which provide the adequate mathematical description of HCN were developed long time ago by purely number theoretical reasons. These are systems of the so-called $m$-adic numbers, $m > 1$ is a natural number. First we note that in mathematical model it would be useful to consider infinite $I$-states:

$$x = (\alpha_0, \alpha_1, \ldots , \alpha_M, \ldots),\ \alpha_j = 0, 1, \ldots , m - 1. \tag{7}$$

Such an $I$-state $x$ can be generated by an ideal infinite HCN $\mathcal{N}$. Denote the set of all vectors (7) by the symbol $\mathbf{Z}_m$. This is an ideal $I$-space, $X_I = \mathbf{Z}_m$. On this space we introduce a metric $\rho_m$ corresponding to the hierarchic structure between neurons in chain $\mathcal{N}$ having an $I$-state $x$: two $I$-states $x$ and $y$ are close with respect to $\rho_m$ if initial (sufficiently long) segments of $x$ and $y$ coincide. If $x = (\alpha_0, \ldots , \alpha_M, \ldots)$, $y = (\beta_0, \ldots , \beta_M \ldots)$, and $\alpha_0 = \beta_0, \ldots , \alpha_{k-1} = \beta_{k-1}$, but $\alpha_k \neq \beta_k$, then $\rho_m(x, y) = \frac{1}{m^k}$. Such a metric is well know in number theory. This is an ultrametric: the strong triangle inequality

$$\rho_m(x, y) \leq \max[\rho_m(x, z), \rho_m(x, y)] \tag{8}$$

holds true. This inequality has the simple $I$-meaning. Let $\mathcal{N}, \mathcal{M}, \mathcal{L}$ be HCNs having $I$-states $x, y, z$, respectively. Denote by $k(\mathcal{N}, \mathcal{M})$ ($k(\mathcal{N}, \mathcal{L})$ and $k(\mathcal{M}, \mathcal{L})$) length of an initial segment in chains $\mathcal{N}$ and $\mathcal{M}$ ($\mathcal{N}$ and $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{L}$) such that neurons in $\mathcal{N}$ and $\mathcal{M}$ have the same level of exciting. Then it is evident that

$$k(\mathcal{N}, \mathcal{M}) \geq \min[k(\mathcal{N}, \mathcal{L}), k(\mathcal{L}, \mathcal{M})]. \tag{9}$$

But this gives inequality (8). As in the every metric space, in $(\mathbf{Z}_m, \rho_m)$ we can introduce balls, $U_r(a) = \{\, x \in \mathbf{Z}_m\ :\ \rho_m(a, x) \leq r\,\}$ and spheres $S_r(a) = \{\, x \in \mathbf{Z}_m\ :\ \rho_m(a, x) = r\,\}$ (with centre at $a \in \mathbf{Z}_m$ of radius $r > 0$). There is one to one correspondence between balls and associations. Let $r = \frac{1}{p^l}$ and $a = (a_0, a_1, \ldots , a_{l-1}, \ldots)$. The $U_r(a) = \{\, x = (x_0, x_1, \ldots , x_{l-1}, \ldots)\ :\ x_0 = a_0, x_1 = a_1, \ldots , x_{l-1} = a_{l-1}\,\} = A_{a_0 a_1 \ldots a_{l-1}}$. The space of associations $X_A$ coincides with the space of all balls. The space of ideas $X_{ID}$ is the space which elements are families of balls.

$I$-dynamics on $\mathbf{Z}_m$ is generated by maps $f\colon \mathbf{Z}_m \to \mathbf{Z}_m$. We are interested in maps which belong to the class $\mathcal{A}(\mathbf{Z}_m)$. They map a ball onto a ball: $f(U_r(a)) = U_{r'}(a')$. To give examples of such maps, we use the standard algebraic structure on $\mathbf{Z}_m$. We study mathematical models for $p$-adic numbers [4]. Let $f(x) = x^n$, $n = 2, 3, 4, \dots$ . Then $f$ belongs to the class $\mathcal{A}(\mathbf{Z}_m^\star)$, where $\mathbf{Z}_m^\star = \mathbf{Z}_m \setminus \{0\}$. Hence associations are transformed into associations and each monomial map generates dynamics of associations as well as ideas.

**4. Stochastic Model.** Deterministic $I$-model (3)–(6), does note provide the right description of complex $I$-processes. It seems that a new $I$-state depends not only on the previous $I$-state, but also on a choice of a new map $f\colon X_I \to X_I$ (to perform a new iteration). What is a basis of such a game? The contemporary level of neurophysiologic research is not sufficient to obtain the definite answer to this question. One of possibilities is that randomness of $I$-evolution of cognitive systems has the same origin as randomness of evolution of quantum systems. Such a viewpoint is very attractive (despite rather speculative character of cognitive quantum arguments). However, in this paper we shall consider classical random models which generalize the deterministic model of section.

Suppose that $\tau$ has $N$ different $I$-processors $\pi_1, \dots, \pi_N$, with dynamical functions $f_z$, $z = 1, 2, \dots, N$. The $\tau$ uses different blocks for processing of an $I$-state. At each instant of time $t = 0, 1, \dots,$ $\tau$ chooses some processor $\pi_t$ and performs a new iteration:

$$x_{n+1} = f_z(x_n). \tag{10}$$

How does $\tau$ choose a sequence of processes $\pi_{z_1}, \pi_{z_2}, \dots, \pi_{z_{n+1}}, \dots$? The simplest model is a model of the deterministic* choice:

$$z_{n+1} = g(z_n). \tag{11}$$

However, such a system $\tau$ will exhibit rather simple $I$-behaviour. A $\tau$ whose choice mechanism is used ruled by a deterministic law (11) could not change its thinking blocks depending on the previous $I$-state $x_n$.

Higher level $I$-systems do not just perform «algorithms» (11). Their choice depends essentially on the previous $I$-state $x_n$:

$$z_{n+1} = g(z_n, x_n). \tag{12}$$

On the next level of complexity $\tau$ uses a random selection mechanism:

$$z_{n+1} = g(z_n, x_n, \omega), \tag{13}$$

*However, we do not follow ideas of Turing. A choice function need not be a recursive function. So it need not be performed by a Turing machine. Such a choice function can have a hardware realization which totally differs from the hardware of ordinary computers.

where $\omega$ is a «choice parameter». This is a random evolution. Here the implicit value $g(z, x, \omega)$ is not so important. $I$-dynamics of $\tau$ is statistical dynamics:

$$x_{n+1}(\omega) = f_{z_{n+1}}(x_n(\omega)). \tag{14}$$

Here a value $x_{n+1}(\omega)$ of a new $I$-state of $\tau$ depends on a choice of $\omega$.

A chance parameter $\omega$ can also evolve with time: $\omega = \theta\omega$, where $\theta\colon \Omega \to \Omega$ is a law of evolution and $\Omega$ is a space of chance parameters. Thus $z$ evolves as $z_0, z_1 = g(z_0, x_0, \omega), z_2 = g(z_1, x_1, \theta^2\omega), z_3 = g(z_2, x_2, \theta\omega), \ldots$ . Finally we have:

$$z_{n+1} = g(z_n, x_n, \theta^{n-1}\omega), \tag{15}$$

$$x_{n+1} = f_{z_{n+1}}(x_n). \tag{16}$$

Roughly speaking $\tau$ does not try to find a «right decision» for each triply $(z, x, \omega)$; $\tau$ tries only to control its behaviour statistically. So statistical $I$-behaviour is determined by probabilities, namely conditional probabilities, $P(x_{n+1} = y/\text{previous})$, to obtain at the next step an $I$-state $y$ on the basis of information about previous information states.

One of the distinguishing features of random dynamics (15), (16) is that such a stochastic process is in general non-Markovian. We recall that a stochastic process (chain) $\{x_n(\omega)\}_{n=0}^{\infty}$ has a Markov property if

$$P(x_{n+1} = y | x_n = k, x_{n-1} = v, \ldots, x_0 = \lambda) = P(x_{n+1} = y | x_n = n). \tag{17}$$

Here the probability of obtaining a new state $x_{n+1} = y$ depends only on the previous state $x_n = n$ of the system (and it does not depend on the evolution $x_0 = \lambda, \ldots, x_{n-1} = v$). A detailed mathematical investigation demonstrated that Markov property of random evolution (15), (16) depends strongly on the initial $I$-state $x_0{=}\lambda$ and the structure of random evolution law $\theta$. For some $\theta$ $I$-dynamics is Markovian for any choice of $x_0 = \lambda$. Such a cognitive system $\tau$ does not use a memory on a long range evolution to create a new $I$-state $x_{n+1} = y$. Here the previous $I$-state $x_n = k$ determines (but, of course, only statistically) the next state $x_{n+1} = y$. Moreover, some $\theta$ (the so-called Bernoulli process) induces an $I$-dynamics which does not have even one step memory: $P(x_{n+1} = y | x_n = n) = P(y)$. Here the randomness of $\theta$ is so strong that stochastically destroys even one step memory. However, the most interesting feature of dynamics (15), (16) is that, for a wide class of $\theta$, a $\tau$ can demonstrate Markovian as well as non-Markovian behaviour depending on the initial $I$-state $x_0 = \lambda$. Some $I$-states $\lambda$ are proceeded with one step memory, but other are proceeded with the long range memory. In the latter case to determine $x_{n+1} = y$, $\tau$ uses all information which was collected in the previous $I$-evolution, $x_0 = \lambda, x_1 = q, \ldots, x_{n-1} = v, x_n = w$. Another interesting feature of this model is that Markovness of $I$-evolution depends on the base $m$ of the coding system.

If, for each $z$, a map $f_z$ belongs to the class $\mathcal{A}(X_I)$, then random $I$-dynamics (15), (16) induces $I$-dynamics:

$$A_{n+1} = f_{z_{n+1}}(A_n) \tag{18}$$

of random associations $A_n = A_n(\omega)$. $I$-dynamics (18) induces automatically $I$-dynamics $J_{n+1} = f_{z_{n+1}}(J_n)$ of random ideas, $J_n = J_n(\omega)$.

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