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DUALITY SUMMETRY OF THE 2D Φ^4 FIELD MODEL

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We show that the exact beta-function $\beta(g)$ of the continuous 2D $g\Phi^4$ model in the strong coupling regime $g > g_+^*$ possesses the Kramers–Wannier duality symmetry. The duality symmetry transformation $\tilde{g} = d(g)$ such that $\beta(d(g)) = d'(g)\beta(g)$ is constructed. The approximate values of the fixed point g_+^* computed from the duality equation $d(g_+^*) = g_+^*$ are shown to agree with those obtained from the strong coupling expansion and with available numerical results.

1. INTRODUCTION

The 2D Ising model and some other lattice spin models are known to possess the remarkable Kramers–Wannier(KW) duality symmetry, playing an important role both in statistical mechanics and in quantum field theory [1–3]. The self-duality of the isotropic 2D Ising model means that there exists an exact mapping between the high-T and low-T expansions of the partition function [3]. In the transfer-matrix language this implies that the transfer-matrix of the model under discussion is covariant under the duality transformation. If we assume that the critical point is unique, the KW self-duality would yield the exact Curie temperature of the model. This holds for a large set of lattice spin models including systems with quenched disorder (for a review see [3, 4]).

In this paper we study mainly the symmetry properties of the beta-function $\beta(g)$ for the 2D $g\Phi^4$ theory, regarded as a continuum limit of the exactly solvable 2D Ising model. In contrast to the latter, the 2D $g\Phi^4$ theory is known not to be an integrable quantum field theory.

The beta-function $\beta(g)$ of the continuum limit theory is known to date only in the four-loop approximation within the framework of conventional perturbation theory at fixed dimension $d = 2$ [5]. (Five-loop RG calculations have also been recently completed [6]). Calculations of beta-functions are of great interest in statistical mechanics and quantum field theory. The beta-function contains the essential information on the renormalized coupling constant g_+^*, this being important for constructing the equation of state of the 2D Ising model. Duality is known to impose some important constraints on the exact beta-function [7].

The paper is organized as follows. In Sect. II we set up basic notations and define both the correlation length and the beta-function $\beta(g)$. In Sect.III the duality symmetry transformation $\tilde{g} = d(g)$ is derived. Then it is proved that $\beta(d(g)) = d'(g)\beta(g)$. An approximate expression for $d(g)$ is also found. Sect. IV contains some concluding remarks.

2. CORRELATION LENGTH AND COUPLING CONSTANT

We begin by considering the standard Hamiltonian of the 2D Ising model (in the absence of an external magnetic field), defined on a square lattice with periodic boundary conditions; as usual:

$$H = -J \sum_{<i,j>} \sigma_i \sigma_j, \tag{2.1}$$

where $< i, j >$ indicates that the summation is over all nearest-neighboring sites; $\sigma_i = \pm 1$ are spin variables and J is the spin coupling. The standard definition of the spin-pair correlation function reads:

$$G(R) =< \sigma_{\mathbf{R}} \sigma_{\mathbf{0}} >, \tag{2.2}$$

where $< ... >$ stands for the thermal average.

The statistical mechanics definition of the correlation length is given by [8]

$$\xi^2 = \frac{d \ln G(p)}{dp^2}|_{p=0}. \tag{2.3}$$

The quantity ξ^2 is known to be conveniently expressed in terms of the spherical moments of the spin correlation function itself, namely

$$\mu_l = \sum_{\mathbf{R}} (R/a)^l G(\mathbf{R}) \tag{2.4}$$

with a being some lattice spacing. It is easy to see that

$$\xi^2 = \frac{\mu_2}{2d\mu_0} \tag{2.5}$$

where d is the spatial dimension (in our case $d = 2$).

In order to extend the KW duality symmetry to the continuous field theory we have need for a «lattice» model definition of the coupling constant g, equivalent to the conventional one exploited in the RG approach. The renormalization coupling constant g of the $g\Phi^4$ theory is closely related to the fourth derivative of the «Helmholtz free energy», namely $\partial^4 F(T,m)/\partial m^4$, with respect to the order parameter $m = \langle\Phi\rangle$. It may be defined as follows (see [8])

$$g(T,h) = -\frac{(\partial^2\chi/\partial h^2)}{\chi^2\xi^d} + 3\frac{(\partial\chi/\partial h)^2}{\chi^3\xi^d}, \tag{2.6}$$

where χ is the homogeneous magnetic susceptibility

$$\chi = \int d^2x G(x). \tag{2.7}$$

It is in fact easy to show that $g(T,h)$ in Eq.(2.6) is merely the standard four-spin correlation function taken at zero external momenta. The renormalized coupling constant of the critical theory is defined by the double limit

$$g^* = \lim_{h\to 0}\lim_{T\to T_c} g(T,h) \tag{2.8}$$

and it is well known that these limits do not commute with each other. As a result, g^* is a path-dependent quantity in the thermodynamic (T,h) plane [8].

Here we are mainly concerned with the coupling constant on the isochore line $g(T > T_c, h = 0)$ in the disordered phase and with its critical value

$$g^*_+ = \lim_{T\to T_c^+} g(T,h=0) = -\frac{\partial^2\chi/\partial h^2}{\chi^2\xi^d}|_{h=0}. \tag{2.9}$$

The «lattice» coupling constant g^*_+ defined in Eq. (2.9) is of course some given function of the temperature T_c.

3. DUALITY SYMMETRY OF THE BETA-FUNCTION

The standard KW duality tranformation is known to be as follows [1–3]

$$\sinh(2\tilde{K}) = \frac{1}{\sinh(2K)}. \tag{3.1}$$

We shall see that it will be more convenient to deal with a new variable $s = \exp(2K)\tanh(K)$, where $K = J/T$.

It follows from the definition that s transforms as $\tilde{s} = 1/s$; this implies that the correlation length of the 2D Ising model given by $\xi^2 = \frac{s}{(1-s)^2}$ is a self-dual quantity [9]. Now, on the one hand, we have the formal relation

$$\xi\frac{ds(g)}{d\xi} = \frac{ds(g)}{dg}\beta(g), \tag{3.2}$$

where $s(g)$ is defined as the inverse function of $g(s)$, i.e., $g(s(g)) = g$ and the beta-function is given, as usual, by

$$\xi \frac{dg}{d\xi} = \beta(g). \tag{3.3}$$

On the other hand, it is known from [9] that

$$\xi \frac{ds}{d\xi} = \frac{2s(1-s)}{(1+s)}. \tag{3.4}$$

From Eqs. (3.2)–(3.4), a useful representation of the beta-function in terms of the $s(g)$ function thus follows

$$\beta(g) = \frac{2s(g)(1-s(g))}{(1+s(g))\,(ds(g)/dg)}. \tag{3.5}$$

Let us define the dual coupling constant $\tilde{g}$ and the duality transformation function $d(g)$ as

$$s(\tilde{g}) = \frac{1}{s(g)}; \qquad \tilde{g} \equiv d(g) = s^{-1}(\frac{1}{s(g)}), \tag{3.6}$$

where $s^{-1}(x)$ stands for the inverse function of $x = s(g)$. It is easy to check that a further application of the duality map $d(g)$ gives back the original coupling constant, i.e., $d(d(g)) = g$, as it should be. Notice also that the definition of the duality transformation given by Eq. (3.6) has a form similiar to the standard KW duality equation, Eq. (3.1).

Consider now the symmetry properties of $\beta(g)$. We shall see that the KW duality symmetry property, Eq. (3.1), results in the beta-function being covariant under the operation $g \to d(g)$:

$$\beta(d(g)) = d'(g)\beta(g). \tag{3.7}$$

To prove it let us evaluate $\beta(d(g))$. Then Eq.(3.5) yields

$$\beta(d(g)) = \frac{2s(\tilde{g})(1-s(\tilde{g}))}{(1+s(\tilde{g}))\,(ds(\tilde{g})/d\tilde{g})}. \tag{3.8}$$

Bearing in mind Eq. (3.6) one is led to

$$\beta(d(g)) = \frac{2s(g)-2}{s(g)(1+s(g))\,(ds(\tilde{g})/d\tilde{g})}. \tag{3.9}$$

The derivative in the r.h.s. of Eq. (3.9) should be rewritten in terms of $s(g)$ and $d(g)$. It may be easily done by applying Eq. (3.6):

$$\frac{ds(\tilde{g})}{d\tilde{g}} = \frac{d}{d\tilde{g}}\frac{1}{s(g)} = -\frac{s'(g)}{s^2(g)}\frac{1}{d'(g)}. \tag{3.10}$$

Substituting the r.h.s. of Eq. (3.10) into Eq. (3.9) one obtains the desired symmetry relation, Eq. (3.7).

Therefore, the self-duality of the model allows us to determine the fixed point value in another way, namely from the duality equation $d(g^*) = g^*$.

Making use of a rough approximation for $s(g)$, one gets [9]

$$s(g) \simeq \frac{2}{g} + \frac{24}{g^2} \simeq \frac{2}{g}\frac{1}{1-12/g} = \frac{2}{g-12}. \tag{3.11}$$

Combining this Padé-approximant with the definition of $d(g)$, Eq. (3.6), one is led to

$$d(g) = 4\frac{3g-35}{g-12}. \tag{3.12}$$

The fixed point of this function, $d(g^*) = g^*$, is easily seen to be $g_+^* = 14$. The recent numerical and analytical estimates yield $g_+^* = 14.69$ (see [9–11] and references therein).

It is worth mentioning that the above-described approach may be regarded as another method for evaluating g_+^*, fully equivalent to the standard beta-function method.

4. CONCLUDING REMARKS

We have proved the existence of the duality symmetry transformation $d(g)$ in the $2D$ $g\Phi^4$ theory such that $\beta(d(g)) = d'(g)\beta(g)$. Actually, this symmetry property was shown to result from the KW duality of the 2D lattice Ising model.

It would be tempting but wrong to regard $d(g)$ as a function connecting the weak-coupling and strong coupling regimes. As a matter of fact, our proof is based on the properties of $g(s), s(g)$ defined only for $0 \leq s < \infty, g_+^* \leq g < \infty$ and therefore does not cover the weak-coupling region, $0 \leq g \leq g^*$. The main statement is that the beta-function $\beta(g)$ does have the dual symmetry only in the strong-coupling region, in contrast to the weak-coupling regime where that symmetry is dynamically broken.

In contrast to widely held views, the duality symmetry imposes only mild restrictions on $\beta(g)$. It means that this symmetry property fixes only even derivatives of the beta-function, $\beta^{(2k)}(g_+^*)(k = 0, 1, ...)$, at the fixed point, leaving the odd derivatives free. The duality equation $d(g) = g$ provides yet another method for determining the fixed point, independently of the approach based on the equation $\beta(g) = 0$. Another open problem is also that of finding a systematic approach for calculating $d(g)$.

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