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The aim of this article is to survey advances made in investigating the properties of the Fermi (F) and symmetrized Fermi (SF) functions and in using them as approximants for basic physical inputs in various applications in nuclear physics and related areas, such as the physics of hypernuclei and of metal clusters. The evaluation of the F- and SF-type integrals, taking also into account more general limits, is considered on the basis, either of the Sommerfeld approximation or beyond that, when, e.g., rapidly oscillating functions are involved in the integrand. Particular attention is paid to the «small exponential terms» and topics such as the Fourier and Bessel transforms of the F and SF functions, their analytic properties, the Dingle representation of the F function, etc. Applications refer to the nuclear diffraction in scattering of particles by nuclei, generalized expressions of the harmonic oscillator (HO) energy level spacing for its variation with the particle number, study of the Woods–Saxon (WS)-type potentials and their use in problems of hypernuclei and metal clusters.

Цель данной работы — дать обзор достижений в исследовании свойств ферми- (Ф) и симметризованной ферми-функций (СФ), использовании их как основы в постановке и решении различных задач ядерной физики и связанных с ней областей, таких как физика гиперядер и металлических кластеров. Рассматриваются способы вычисления интегралов, в том числе с произвольными пределами, включающими Ф- и СФ-функции, на основе приближения Зоммерфельда, а также в более общей постановке, когда, например, одна из подынтегральных функций быстро осциллирует. Особое внимание уделено роли появляющихся в таких задачах «экспоненциально малых вкладов», а также преобразованиям Фурье и Бесселя, представлению Дингла для Ф- и СФ-функций, методам, учитывающим аналитические свойства этих функций, и др. Приложения связаны с проблемами ядерного дифракционного рассеяния, обобщением зависимости от атомного номера распределений энергий уровней в потенциале гармонического осциллятора, изучением возможностей потенциала типа Вудса–Саксона (ВС) в исследовании проблем физики гиперядер и металлических кластеров.

1. INTRODUCTION

The Fermi function (F function):

$$f_{\mathrm{F}}(r) = \frac{1}{1 + \mathrm{e}^{(r-c)/a}} \tag{1.1}$$

or

$$f_{\mathrm{F}}(r) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{c-r}{2a}\right) \tag{1.2}$$

has many applications in physics (see [1, 2] and refs. therein). It has been extensively used [3–6] by the Stanford group to represent the charge density $\rho_{\mathrm{F}}(r) = \rho_0 f_{\mathrm{F}}(r)$ of nuclei for a wide range of mass numbers. Then, beginning with [7] it was often used in the so-called high-energy approximation in calculating the charge form factor (FF) of nuclei. Furthermore, the «form factor» of the conventional Woods–Saxon potential [8], which is a fair first approximation to the self-consistent single-particle potential, is an F function. Among other applications of the F function we mention its use in connection with the strong absorption models [9–14]. Another function which is closely related to $f_{\mathrm{F}}(r)$ is the symmetrized Fermi function (SF function) (see, e.g., [15, 16]):

$$f_{\mathrm{SF}}(r) = \frac{1}{1 + \mathrm{e}^{(r-c)/a}} + \frac{1}{1 + \mathrm{e}^{-(r+c)/a}} - 1. \tag{1.3}$$

The function $f_{\mathrm{SF}}(r)$ has the property $f_{\mathrm{SF}}(-r) = f_{\mathrm{SF}}(r)$ and may also be written in the following forms:

$$f_{\mathrm{SF}}(r) = \frac{1}{1 + \mathrm{e}^{(r-c)/a}} - \frac{1}{1 + \mathrm{e}^{(r+c)/a}}, \tag{1.4}$$

$$f_{\mathrm{SF}}(r) = \frac{\sinh(c/a)}{\cosh(r/a) + \cosh(c/a)}, \tag{1.5}$$

$$f_{\mathrm{SF}}(r) = \frac{1}{2}\left[\tanh\left(\frac{c+r}{2a}\right) + \tanh\left(\frac{c-r}{2a}\right)\right]. \tag{1.6}$$

It is evident, since $f_{\mathrm{SF}}(r)$ is an even analytic function, that it can be expanded in even powers of r, and has a zero slope at the origin $f'_{\mathrm{SF}}(0) = 0$. Furthermore, it has certain analytic advantages. For light nuclei with $c/a > 1$, it resembles a Gaussian function while for heavier ones it goes over to the Fermi distribution. Thus, it might be said that it is quite appropriate to be considered as a «universal» nuclear density. In practice, however, it leads for medium and heavy nuclei to

results very similar to those of the usual Fermi distribution. We may also recall that the so-called «cosh» [17] and the SF potentials [18] are appropriate to represent cluster model potentials [17].

In a recent publication [19] the «expansion of the Fermi distribution» has been obtained in terms of derivatives of the δ function in an alternative way to the traditional one:

$$\frac{1}{1+\mathrm{e}^{(r-c)/a}} = \Theta(c-r) - \sum_{k=0}^{\infty} \delta^{(2k+1)}(r-c) a^{2k+2} A_{2k+1} \tag{1.7}$$

with the coefficients $A_n = A_{2k+1}$ expressed through the Bernoulli numbers. In the above expansion both sides should be understood under the integral sign, with a well-behaved function $q(r)$. The left–side integral has been discussed in [4] (for $q(r) = r^n$) and called «the Fermi-type integral». In those cases when Eq.(1.7) has meaning, the corresponding integrals are corrected by the exponentially small terms of the order $\exp(-c/a)$. They have been omitted in [19] as well as in other studies (e.g., [20,21]), where only the first terms of (1.7) have been derived. Exact formulae and estimations for omitted terms have been given in [22] and some examples where their contribution can be important have been considered.

We have seen very recently [23] some operator representation for the integrals with the SF function. Similar result for Fermi integrals*

$$I_{\mathrm{F}}(\alpha) = \int_0^{\infty} \frac{G(x)}{1+\mathrm{e}^{x-\alpha}} dx, \tag{1.8}$$

where the function $G(x)$ meets the conditions of Sommerfeld's lemma [24] (see Sect. 2), is well known in solid state physics [2] (cf. integrals involving (1.1) by setting $(r/a) = x$ and $(c/a) = \alpha$).

The aim of this survey is to present and compare various analytic methods and results which have been used in those problems of nuclear and hypernuclear physics, where the significant theoretical ingredients such as nucleon density distributions, particle–nucleus interactions and mean field potentials are approximated by the F function or the Fermi-type functions. Our exposition is organized as follows.

First of all, the methods under consideration are divided into two categories, viz., those which essentially rely on the Sommerfeld lemma (Sect. 2) and those which may be helpful beyond it (Sect. 3). In accordance with this, in Subsect. 2.1 we examine the conditions of validity for expansions like (1.7) and write them in a compact operator form which is equivalent to Sommerfeld's result known in the electron theory of metals. Applicability of the result for calculation of the model

*Sometimes, they are called the Fermi–Dirac integrals.

nuclear form factors is specified. In Subsect. 2.2, we allow for more general integration limits, namely from $R_i < c$ to $R_f > c$, including in the expansion the exponential terms. The same procedure is applied to the SF function and the results for both distributions are obtained in a unified way.

In Sect. 3 an alternative treatment is carried out on the basis of Fourier transforms and the properties of the hypergeometric functions. Special attention is paid to the Dingle representation for the F function, apparently little known to the nuclear physics community.

In Sect. 4 we show that the results obtained in the preceding paragraphs in combination with the Watson–Sommerfeld method, the Abel–Plana summation procedure and the contour integration method become very useful tools in studying the scattering of fast particles (electrons, hadrons, mesons, and heavy ions) by nuclei. Strong emphasis is given to the exponentially small contributions to the scattering amplitudes for strongly absorbing particles.

Section 5 is devoted to applications as well. The F function is used as an approximation to the nuclear densities in order to obtain improved expressions for the variation of the harmonic oscillator (HO) energy level spacing $\hbar\omega$ with the mass number A. This leads to additional terms with powers of A in the standard expression $\hbar\omega = \text{const}\, A^{-1/3}$. A considerable part of Sect. 5 is devoted to a discussion and use of «Woods–Saxon-type» potentials in approximating the nucleon and Λ-nucleus interactions which appear in the so-called folding model. Attention is paid to the dependence of the potential radius R on the particle number. The problem of determination of the variation of $\hbar\omega$ with this number on the basis of WS-type potentials is also discussed. In addition, the question of the approximate (semi)analytic treatment of the s-state energies for the WS potential is dicussed and comparison is made in one case with the corresponding numerical result. Reference to the use of those approaches in the study of atomic (metal) clusters is also made.

2. EVALUATION OF SOME FERMI AND SYMMETRIZED FERMI INTEGRALS. THE SOMMERFELD LEMMA

First of all, we consider those cases in which the integrand in (1.8) satisfies the following conditions:

(i) The «degeneracy» parameter* α (if one uses the terminology adopted in the physics of metals and semiconductors (see, e.g., [1]), is much larger than unity: $\alpha \gg 1$.

*In many applications to nuclear physics problems, this parameter is the radius-to-diffuseness ratio $\alpha = c/a$.

(ii) The function $G(x)$ is slowly varying near $x = \alpha$ and possessing a Taylor series expansion about that point.

Then, it is possible to apply the Sommerfeld method [24] to obtain an asymptotic expansion in ascending powers of α^{-1}. The latter may be written in the form [2]

$$I_{\mathrm{F}}(\alpha) = \pi \, \mathrm{csc}\,(\pi \hat{D}_\alpha) G(x), \tag{2.1}$$

where the operator $\mathrm{csc}\,(\pi \hat{D}_\alpha)$ indicates the Laurent expansion of the cosecant about zero with

$$\hat{D}_\alpha = \frac{d}{dx}\Big|_{x=\alpha}$$

and

$$\hat{D}_\alpha^{-1} = \int\limits_0^\alpha dx.$$

The above two conditions and the result (2.1) are a statement known as the Sommerfeld lemma.

One should keep in mind that even if these conditions are satisfied, Eq. (2.1) holds with an accuracy to the $\exp(-\alpha)$-order terms. Afterwards, more general expansions for the integrals in question will bring the Sommerfeld approximation as a byproduct together with explicit expressions for the corresponding «correction» terms. It turns out that the condition (ii) may be relaxed whereas result (2.1) remains unchanged. We shall exemplify it below but beforehand it is clear that the «best» alternative to the lemma with $G(x) = \exp(\imath\omega x)$ permits the series expansion for an arbitrary «frequency» ω. In this connection, one has to specify the meaning of a «slowly varying» function in the above formulation (see next subsection).

2.1. Expansions in Powers of the «Diffuseness» Parameter. A general integral containing the SF function can be written as [22]

$$I_{\mathrm{SF}} = \int_0^\infty f_{\mathrm{SF}}(r) q(r) dr = I_{\mathrm{F}} - \mathcal{J}^{(+)}, \tag{2.2}$$

where the «standard Fermi integral» considered previously in [19] is

$$I_{\mathrm{F}} = \int_0^\infty \frac{q(r)}{1 + \mathrm{e}^{(r-c)/a}} dr, \tag{2.3}$$

and $\mathcal{J}^{(+)}$ is one of the forms

$$\mathcal{J}^{(\pm)} = \int_0^\infty \frac{q(\pm r)}{1 + \mathrm{e}^{(r+c)/a}} dr = a \int_{c/a}^\infty \frac{q\,(\pm(az - c))}{1 + \mathrm{e}^z} dz. \tag{2.4}$$

As shown in [22], the integrals of interest can be represented as follows,

$$I_{\rm F} = I_s + I_{as} + \mathcal{J}^{(-)}, \tag{2.5}$$

$$I_{\rm SF} = I_s + I_{as} + \mathcal{J}, \tag{2.6}$$

where

$$I_s = a\int_0^{c/a} q(c-az)dz = \int_0^{\infty} \Theta(c-r)q(r)dr, \tag{2.7}$$

$$I_{as} = a\int_0^{\infty} \frac{q(c+az)-q(c-az)}{1+{\rm e}^z}dz, \tag{2.8}$$

$$\mathcal{J} = \mathcal{J}^{(-)} - \mathcal{J}^{(+)}, \tag{2.9}$$

and $\Theta(x)$ is the unit step function as defined in [22].

The representation for the F and SF integrals (2.5) and (2.6) is rather instructive. Indeed, the first term I_s contains the very simple sharp cutoff function in an integrand. The second term I_{as} includes an «antisymmetric» function $g(z) = q(c+az) - q(c-az)$. The property $g(z) = -g(-z)$ enables one to simplify considerably its evaluation. Finally, the integrals $\mathcal{J}^{(\pm)}$ and $\mathcal{J}$ are usually exponentially small since merely the integration from a large number $z = c/a \gg 1$ (when condition (i) is valid) to ∞, where only the tail of the integrand function $(1+{\rm e}^z)^{-1} \simeq {\rm e}^{-z} \ll 1$ contributes to them, is involved.

Now, when calculating the I_{as} integral we assume that $q(c \pm az)$ can be expanded in the series (cf. condition (ii))

$$q(c \pm az) = q(c) + \sum_{n=1}^{\infty} (\pm 1)^n a^n \frac{q^{(n)}(c)}{n!} z^n. \tag{2.10}$$

Inserting (2.10) into (2.8) and then changing the order of integration and summation (which is assumed to be valid) we obtain:

$$I_{as} = a\sum_{n=1}^{\infty} D_n a^n q^{(n)}(c), \tag{2.11}$$

where the coefficients D_n ($n = 1, 2, 3, ...$) are related to the Bernoulli numbers (see, e.g., [25,26]) B_{n+1} and are given by:

$$D_n = \frac{1-(-1)^n}{n!}\int_0^{\infty} \frac{z^n}{1+{\rm e}^z}dz = \begin{cases} 0 & \text{for even } n \\ \dfrac{2}{n!}\dfrac{\pi^{n+1}}{(n+1)}\left(2^n-1\right)\left|B_{n+1}\right| & \text{for odd } n. \end{cases} \tag{2.12}$$

Equivalently, they may be expressed in terms of the Riemann ζ function. Thus, one can obtain, for example, the first coefficients:

$$D_1 = \frac{\pi^2}{6}, \quad D_3 = \frac{7}{4}\,\frac{\pi^4}{90}, \quad D_5 = \frac{31}{16}\,\frac{\pi^6}{945}. \tag{2.13}$$

Hence, under these assumptions we find,

$$I_s + I_{as} = \int_0^c q(r)dr + a\sum_{n=1}^{\infty} D_n a^n q^{(n)}(c) \tag{2.14}$$

or, by using the relation 1.411(11) from [26]:

$$H(z) \equiv \csc(z) - z^{-1} = \frac{1}{\pi}\sum_{n=1,\text{odd}} D_n \left(\frac{z}{\pi}\right)^n, \qquad |z| < \pi, \tag{2.15}$$

in a more compact form:

$$I_s + I_{as} = \int_0^c q(r)dr + \pi a H(\pi a \hat{D}_c) q(r) \tag{2.16}$$

with

$$\hat{D}_c = \frac{d}{dr}\Big|_{r=c}.$$

Now, if one can neglect integrals $\mathcal{J}\left(\mathcal{J}^{(-)}\right)$ we obtain

$$I_{\mathrm{SF}(F)} = \int_0^{\infty} f_{\mathrm{SF}(F)}(r) q(r) dr = \pi a\, \csc(\pi a \hat{D}_c) q(r), \tag{2.17}$$

that is equivalent to the Sommerfeld approximation (2.1).

Further, accepting the relation

$$q^{(n)}(c) = (-1)^n \int_0^{\infty} \delta^{(n)}(r-c) q(r) dr \qquad (n = 1, 2, 3...) \tag{2.18}$$

as valid for some class of the functions $q(r)$ (see, e.g., [27]) we arrive at the following expressions:

$$I_{\mathrm{SF}(F)} = \int_0^{\infty} \Theta(c-r) q(r) dr - \\ - \sum_{n=1,\text{odd}}^{\infty} D_n a^{n+1} \int_0^{\infty} \delta^{(n)}(r-c) q(r) dr + \mathcal{J}\left(\mathcal{J}^{(-)}\right) \tag{2.19}$$

for integrals $I_{\rm F}$ and $I_{\rm SF}$ expanded in powers of the diffuseness parameter a. To this approximation when one can neglect the last terms in (2.19), the expansions for the SF and F functions coincide with each other, and therefore one can write:

$$f_{\rm SF(F)}(r) = \frac{\sinh(c/a)}{\cosh(r/a)+\cosh(c/a)} = \Theta(c-r) - \sum_{n=1,\text{odd}}^{\infty} a^{n+1} D_n \delta^{(n)}(r-c). \tag{2.20}$$

The explicit form as a series with terms proportional to the odd derivatives of the δ function may be useful for practical calculations. However, in all the cases one needs to keep in mind the conditions of its validity, viz., (i) existence of the expansion (2.10), (ii) capability of the transition from (2.10) to (2.11), (iii) determination of the class of functions, on which the generalized δ function and its derivatives act. In order to do the essential points more transparent let us apply these results in a specific example.

2.1.1. Fourier and Bessel Transforms of F and SF Distributions. The standard expression for the charge FF (neglecting normalization) is given by

$$F_{\rm SF(F)}(p) = \int \mathrm{e}^{i\mathbf{pr}} f_{\rm SF(F)}(r) d\mathbf{r} = \frac{4\pi}{p}\int_0^\infty f_{\rm SF(F)}(r)\ \sin(pr)\ r dr = \\ = -\frac{4\pi}{p}\frac{d}{dp}\ I_{\rm SF(F)}(p), \tag{2.21}$$

where

$$I_{\rm SF(}_{F)}(p) = \int_0^\infty f_{\rm SF(}_{F)}(r)\ \cos(pr)\ dr. \tag{2.22}$$

First, it is easily seen from (2.7) that $I_s = \sin pc/p$. Then, in calculating I_{as} by means of (2.11) we use $d^n \cos pr/dr^n = (-1)^{(n+1)/2} p^n \sin pr$ for $n =$ odd. Thus, we obtain:

$$I_{as} = ai \sin pc \sum_{n=1,\text{odd}} D_n \left(\frac{i\pi pa}{\pi}\right)^n = \pi a \frac{\sin pc}{\sinh \pi pa} - \frac{\sin pc}{p}, \qquad pa < 1. \tag{2.23}$$

Bearing in mind that for the even $\cos pr$-function $\mathcal{J} = 0$, we have:

$$I_{\rm SF}(p) = I_s + I_{as} = \frac{\pi a \sin pc}{\sinh \pi pa}, \tag{2.24}$$

that yields

$$F_{\rm SF}(p) = -\frac{4\pi}{p}\frac{d}{dp}\ \frac{\pi a \sin pc}{\sinh \pi pa}. \tag{2.25}$$

The quantity $F_{\rm F}(p)$ is approximated by the same expression neglecting the terms of order $\exp(-c/a)$ and higher. Their estimation is given in Subsect. 2.2.2.

One should stress an important point, namely that the results have been obtained for the SF and F integrals with the oscillating function $\cos pr$ under the condition $pa < 1$ which ensures the convergence of the series in (2.23). It means that the method used may be applied if the «wavelength» p^{-1} is greater than the thickness a of a «surface layer» of the density distributions. On the other hand, if one evaluates the integral (2.8) by using Eq. 3.911(1) from [26] we obtain

$$I_{as} = a\int_0^\infty \frac{\cos[p(c+az)] - \cos[p(c-az)]}{1+\mathrm{e}^z}dz = \\ = -2a\sin pc\int_0^\infty \frac{\sin paz}{1+\mathrm{e}^z}dz = \pi a\frac{\sin pc}{\sinh \pi pa} - \frac{\sin pc}{p} \quad (2.26)$$

for any values of the effective parameter pa. The r.h.s. of (2.26) may be expanded in the series appearing in (2.23) only under the condition $pa < 1$. Of course, the result (2.26) can be considered as analytic continuation* of (2.23) for values $pa \geq 1$, i.e., for the «rapidly» oscillating function $q(r) = \cos pr$. Unlike the latter, the «slowly» varying function $q(r) = \cos pr$ corresponds to the case where $pa < 1$ even if the wavelength p^{-1} is much smaller than the distribution radius c, i.e., $pc \gg 1$.

In general, this procedure with the intermediate series expansion and subsequent term-by-term integration and summation may be impractical and below we shall describe other more advanced methods for evaluation of the Fermi-type integrals. Nevertheless, we continue our illustrations when dealing with the Bessel transform:

$$B_{\rm SF}(p) = \int_0^\infty f_{\rm SF}(r)J_0(pr)rdr, \quad (2.27)$$

where $p = 2k\sin\dfrac{\theta}{2}$, θ is the scattering angle, and $J_0(x)$ is the cylindric Bessel function of the zeroth order. This transform determines in certain approximation the high-energy scattering amplitudes in the theory of nuclear diffraction with the short-wavelength condition:

$$kc \gg 1. \quad (2.28)$$

*Eq. (2.26) can be derived including complex pa values within band $\left|\Im(pa)\right| < 1$ in an alternative way.

Again, in accordance with prescription (2.16) we find

$$I_s = \int_0^c r J_0(pr) dr = \frac{pcJ_1(pc)}{p^2} \tag{2.29}$$

and

$$I_{as} = \frac{a}{p} \sum_{n=1}^{\infty} D_n (pa)^n [pcJ_1(pc)]^{(n+1)}, \tag{2.30}$$

where we have employed the relation $(d/dz)\{zJ_1(z)\} = zJ_0(z)$ and introduced the notation $[g(z)]^{(n)}$ for the n-th derivative.

In order to calculate the derivatives involved, let us recall the asymptotic expression

$$J_1(pc) \sim \sqrt{\frac{2}{\pi pc}} \cos\left(pc - \frac{3}{4}\pi\right) \tag{2.31}$$

at $pc \gg 1$. This implies not too small scattering angles, in view of strong inequality (2.28). Substituting (2.31) into the r.h.s. of Eq.(2.30) and ignoring terms of order $(pc)^{-\frac{1}{2}}$ and smaller, we have

$$[pcJ_1(pc)]^{(2m)} = (i)^{2m} [pcJ_1(pc)], \qquad m = 1, 2, \ldots, \tag{2.32}$$

and therefore

$$I_{as} = i\frac{a}{p} pcJ_1(pc) \sum_{n=1}^{\infty} D_n (ipa)^n. \tag{2.33}$$

At $pa < 1$ the series in (2.33) is convergent and, as in the previous example, one can apply relation (2.15), so that

$$I_{as} = \left[\frac{\pi pa}{\sinh \pi pa} - 1\right] \frac{pcJ_1(pc)}{p^2}. \tag{2.34}$$

Finally, by adding expressions (2.29) and (2.34) we obtain

$$B_{\mathrm{SF}}(p) = I_s + I_{as} = \frac{\pi pa}{\sinh \pi pa} \frac{pcJ_1(pc)}{p^2}, \tag{2.35}$$

or

$$B_{\mathrm{SF}}(p) = \frac{\pi pa}{\sinh \pi pa} I_s, \tag{2.36}$$

that coincides with the first term of the asymptotic series obtained in [23].

One should keep in mind that this result has been derived for not too large momentum transfers p, namely, $pa < 1$ omitting the corresponding $\mathcal{J}$ contribution. Unlike the calculation of $F_{\mathrm{SF}}(p)$ the correction term is not equal to zero since we deal with the odd function $q(r) = rJ_0(pr)$.

2.1.2. Correction Terms. When estimating the correction terms determined by Eq. (2.4), it is sufficient to consider the integrals

$$\mathcal{J}_{e,o}^{(\pm)} = \int_0^\infty \frac{q_{e,o}(\pm r)}{1 + \mathrm{e}^{(r+c)/a}} dr \tag{2.37}$$

with (even, odd) functions $q_{e,o}(r)$. In fact, every function is the sum of its even and odd parts.

By definition, $q_{e,o}(-r) = P_{e,o} q_{e,o}(r)$, where the parity factor $P_{e,o} = +1(-1)$ for any even (odd) function, that leads to the following simple relation:

$$\mathcal{J}_{e,o}^{(-)} = P_{e,o} \mathcal{J}_{e,o}^{(+)}. \tag{2.38}$$

Hence for the even functions

$$I_{\mathrm{SF}} = I_s + I_{as}, \qquad I_{\mathrm{F}} = I_s + I_{as} + \mathcal{J}_e^{(+)} = I_{\mathrm{SF}} + \mathcal{J}_e^{(+)}, \tag{2.39}$$

so that the correction is needed only for the Fermi integral.

For the odd functions we have

$$I_{\mathrm{SF}} = I_s + I_{as} - 2\mathcal{J}_o^{(+)}, \qquad I_{\mathrm{F}} = I_s + I_{as} - \mathcal{J}_o^{(+)}, \tag{2.40}$$

where the correction term for I_{F} has opposite sign in comparison to that in the preceding case, while for I_{SF} it is twice that for I_{F}.

In general, as mentioned above, for sufficiently smooth $q(r)$ these terms are thought to be of the order of $\mathrm{e}^{-(c/a)}$. Indeed, the inequality

$$\left| \mathcal{J}^{(+)} \right| \leq \mathrm{e}^{-c/a} \int_0^\infty |q(r)| \ \mathrm{e}^{-r/a} dr \tag{2.41}$$

gives us the simplest estimation. A more refined calculation can be performed with the expansion in ascending powers of $\mathrm{e}^{-c/a}$,

$$\mathcal{J}^{(+)} = -\sum_{n=1}^{\infty} (-1)^n \, \mathrm{e}^{-nc/a} \int_0^\infty q(r) \ \mathrm{e}^{-nr/a} dr, \tag{2.42}$$

which is especially useful if the separate contributions to it can be exactly calculated or easily estimated (see below). Sometimes, when evaluating integrals $\mathcal{J}^{(+)}$, it is convenient to employ the following representation:

$$\int_0^\infty \mathrm{e}^{\phi(r)} dr = \frac{\mathrm{e}^{\phi(r)}}{\phi'(r)} \Big|_0^\infty + \frac{\phi''}{[\phi']^3} \, \mathrm{e}^{\phi(r)} \Big|_0^\infty + ..., \tag{2.43}$$

which can be obtained through integration by parts. For

$$\mathcal{J}^{(+)} \simeq \mathrm{e}^{-c/a} \int_0^\infty q(r) \ \mathrm{e}^{-r/a} dr$$

it follows from (2.43) that with $\phi(r) = \ln q(r) - r/a$ one obtains

$$\mathcal{J}^{(+)} \simeq a\frac{q(0)}{1 - a\dfrac{q'(0)}{q(0)}}\,\mathrm{e}^{-c/a}\left\{1 + a^2\frac{\phi''(0)}{\left[1 - a\dfrac{q'(0)}{q(0)}\right]^2} + \ldots\right\}, \tag{2.44}$$

if the function $q(r)\exp(-r/a)$ tends to zero as $r \to +\infty$. In particular, one can see that for a function $q(r)$ with $a|q'(0)/q(0)| \gg 1$ the additional small factor $q(0)/aq'(0)$ appears in the estimation (2.44).

For instance, let us come back to that calculation of the Fourier transform of the Fermi function

$$I_{\mathrm{F}}(p) = \int_0^\infty \frac{\cos pr}{1 + \exp\left[(r-c)/a\right]}dr. \tag{2.45}$$

With the help of the relation (2.39) between I_{F} and I_{SF} integrals in the case of the even function $\cos pr$ one can write,

$$I_{\mathrm{F}}(p) = \pi a\frac{\sin pc}{\sinh \pi pa} + \mathcal{J}^{(+)}, \tag{2.46}$$

since I_{SF} is determined by Eq. (2.24).

The correction term in (2.46) is readily estimated via Eq. (2.42) with $q(r) = \cos pr$, namely,

$$\mathcal{J}^{(+)} = -a\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2 + p^2a^2}\,\mathrm{e}^{-nc/a} \tag{2.47}$$

or

$$\mathcal{J}^{(+)} \le \mathrm{e}^{-c/a}\frac{a}{1 + p^2a^2}. \tag{2.48}$$

In the case of the Bessel transform with the odd function $q(r) = rJ_0(pr)$, both integrals I_{F} and I_{SF} should be corrected. The respective corrections (see Eqs. (2.40) are determined by the expansion (cf. [23]):

$$\mathcal{J}^{(+)} = \int_0^\infty \frac{rJ_0(pr)}{1 + \mathrm{e}^{(r+c)/a}}dr = -\sum_{n=1}^{\infty}(-1)^n\,\mathrm{e}^{-nc/a}\int_0^\infty rJ_0(pr)\,\mathrm{e}^{-nr/a}dr. \tag{2.49}$$

The integral involved is known (see, e.g., formula 7.7.3(18) in [28]) to be $na^2/(n^2 + p^2a^2)^{3/2}$, and therefore

$$\mathcal{J}^{(+)} = -a^2\sum_{n=1}^{\infty}(-1)^n\frac{n}{(n^2 + p^2a^2)^{3/2}}\,\mathrm{e}^{-nc/a} \tag{2.50}$$

or

$$\mathcal{J}^{(+)} \leq \mathrm{e}^{-c/a} \frac{a^2}{(1+p^2a^2)^{3/2}}. \tag{2.51}$$

By comparison of the first and second terms in the r.h.s. of Eq. (2.46), it is seen that the «correction» terms of order $\exp(-c/a)$ may be comparable and in some cases larger than the oscillating contribution to the form factor. In such cases with quickly changing functions $q(r)$ one needs to develop methods which calculate these contributions in a more satisfactory way. In Sec.3, methods will be demonstrated where the results are expressed through the hypergeometric functions and the corresponding series are, in fact, the decompositions in the small parameter $\exp(-c/a)$.

2.2. A Generalization in the Calculation of the Fermi-Type Integrals. Here we extend our consideration by introducing the integration limits $R_i < c$ and $R_f > c$, so that the «standard Fermi integral» (namely, for $R_i \to 0$ and $R_f \to \infty$) is a special case of the integral we calculate. Such a generalization is not only of mathematical interest but it is also relevant to physical problems. Henceforth, in this section we proceed in the same way as in certain treatments made for more specialized cases [29]. Namely, by splitting the second integral in a form suitable for making use of the well-known formula for the geometrical progression, we find [22]

$$I_{\mathrm{F}}(R_i, R_f) \equiv \int_{R_i}^{R_f} \frac{q(r)}{1+\mathrm{e}^{(r-c)/a}} dr = \int_{R_i}^{c} q(r)dr + \\ + \sum_{m=1}^{\infty} (-1)^m \left[\int_{R_i}^{c} q(r)\,\mathrm{e}^{m(r-c)/a} dr - \int_{c}^{R_f} q(r)\,\mathrm{e}^{-m(r-c)/a} dr \right]. \tag{2.52}$$

2.2.1. Expansion of the «Generalized» Fermi-Type Integral Using a Taylor Series. We now assume that the function $q(r)$ can be expanded in a Taylor series around $r = c$

$$q(r) = \sum_{n=0}^{\infty} q^{(n)}(c) \frac{(r-c)^n}{n!}. \tag{2.53}$$

Substituting (2.53) into (2.52), one can show that

$$I_{\mathrm{F}}(R_i,R_f)=\int_{R_i}^{c}q(r)dr+\sum_{n=0}^{\infty}q^{(n)}(c)a^{n+1}\Bigg\{D_n+ \\ +\sum_{l=0}^{n}\frac{1}{l!}\Bigg[(-1)^{n+1}\left(\frac{c-R_i}{a}\right)^{l}F(-\,\mathrm{e}^{(R_i-c)/a},n{+}1{-}l)+ \\ +\left(\frac{R_f-c}{a}\right)^{l}F(-\,\mathrm{e}^{(c-R_f)/a},n{+}1{-}l)\Bigg]\Bigg\}. \quad (2.54)$$

Here according to [25, p.45] the function $F(z,k)$ is determined by

$$F(z,k)=\sum_{m=1}^{\infty}\frac{z^m}{m^k}=z\Phi(z,k,1), \quad (2.55)$$

where $\Phi(z,k,1)$ has the following integral representation:

$$\Phi(z,k,1)=\frac{1}{\Gamma(k)}\int_0^{\infty}\frac{t^{k-1}\,\mathrm{e}^{-t}}{1-z\,\mathrm{e}^{-t}}dt, \quad (2.56)$$

that is valid if either $|z|\le 1,\ z\ne 1$ and $\operatorname{Re}k>0$ or $z=1$ and $\operatorname{Re}k>1$ (see Eq. (3) in [25], p. 43). Here $\Gamma(k)$ is the ordinary Γ function.

Note a compact form [22],

$$I_{\mathrm{F}}(R_i,R_f)=\int_{R_i}^{c}q(r)dr+\sum_{n=0}^{\infty}q^{(n)}(c)a^{n+1}\times \\ \times\left\{D_n+\frac{(-1)^n}{2}D_n\left(\frac{c-R_i}{a}\right)-\frac{1}{2}D_n\left(\frac{R_f-c}{a}\right)\right\}, \quad (2.57)$$

where

$$D_n(\beta)=\frac{2}{n!}\int_{\beta}^{\infty}\frac{t^n}{\mathrm{e}^t+1}dt \qquad (\beta\ge 0). \quad (2.58)$$

2.2.2. Integrals with the SF *Function.* It is convenient to use form (1.4) of the SF function. Thus we have only to calculate the integral which corresponds to the second term in (1.4). In this case no separation of the interval of integration

is needed and we obtain after some algebra

$$\mathcal{J}^{(+)}(R_i,R_f)\equiv\int_{R_i}^{R_f}\frac{q(r)}{1+\mathrm{e}^{(r+c)/a}}dr=\sum_{n=0}^{\infty}q^{(n)}(c)a^{n+1}\times$$
$$\times\left\{\sum_{s=0}^{n}\frac{1}{s!}\left[\left(\frac{R_f-c}{a}\right)^s F\left(-\mathrm{e}^{-(R_f+c)/a},n+1-s\right)-\right.\right.$$
$$\left.\left.-\left(\frac{R_i-c}{a}\right)^s F\left(-\mathrm{e}^{-(R_i+c)/a},n+1-s\right)\right]\right\}. \quad (2.59)$$

The above result can be combined with the corresponding one of the previous section and therefore we obtain immediately the expansion of the integral with the SF distribution. However, it is more expedient to write the results obtained in a unified way, that is to write in a simple formula the expansion of both the F and SF function, by introducing a factor ϵ, which is equal to 1 in the case of the SF function and to 0 in the case of the usual F. Thus, we write:

$$I(R_i,R_f,\epsilon)=\int_{R_i}^{R_f}q(r)f(r)dr=I_{\mathrm{F}}(R_i,R_f)-\epsilon\mathcal{J}^{(+)}(R_i,R_f), \quad (2.60)$$

where we take

$$f(r)=\frac{1}{1+\mathrm{e}^{(r-c)/a}}-\epsilon\frac{1}{1+\mathrm{e}^{(r+c)/a}}. \quad (2.61)$$

Some general expansions for the integral can be found in [22]. Here we give a simplified result for $R_i\to 0$ and $R_f\to\infty$:

$$I(0,\infty,\epsilon)=\int_0^{\infty}q(r)f(r)dr=\int_0^{c}q(r)dr+\sum_{m=0}^{\infty}q^{(m)}(0)a^{m+1}\times$$
$$\times\left\{\sum_{l=0}^{m}\frac{1}{(m-l)!}D_l\left(\frac{c}{a}\right)^{m-l}+[\epsilon-(-1)^m]F\left(-\mathrm{e}^{-c/a},m+1\right)\right\}. \quad (2.62)$$

In the special case with $q(r)=r^n$, we find:

$$I_n(0,\infty,\epsilon)=\int_0^{\infty}r^n f(r)dr=\frac{c^{n+1}}{n+1}\left\{1+(n+1)!\left(\frac{a}{c}\right)^{n+1}\times\right.$$
$$\left.\times\left[\sum_{l=0}^{n}\frac{1}{(n-l)!}D_l\left(\frac{c}{a}\right)^{n-l}+[\epsilon-(-1)^n]F\left(-\mathrm{e}^{-c/a},n+1\right)\right]\right\}. \quad (2.63)$$

The following remarks can be made regarding this expression. Firstly, in the case of the F distribution ($\epsilon = 0$) it reduces to the result which follows from the general expression of the «Fermi integral» $F_n(k), k = (c/a)$ quoted by Elton (see Appendix of [4]) since $\int_0^\infty r^n f_{\mathrm{F}}(r)dr = a^{n+1}F_n(k)$. We note that generalizations to nonintegral values of n, etc., in moment calculations have been discussed in literature [30–32]. Secondly, in the case of the SF distribution ($\epsilon = 1$), there are no exponential terms when n is even. Thus, the use of the SF distribution has the advantage all its even moments to be free of exponential terms. This property simplifies their treatment.

3. TWO TECHNIQUES FOR EVALUATION OF THE FERMI-TYPE INTEGRALS BEYOND THE SOMMERFELD APPROXIMATION

3.1. Connection with the Gauss Hypergeometric Function. Separation of Exponentially Small Contributions. The previous results have been based on the assumption that the function $q(r)$ may be expanded in a power series at a vicinity of the radius $r = c$. In this section we shall relax this assumption and consider the exponential Fourier transform*:

$$q(r) = \mathcal{F}\{\bar{q}(p); r\} = (1/2\pi)\int_{-\infty}^{\infty} \bar{q}(p)\,\mathrm{e}^{irp}dp. \tag{3.1}$$

In calculating the Fermi-type integrals with such functions $q(r)$, one can use the following representation for the Gauss hypergeometric function $F(a, b; c; z)$ ([26], p. 319):

$$\int_0^\infty (1-\mathrm{e}^{-x})^{\nu-1}(1-\beta\,\mathrm{e}^{-x})^{-\rho}\,\mathrm{e}^{-\mu x}dx = B(\mu,\nu)F(\rho,\mu;\nu+\mu;\beta), \tag{3.2}$$

where

$$\mathrm{Re}\,\mu > 0, \quad \mathrm{Re}\,\nu > 0, \quad |\arg(1-\beta)| < \pi,$$

and $B(x, y)$ is the beta function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Let us set $q(r) = \mathrm{e}^{ipr}$ and calculate the integral $\int_0^\infty q(r)f_{\mathrm{F}}(r)dr$. Obviously, this is the case when in the more general expression (3.2) one should put $\mu =$

*What follows is easily extended to the sine- and cosine Fourier transforms and the Laplace one.

$1 - ipa, \nu = 1, \rho = 1$ and $\beta = -\mathrm{e}^{c/a}$. Therefore, one can obtain [33,34]:

$$A_{\mathrm{F}}(p) = \int_0^\infty \frac{\mathrm{e}^{ipr}}{1 + \mathrm{e}^{(r-c)/a}} dr = aB(1 - ipa, 1)\, \mathrm{e}^{c/a} F(1, 1 - ipa; 2 - ipa; -\mathrm{e}^{c/a}). \tag{3.3}$$

Furthermore, because for the applications in question $\exp(c/a) > 1$ (or even $\mathrm{e}^{c/a} \gg 1$), it is pertinent to transform (3.3) into

$$A_{\mathrm{F}}(p) = \frac{\pi a}{i \sinh \pi pa}\, \mathrm{e}^{ipc} + ip^{-1} F(1, ipa; 1 + ipa; -\mathrm{e}^{-(c/a)}). \tag{3.4}$$

Thus, the Fourier transform of the Fermi distribution has been expressed in terms of functions of well-known properties. One should emphasize that the exact result (3.4) reflects explicitly the interplay between the physical parameters involved, namely, the «radius» c, the «diffuseness» parameter a and the «incident frequency» p. In many applications the latter plays the role of momentum transfer.

Formula (3.4) enables one to separate all at once the oscillating part of the form factor $A_{\mathrm{F}}(p)$ (the first term in the r.h.s. of (3.4)) and a comparatively smooth p-dependence that is determined by its second term. Note, that the separation has been achieved without those constraints inherent to the previous approaches (see Sect. 2). We see that the corresponding oscillations at $pc > 1$ (the «edge» effect) have an exponential fall-off generated by the factor $[\sinh \pi pa]^{-1} \sim \exp(-\pi pa)$ at $pa \geq 1$ (the «surface diffuseness» effect).

Further, by using the definition

$$F(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots \tag{3.5}$$

of the Gauss series, the smooth contribution to $A_{\mathrm{F}}(p)$ can be split into the pole term p^{-1} and an expansion in ascending powers of an «effective» parameter $\exp(-c/a) < 1$. The former is cancelled at $p = 0$ with the same term which stems from $-i\pi a[\sinh \pi pa]^{-1} \exp(ipc)$, while the latter may not be disregarded even for the values of $c/a \gg 1$. In fact, at high frequencies with $\pi ap \sim c/a$ the contribution of these exponentially small terms becomes comparable to the contribution of the remaining terms in (3.4) and the formula gives a systematic way to calculate each of them. We shall come back to this point in Subsect. 4.4.2.

Now, we apply this result to evaluate the integral considered in Subsect. 2.2:

$$I_{\mathrm{F}}(R_i, R_f) = \int_{R_i}^{R_f} \frac{q(r)}{1 + \mathrm{e}^{(r-c)/a}} dr = I_{\mathrm{F}}(R_i, \infty) - I_{\mathrm{F}}(R_f, \infty) \tag{3.6}$$

with finite lower R_i and upper R_f limits which satisfy the condition $R_i < c < R_f$. Here

$$I_{\mathrm{F}}(R, \infty) = \int_R^\infty \frac{q(r)}{1 + \mathrm{e}^{(r-c)/a}} dr = \frac{1}{2\pi} \int_{-\infty}^\infty dp \bar{q}(p) A_{\mathrm{F}}(p, R) \tag{3.7}$$

with the function $q(r)$ being replaced by its exponential Fourier transform. Again the problem reduces to the following:

$$A_{\mathrm{F}}(p,R)=\int_R^\infty \frac{\mathrm{e}^{ipr}}{1+\mathrm{e}^{(r-c)/a}}dr=\mathrm{e}^{ipR}\int_0^\infty \frac{\mathrm{e}^{ipr}}{1+\mathrm{e}^{(r-c+R)/a}}dr. \tag{3.8}$$

By using (3.3) one gets

$$A_{\mathrm{F}}(p,R)=aB(1-ipa,1)\,\mathrm{e}^{ipR}\,\mathrm{e}^{(c-R)/a}F(1,1-ipa;2-ipa;-\mathrm{e}^{(c-R)/a}). \tag{3.9}$$

Two cases should be considered, namely: $R<c$ and $R>c$.

Case i) $R<c$**:**

In this case it is convenient to convert the hypergeometric function of (3.9) into the corresponding hypergeometric series (cf., the transition from (3.3) to (3.4)), and so we have

$$A_{\mathrm{F}}(p,R)=\mathrm{e}^{ipR}\left\{\frac{\pi a}{\sin(\pi ipa)}\,\mathrm{e}^{ip(c-R)}+\frac{i}{p}F(1,ipa;1+ipa;-\mathrm{e}^{-(c-R)/a})\right\}=$$

$$=\mathrm{e}^{ipR}\left\{\frac{\pi a}{\sin(\pi ipa)}\,\mathrm{e}^{ip(c-R)}-\frac{1}{ip}+\frac{a}{1+ipa}\,\mathrm{e}^{-(c-R)/a}+O\left(\mathrm{e}^{-2(c-R)/a}\right)\right\}, \tag{3.10}$$

or omitting the terms of higher order in $\mathrm{e}^{-(c-R)/a}$ we obtain

$$A_{\mathrm{F}}(p,R)=\pi aH(\pi ipa)\,\mathrm{e}^{ipc}+\int_R^c \mathrm{e}^{ipr}dr+\mathrm{e}^{ipR}\frac{a}{1+ipa}\,\mathrm{e}^{-(c-R)/a}, \tag{3.11}$$

where the function $H(z)=\sin^{-1}z-z^{-1}$ is the function considered in Sec. 2. Substituting (3.11) into (3.6) and preserving the exponential Fourier transform in r space we arrive at the expression

$$I_{\mathrm{F}}(R,\infty)=\pi a\mathcal{F}\{\bar{q}(p)H(\pi ipa);c\}+\int_R^c q(r)dr+a\mathcal{F}\Big\{\frac{\bar{q}(p)}{1+ipa};R\Big\}\mathrm{e}^{-(c-R)/a}. \tag{3.12}$$

Case ii) $R>c$**:**

In this case Eq. (3.9) includes the hypergeometric series directly from the beginning and therefore

$$A_{\mathrm{F}}(p,R)=\frac{a}{1-ipa}\,\mathrm{e}^{ipR}\,\mathrm{e}^{-(R-c)/a}F(1,1-ipa;2-ipa;-\mathrm{e}^{-(R-c)/a}). \tag{3.13}$$

If the parameters involved meet the inequality $\mathrm{e}^{-(R-c)/a} \ll 1$, we find

$$A_{\mathrm{F}}(p,R) = \frac{a}{1-ipa}\,\mathrm{e}^{ipR}\,\mathrm{e}^{-(R-c)/a}, \tag{3.14}$$

and finally making the same substitutions as in case i) we get for $I_{\mathrm{F}}(R,\infty)$:

$$I_{\mathrm{F}}(R,\infty) = a\mathcal{F}\Big\{\frac{\bar{q}(p)}{1-ipa};R\Big\}\ \mathrm{e}^{-(R-c)/a}. \tag{3.15}$$

Combining Eq. (3.12) and Eq. (3.15) we get

$$I_{\mathrm{F}}(R_i,R_f) = \pi a\mathcal{F}\{\bar{q}(p)H(\pi ipa);c\} + \int_{R_i}^{c} q(r)dr +$$

$$+a\mathcal{F}\left\{\frac{\bar{q}(p)}{1+ipa};R_i\right\}\mathrm{e}^{-(c-R_i)/a} - a\mathcal{F}\left\{\frac{\bar{q}(p)}{1-ipa};R_f\right\}\mathrm{e}^{-(R_f-c)/a}. \tag{3.16}$$

Similarly, evaluation of the generalized SF integral

$$I_{\mathrm{SF}}(R_i,R_f) = \int_{R_i}^{R_f} q(r)f_{\mathrm{SF}}(r)dr \tag{3.17}$$

can be reduced to the following substraction of the Fermi-type integrals:

$$\begin{aligned} A_{\mathrm{SF}}(p,R) &= A_{\mathrm{F}}(p,R) - \mathcal{A}^{(+)}(p,R) = \\ &= \mathrm{e}^{ipR}\Big\{\frac{\pi a}{i\sinh\,\pi pa}\,\mathrm{e}^{ip(c-R)} + \frac{i}{p} + \frac{a}{1+ipa}\times \\ &\times \mathrm{e}^{-(c-R)/a}F(1,1+ipa;2+ipa;-\mathrm{e}^{-(c-R)/a}) - \\ &- \frac{a}{1-ipa}\,\mathrm{e}^{-(c+R)/a}F(1,1-ipa;2-ipa;-\mathrm{e}^{-(c+R)/a})\Big\}, \end{aligned} \tag{3.18}$$

where

$$\mathcal{A}^{(+)}(p,R) = \int_{R}^{\infty} \frac{\mathrm{e}^{ipr}}{1+\,\mathrm{e}^{(r+c)/a}}dr.$$

Putting in (3.18) $R=0$ we find for the «standard» form factors the expression:

$$I_{\mathrm{SF}}(p) = \mathrm{Re}\,A_{\mathrm{SF}}(p;0) = \pi a\frac{\sin\,pc}{\sinh\,\pi pa}. \tag{3.19}$$

The approach described in this section is an alternative way to evaluate the integrals in question. The following comments can be made: (a) It is relied on the well-known results of the theory of special functions and can be presented in a mathematically compact form; (b) We have managed to bypass too strong assumptions (in particular, condition (ii) of Sommerfeld's lemma) leading to (2.11); (c) The corrections of any order in $\exp(-c/a)$ may be evaluated in a systematic manner.

3.2. The Dingle Representation for the Fermi Function. In the previous subsection we have shown a possible way of handling the Fermi-type integrals whose integrands may not meet, at least, condition (ii) of the Sommerfeld lemma. In such situations when $q(r)$ may oscillate rapidly or may have a branch point another useful technique is based upon application of the identity

$$\frac{1}{\mathrm{e}^x + 1} = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \pi \csc(\pi t)\, \mathrm{e}^{-xt} dt, \qquad (0 < \beta < 1), \tag{3.20}$$

which can be proved closing the contour to the left if $x < 0$ and to the right if $x \geq 0$.

This representation has been introduced by Dingle [35] in his calculations of the generalized moments of the Fermi distribution, and we shall call it the Dingle representation or the Dingle integral*. It has been shown (see [2] and refs. therein) that this representation is a sufficiently flexible tool allowing to reduce many F integrals which occur in solid state physics to tabulated Laplace transforms and their inverses or sometimes to tractable exercises in residue theory.

When applying the Dingle result to nuclear physics problems let us return to the FF $A_{\mathrm{F}}(p)$ defined in Subsect. 3.1. In this case, inserting the corresponding expression (3.20) in the integrand of integral (3.3) and integrating over r we obtain,

$$A_{\mathrm{F}}(p) = \frac{a}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \pi \csc(\pi t) \frac{\exp\left(\frac{c}{a} t\right)}{t - ipa} dt. \tag{3.21}$$

Further, the contour in (3.21) may be closed by a very large semicircle to the left picking up the residues from the point $t = ipa$ and the simple poles of the meromorphic function $\pi \csc(\pi t)$. It is easily seen that due to the Cauchy theorem this procedure leads to the known formula (3.4).

One more illustration is related to evaluation of the eikonal phase integral

$$G_{\mathrm{F}}(b) = \int_0^\infty \frac{dz}{1 + \exp\left(\dfrac{\sqrt{z^2 + b^2} - c}{a}\right)}, \tag{3.22}$$

that determines the scattering amplitude in the high-energy approximation at small angles [36,37] for the Fermi density distribution, so that here c is the half-density radius.

*In the introduction of the interesting paper by Dingle a survey of early works on the Fermi integrals is also given and relevant contributions are pointed out.

Again, making use of Eq. (3.20) we get,

$$G_{\mathrm{F}}(b) = \frac{b}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \pi \csc(\pi t) K_1\left(\frac{b}{a}t\right) \exp\left(\frac{c}{a}t\right) dt, \tag{3.23}$$

where the McDonald function $K_1(\frac{b}{a}t)$ is determined by

$$bK_1\left(\frac{b}{a}t\right) = \int_0^\infty \exp\left(-\frac{\sqrt{z^2+b^2}}{a}t\right) dz.$$

At this point we shall confine ourselves to the case of peripheral collisions with $b > c$.

It is pertinent to write down a Mittag–Leffler expansion (see, e.g., [38], p. 402)

$$\pi \csc \pi t = \frac{1}{t} + \sum_{m=1}^{\infty} (-1)^m \left[\frac{1}{t+m} + \frac{1}{t-m}\right]. \tag{3.24}$$

Taking into account this equation we find with the help of Jordan's lemma

$$G_{\mathrm{F}}(b) = b \sum_{m=1}^{\infty} (-1)^{m+1} K_1\left(m\frac{b}{a}\right) \exp\left(m\frac{c}{a}\right). \tag{3.25}$$

Of course, the same result could be derived directly using the geometrical progression expansion for the integrand in (3.22).

Because of the condition $c/a \gg 1$, one may replace the function K_1 in the r.h.s. of Eq. (3.25) by its asymptotic expression

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}}\, \mathrm{e}^{-z} \tag{3.26}$$

for $|z| \gg 1$ and $-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}$ (see, e.g., [28], Eq. 7.4.1(4)). This leads to

$$G_{\mathrm{F}}(b) \sim \sqrt{\frac{\pi}{2}ab} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\exp\left(-m\frac{b-c}{a}\right)}{\sqrt{m}} \tag{3.27}$$

or

$$G_{\mathrm{F}}(b) \sim -\sqrt{\frac{\pi}{2}ab} F\left(-\exp\left(-\frac{b-c}{a}\right), \frac{1}{2}\right), \tag{3.28}$$

where we have employed definition (2.55).

At large values of the impact parameter b, when

$$\exp\left(-\frac{b-c}{a}\right) \ll 1, \tag{3.29}$$

we obtain the asymptotic result:

$$G_{\rm F}(b) \sim \sqrt{\frac{\pi}{2}ab}\,\exp\left(-\frac{b-c}{a}\right). \tag{3.30}$$

In this context one should note that an exponential decrease of the eikonal phase with b increasing is expected.

4. TREATMENT ON THE BASIS OF ANALYTIC PROPERTIES OF THE FERMI FUNCTION

4.1. A Contour Integration Method for Form Factors Calculations. We start from a simple application of the Cauchy theorem to evaluate some Fermi integrals, the so-called contour integration method, based on the property of the F function to have simple poles in the complex plane. So, this method has been utilized for evaluation of nuclear FF's in the so-called high-energy approximation (HEA) [7] (see, also, [39]). As an example, we consider their calculation in the Born approximation (BA) with charge density distributions $\rho_{\rm F}(r) = \rho_0\ f_{\rm F}(r)$ and $\rho_{\rm SF}(r) = \rho_0\ f_{\rm SF}(r)$. The corresponding FF's are determined by the integrals (omitting for simplisity the renormalization constant)

$$I_{\rm F(SF)}(p) = {\rm Re}\, A_{\rm F(SF)}(p), \tag{4.1}$$

where $A_{\rm F}(p)$ is the integral

$$A_{\rm F}(p) = \int_0^\infty \frac{{\rm e}^{ipr}}{1+{\rm e}^{(r-c)/a}}dr,$$

and $A_{\rm SF}(p)$ is given by

$$A_{\rm SF}(p) = A_{\rm F}(p) - \mathcal{A}^{(+)}(p) \tag{4.2}$$

with

$$\mathcal{A}^{(+)}(p) = \int_0^\infty \frac{{\rm e}^{ipr}}{1+{\rm e}^{(r+c)/a}}dr.$$

In order to calculate $A_{\rm F}(p)^*$ it is convenient to consider the integral

$$I_{\rm F}^{(1)} = \oint\limits_{C_1} W_{\rm F}(r)dr, \quad W_{\rm F}(r) = {\rm e}^{ipr} f_{\rm F}(r),$$

where the contour C_1, lying in the first quadrant of the complex r plane and proceeding counter-clockwise, consists of the three parts, namely, the piece $[0, R]$ of the positive real axis, a quarter-circle of the radius R, with the centre in the origin, and the piece $[iR, 0]$ of the imaginary axis. In the limit $R \to \infty$ the Cauchy theorem allows us to write

$$\lim_{R\to\infty} I_{\rm F}^{(+)} = 2\pi i \sum_{n=1}^{\infty} {\rm res}\, W_{\rm F}(r_n^{(+)}) \equiv \Sigma_{\rm res}^{(+)}, \tag{4.3}$$

so that

$$A_{\rm F}(p) = \Sigma_{\rm res}^{(+)} + i\int_0^{\infty} W_{\rm F}(iy)dy, \tag{4.4}$$

where the poles of the F function in the first quadrant are

$$r_n^{(+)} = c + i(2n-1)\pi a \quad (n = 1, 2, 3, ...). \tag{4.5}$$

While the sum of residues in these poles is equal to

$$\Sigma_{\rm res}^{(+)} = -2\pi a i\, {\rm e}^{ipc} \sum_{n=1}^{\infty} {\rm e}^{-(2n-1)\pi pa} = \frac{\pi a}{i\sinh(\pi pa)}\, {\rm e}^{ipc}, \tag{4.6}$$

the integral in the r.h.s. of Eq. (4.4) is the following expansion in ascending powers of $\exp(-c/a)$:

$$\int_0^{\infty} W_{\rm F}(iy)dy = a\int_0^{\infty} \frac{{\rm e}^{-pas}}{1+{\rm e}^{-c/a}\,{\rm e}^{is}} ds = ia\sum_{m=0}^{\infty} \frac{(-1)^m}{m+ipa}\, {\rm e}^{-m\,c/a}. \tag{4.7}$$

In its turn, this expansion is equivalent to one of the Gauss series from Subsect.3.1, viz., it is $p^{-1}F(1, ipa; 1+ipa; -\exp(-c/a))$. Now, it is seen that the method gives an alternative derivation of the result (3.4).

As to the case with the SF distribution, one should note the property,

$$\mathcal{A}^{(+)}(p) = -\int_0^{\infty} \frac{{\rm e}^{-pr}}{1+{\rm e}^{(r+c)/a}} dr. \tag{4.8}$$

*Although explicit expressions for this integral have been given in the preceding sections, we follow the approach that has been used for the first time in treatment [7] of the electron scattering by nuclei.

Moreover, owing to the identity $(1+\exp\,z)^{-1}=1-[1+\exp\,(-z)]^{-1}$ one can write,

$$\mathcal{A}^{(+)}(p)=-ip^{-1}+i\int_0^\infty W_{\rm F}(-iy)dy. \tag{4.9}$$

Obviously, with the help of these relations we are able to reproduce formula (3.19).

Doing so we obtain the form factor for the SF charge density distribution [15, 16]

$$F_{\rm SF}(p)=\frac{4\pi^2ac\rho_0}{p\sinh\,\pi pa}\left[\cos\,pc-\frac{\pi a}{c}\sin\,pc\coth\pi pa\right], \tag{4.10}$$

Here $\rho_0=3/4(\pi c)^{-3}[1+(\pi a/c)^2]^{-1}$.

The contour integration method has successfully been employed in the description of the electron scattering by medium and heavy nuclei when the Coulomb distortion effects are significant, so that it becomes necessary to go out beyond the BA. In this connection, within the HEA one has to deal with the eikonal phases $\beta(\mathbf{r})$ and distorted amplitudes $g(\mathbf{r})$ that determine the electron wave functions instead of those calculated by numerical solution of the respective Dirac equation. As a result, the form factor, similar to the preceding one, has been obtained for nuclei with the SF density distribution

$$F_{\rm SF}(p)=\frac{4\pi^2ac\rho_0}{p\sinh\,\pi pa}D(p)\left[\cos\,(pc+\Phi)-\frac{\pi a}{c}\sin\,(cR+\Phi)\coth\pi pa\right], \tag{4.11}$$

where D and Φ are expressed in terms of the distortion functions $\beta(\mathbf{r})$ and $g(\mathbf{r})$ [39]. With the help of this expression, in [40] the treatment of available data for many nuclei from ^{4}He to ^{208}Pb has been undertaken, and the corresponding charge density distributions have been extracted taking into account the radial variations of them.

4.2. The Fermi-Like Functions in the Strong Absorption Models for the S Matrix of Diffraction Scattering. Many features of diffractive processes studied in nuclear physics at intermediate and high energies can be rather well described using the so-called strong absorption assumption. To be more specific let us consider the elastic scattering amplitude for a spinless particle incident on a spin-zero target nucleus

$$f(\theta)=\sum_{l=0}^{\infty}F_l(\theta), \tag{4.12}$$

$$F_l(\theta)=(2ik)^{-1}(2l+1)(S_l-1)P_l(\cos\,\theta)\exp\,\left[-\gamma\left(l+\frac{1}{2}\right)\right], \tag{4.13}$$

where k^{-1} is the corresponding wave number, and the S matrix elements

$$S_l = S_N(l) = \eta_l \exp[2i\delta_l] \tag{4.14}$$

are determined merely by the nuclear interaction*. Our discussion refers at the moment (in order to avoid some complications associated with inclusion of the Coulomb interaction) to the scattering of neutral particles. We shall come back to this point later.

It is implied herewith that the short-wavelength condition

$$kR \gg 1, \tag{4.15}$$

where R is the radius of nuclear interaction, is fulfilled. This is a general condition for nuclear diffraction.

Then, the absorption coefficient $\eta_l \leq 1$ is expected to increase smoothly and monotonically with l, from small values to unity. It reaches this value near l_0, the so-called grazing angular momentum. The rise occurs within a transition region of width Δ around l_0, which defines the diffuseness**. The l_0 value is related to the «absorption radius» R as follows:

$$L = l_0 + \frac{1}{2} = kR. \tag{4.16}$$

Strong absorption may be defined as the situation for which the surface diffuseness region is very small compared to the grazing angular momentum, i.e.,

$$\Delta \ll l_0. \tag{4.17}$$

This strong inequality plays an essential role in deriving important asymptotic expressions.

Let us remind that such a behaviour of η_l reflects the scattering situation with many open inelastic channels, in which all incoming waves $l \leq l_0$ are almost completely absorbed, whereas the remaining waves are slightly affected by the nuclear interaction. This concept (from the semiclassical point of view, a black disk with a grey, partially transparent edge) that implies the scattering off a nucleus with smooth boundary, was realized many years ago within the strong absorption model (SAM), put forward in [9],

$$S_N(l) = \eta(l) = \left[1 + \exp\frac{l_0 - l}{\Delta}\right]^{-1} = 1 - \eta_{\mathrm{F}}(l), \tag{4.18}$$

where the parameters obey Eq. (4.17).

---

*For convenience, we have introduced a cutoff factor $\exp[-\gamma(l + 1/2)]$ where the parameter $\gamma(> 0)$ should be set equal to zero at the end of the calculations.

**In particular, this means that the S_l-matrix elements are defined not only for the discrete integer l values as in the case of ordinary potential scattering theory.

Many analyses have been carried out by using parametrization [11]:

$$S_N(l) = \eta(l) + i\mu_0 \frac{d\eta(l)}{dl} \tag{4.19}$$

with the same function $\eta(l)$ as in Eq. (4.18). The inclusion of the term involving the derivative is equivalent to an allowance of nuclear refraction.

In another model [13] often used for the description of the scattering of strongly absorbing particles (e.g., heavy ions at intermediate energies [41, 42], and pions in the GeV-region [43]), one writes $S_N(l)$ containing three parameters:

$$S_N(l) = \eta(l) \exp\left[2i\delta(l)\right] = \left[1 + \exp\,\frac{\bar{l_0} - l}{\Delta}\right]^{-1}, \qquad \bar{l_0} = l_0 + il_I. \tag{4.20}$$

Of course, unitarity, i.e., the condition

$$|S_N(l)| \leq 1 \tag{4.21}$$

for all non-negative integer l values, imposes some constraints on the parameters l_I and Δ, viz.,

$$\cos\,\frac{l_I}{\Delta} \geq 0 \tag{4.22}$$

or

$$\left|\frac{l_I}{\Delta}\right| \leq \frac{\pi}{2}. \tag{4.23}$$

The latter is valid since the nuclear phase takes on not too large values for the peripheral partial waves with $l \geq l_0 \gg 1$.

This recollection of various diffraction models would be incomplete without mentioning the expression

$$S(l) = \left[1 - \eta_{\mathrm{F}}(l)\right] \exp\left[2i\delta_0 \eta_{\mathrm{F}}(l)\right] \tag{4.24}$$

proposed in [10] for the analysis of the α-particle elastic scattering experiments. Later, this model and its modifications (see, e.g., in [44]) were applied to treatment of heavy-ion elastic scattering and are still popular. Its attractive feature as it is shown below, is the possibility of getting a closed expression for the corresponding amplitude [14, 34] that enables one to understand better the true role of the refractive properties of the nuclear medium in a diffraction scattering.

Note that the three-parameter form (4.24) follows from the prescription (4.21) for a «weak» nuclear attraction around the nuclear surface

$$\delta(l_0) \ll 1 \tag{4.25}$$

with $\delta(l_0) = |l_I|/4\Delta \simeq \frac{1}{2}\delta_0$, where the second relation is valid to order $\exp(-l_0/\Delta)$. However, although Eq. (4.24) is a good approximation to the form (4.21) for positive real l, their analytical properties are rather different. We have a line of simple poles in the l plane for the Ericson model vs. the essential singular points $l_m = l_0 \pm i\pi\Delta(2m-1) \quad (m = 1, 2...)$ for the function (4.24).

4.2.1. Distributions with Arbitrary Number of Parameters. Generalized Phase-Shift Analysis. The Fermi function has the specific analytic properties not only itself. It has been shown [45] that this function and its derivatives form a complete set for the class of quadratically integrable functions. To do this explicit let us employ formula (1.2) to write down

$$\eta_{\mathrm{F}}(l) = \frac{1}{2}\left[1 - \tanh\frac{l-l_0}{2\Delta}\right] \tag{4.26}$$

and construct the set of functions

$$\phi_0(l) = 1, \qquad \phi_k(l) = \frac{d^{k-1}}{dl^{k-1}}\tanh\frac{l-l_0}{2\Delta}, \qquad (k = 1, 2, ...), \tag{4.27}$$

defined in the interval of $l \in [-\infty, \infty]$. One can verify that the function $\phi_k(l)$ is a linear combination of the Legendre polynomials $P_n(y)$ of the variable $y = \tanh\frac{l-l_0}{2\Delta}$ with the maximum index value n equal to k. For instance, one has

$$\begin{aligned} \phi_2(l) &= \frac{dy}{dl} = \frac{1}{3\Delta}\{P_0(y) - P_2(y)\}, \\ \phi_3(l) &= -\frac{P_1(y)}{\Delta}\frac{dy}{dl}, \qquad \phi_4(l) = \frac{P_2(y)}{\Delta^2}\frac{dy}{dl}. \end{aligned} \tag{4.28}$$

Obviously, the substitution $y = \tanh\frac{l-l_0}{2\Delta}$ maps the l interval $[-\infty, \infty]$ into the y interval $[-1, 1]$ where the Legendre polynomials form a complete set.

The associated orthogonal functions $\tilde{\phi}_k(l) = \sqrt{k+\frac{1}{2}}P_k(y)$ satisfy the relations

$$\int_{-\infty}^{\infty} \tilde{\phi}_k(l)\tilde{\phi}_{k'}(l)w(l)dl = \delta_{kk'}, \qquad (k, k' = 0, 1, 2, ...) \tag{4.29}$$

with the weight function $w(l) = dy/dl$. The coefficients in the expansion

$$\chi(l) = \sum_{k=0}^{\infty} a_k\tilde{\phi}_k(l) \tag{4.30}$$

for a given function $\chi(l)$ are determined by*

$$a_k = \int_{-\infty}^{\infty} \tilde{\phi}_k(l)\chi(l)w(l)dl. \tag{4.31}$$

Similar arguments were used in [45] for the unification of the models for the S matrix of diffraction scattering. In general, assuming for the S matrix that

$$S_N(l) = u(l) + iv(l), \tag{4.32}$$

where the real (imaginary) part $u(l)\Big(v(l)\Big)$ is like the Fermi function (δ function), the corresponding expansions can be written as

$$u(l) = D^{(1)}\left(\frac{\partial}{\partial l_1}\right)\eta^{(1)}(l), \tag{4.33}$$

$$v(l) = D^{(2)}\left(\frac{\partial}{\partial l_2}\right)\eta^{(2)}(l) \tag{4.34}$$

with

$$\eta^{(1,2)}(l) = \frac{1}{2}\left[1 + \tanh\frac{l_{1,2} - l}{2\Delta_{1,2}}\right]. \tag{4.35}$$

The differential operators $D^{(j)}(z)$ are generated by the series

$$D^{(j)}(z) = \sum_{k=0}^{\infty} C_k^{(j)} z^k, \qquad (j = 1, 2). \tag{4.36}$$

Furthermore, the parameters l_1, l_2, Δ_1 and Δ_2 can be chosen in an «optimum» way so that

$$C_0^{(1)} = 1, \qquad C_1^{(1)} = C_2^{(1)} = 0\ , \tag{4.37}$$

$$C_0^{(2)} = C_2^{(2)} = C_3^{(2)} = 0. \tag{4.38}$$

Unitarity $u_l^2 + v_l^2 \leq 1$ imposes some extra constraints.

Thus, keeping several first terms in the r.h.s of Eq. (4.30) one can introduce the necessary number of parameters in the treatment of diffraction scattering. In combination with the asymptotic results shown underneath the approach called in [45] the generalized phase-shift analysis is sufficiently flexible allowing to evaluate the contribution of each separate term into the scattering amplitude.

*This implies that the series (4.30) converges in the mean, i.e., the function $\chi(l)$ is quadratically integrable, etc.

4.3. The Complex Angular Momentum Method in the Theory of Nuclear Diffraction. The Watson–Sommerfeld Transformation. Analytic properties of the F function gave an essential impetus to the development of several fruitful approaches within the models described above. One of them is the so-called complex angular-momentum method, developed by Inopin [12] and Ericson [13] and based on the Watson–Sommerfeld (WS) transformation [46]. It enables one to find a simple connection between the properties of the S matrix in the complex l plane and the corresponding cross sections in the «shadow» region, i.e., for scattering angles $\theta > \theta_c$. The Coulomb or grazing angle θ_c is the classical deflection angle for a charged particle moving along the grazing Coulomb trajectory with the distance of the closest approach from the field centre equal to R

$$kR = n\left(1 + \csc\frac{\theta_c}{2}\right), \tag{4.39}$$

where n is the Sommerfeld parameter. In what follows we confine ourselves to the consideration of the elastic scattering at energies E above the Coulomb barrier $B = Z_1 Z_2 e^2/R$, where Z_1 and Z_2 are the charges of the colliding nuclei.

It is generally accepted to separate out the Rutherford scattering contribution

$$S_R(l) = \exp\left[2i\sigma(l)\right] = \frac{\Gamma(l+1+in)}{\Gamma(l+1-in)} \tag{4.40}$$

from the total S matrix

$$S(l) = S_N(l)\exp\left[2i\sigma(l)\right]. \tag{4.41}$$

We assume that the «nuclear» part $S_N(l)$ is one of the distributions considered in Subsection 4.2.

In order to evaluate a sum like (4.12) with a large number of terms it is often convenient to convert the sum into an integral. This can be done exactly using the Cauchy theorem provided one can find an analytic function $F(l;\theta)$ in a neighbourhood of the positive real l axis, which takes on the values $F_l(\theta)$ at the positive integer l values. The sum (4.12) can be transformed into an integral in the complex l plane

$$f(\theta) = \frac{i}{2}\int\limits_{\Gamma}\frac{F(l;\pi-\theta)}{\sin\pi l}dl, \tag{4.42}$$

where Γ is the contour which proceeds clockwise around the positive real axis, and may be infinitesimally close to it. This representation referred to as the Watson transform is based on calculating the residues of the integrand in (4.42) at the zeros of $\sin\pi l$.

The contour Γ is then deformed into the line $\mathrm{Re}\, l = -\frac{1}{2}$ closed by a very large semicircle to the right, picking up the contributions from the poles $l_n = l_0 \pm i\pi\Delta(2n-1)$ $(n = 1, 2...)^*$ of $\eta(l)$ (in general, from the possible (isolated) singularities of $S(l)$ which are to the right of the line $\mathrm{Re}\, l = -\frac{1}{2}$ in the complex l plane). Denoting the residues in these poles through β_n (equal to Δ for the absorption coefficient (4.18)) we obtain

$$f(\theta) = \frac{i\pi}{2k}\sum_n (2l_n+1)\frac{\beta_n}{\sin\,\pi l_n} P_{l_n}(-\cos\,\theta)\,\mathrm{e}^{-\gamma(l_n+\frac{1}{2})} + I_B(\theta), \tag{4.43}$$

$$I_B(\theta) = (2k)^{-1}\int_{-\infty}^{+\infty} \frac{y\left[\eta\left(-\frac{1}{2}+iy\right)-1\right]}{\cosh\,\pi y} P_{-\frac{1}{2}+iy}(-\cos\,\theta)\,\mathrm{e}^{-i\gamma y}dy. \tag{4.44}$$

The Watson–Sommerfeld representation (4.43) becomes especially useful when the so-called background integral $I_B(\theta)$ is negligible and one can find efficient ways for calculating the sum in (4.43).

4.3.1. The Two-Pole Approximation. Exponential Falloff of the Cross Sections of Diffraction Scattering. An important consequence of this pole expansion is that the typical pole contribution in Eq. (4.43) outside the small angle region is characterized by

$$\frac{P_{l_n}(-\cos\,\theta)}{\sin\,\pi l_n} \simeq -\left[\frac{2}{\pi\left(l_n+\frac{1}{2}\right)\sin\,\theta}\right]^{1/2} \times \times \begin{cases} \exp\left[i\left(l_n+\frac{1}{2}\right)\theta + i\frac{\pi}{4}\right] & \text{for } \mathrm{Im}\, l_n > 0 \\ \exp\left[-i\left(l_n+\frac{1}{2}\right)\theta - i\frac{\pi}{4}\right] & \text{for } \mathrm{Im}\, l_n < 0, \end{cases} \tag{4.45}$$

if the «falloff» factors $\exp\left[-|\mathrm{Im}\ l_n|\theta\right]$ and $\exp\left[-|\mathrm{Im}\ l_n|(\pi-\theta)\right]$ are small. Thus, the magnitude of a pole contribution is proportional to $\exp\left[-|\mathrm{Im}\ l_n|\theta\right]$,

*This means that, according to the Jordan lemma, the contribution from the integral along a semicircle of radius R, with centre in $-\frac{1}{2}$, must tend to zero as $R \to +\infty$. This is the case for the Fermi-like functions considered here.

so that the influence of poles far from the real axis decreases rapidly with increasing θ. Of course, such estimates may be modified by other factors and in particular by the weight factors β_n. For the grey disk absorption model the background integral (4.44) is proportional to $O(\exp[-l_0/\Delta])$ except for the extreme forward-scattering region, where the conical function $P_{-(1/2)+iy}(-\cos\theta)$ has a logarithmic singularity. Therefore, unless the pole contributions fall to very small values, the background term in Eq. (4.43) is in practice negligible.

For a not too sharp boundary there is a large angular region for which $\exp\left[-2\pi\Delta\theta\right] \ll 1$ and $\exp\left[-2\pi\Delta(\pi-\theta)\right] \ll 1$. This region is dominated by the two poles nearest to the real axis. The scattering amplitude can then be written as

$$f(\theta) = \frac{i\pi\Delta}{2k}\left[(2l_1+1)\frac{P_{l_1}(-\cos\theta)}{\sin\pi l_1} + (2l_1^*+1)\frac{P_{l_1^*}(-\cos\theta)}{\sin\pi l_1^*}\right] \tag{4.46}$$

or

$$f(\theta) = 2i\pi\Delta\frac{L}{k}\exp\left(-\pi\Delta\theta\right)\left[\frac{\theta}{\sin\theta}\right]^{\frac{1}{2}} J_1(L\theta) \tag{4.47}$$

for

$$(\pi\Delta)^{-1} < \theta < \pi - (\pi\Delta)^{-1}. \tag{4.48}$$

Here the cylindric Bessel function $J_1(L\theta)$ should be replaced by its asymptotic expression at $L\theta \gg 1$.

The corresponding cross section is

$$\sigma(\theta) = 4\pi^2\Delta^2\frac{L^2}{k^2}\exp\left(-2\pi\Delta\theta\right)\frac{\theta}{\sin\theta}\left[J_1(L\theta)\right]^2. \tag{4.49}$$

At this point we can already conclude that as a general feature the diffuseness gives rise to an exponential overall decrease of the cross section with angle. Regular oscillations of Fraunhofer type are superimposed on this decrease. The damping becomes stronger when the parameter Δ increases. This general result has been also obtained using other analytical methods [11]. The present phenomenon has been observed in many experiments with nuclear particles at intermediate energies that give a strong confirmation of the theory based on the SAM with Fermi-type distributions.

The idea of pole dominance and the two-pole approximation have been a spring of inspiration for further elaborations [41,47–49]. We shall come back to this point later.

4.3.2. Inclusion of Nuclear Refraction. Dip Phenomenon. Another application of the method is connected with the inclusion of the refractive properties of the S matrix. We show this within the ansatz (4.24) which is equivalent to

$$S_N(l) = \frac{1}{2}\left[1+\tanh\frac{l-l_0}{2\Delta}\right]\exp\left[i\delta_0\left(1-\tanh\frac{l-l_0}{2\Delta}\right)\right], \tag{4.50}$$

in order to exhibit extra advantages of the method and avoid some unnecessary simplifications.

As mentioned above, the function $S_N(l)$ has isolated singularities associated with the poles of the phase $\delta(l)$, i.e., the essential singular points $l_m = l_0 \pm i\pi\Delta(2m-1)\,(m = 1, 2, ...)$. It was first shown in [14] that the Watson–Sommerfeld transformation makes it possible to get a closed expression for the scattering amplitude in this case as well. In fact, if we neglect the respective background integral, the method yields

$$\begin{aligned} f(\theta) &= \sum_m \left[f^{(m)}(\theta)\Big|_{\mathrm{Im}\, l_m > 0} + f^{(m)}(\theta)\Big|_{\mathrm{Im}\, l_m < 0} \right], \\ f^{(m)}(\theta) &= \frac{i}{2} \oint_{l_m} \frac{F(l; \pi - \theta)}{\sin\, \pi l} dl, \end{aligned} \tag{4.51}$$

where the contribution of the m-th singularity is given by an integral along a circle of an infinitesimally small radius, with centre at l_m, when moving along the path in a counter-clockwise direction. Here the function $S_N(l)$ involved in the partial scattering amplitude $F(l; \pi - \theta)$ is given by Eq. (4.50) with an arbitrary parameter δ_0.

Following [14] we obtain

$$f^{(m)}(\theta) = \frac{1}{2k} \oint_{l_m} \left(l + \frac{1}{2} \right) S_N(l) \exp\left[2i\sigma(l) \right] P_l(-\cos\,\theta) \frac{dl}{\sin\, \pi l} \simeq$$

$$\simeq -\frac{1}{k\sqrt{2\pi \sin\,\theta}} \oint_{l_m} S_N(l) g_m(l) dl, \tag{4.52}$$

with

$$g_m(l) = \sqrt{l + \frac{1}{2}} \exp\left\{ 2i\sigma(l) + i \left[\left(l + \frac{1}{2} \right) \theta + \frac{\pi}{4} \right] \mathrm{sign}\,(\mathrm{Im}\, l_m) \right\}$$

and the l values being in neighborhood of the point $l = l_m$.

Further, let us consider the integral

$$C_m(x) = \oint_{l_m} \exp\left[-ix \tanh \frac{l - l_0}{2\Delta} \right] g_m(l) dl. \tag{4.53}$$

The integral in the r.h.s. of (4.52) is simply related to the value $C_m(\delta_0)$, and it turns out that the quantity $C_m(x)$ satisfies the homogeneous differential equation

$$\frac{d^2 C_m(x)}{dx^2} + \left\{ 1 - \frac{2\Delta}{x} \left[2\sigma'(l_m) + \mathrm{sign}\,(\mathrm{Im}\, l_m) \right] \right\} C_m(x) = 0, \tag{4.54}$$

if one neglects small variations of the square root $\sqrt{l+\frac{1}{2}}$ and the Coulomb phase derivative $\sigma'(l)$ near the point $l = l_m$.

The substitution $x = -\frac{1}{2}iy$ reduces (4.54) to the standard Whittaker form

$$\frac{d^2Z(y)}{dy^2} + \left[-\frac{1}{4} + \frac{k_m}{y}\right] Z(y) = 0 \tag{4.55}$$

with the parameter

$$k_m = i\Delta\big[2\sigma'(l_m) + \operatorname{sign}(\operatorname{Im} l_m)\big].$$

All we need is to write down its solution which satisfies the boundary conditions

$$Z(y)\big|_{y=0} = 0, \quad Z'(y)\big|_{y=0} = -2\pi i\Delta g_m(l_m).$$

The respective result is (cf. [14])

$$Z(y) = -2\pi i\Delta g_m(l_m) y \exp\left(-\frac{y}{2}\right) \Phi(1-k_m, 2; y), \tag{4.56}$$

where $\Phi(a, b; x)$ is the confluent hypergeometric function.

It follows from (4.56) that the contribution of interest is

$$f^{(m)}(\theta) = \frac{1}{ik}\left[\frac{2\pi}{\sin\theta}\right]^{1/2} \Delta g_m(l_m)\Phi(-k_m, 1; 2i\delta_0). \tag{4.57}$$

Therefore, to the two-singular-point approximation

$$f(\theta) \simeq f^{(1)}(\theta)\Big|_{\operatorname{Im} l_1>0} + f^{(1)}(\theta)\Big|_{\operatorname{Im} l_1<0}, \tag{4.58}$$

we find

$$f(\theta) = \frac{\Delta}{ik}\left[\frac{8\pi L}{\sin\theta}\right]^{1/2} \exp(-\pi\Delta\theta)\exp[2i\operatorname{Re}\sigma(l_1)] \times$$

$$\times \big[\Phi(-i\Delta\theta_+, 1; 2i\delta_0)\Phi(i\Delta\theta_-, 1; 2i\delta_0)\big]^{1/2} \cos\big[L\theta + \rho(\theta) + i\xi(\theta)\big], \tag{4.59}$$

where

$$\rho(\theta) = \frac{\pi}{4} + \frac{1}{2}\arg\frac{\Phi(-i\Delta\theta_+, 1; 2i\delta_0)}{\Phi(i\Delta\theta_-, 1; 2i\delta_0)}, \tag{4.60}$$

$$\xi(\theta) = 2\operatorname{Im}\sigma(l_1) - \frac{1}{2}\ln\left|\frac{\Phi(-i\Delta\theta_+, 1; 2i\delta_0)}{\Phi(i\Delta\theta_-, 1; 2i\delta_0)}\right|, \tag{4.61}$$

with $\theta_\pm = \theta \pm \theta_c$. The corresponding cross section is

$$\sigma(\theta) = \frac{8\pi L}{k^2}\Delta^2 \frac{|\Phi(-i\Delta\theta_+, 1; 2i\delta_0)\Phi(i\Delta\theta_-, 1; 2i\delta_0)|}{\sin\theta} \times$$
$$\times \exp\left(-2\pi\Delta\theta\right)\left\{\cos^2\left[L\theta + \rho(\theta)\right] + \sinh^2\xi(\theta)\right\}. \quad (4.62)$$

The series

$$\Phi(\mp i\Delta\theta_\pm, 1; 2i\delta_0) \equiv \sum_{n=0}^{\infty} \frac{\Gamma(\mp i\Delta\theta_\pm + n)}{\Gamma(\mp i\Delta\theta_\pm)} \frac{[2i\delta_0]^n}{[n!]^2} =$$
$$= 1 \pm 2\Delta\theta_\pm\delta_0 \pm \frac{i}{4}\Delta\theta_\pm(1 \mp i\Delta\theta_\pm)\delta_0^2 + ..., \quad (4.63)$$

introduced in the diffraction theory [14] accumulates the refractive effects in all orders in the phase parameter δ_0. It follows from (4.63) that a true role of the nuclear refraction is characterized by the products $2\Delta\theta_\pm\delta_0$ but not the value of δ_0 itself.

Moreover, if the physical parameters obey the inequality

$$2\Delta\theta_+|\delta_0| < 1, \quad (4.64)$$

one can put

$$\Phi(\mp i\Delta\theta_\pm, 1; 2i\delta_0) \simeq 1 \pm 2\Delta\theta_\pm\delta_0. \quad (4.65)$$

Then, the formula (4.62) predicts an extreme swing of the cross-section oscillations near a scattering angle $\theta = \theta_1$, where

$$\theta_1 = \left(p - \frac{1}{4}\right)\frac{\pi}{L} = (1 + 2\Delta\theta_c\delta_0)\frac{\tanh\pi\Delta\theta_c}{2\Delta\delta_0} \quad (4.66)$$

and p is a half-integer number. For this value, both the cosine and the filling factor $\sinh^2\xi(\theta)$ in (4.62) become equal to zero, i.e., the cross section has a very deep minimum. This «dip» phenomenon may take place even under the condition

$$\exp\left(2\pi\Delta\theta_c\right) \gg 1, \quad (4.67)$$

in which case the nuclear phase is not included ($\delta_0 = 0$) and the Fraunhofer-type oscillations are suppressed (the so-called Coulomb damping (see [50] and refs. therein)).

For a weak Coulomb repulsion ($\theta_c \simeq 2n/L \ll 1$) we have instead of Eq. (4.66)

$$\theta_1 = \left(p - \frac{1}{4}\right)\frac{\pi}{L} = \frac{\pi}{2\delta_0}\theta_c. \quad (4.68)$$

This angle was introduced in Refs. [51, 14], where we proposed to use the dip phenomenon, if any, to extract some information on the nuclear phase function for peripheral collisions.

One should note that the approximation (4.62) works well in the shadow region at $\theta_c + (\pi\Delta)^{-1} < \theta < \pi$. The asymptotic expression, which is applied in the illuminated region too, will be given below (see Subsect. 4.4.2).

4.4. The Abel–Plana Summation Procedure and Its Extension. The WS method has been employed successfully [12, 13] to evaluate the corresponding amplitude for $\theta > \theta_c$ using a few poles only. However, at angles θ close to the grazing angle θ_c this method becomes ineffective (cf. [42]) since the series of the residues in Eq. (4.43) with the Coulomb repulsion included is not rapidly convergent, so that the entire pole line begins to contribute. In order to overcome such obstacles one can use another approach [52] based on the following representation:

$$\sum_{l=0}^{\infty} F_l = \int_0^{\infty} F\left(s-\frac{1}{2}\right) ds - i \int_0^{\infty} \frac{F\left(is-\frac{1}{2}\right) - F\left(-is-\frac{1}{2}\right)}{\exp(2\pi s)+1} ds - \\ - \pi \sum_k \operatorname{res}\left[\frac{\mathrm{e}^{i\pi l}}{\sin \pi l} F(l)\right]_{l=l_k} (\operatorname{Im} l_k > 0) - \\ - \pi \sum_k \operatorname{res}\left[\frac{\mathrm{e}^{-i\pi l}}{\sin \pi l} F(l)\right]_{l=l_k} (\operatorname{Im} l_k < 0). \quad (4.69)$$

This result is an extension of the Abel–Plana formula (cf. Eq. 1.9(11) in [25]. See also [53]). Unlike the latter, Eq. (4.69) can be employed both, for functions $F(l)$ regular on the right of the line $\operatorname{Re} l = -\frac{1}{2}$, as well as for those which have isolated singularities in that part of the complex l plane.

This approach takes an intermediate place between the WS method focused upon an analytical continuation of F_l and the Poisson procedure [54] where an initial sum is replaced by an infinite series of the integrals along the real axis, viz.,

$$f(\theta) = \int_0^{\infty} F\left(s-\frac{1}{2}\right) ds + 2 \sum_{m=0}^{\infty} \int_0^{\infty} F\left(s-\frac{1}{2}\right) \cos\left[2\pi m\left(s-\frac{1}{2}\right)\right] ds. \quad (4.70)$$

The latter turned out to be successful whenever only the first terms contribute significantly to (4.70) owing to the frequently oscillating factors $\cos\left[2\pi m\left(s-\frac{1}{2}\right)\right]$.

The problem was investigated in [11,41], where this approach was justified using cumbersome tricks. In particular, these authors have shown that the approximation

$$f(\theta) = \int_0^\infty F\left(s - \frac{1}{2}\right) ds \tag{4.71}$$

is rather good as long as the cutoff is a smooth function.

It is noteworthy that Eq. (4.71) itself and some exact conditions for its validity can be obtained simply starting from Eq. (4.69) and using the fact that for

$$\Delta \sim 1 \tag{4.72}$$

the contributions from all the poles are suppressed by the factors $\exp(\pm i\pi l)/\sin \pi l$, which are exponentially small*. Moreover, the second integral in (4.69) can be estimated usually without complications. Of course, one should keep in mind that the contributions from the singularities of the S matrix in Eq. (4.69) are attenuated by extra factors $\exp(-|\mathrm{Im}\, l_k|)$ in comparison with the WS method. Thus, their relative role is to a great extent reduced even if the singular points are not too far from the real axis.

4.4.1. Diffraction Scattering of Charged Particles. Since the competition between the Coulomb repulsion and the nuclear absorption is typical for nucleus-nucleus interactions at the collision energies above the Coulomb barrier, it is interesting to unite them within a realistic model for the absorption coefficient η_l. Here we shall parametrize the S matrix elements (4.41) following (4.18), where the grazing angular momentum l_0 is now given by

$$L = l_0 + \frac{1}{2} = kR\sqrt{1 - \frac{B}{E}} = kR\sqrt{1 - \frac{2n}{kR}}. \tag{4.73}$$

Combining Eq. (4.73) with Eq. (4.39) one has

$$\frac{n}{L} = \tan\frac{\theta_c}{2}. \tag{4.74}$$

At this point, the amplitude can be written as [34]

$$f(\theta) = -(ik)^{-1} \lim_{\gamma \to 0^+} \frac{\partial}{\partial \gamma}\left[S_\gamma(\theta) - \delta_\gamma(\theta)\right], \tag{4.75}$$

*Exponentially small contributions of different origin are compared to one another in Subsect. 4.4.3 with significant physical consequences.

$$S_\gamma(\theta) = \sum_{l=0}^{\infty} \eta_l \exp\,(2i\sigma_l) P_l(\cos\,\theta)\,\mathrm{e}^{-\gamma(l+\frac{1}{2})}, \tag{4.76}$$

$$\delta_\gamma(\theta) = \sum_{l=0}^{\infty} \mathrm{e}^{-\gamma(l+\frac{1}{2})} P_l(\cos\,\theta) = \frac{1}{2}\left[\sinh^2\frac{\gamma}{2} + \sin^2\frac{\theta}{2}\right]^{-\frac{1}{2}}. \tag{4.77}$$

When applying formula (4.69) to calculate $S_\gamma(\theta)$, one can find that the respective «background» integral (the second term in the r.h.s. of (4.69)) is of the order of $\exp\,(-L/\Delta) \ll 1$. Moreover, if

$$\exp\,(-\pi^2\Delta) \ll 1, \tag{4.78}$$

which means a moderately sharp cutoff implied, for instance, when condition (4.72) is fulfilled, the pole line of the Fermi function contributes as

$$\Sigma_\gamma^{\mathrm{pole}}(\theta) \simeq 2\pi\Delta i\,\mathrm{e}^{-2\pi^2\Delta}\left[\mathrm{e}^{2\pi i l_0}\,\mathrm{e}^{-(l_1+\frac{1}{2})\gamma}\,\mathrm{e}^{2i\sigma(l_1)} P_{l_1}(\cos\,\theta) - \right.$$
$$\left. -\mathrm{e}^{-2\pi i l_0}\,\mathrm{e}^{-(l_1^*+\frac{1}{2})\gamma}\,\mathrm{e}^{2i\sigma(l_1^*)} P_{l_1^*}(\cos\,\theta)\right].$$

On these conditions, one can replace the sum (4.76) by the integral,

$$S_\gamma(\theta) \simeq A_\gamma(\theta) = \int_0^\infty S\left(l-\frac{1}{2}\right) P_{l-\frac{1}{2}}(\cos\,\theta)\,\mathrm{e}^{-l\gamma} dl, \tag{4.79}$$

where

$$S\left(l-\frac{1}{2}\right) = \frac{\exp\left[2i\sigma\left(l-\frac{1}{2}\right)\right]}{1+\exp\,\dfrac{L-l}{\Delta}}.$$

It has been shown that for realistic values of the parameters involved, Eq. (4.79) gives a fair approximation apart from extremely large scattering angles.

In order to proceed in obtaining an asymptotic expression for the integral $A_\gamma(\theta)$, let us employ the Mehler–Dirichlet formula,

$$P_l(\cos\,\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos\left(l+\frac{1}{2}\right)v}{(\cos\,v-\cos\,\theta)^{1/2}} dv, \tag{4.80}$$

and the expression for the Coulomb phase factor,

$$\exp\,[2i\sigma(l)] = B(l+1+in,-2in)/\Gamma(-2in) =$$
$$= h(n)\int_0^\infty \left[\sinh\frac{s}{2}\right]^{-1-2in} \mathrm{e}^{-(l+(1/2))s} ds, \tag{4.81}$$

which is equivalent to

$$\exp\left[2i\sigma(l)\right] = \frac{1}{2}\frac{h(n)}{\sinh\,\pi n}\int\limits_0^{2\pi}\left[\sin\,\frac{\tau}{2}\right]^{-1-2in}\,\mathrm{e}^{i(l+(1/2))\tau}d\tau, \tag{4.82}$$

being one of the Nielsen results (cf. Eqs. 1.5.1(25,29) in [25]). It leads directly to the integral considered in Subsect. 3.1 (cf. Eq. (3.3)),

$$\int\limits_0^{\infty}\frac{\exp\left(-\mu l\right)}{1+\exp\,\dfrac{L-l}{\Delta}}dl = \pi\Delta\frac{\exp\left(-L\mu\right)}{\sin\,\pi\Delta\mu} + \\ + \frac{\Delta\exp\left(-L/\Delta\right)}{\Delta\mu-1}F\big(1,1-\Delta\mu;2-\Delta\mu;-\exp\left(-L/\Delta\right)\big), \tag{4.83}$$

where $\mu = \gamma - i\tau \pm iv$.

Along this guideline, going back to the Legendre polynomials and the Coulomb phase shift via Eqs. (4.80) and (4.82), one can show that if along with the strong inequality (4.78) one has

$$\exp\left(-L/\Delta\right) \ll 1, \tag{4.84}$$

at $l_0\theta \gg 1$, the asymptotic formulae obtained in [34, 52] yield the following expression for the elastic scattering amplitude:

$$f(\theta) = f_{\mathrm{Fre}}(\theta) + \tilde{f}^{(+)}(\theta) + \tilde{f}^{(-)}(\theta) \tag{4.85}$$

with the Fresnel-type part

$$f_{\mathrm{Fre}}(\theta) = f_R(\theta)\left\{G(\theta) + \mathrm{sign}\,(\theta_c-\theta)\frac{\exp\,\left[-ix - i\dfrac{\pi}{4}\right]}{2\sqrt{\pi x}}\right\}\exp\left[2i(\sigma_0^{as}-\sigma_0)\right], \tag{4.86}$$

where

$$G(\theta) = \frac{1}{2}\left[1 + \frac{2}{\sqrt{\pi}}\mathrm{sign}\,(\theta_c-\theta)\mathrm{Er}\,f(\sqrt{ix})\right],$$

$$x = n\left\{(\theta-\theta_c)\cot\frac{\theta_c}{2} + 2\ln\frac{\sin\left(\theta_c/2\right)}{\sin\left(\theta/2\right)}\right\},$$

$$\mathrm{Er}\,f(y) = \int\limits_0^{y}\exp\left(-t^2\right)dt,$$

and the Fraunhofer-type branches

$$\tilde{f}^{(\pm)}(\theta) = \frac{1}{k}\Big[\frac{L}{2\pi \sin\,\theta}\Big]^{1/2} \frac{\pi\Delta}{\sinh\big[\pi\Delta(\theta_c \pm \theta)\big]} \exp\left[2i\sigma\Big(L - \frac{1}{2}\Big) \pm iL\theta \mp i\frac{\pi}{4}\right]. \tag{4.87}$$

Here $f_R(\theta)$ is the exact Rutherford amplitude,

$$f_R(\theta) = -\frac{n}{2k} \frac{\exp\left[2i\sigma_0 - 2in\,\ln\left(\sin\,\dfrac{\theta}{2}\right)\right]}{\sin^2\dfrac{\theta}{2}}. \tag{4.88}$$

By σ_0^{as} we denote the asymptotic Coulomb phase shift σ_0 at $n \gg 1$,

$$\sigma_0^{as} = \frac{\pi}{4} + n(\ln n - 1).$$

Expression (4.85) gives a solution of the diffraction problem for arbitrary values of the Sommerfeld parameter in the SAM with nuclear refraction not included.

4.4.2. Application to Heavy-Ion Scattering at Intermediate Energies. The Coulomb-Nuclear Interference in Different Regimes of Nuclear Diffraction. These analytic results reflect a rich physics, inherent to the interplay between the Coulomb repulsion and the properties of nuclear interaction (in particular, its diffuseness). To show it more prominently let us recall the following expression

$$f_{\rm sco}(\theta) = f_{\rm Fre}(\theta) + f^{(+)}(\theta) + f^{(-)}(\theta), \tag{4.89}$$

$$f^{(\pm)}(\theta) = \frac{1}{k}\left[\frac{l_0}{2\pi \sin\,\theta}\right]^{\frac{1}{2}} \frac{\exp\left[2i\sigma\left(l_0 - \dfrac{1}{2}\right) \pm il_0\theta \mp i\dfrac{\pi}{4}\right]}{2\,\sin\,\dfrac{\theta_c \pm \theta}{2}}, \tag{4.90}$$

obtained in [34] for the scattering amplitude $f_{\rm sco}(\theta)$ in the sharp cutoff limit ($\Delta \to 0$). Of course, Eq. (4.89) cannot be derived directly from Eq. (4.85) because of the restriction to the Δ values, imposed by inequality (4.78).

Also note that the factor $G(\theta)$ can be expressed through the Fresnel integrals $C(x)$ and $S(x)$:

$$G(\theta) = \frac{1}{\sqrt{2}}\left\{\left[\frac{1}{2} + \text{sign}\,(\theta_c - \theta)C(x)\right] + i\left[\frac{1}{2} + \text{sign}\,(\theta_c - \theta)S(x)\right]\right\}, \tag{4.91}$$

where, by definition,

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt, \qquad S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt. \tag{4.92}$$

Comparing expressions (4.89) and (4.85) we see that the diffuseness of nuclear surface does not change the structure of the scattering amplitudes, which, in particular, retains its Fresnel part. The principal difference between them is related to the Fraunhofer branches where the attenuation factors $\pi\Delta/\sinh[\pi\Delta(\theta_c \pm \theta)]$ appear instead of $\left[2\sin\frac{\theta_c \pm \theta}{2}\right]^{-1}$. This change causes significant distortions of the diffractive patterns. Thus, one of the factors (corresponding to the + sign) contributes negligibly (for $\Delta \sim 1$ and $\theta_c \sim 1$) to the amplitude (4.85) in the illuminated region ($\theta \leq \theta_c$), where the cross section is determined mainly by the interference of the Fresnel part in Eq. (4.85) and the «negative» Fraunhofer branch $\tilde{f}^{(-)}(\theta)$. If one uses the concepts geometrical interpretation of diffractive phenomena [55], one may say that the diffuseness of nuclear boundary leads to a suppression of the interference with waves scattered from the «far» side of the nuclear surface. Therefore, in the illuminated region with $\sqrt{x} \gg 1$ we find the ratio of the elastic scattering cross section $\sigma(\theta) = |f(\theta)|^2$ to the Rutherford cross section $\sigma_R(\theta) = |f_R(\theta)|^2$:

$$\frac{\sigma(\theta)}{\sigma_R(\theta)} = \left|1 + \frac{\tilde{f}^{(-)}(\theta)}{f_R(\theta)}\right|^2, \tag{4.93}$$

so that the exponential decrease of $|\tilde{f}^{(-)}(\theta)|$ while moving away from θ_c gives an exponential falloff of the envelopes of the oscillations inside the region. In other words, at $\theta \leq \theta_c$ the $\tilde{f}^{(+)}(\theta)$ damping which is more rapid in comparison with $\tilde{f}^{(-)}(\theta)$ results in smoothing the angular dependence of σ/σ_R (any shallow oscillations induced by interference of $f_{\text{free}}(\theta)$ with both the branches $f^{\pm}(\theta)$ in $f_{\text{sco}}(\theta)$ vanish).

These peculiarities of the diffractive patterns are illustrated in Fig. 1, taken from [56]. A more detailed discussion of the diffuseness effects can be found in [34] and [56].

In the region $\theta > \theta_c$, the two branches $\tilde{f}^{(+)}(\theta)$ and $\tilde{f}^{(-)}(\theta)$ begin to contribute approximately equally, generating the «Fraunhofer» regime of the diffraction. It is clear, however, that on the condition

$$\exp(2\pi\Delta\theta_c) \gg 1$$

the purely Fraunhofer picture of oscillations is not able to develop, and in the shadow region the cross section decreases rapidly without oscillations.

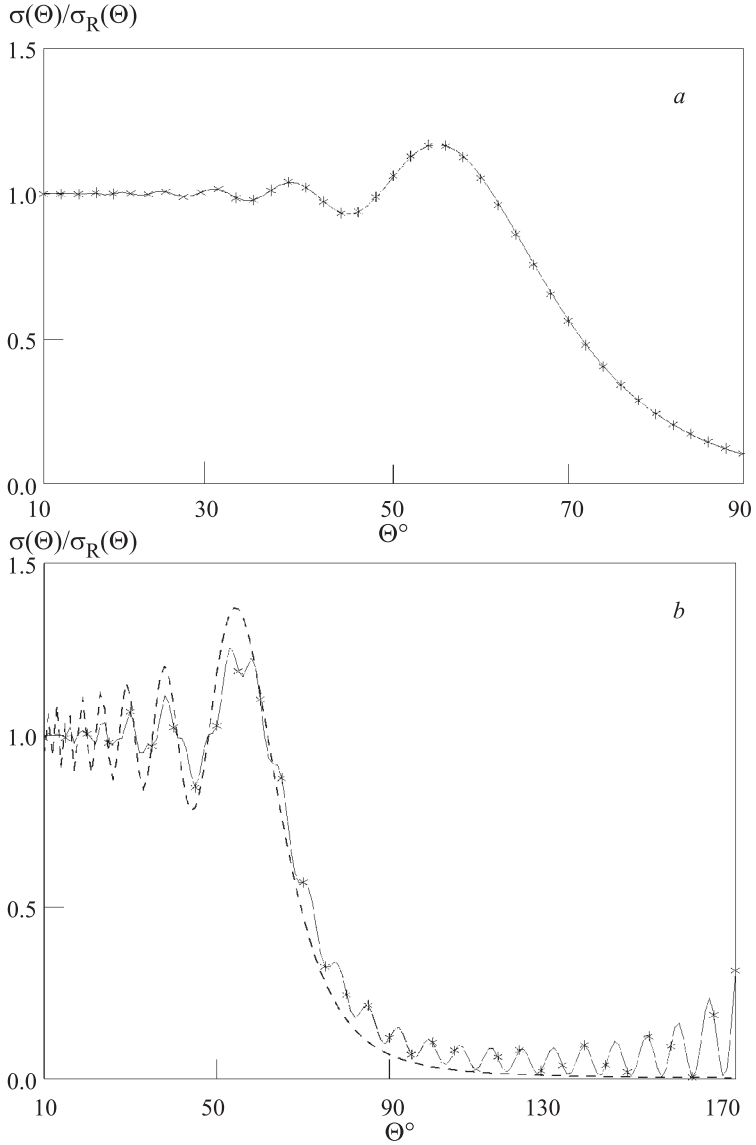

Fig. 1. Ratio of the scattering cross section to the Rutherford cross section. *a*) Calculations in the smooth (sharp) cutoff diffraction model. *b*) Calculations with the uniform asymptotic method (solid curve), the Fresnel approximation (dashed curve) and numerical summation (points). The parameters involved are: $l_0 = 25$, $n = 20$, and $\Delta = 1$

Assuming $n = 0$ (the N limit according to [57]) in (4.85), we find

$$f(\theta) = i\frac{L}{k}\left[\frac{\theta}{\sin\,\theta}\right]^{1/2}\frac{\pi\Delta}{\sinh\,\pi\Delta\theta}J_1(L\theta). \tag{4.94}$$

This is equivalent, on the condition $\exp(-2\pi\Delta\theta) \ll 1$, to the result (4.47) derived in the two-pole approximation.

Finally, by using the Abel–Plana summation and other analytic procedures (in particular, formula (3.2) with integer values of the index ρ), one can obtain the following expression:

$$f(\theta) = f_{\text{Fre}}(\theta) + \bar{f}^{(+)}(\theta) + \bar{f}^{(-)}(\theta), \tag{4.95}$$

$$\bar{f}^{(\pm)}(\theta) = \Phi(\mp i\Delta\theta_{\pm}, 1; 2i\delta_0)\tilde{f}^{(\pm)}(\theta) \tag{4.96}$$

for the scattering amplitude within the SAM (see [10]). Thus, with the inclusion of nuclear refraction the Fresnel part of the scattering amplitude remains unchanged. The asymptotic formula is a good approximation to the initial partial-wave expansion both, in the illuminated and shadow regions (in the latter, at not too large scattering angles when one can neglect other exponentially small contributions shown in the next subsection).

4.4.3. On Exponentially Small Contributions to the Amplitudes of Diffraction Scattering. We have seen that the exponential falloff of the envelope of diffraction maxima in the elastic scattering cross sections is regulated by the diffuseness parameter Δ which is proportional to the thickness parameter a of the surface layer of the nucleus. The corresponding contribution to the amplitude is determined by the factor $\exp(-\pi\Delta\theta) \simeq \exp(-\pi ka\theta)$. Here we show some satellite terms (for instance, of the $\exp(-l_0/\Delta)$–order), which are usually disregarded keeping in mind strong absorption condition (4.17). Following [58] (see also [59]) let us rewrite the sum of interest:

$$\Sigma(z) = \sum_{l=0}^{\infty}(2l+1)\,g\left(\frac{l-l_0}{\Delta}\right)P_l(z) \tag{4.97}$$

in the form

$$\Sigma(z) = \frac{d}{dz}\sum_{l=0}^{\infty}\left[g\left(\frac{l-l_0-1}{\Delta}\right) - g\left(\frac{l-l_0+1}{\Delta}\right)\right]P_l(z), \qquad (z = \cos\,\theta), \tag{4.98}$$

where the function $g\left(\dfrac{l-l_0}{\Delta}\right)$ gives an extrapolation $S(l) = g\left(\dfrac{l-l_0}{\Delta}\right)$ of the S matrix in l plane and satisfies all the necessary analytical requirements (in particular, it has a well-defined behaviour at infinity).

By using some relations and tricks shown when deriving the asymptotics for the amplitudes in the case of scattering of charged particles, it has been found [58,59], when using

$$S(l) = g_{\mathrm{F}}\left(\frac{l-l_0}{\Delta}\right) = 1 - \left[1 + \exp\,\frac{l-l_0}{\Delta}\right]^{-1}, \tag{4.99}$$

the following expression for the scattering amplitude of interest (with an accuracy to contributions of $\exp\,(-2\pi^2\Delta)$ – and $\exp\,(-2L/\Delta)$–order):

$$f(\theta) = f_D(\theta) + f_{ND}(\theta) + f_{\mathrm{pole}}(\theta), \tag{4.100}$$

$$f_D(\theta) = \frac{\sqrt{2}\Delta}{ik}\,\frac{1}{\sin\,\theta}\,\frac{d}{d\theta}\,\mathrm{Im}\,I_D(\theta), \tag{4.101}$$

$$f_{ND}(\theta) = \frac{1}{i\sqrt{2}k}\,\frac{\sinh\,(1/\Delta)}{[\cosh\,(1/\Delta) - \cos\,\theta]^{\frac{3}{2}}}\,\mathrm{e}^{-L/\Delta}, \tag{4.102}$$

$$f_{\mathrm{pole}}(\theta) = \frac{2\pi\Delta}{ik}\,\mathrm{Im}\,\left[\left(2l_1^+ + 1\right)\exp\,\left(2\pi i l_1^+\right)\,P_{l_1^+}(\cos\,\theta)\right]. \tag{4.103}$$

A careful investigation shows that the diffraction integral $I_D(\theta)$

$$I_D(\theta) = \exp\,(iL\theta)\int\limits_0^\infty \mathrm{e}^{-Lu}\,\frac{\sin\,(\theta+iu)}{\sinh\,[\pi\Delta(\theta+iu)]}\,\frac{du}{\sqrt{\cos\,(\theta+iu) - \cos\,\theta}} \tag{4.104}$$

can be approximated either by

$$I_D(\theta) = \sqrt{2}\frac{\sin\,\theta}{\sinh\,(\pi\Delta\theta)}Q_{L-\frac{1}{2}}(\cos\,\theta - i0), \tag{4.105}$$

or by

$$I_D(\theta) = -i\sqrt{2}\left[Q_{l_1^+ +1}(\cos\,\theta - i0) - Q_{l_1^+ -1}(\cos\,\theta - i0)\right], \tag{4.106}$$

where $Q_\nu(x - i0)$ is the Legendre function of second kind. Formula (4.105) works asymptotically at $L \gg 1$, i.e., owing to the short wavelength condition (4.15), over the band $0 \le \theta < \pi - L^{-1}$. At the same time the r.h.s. of Eq. (4.106) is the leading term of the expansion in powers of the factor $\exp\,(-\pi\Delta\theta)$ and it provides a good approximation to $I_D(\theta)$ in the band $(\pi\Delta)^{-1} < \theta \le \pi$. One can

verify that these approximations give rise to the same angular dependence on an intersection of the bands.

Now, it can be seen that the part $f_D(\theta)$ yields a typical diffractive distribution. In fact, let us replace $I_D(\theta)$ by (4.105) and use the discontinuity relation,

$$\pi i P_\nu(\cos\theta) = Q_\nu(\cos\theta - i0) - Q_\nu(\cos\theta + i0). \tag{4.107}$$

Then

$$f_D(\theta) = \frac{\pi\Delta}{ik}\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\frac{\sin\theta}{\sinh(\pi\Delta\theta)}P_{l_0}(\cos\theta)\right]. \tag{4.108}$$

Whence at $L \gg 1$ it follows the well-known result [11] (cf. also Eq. (4.94))

$$f_D(\theta) = i\frac{L}{k}\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\frac{\pi\Delta}{\sinh(\pi\Delta\theta)}J_1(L\theta). \tag{4.109}$$

Being concentrated in the forward-scattering diffractive cone $0 \le \theta < x_1/L$, when x_1 is the first zero of $J_1(x)$, the distribution $|f_D(\theta)|^2$ has the oscillations of the Fraunhofer type beyond this cone. The envelope of their maxima is determined by the factor $\exp(-2\pi\Delta\theta)$.

This diffractive picture occurs against the background of smooth angular dependence of the nondiffractive contribution $f_{ND}(\theta)$. Its magnitude is governed by the other factor $\exp(-L/\Delta)$*. Obviously, the values can be comparable if

$$\exp(-\pi\Delta\theta) \sim \exp(-L/\Delta), \tag{4.110}$$

i.e., at $\pi\Delta^2\theta \simeq L$.

Finally, the pole contribution has the asymptotic behaviour:

$$f_{\text{pole}}(\theta) \sim \frac{2i\Delta}{k}\left(\frac{2\pi L}{\sin\theta}\right)^{1/2} \mathrm{e}^{-(2\pi-\theta)\pi\Delta}\sin\left[L(2\pi-\theta)+\frac{\pi}{4}\right], \quad L^{-1} \ll \theta < \pi - L^{-1}, \tag{4.111}$$

so that the distribution $|f_{\text{pole}}(\theta)|^2$ oscillates inside a backward-scattering diffractive cone where the respective envelope has the $\exp[-2(2\pi-\theta)\pi\Delta]$-falloff while moving from $\theta = \pi$ towards smaller angles.

A typical interplay between the three contributions which can simultaneously take on exponentially small values is shown in Fig. 2, taken from [59]. We see

*In fact, we are able to determine a whole hierarchy of such contributions classified by ascending powers of the exponentially small factor.

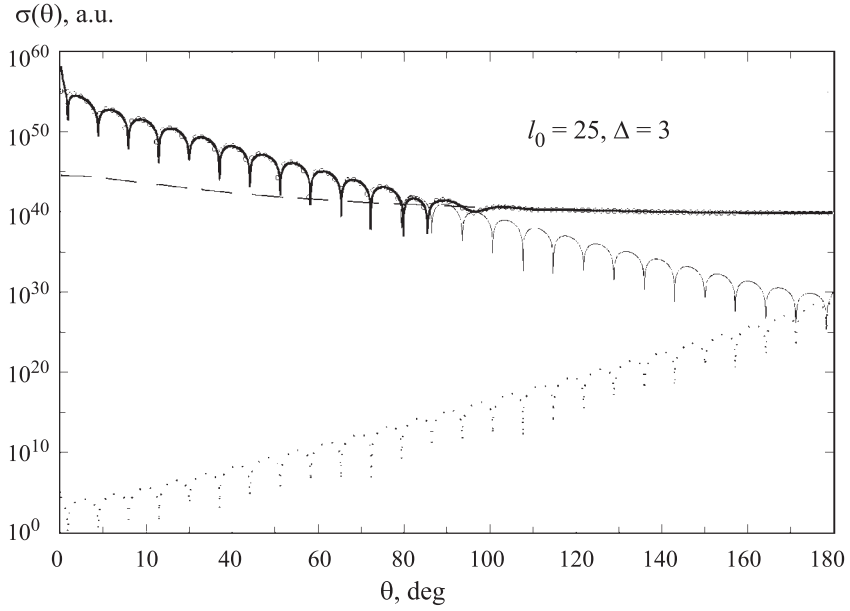

Fig. 2. Exponentially small contributions to the cross section: diffractive (tiny solid line), nondiffractive (dashed), pole (dots) and their sum (thick solid curve). The circles are obtained by summing the partial wave expansion. Here $l_0 = 25$ and $\Delta = 3$

that the nondiffractive effects which are closely related to the smooth cutoff in the absorption become prevalent for scattering angles in the backward hemisphere. At this point, let us remind that for nucleus–nucleus scattering at intermediate energies (e.g., 10 MeV per nucleon) the ratio L/Δ is approximately equal to 15–20 and the nondiffractive contribution can be comparatively small. However, this is not the case for hadron–nucleus scattering in the GeV-region. See, e.g., Table 1 of [43] where for scattering of 800 MeV/c pions from ^{12}C the adjustable parameters are such that $L/\Delta \simeq 4$ with $\Delta \simeq 2.5$. Then one can expect that $f_{ND}(\theta)$ contributes starting from $\theta \geq \pi/6$.

We distinguish two diffractive contributions. One of them yields the angular distributions concentrated inside the forward-scattering cone while the other has a prominent peak at backward-scattering angles. Envelopes of their maxima are determined by the factors $\exp(-\pi\Delta\theta)$ and $\exp[-(2\pi - \theta)\pi\Delta]$, respectively. Therefore, they become comparable for the extremely large scattering angles where their interference turns out to be destructive because of some differences in the corresponding pre-exponentials.

5. THE FERMI FUNCTION IN STUDYING THE PROPERTIES OF NUCLEI, HYPERNUCLEI AND METAL CLUSTERS

5.1. The Fermi Distribution for Nuclear Density and the Determination of the Harmonic Oscillator Energy Level Spacing. One of the applications of the F function approximating the density distribution of nuclei is its use in determining the harmonic oscillator (HO) energy level spacing $\hbar\omega$ and its variation with the mass number A of the nucleus. A pertinent study has been undertaken in [60], where nuclei with $Z = N$ were considered. This approach follows the general procedure usually developed for the uniform distribution (see Refs. 61, 62), but the possibility of treating in a way nuclei with n valence nucleons in the spirit of [8, 63] has been considered without making, however, any approximation in relating the number of the highest filled shell K to the mass number.

The average HO shell model (SM) mean square (m. s.) radius may be written as [60]

$$\overline{\langle r^2\rangle}_{K+n} = \frac{\hbar}{M\omega}\frac{4\sum\limits_{p=1}^{K}\left(p+\frac{1}{2}\right)N(p)+\left(K+\frac{3}{2}\right)n}{4\sum\limits_{p=1}^{K}N(p)+n} = \\ = (\hbar/M\omega)[(K+1)(3A+n)+2n]/4A, \tag{5.1}$$

where K is determined by the solution of the equation

$$4\sum_{p=1}^{K}N(p)+n = 2\sum_{p=1}^{K}(p^2+p)+n = A. \tag{5.2}$$

We have therefore

$$\frac{2}{3}K(K+1)(K+2)+n = A. \tag{5.3}$$

This equation can be solved exactly. Its only real solution is

$$K+1 = v^{1/3}[(1+u)^{1/3}+(1-u)^{1/3}], \\ v = \frac{3}{4}(A-n), \quad u = \left(1-\frac{1}{27}v^{-2}\right)^{1/2}. \tag{5.4}$$

In deriving the asymptotic expression $\hbar\omega = 41A^{-1/3}$ one equates the average m. s. radius $\overline{\langle r^2\rangle}$ to that corresponding to a uniform distribution. It is more appropriate, however, to consider here (for nuclei with $N = Z$ or $N \neq Z$) a Fermi distribution for the density of nuclear matter

$$\rho(r) = \rho_{\mathrm{F}}(r) = \rho_0\{1+\exp[(r-c)/a]\}^{-1}. \tag{5.5}$$

Such a density is not, of course, suitable for nuclei with small A. We assume, as in certain other studies (see, e.g., [4]), that $\rho(r)$ is the same with the charge density distribution $\rho_{\rm ch}(r)$ (as well as that the point-nucleon density distribution is the same with the point proton one), apart from a normalization constant, that is $\rho_{0,\rm ch} = (Z/A)\rho_0$. Such an assumption implies equality of the corresponding m.s. radii for a given nucleus, which is only approximately correct [5, 6]. Here we adopt the assumptions made in [4]. One of them is that the central density $\rho(0) \simeq \rho_0$ is independent of the mass number, which was argued to be expected to be true, apart from the lightest nuclei, because of the saturation properties of nuclear forces. The influence of the Coulomb forces which do not saturate and of the exclusion principle which has opposite effects on neutrons and protons is neglected. On the basis of the previous assumption ρ_0 is taken in this study to be constant, which (as well as a) is fixed by a fitting procedure (see below). The value of c follows then from the normalization condition of $\rho(r)$. An alternative possibility which has also been followed in the past for the determination of the parameters of the Fermi distribution is to choose a value of a and a dependence of c on A which fit the results of elastic electron scattering experiments. Such an approach appears interesting since ρ_0, which is determined now by the normalization condition, becomes dependent on A. It seems, however, that there is still a sort of ambiguity as far as the functional dependence of c on A is concerned, since the choice for this dependence does not appear to be unique. We may also note that the dependence of ρ_0 on the mass number which comes out in this way with $a = 0.54$ fm, and the dependence of c on A given by Eq. (2.29) in Ref. 64, namely: $c = (0.978 + 0.0206A^{1/3})A^{1/3}$ is not strong. The difference between the maximum and minimum value of ρ_0, in the region $12 < A < 208$ is less than 16 % of its maximum value.

By using expression (5.1), the following formula for $\hbar\omega$ is obtained

$$\hbar\omega = \frac{3}{4}\frac{\hbar^2}{MA}\Big[(K+1)\Big(A+\frac{1}{3}n\Big)+\frac{2}{3}n-2\Big]\times \\ \times\Big[\langle r^2\rangle-(\langle r^2\rangle_p+\langle r^2\rangle_n)\Big]^{-1}, \quad (5.6)$$

where a value of $(\langle r^2\rangle_p + \langle r^2\rangle_n)$ is 0.659 fm^2 [65]. In the above expression the correction to $\langle r^2\rangle = \langle r^2\rangle_{\rm ch}$ due to the centre-of-mass motion [66] has also been taken into account, in addition to proton and neutron finite size effects.

We may remark that the «half-way» radius is given in terms of A, a, and ρ_0 by the expression

$$c = \left(\frac{1}{2}\right)^{1/3} bA^{1/3}\{[1+\delta]^{1/3}+[1-\delta]^{1/3}\}, \quad \delta = \left[1+\frac{4}{27}\left(\frac{\pi a}{bA^{1/3}}\right)^6\right]^{1/2} \quad (5.7)$$

with

$$b = (3/(4\pi\rho_0))^{1/3}.$$

It is an «almost exact relation» that follows from the normalization condition for ρ when omitting terms of $\exp(-c/a)$ and of higher order.

The m. s. radius of the Fermi distribution with an accuracy to the small exponential terms is equal to

$$\langle r^2\rangle = \frac{3}{5}\left(c^2 + 7\pi^2\frac{a^2}{3}\right), \tag{5.8}$$

where c is given by (5.7). Expansion of the quantity in the curly brackets in (5.7) in ascending powers of $A^{-1/3}$ leads to successive approximations to c and to the radius of an equivalent uniform distribution,

$$R_u = \left(\frac{5}{3}\langle r^2\rangle\right)^{1/2}, \tag{5.9}$$

given in [4].

On the basis of the above discussion the following (approximate) formula for $\hbar\omega$ may be proposed:

$$\hbar\omega = \frac{5}{4}\frac{\hbar^2}{MA}\left[(K+1)\left(A+\frac{1}{3}n\right)+\frac{2}{3}n-2\right]\left(c^2+\lambda\right)^{-1}, \tag{5.10}$$

where

$$\lambda = \frac{7}{3}(\pi a)^2 - \frac{5}{3}\left(\langle r^2\rangle_p + \langle r^2\rangle_n\right). \tag{5.11}$$

Expression (5.10) can be expanded in powers of $A^{-1/3}$, the leading term of the expansion being the well-known result: const $A^{-1/3}$. Thus, we have

$$\hbar\omega = \frac{5}{4}\left(\frac{3}{2}\right)^{1/3}\left(\frac{\hbar^2}{M}\right)b^{-2}A^{-1/3}\left(1+c_1A^{-2/3}+c_2A^{-4/3}+...\right), \tag{5.12}$$

where

$$c_1 = \frac{1}{3}\left(\frac{2}{3}\right)^{2/3} + \left[\frac{2}{3} - \frac{\lambda}{(\pi a)^2}\right](\pi a/b)^2, \tag{5.13}$$

$$c_2 = \left(\frac{2}{3}\right)^{1/3}\left(\frac{2}{3}n-2\right) + \frac{1}{3}\left(\frac{2}{3}\right)^{2/3}(\pi a/b)^2 - \\ - \frac{1}{3}\left[\left(\frac{2}{3}\right)^{2/3} + 4(\pi a/b)^2\right]\lambda b^{-2} + \lambda^2 b^{-4} + \frac{1}{3}(\pi a/b)^4. \tag{5.14}$$

It is seen from the above expansion that the first two terms are independent of the number of the valence nucleons. Obviously, for closed shell nuclei all the terms are independent of n. In addition, we may remark that the centre-of-mass correction does not affect the first two terms of the expansion.

The values of ρ_0 and a needed for the computation of $\hbar\omega$ have been obtained by fitting expression (5.8) (with c given by Eq. (5.7)) for the r.m.s radius to the experimental r.m.s radii of the charge distributions of individual nuclei, taken from Ref. 67 (neglecting the errors quoted there). The $\langle r^2\rangle_{\rm exp}^{1/2}$ for the calcium isotopes, which were used are the same with those used in Ref. 68. The best fit values found (by fitting in the region $12 \leq A \leq 208$) are $\rho_0 = 0.156$ fm^{-3} and $a = 0.492$ fm.

Alternatively, the values ρ_0 and a can be determined by fitting the theoretical expression (5.10) to the value of $\hbar\omega$, which reproduce by means of (5.6) the experimental values of the r.m.s radii of the charge distributions of the relevant nuclei, as these are determined from the elastic electron scattering experiments. Such a fit (for nuclei with $N = Z$ in the region $12 \leq A \leq 40$) gave the values $\rho_0 = 0.123$ fm^{-3} and $a = 0.369$ fm.

The results of the calculations, in the framework of the approximate scheme described, show discontinuities in the slope of $\hbar\omega$ at closed shells. These discontinuities die out as the mass number increases (see Fig. 1 in [60]).

Subsequent work for the determination of $\hbar\omega$ on the basis of a density distribution were made in Ref. 69, where the trapezoidal density was used, and in Refs. 70–73. In Ref. 70 the problem of the dependence of the harmonic oscillator spacings $\hbar\omega_n(\hbar\omega_p)$ for neutrons (protons) on $N(Z)$ was addressed using the semiphenomenological distribution from Refs. 74, 75 for $\rho_n(r)$, $\rho_p(r)$ which is rather complex but has certain desirable features. A comparison of the results for $\hbar\omega(=\hbar\omega_{\rm nucleon}(A))$ obtained with the density from [74, 75] and those with the Fermi density of [60] (leading to larger values of $\hbar\omega$) was made in [71]. At last, in Refs. 72 and 73 isospin dependent oscillator spacings were considered. In Ref. 72 a SF density was also parametrized and expressions for the radius and $\hbar\omega$ dependent on A and $N - Z$ were obtained. In Ref. 73 the dependence of $\hbar\omega_n(\hbar\omega_p)$ on N and Z was established employing a parametrization of the m. s. nuclear radius obtained in HF-BCS calculations. The parametrization had been introduced in [76] with a Fermi-like shape assumed for the neutron and proton densities.

Finally, we mention that a Fermi distribution was also used recently [77, 78] in determining the HO energy level spacing $\hbar\omega$ for metal clusters, within the jellium model in a way analogous to the one described in this section. Neutral sodium clusters were considered and parameters of the Fermi distribution were determined by means of the jellium model results for the electronic densities obtained in the detailed local density calculations in Ref. 79. An «overall» least

squares fit was made of the expression for $\rho_{\rm F}(r)/\rho_0$ to the results of [79] for a number of clusters which gave $b = 2.08$ Å and $a = 0.39$ Å. The variation of $\hbar\omega$ with the particle number N (the number of the valence electrons of the atoms in the jellium model) showed again «slope discontinuities» (kinks) at the closed shells, under the assumptions made (see Figs. 1–3 of [78]). These are somewhat more apparent in the plot of $\Delta\hbar\omega/\Delta N$ (see Fig. 4 of [78]).

5.2. «Woods–Saxon-Type» Potentials. The aim of this paragraph is to discuss certain potentials which are quite similar among themselves and we use for them the term «Woods–Saxon-type» potentials. They include in addition to the usual Woods–Saxon (WS) potential, the so-called «cosh» potential $V_c(r)$ and the «symmetrized Woods–Saxon» one: $V_{\rm SWS}$ (having as its radial dependence the SF function). Potentials of this type have been extensively used in nuclear and hypernuclear physics calculations and also in studying metal clusters.

5.2.1. The Potential Radial Dependences, Depths and Radii. The well-known WS potential

$$V_{\rm WS}(r) = -V_0 f_{\rm F}(r) = \frac{-V_0}{1 + \exp\left[(r - R)/a\right]}, \tag{5.15}$$

being a very helpful approximation in nuclear physics, has been discussed in many monographs and textbooks (see, e.g., [8, 80]). Such a potential does not satisfy the physical requirement which is usually imposed [81] that the force experienced by a particle at the centre of a spherically symmetric potential be zero, though the value of $dV_{\rm WS}(r)/dr|_{r=0}$ is very small for $R/a \gg 1$, that is apart from the light nuclei.

An alternative potential considered in [17], the so-called «cosh» potential, which is a symmetrized form of the WS one, namely,

$$V_c(r) = -V_0 f_c(r) = -V_0 \frac{1 + \cosh\left(R/a\right)}{\cosh\left(r/a\right) + \cosh\left(R/a\right)} \tag{5.16}$$

has the desirable feature: $(dV_c/dr|_{r=0}) = 0$. Note that we usually drop the corresponding subscripts in the potential parameters in order to simplify the notation. The radial dependence of this potential (its FF) is very similar to that which has been put forward in [15] (cf. 1.3) to describe the densities of nuclei for a wide range of the mass number. In Ref. 17, the potential (5.16) has been used as the central part of a cluster-core potential, and by suitably choosing parameters R and a, potential shapes can be obtained, which are remarkably similar to the folding potentials used in [82], and hence they also exhibit rotational spectra.

Subsequently, in Refs. 83, 84 the potential (5.16) was used in connection with a semiclassical method of quantization for a particle in a nuclear potential and for comparison of the results with those obtained with the Schrödinger equation. In this case, the WS potential is inadequate (according to [84]) to calculate the

classical trajectories since the numerical methods used for the integration of the equations of motion are inapplicable in the vicinity of $r = 0$ for this potential.

One may employ, instead of potential (5.16), a symmetrized WS potential in the form which has been used in the case of nuclear densities (and in the analysis of hypernuclei),

$$V_{\rm SWS}(r) \equiv -V_0 f_{\rm SF}(r) = -V_0 \frac{\sinh(R/a)}{\cosh(r/a) + \cosh(R/a)}. \tag{5.17}$$

Following Ref. 18, we would like to note that potentials (5.16) and (5.17) are of SF form and have zero slope at the origin, but their FF's differ, their ratio being:

$$\frac{f_{\rm SF}}{f_c} = \sinh(R/a)\,[1 + \cosh(R/a)]^{-1} \tag{5.18}$$

with the same values of R and a in both cases. This ratio is almost unity for $R/a \gg 1$, that is for the heavier elements for which both potentials are very close to the WS one.

The choice of the numerator in potential (5.16) seems to originate from the requirement that the form factor to become unity at the origin, where the potential has its minimum. Thus, V_0 is the actual depth of the «cosh» potential: $D_c = V_0$. This is not the case for the other two potentials for which the depths are:

$$\begin{aligned} D_{\rm WS} &= |V_{\rm WS}(0)| = V_0[1 + \exp(-R/a)]^{-1}, \\ D_{\rm SWS} &= |V_{\rm SWS}(0)| = V_0[1 - \exp(-R/a)][1 + \exp(-R/a)]^{-1}. \end{aligned} \tag{5.19}$$

Therefore, for each of these potentials V_0 is a «depth parameter», which determines its depth to a large extent, but not entirely, since D depends also upon R and a.

Analogous remarks can be made regarding the meaning of R. For the WS potential the parameter R is the distance from the origin at which the absolute value of the potential becomes half of its depth parameter V_0 (not its depth). It is a «half-depth» parameter radius since

$$|V_{\rm WS}(R)| = \frac{1}{2}V_0 = \frac{1}{2}D_{\rm WS}[1 + \exp(-R/a)] > \frac{1}{2}D_{\rm WS}. \tag{5.20}$$

For the «cosh» potential we have

$$|V_c(R)| = \frac{1}{2}D_c\left\{1 + 2\exp(-R/a)[1 + \exp(-2R/a)]^{-1}\right\} > \frac{1}{2}D_c. \tag{5.21}$$

The same expression holds for $V_{\rm SWS}(r)$. Therefore, for all three potentials, R is a quantity which characterizes the range of the potential, but it is somehow smaller than the «half-depth» radius (see the Table).

Potential	Depth of the potential	Value of the potential at R
Woods–Saxon $V_{\mathrm{WS}}(r) = \dfrac{-V_0}{1+\exp\left[(r-R)/a\right]}$	$D_{\mathrm{WS}} = V_0[1+\exp\,(-R/a)]^{-1}$	$\lvert V_{\mathrm{WS}}(R)\rvert = \dfrac{1}{2}V_0 =$ $= \dfrac{1}{2}D_{\mathrm{WS}}[1+\exp\,(-R/a)] > \dfrac{1}{2}D_{\mathrm{WS}}$
«Cosh» potential $V_c(r) =$ $= -V_0\dfrac{1+\cosh\,(R/a)}{\cosh\,(r/a)+\cosh\,(R/a)}$	$D_c = V_0$	$\lvert V_c(R)\rvert =$ $= \dfrac{1}{2}D_c\left[1+\dfrac{2\,\exp\,(-R/a)}{1+\exp\,(-2R/a)}\right] > \dfrac{1}{2}D_c$
Symmetrized Woods–Saxon $V_{\mathrm{SWS}}(r) =$ $= -V_0\dfrac{\sinh\,(R/a)}{\cosh\,(r/a)+\cosh\,(R/a)}$	$D_{\mathrm{SWS}} = V_0\dfrac{1-\exp\,(-R/a)}{1+\exp\,(-R/a)}$	$\lvert V_{\mathrm{SWS}}(R)\rvert =$ $= \dfrac{1}{2}D_{\mathrm{SWS}}\left[1+\dfrac{2\,\exp\,(-R/a)}{1+\exp\,(-2R/a)}\right] > \dfrac{1}{2}D_{\mathrm{SWS}}$

The above remarks as well as the following one are pertinent to the cases in which the condition $R/a \gg 1$ is not quite satisfied, that is, to the lighter nuclei.

The choice of the numerator in the «cosh» potential discussed earlier seems, however, to have another implication. The volume integral of the potential becomes a transcendental function of the radius R:

$$|\overline{V}_c| = \left| 4\pi \int_0^\infty V_c(r) r^2 dr \right| = \frac{4\pi V_0}{3} R^3 \left(1 + \frac{(\pi a)^2}{R^2} \right) \left(\frac{1 + \cosh(R/a)}{\sinh(R/a)} \right), \tag{5.22}$$

and it is not easy to use it in order to express R as a function of the mass number of the nucleus. Transcendental terms appear also in the volume integral of the WS potential [4]. On the contrary, for the $V_{\mathrm{SWS}}(r)$ these terms are absent as the results given in [15] for the SF density show and detailed calculations [18] have verified (see also Sects. 2 and 3). One can therefore write on the basis of a folding model and the well-known properties of the convolution (in analogy, for example, with the approximate treatment of [86] (see below Subsect. 5.3) for a WS):

$$\frac{4\pi V_0}{3} R^3 \left(1 + \frac{(\pi a)^2}{R^2} \right) = A_c |\overline{V}_{NN}|, \tag{5.23}$$

where, in the case of a bound nucleon, $A_c = A - 1$, A being the mass number of the nucleus, $|\overline{V}_{NN}|$ is the volume integral of the spin-average (central) nucleon-nucleon potential$|\overline{V}_{NN}| = |\int \overline{V}_{NN}(r) d\mathbf{r}|$. The above equation is a third-order equation with respect to R and can be solved exactly. In fact, it has the same structure as the equation arising from the normalization integral for a trapezoidal distribution [69, 87]. The corresponding equation for the Fermi distribution [60] (and the WS potential) is approximate.

Thus, in the case of the V_{SWS} potential, one may obtain the following exact expression of the radius R in terms of the mass number:

$$R = \frac{1}{2^{1/3}} r_0 A_c^{1/3} \{ [1 + \delta_c]^{1/3} + [1 - \delta_c]^{1/3} \} =$$

$$= r_0 A_c^{1/3} \left[1 - \frac{1}{3} \left(\frac{\pi a}{r_0 A_c^{1/3}} \right)^2 + \frac{1}{81} \left(\frac{\pi a}{r_0 A_c^{1/3}} \right)^6 + \frac{1}{243} \left(\frac{\pi a}{r_0 A_c^{1/3}} \right)^8 + \dots \right],$$

$$\delta_c = \left[1 + \frac{4}{27} \left(\frac{\pi a}{r_0 A_c^{1/3}} \right)^6 \right]^{1/2} \tag{5.24}$$

where $r_0 = (3|\overline{V}_{NN}|/4\pi V_0)^{1/3}$. The leading term of this expansion (assuming also a suitable dependence of V_0 on A, such as the one in Sect. 4 of Ref. 18):

$R \simeq r'_0(A-1)^{1/3}$ is of the form which is often used for the radius of the WS potential. In the treatment of Ref. 88, however, the r'_0 in such an expression was dependent on A.

It should be noted that the higher even moments of potential $V_{\mathrm{SWS}}(r)$ may also be given analytically and contain no exponential terms. The same holds for the «cosh» potential but not for the WS one. The respective expression for the «nth» moment is determined by Eq. (2.63) in Subsect. 2.2.1.

We would like also to point out that some findings which have been described, permit to give rather transparent qualitative predictions. In particular, it has been shown [89] that an expansion of the type (1.7) or (2.20), truncated at the term with the first derivative of δ function, may be helpful in certain cases. Thus, one may write the WS potential in the Schrödinger equation approximately as

$$V(r) = -V_0\Theta(R-r) + V_0\frac{\pi^2}{6}a^2\frac{d}{dr}\delta(r-R), \tag{5.25}$$

considering the corresponding eigenvalue problem. In particular, for bound s states, this leads to the following analytic eigenvalue equation:

$$\frac{X}{X_0} = \frac{|\sin X|}{(1-g^2\cos^2 X)^{1/2}}, \tag{5.26}$$

where

$$X = KR = [(2\mu/\hbar^2)(V_0 - |E|)]^{1/2}R, \tag{5.27}$$

$$X_0 = K_0R = [(2\mu/\hbar^2)V_0]^{1/2}R \tag{5.28}$$

and

$$g^2 = 1 - \left(\frac{1-\xi}{1+\xi}\right)^2, \qquad \xi = (\mu/\hbar^2)(V_0/6)\pi^2a^2. \tag{5.29}$$

It is seen immediately that for the diffuseness parameter a going to zero, Eq. (5.26) yields the usual eigenvalue relation for the square-well potential (see, e.g., paragraph 3.2 in [90]).

Numerical or graphical solution of (5.26) determines the energy eigenvalues. Application of this equation to the ground state binding energy of the Λ particle in the hypernucleus ${}^{89}_{\Lambda}\mathrm{Y}$ gave for it the value $B_\Lambda = 23.8$ MeV while the Shrödinger equation solved numerically for the WS potential gave the value $B_\Lambda = 24.3$ MeV. The parameters used were $V_0 = 29.8$ MeV, $r_0 = 1.2$ fm, and $a = 0.6$ fm. We see that these binding energy values are quite close.

5.2.2. The Estimate of the Potential Parameters. Firstly, one may point out, following a proposal made in a study of inclusive pion nuclear reactions [91], that the radius R and the diffuseness parameter a for a point proton density distribution of the Fermi-type can be obtained approximately from the corresponding parameters R_e and a_e for the same type charge density distribution and the m. s. radius for the charge density distribution of the proton r_p^2. The pertinent expressions were given by formulae (18) and (19) of Ref. 18, where an improvement is also pointed out.

The corresponding approximate depth, radius and diffuseness parameter for the SWS potential are derived under the assumption this potential to be given by the folding model expression [92, 93], viz.,

$$V(r) = A_c \int \rho(|\mathbf{r} - \mathbf{r}'|) V_{NN}(r') d\mathbf{r}' =$$

$$= \left[A_c \int V_{NN}(r) d\mathbf{r} \right] \left(1 + \sum_{n=1}^{\infty} \frac{\langle r'^{2n} \rangle_{NN}}{(2n+1)!} \Delta^n \right) \rho(r), \quad (5.30)$$

where $\Delta^n \rho(r) = [(d/dr)^{2n} + (2n/r)(d/dr)^{2n-1}]\rho(r)$, and $\rho(r)$ is the point proton density, which is taken to be the same with the nucleon one.

The above expansion is valid for large r, compared with the range of the nucleon–nucleon potential V_{NN}, and for sufficiently short range V_{NN} only the first term in the sum may be taken into account, so that, for instance,

$$-V_0 \simeq \left(A_c \rho_0 \int V_{NN}(r) d\mathbf{r} \right) \left(1 - \frac{\langle r^2 \rangle_{NN}}{6Ra} \right), \quad (5.31)$$

where R and a are the radius and diffuseness parameters of the (S)F distribution for point nucleons.

The parameters (R_V, a_V) for the (S)WS potential are also given by approximate analytic expressions (formulae (28), (29) or (32) and (33) of Ref. 18).

Another possible way of determining the parameters R_V and a_V is the one which is outlined in [94] (see also [95]) in the case of the Λ nucleus potential. According to this approach one uses the relation

$$\langle r^2 \rangle + r_p^2 = \langle r^2 \rangle_e \quad (5.32)$$

between the second moments of the SF distribution and the corresponding one between the fourth moments (the factor $7/3$ in expression (A3) of [94] should be written as $10/3$) [94, 96]:

$$\langle r^4 \rangle_e = \langle r^4 \rangle + r_p^4 + \frac{10}{3} \langle r^2 \rangle r_p^2. \quad (5.33)$$

Analogous relations are used between the moments of the potential and the point-nucleon distribution. The analytic expressions for $\langle r^n \rangle_{\mathrm{SF}}$ can be found by means of general relation (2.63).

Again in this approach the parameters R and a are first determined from R_e and a_e. From the $\langle r^2 \rangle_e$ and $\langle r^4 \rangle_e$, which are known in terms of R_e and a_e, the $\langle r^2 \rangle$ and $\langle r^4 \rangle$ (and the ratio $\frac{21}{25}\langle r^4 \rangle / \langle r^2 \rangle^2 = \lambda^2$) are determined by means of (5.32) and (5.33). Subsequently, the quantity $y = (\pi a/R)^2$ is expressed in terms of λ^2, using the expression of the second and the fourth moment for SF:

$$y = \left(\frac{\pi a}{R}\right)^2 = \frac{3}{49\lambda^2 - 93}\left(-7\lambda^2 + 9 \pm 2\sqrt{4\lambda^2 - 3}\right). \tag{5.34}$$

Thus, using (5.32) and (5.34), the expressions of R and a are

$$R = \left(\frac{5}{3}\frac{\langle r^2 \rangle_e - r_p^2}{1 + \frac{7}{3}y}\right)^{1/2} \tag{5.35}$$

and

$$a = \frac{y^{1/2}R}{\pi} = \left(\frac{5y(\langle r^2 \rangle_e - r_p^2)}{3\pi^2(1 + \frac{7}{3}y)}\right)^{1/2}. \tag{5.36}$$

The expressions of R_V and a_V are quite analogous. The λ^2 refers now to the ratio of $\frac{21}{25}\langle r^4 \rangle_V / \langle r^2 \rangle_V^2$ and instead of r_p^2 and r_p^4 the corresponding moments of the nucleon–nucleon potential appear in the various expressions.

It should be clear that in both approaches described above the approximation is made, as in [94], that the convolution of a SF distribution is a SF distribution. This should be reasonable as long as the «folding» distribution (either proton charge density or nucleon–nucleon potential) are of sufficiently short range compared with the folded distribution. Having determined R_V and a_V in one way or the other the potential depth parameter V_0 is adjusted so that the values of the experimental single particle energies are reproduced on solving the Schrödinger equation.

Finally, all parameters could be determined by a suitable least-squares fit to the experimental single-particle energies. It appears, however, that in most cases such a procedure would not be satisfactory, mainly because of the large experimental uncertainties. For this reason, one of the parameters, namely a_V (on which the single-particle energies, at least for the ground state [86], are not expected to depend strongly, unless A_c is small), is fixed from our previous experience and the remaining parameters could be attempted to be determined by a least-squares fit.

5.2.3. Numerical Results and Discussion. Numerical estimates of the SWS potential parameters (R_V, a_V, and V_0) have been obtained on the basis of the procedures that have been outlined briefly in the preceding paragraph (see [18] for more details) considering the nuclei, $^{12}_{6}$C, $^{16}_{8}$O, $^{24}_{12}$Mg, $^{28}_{14}$Si, $^{32}_{16}$S, $^{40}_{20}$Ca, $^{56}_{26}$Fe, and $^{58}_{28}$Ni. The parameters used for the charge density distribution for these and certain other nuclei were determined in [15, 16] from the analysis of the elastic electron scattering experiments. Both approaches discussed were applied, more exactly, an improved version of the approach in [91] and also the approach of the second and fourth moments following [94, 95].

The second and fourth moments of the nucleon–nucleon potential were estimated by using interaction 4 of Table 1 of [97]. This potential is of short range that is essential for the applicability of the method. The results for R_V and a_V are displayed in Tables 1 and 2, respectively, of Ref. 18 and they do not differ between the two methods appreciably. The first approach leads to larger values of R_V and to smaller values of a_V compared with those of the second approach. In the same tables, the values of the potential depth parameter V_0 determined through the values which have been found for R_V and a_V and the nuclear contributions to the proton energy for various states are also displayed. These energies have been estimated from the experimental proton energies given in [98] (or by the corresponding curves in Fig. 117 of [99]) subtracting approximate values for the Coulomb energy E_c. The same expression for E_c, as in [85], has been used for the present rough estimates relying upon the values of Table 1 of [18] for R_e and a_e. The values of E_c obtained in this way are quite close to those of [85] in which the R_e and a_e for the corressponding (or neighbouring) nuclei have been taken from [100], [67]. A marked state dependence of V_0 has been observed. V_0 is smaller for the $1p$ states and (normally) even smaller for the $1d$ or $2s$ states.

We may note that the nonorthogonality between the $1s$ and $2s$ states, because of the state dependence of the potential, can be treated within the relevant formalism [101]. In practice, it is easier, however, to apply an orthogonalization procedure like the Gram–Schmidt one. Such an approach has been developed in [102] using the WS potential to calculate the charge density of ^{58}Ni. It led to the interesting observation that under certain general conditions the occupation numbers remain approximately unchanged.

It should be also pointed that the folding model approach with an average spin and isospin independent nucleon–nucleon potential is essentially limited to the case of light symmetric nuclei. Although the described approach could perhaps be also attempted for some asymmetric nuclei, its validity in these cases should be in general quite doubtful.

A couple of additional remarks are relevant. As in the case with the WS potential [103], the solution of the Schrödinger eigenvalue problem with an energy-dependent SWS potential is equivalent to the solution of this problem with an

energy-independent potential $\tilde{V}_{\rm SWS}$ but with radial dependent effective mass:

$$m^*(r,\epsilon) = m\left(1 - \frac{\partial V_{\rm SWS}(r,\epsilon)}{\partial \epsilon}\right). \tag{5.37}$$

By assuming a linear energy dependence of the potential depth parameter of $V_{\rm SWS}$

$$V_0(\epsilon) = V_0 + \left(1 - \frac{m_0^*}{m}\right) f_{\rm SF}^{-1}(0)\epsilon, \tag{5.38}$$

where m_0^* is the value of the effective mass for $r = 0$, the effective mass (5.37) becomes energy independent:

$$m^*(r) = m\left[1 - \left(1 - \frac{m_0^*}{m}\right)\frac{f_{\rm SF}(r)}{f_{\rm SF}(0)}\right]. \tag{5.39}$$

The corresponding energy-independent potential is

$$\tilde{V}_{\rm SWS}(r) = -\frac{m}{m^*(r)} V_0 f_{\rm SF}(r). \tag{5.40}$$

The value of m_0^*/m was estimated for $^{40}_{20}$Ca and $^{58}_{28}$Ni, using equation (5.38) and the values of $V_0(\epsilon)$ (from Table 2) of [18] for the various occupied states (apart from the last one). For $^{40}_{20}$Ca it was found $m_0^*/m = 0.6$; while for $^{58}_{28}$Ni, $m_0^*/m = 0.5$.

We may also note that a spin-orbit term $V^{\rm SO}$ should be added to the central SWS potential

$$\begin{aligned} V_{\rm SWS}^{\rm SO} &= V_0^{\rm SO}\left(\frac{\hbar}{m_\pi c}\right)^2 \frac{1}{r}\frac{df_{\rm SF}(r)}{dr}\mathbf{l}\cdot\boldsymbol{\sigma} = \\ &= -\frac{V_0^{\rm SO}}{\alpha_V}\left(\frac{\hbar}{m_\pi c}\right)^2 \frac{1}{r}\frac{\sinh\left(R_V/\alpha_V\right)\sinh\left(r/\alpha_V\right)}{\left(\cosh\left(R_V/\alpha_V\right) + \cosh\left(r/\alpha_V\right)\right)^2}\mathbf{l}\cdot\boldsymbol{\sigma}. \end{aligned} \tag{5.41}$$

The strength $V_0^{\rm SO}$ of this term may be estimated empirically from the experimental splitting of the single-particle levels due to the spin-orbit force. The values of V_0 and $V_0^{\rm SO}$ for a number of nuclei are displayed in Table 3 of [18]. In all those estimates the values of R_V and a_V were taken to be those displayed in Table 2 of that reference.

In conclusion, we should make clear that only a rough estimate of the potential parameters has been attempted. This is mainly due to the approximations involved, to the rough estimate of the Coulomb energies and to the rough experimental values for the single-particle energies. In order to become free of the second ambiguity and diminish the third one, were used the comparatively more

recent neutron separation energies reported in [104]. Unfortunately, only the ^{12}C and ^{16}O nuclei (and some lighter ones) have been studied in that reference. The values of the potential depth parameters obtained from the experimental values for the energies in the $1s$ and $1p$ states are somehow different from those of Tables 1 and 2 of [18]. The differences are, however, of the order of about 10 % or less.

5.3. Woods–Saxon-Type Potentials in Analyses of Hypernuclei and Metal Clusters. *5.3.1. The* WS Λ*-Nucleus Potential.* The WS potential has also been employed in studying hypernuclei such as Λ and Ξ hypernuclei. Initially and in a number of calculations the half-way radius R was related to the mass number of the core-nucleus $A_c = A$ by the simple expression $R = r_0 A^{1/3}$ [94, 95, 105].

An improved relation [86] between R and A may be derived by considering the «rigid-core model» (folding model) [106] expression for the Λ-nucleus potential, assumed to be of the WS form,

$$V_{\Lambda A}(r) = \frac{-V_0}{1 + \exp\left[(r - R)/a\right]}. \tag{5.42}$$

Thus,

$$V_{\Lambda A}(r) = A \int V_{\Lambda N}(|\mathbf{r} - \mathbf{r}'|)\rho_A(r')d\mathbf{r}', \tag{5.43}$$

where $V_{\Lambda N}$ is the (spin-average) Λ-nucleon potential and ρ_A the normalized to unity point nucleon density distribution of the core nucleus. Using the well-known property of the convolution, we may write

$$4\pi \int_0^\infty V_{\Lambda A}(r) r^2 dr = -A\overline{V}, \tag{5.44}$$

where the volume integral of the (spin-average) Λ-nucleon potential is denoted here by $\overline{V} > 0$.

If $V_{\Lambda A}(r)$ is assumed to be of the form (5.42), the above integral may be evaluated analytically and the following equation for R is obtained:

$$R^3 + (\pi a)^2 R - \frac{3\overline{V}}{4\pi V_0} A = 0, \tag{5.45}$$

provided that $\exp\left[-R/a\right] \ll 1$. The latter condition is usually well satisfied. The third-order equation is of the same form with the one which follows from the normalization condition of a trapezoidal distribution for the nuclear density and

has been solved analytically [87]. The expression for R obtained in this way is

$$R = \left(\frac{1}{2}\right)^{1/3} r_0 A^{1/3} \left\{ \left[1 + \left(1 + \frac{4}{27} \left(\frac{\pi a}{r_0 A^{1/3}} \right)^6 \right)^{1/2} \right]^{1/3} + \right.$$

$$\left. + \left[1 - \left(1 + \frac{4}{27} \left(\frac{\pi a}{r_0 A^{1/3}} \right)^6 \right)^{1/2} \right]^{1/3} \right\}, \quad (5.46)$$

where $r_0 = (3\overline{V}/4\pi V_0)^{1/3}$. It is easily seen from (5.46) that R may be expressed as an expansion in powers of $A^{1/3}$, which is the same (if we replace $V_0/\overline{V}$ by ρ_0) with the corresponding expansion of R for a Fermi-type nuclear density distribution.

If we wish to make an estimate of V_0, the Schrödinger equation must be solved numerically with potential (5.42). Alternatively, one might consider the relevant eigenvalue equation for B_Λ, which is known in the case $\exp[-R/a] \ll 1$, to be (for the ground (1s) state) [107, 108]

$$R\xi + \arctan\frac{\xi}{\eta} + \arg\Gamma(1 + 2ia\xi) - 2\arg\Gamma(1 + a\eta + ia\xi) = \pi, \quad (5.47)$$

where

$$\xi = \left[\frac{2\mu_{\Lambda A}}{\hbar^2}(V_0 - B_\Lambda) \right]^{1/2}, \qquad \eta = \left(\frac{2\mu_{\Lambda A}}{\hbar^2} B_\Lambda \right)^{1/2}. \quad (5.48)$$

It is shown in [108] that a Walecka-type expression (see [109]), which is appropriate for sufficiently large A, can be written for B_Λ,

$$B_\Lambda \simeq V_0 - \frac{\hbar^2\pi^2}{2\mu_{\Lambda A} R^2} \left(1 - \frac{2}{S} + \frac{3}{S^2} \right). \quad (5.49)$$

The quantity $1/S$ is determined by

$$\frac{1}{S} = \frac{1}{Rk_0} \left\{ 1 - 2ak_0 \left[\gamma + \Psi(1 + ak_0) \right] \right\}, \qquad k_0 = \left(\frac{2\mu_{\Lambda A} V_0}{\hbar^2} \right)^{1/2}, \quad (5.50)$$

where γ is Euler's constant ($\gamma = 0.577...$) and Ψ the logarithmic derivative of the Γ function. For comments on (5.49) see Ref. 108.

We may remark that a somewhat different approximate expression for B_Λ is [86]

$$B_\Lambda \simeq V_0 - \left(\frac{\hbar^2}{2\mu_{\Lambda A}} \right) \left(\frac{\pi}{R} \right)^2 \left(1 + \frac{1}{S} \right)^{-2}, \quad (5.51)$$

which has the same form with the one in the square-well case, where $S = Rk_0$ (see, e.g., [110]).

We should note that equation (5.51) gives very similar results with (5.49), at least apart from the comparatively small values of A for which the results of these expressions deviate more or they are even quite different from those obtained with the exact expression for B_Λ.

Expression (5.51) for B_Λ is an approximate semiempirical mass formula which has the attractive feature (as the truncated expression (5.49) too) of being rather simple and of giving also quite accurate estimates for B_Λ, apart from the cases of light hypernuclei. It may, therefore, be used for the heavy hypernuclei as an alternative to the evaluation of B_Λ either by solving numerically the Schrödinger equation or the analytic eigenvalue equation (5.47) since these procedures, particularly the first one, are much more time consuming. B_Λ, given by expression (5.51), may be written as an expansion in powers of A, if the relevant expansion for R given in [4] is used:

$$R = r_0 A^{1/3}\left[1 - \frac{1}{3}\left(\frac{\pi a}{r_0}\right)^2 A^{-2/3} + \frac{1}{81}\left(\frac{\pi a}{r_0}\right)^6 A^{-2} + \right.$$

$$\left. + O\left(A^{-8/3}\right)\right] = r_0'(A)A^{1/3}. \quad (5.52)$$

The parameters V_0 and a are assumed here to be independent of A. The final result at which we arrive after some algebra is the following:

$$B_\Lambda \simeq V_0 - \frac{\hbar^2\pi^2}{2m_\Lambda} r_0^{-2} A^{-2/3}\left\{1 - 2\tilde{S}^{-1}A^{-1/3} + \right.$$

$$+ \left[3\tilde{S}^{-2} + \frac{2}{3}\left(\frac{\pi a}{r_0}\right)^2\right] A^{-2/3} - \left[4\tilde{S}^{-3} + 2\tilde{S}^{-1}\left(\frac{\pi a}{r_0}\right)^2 - \frac{m_\Lambda}{m_N}\right] A^{-1} +$$

$$+ \left[5\tilde{S}^{-4} + 4\tilde{S}^{-2}\left(\frac{\pi a}{r_0}\right)^2 + \frac{1}{3}\left(\frac{\pi a}{r_0}\right)^4 - 2\left(\frac{m_\Lambda}{m_N}\right) \times \right.$$

$$\left.\left. \times \left(\tilde{S}^{-1} + \frac{a^2}{r_0}\tilde{k}_0\Psi'(1 + a\tilde{k}_0) - \frac{1}{4r_0\tilde{k}_0}\right)\right] A^{-4/3} + O(A^{-5/3})\right\}, \quad (5.53)$$

where

$$\tilde{S}^{-1} = r_0^{-1}(\tilde{k}_0^{-1} - 2a(\gamma + \Psi(1 + a\tilde{k}_0))), \qquad \tilde{k}_0 = \left(\frac{2m_\Lambda}{\hbar^2}V_0\right)^{1/2} \quad (5.54)$$

and $\Psi'(1 + a\tilde{k}_0)$ is the derivative of $\Psi(1 + a\tilde{k}_0)$ with respect to $a\tilde{k}_0$.

The function $\Psi(1+a\tilde{k}_0)$ may be expressed as a series expansion

$$\Psi(1+a\tilde{k}_0) = -\gamma + \sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)(a\tilde{k}_0)^n, \qquad a\tilde{k}_0 < 1, \tag{5.55}$$

where $\zeta(n+1)$, $n = 1, 2, ...$ is the Riemann's zeta function ($\zeta(2) = 1.6449$, $\zeta(3) = 1.2021$, $\zeta(4) = 1.0823$ etc.). The function $\Psi'(1+a\tilde{k}_0)$ may also be given as a series expansion.

We may note that the surface diffuseness effects ($a \neq 0$) influence the coefficients of the third and higher terms of the expansion for B_Λ, while the use of the reduced mass $\mu_{\Lambda A}$ has resulted in additional correction terms in the coefficients of the fifth ($\sim A^{-5/3}$) and higher terms. In Ref. 86 preliminary numerical calculations of B_Λ were made using the WS potential with the complex expression for R and the earlier experimental values for B_Λ (from the emulsion data [105, 111]). The well-depth $D = V_0$, the diffuseness parameter a and $\overline{V}$ were treated as adjustable parameters. The values turned out to be around $D \simeq 29$ MeV, $a \simeq 0.6$ fm and $\overline{V} \simeq 325$ MeV $\cdot$ fm^3 ($r_0 = 1.4$ fm) (for more details see [86]). Also, for $^{13}_{\Lambda}$C, the value of the quantity $(R - r_0 A^{1/3})/R$ is indicated to be roughly (10–15) %.

The WS and SWS potentials were used in the same spirit in the analysis of Ξ hypernuclei (see [112]).

Finally, we recall that the WS potential was also used in the analysis of the Λ energies not only in the ground state but also in its excited states. Significant progress in measurements of the Λ-single particle energies for various hypernuclei in their ground and excited states was made during the last decades [113]. We mention, in particular, Ref. 114 (and references therein). There, the radius parameter r_0 was also taken to be dependent on the mass number, but it was equal to $1.128 + 0.43A^{-2/3}$, where A is the mass number of the hypernucleus. In addition, more complex potentials were employed including those containing a quadratic form of the Fermi function. In this context, we would like to note Ref. 115 where the SWS potential had state-dependent depth and radius parameter r_0 (in addition to the A dependence of the latter). Also, let us recall applications of the WS form in relativistic energy calculations of the Λ particle on the basis of the Dirac equation with attractive and repulsive potentials [116].

5.3.2. WS-*Type Potentials for Metal Clusters.* In this subsection we would like to point out that WS-type potentials were used in the study of atomic (metal) clusters as well. This research area was much developed in the last two decades after it had been found in 1984 that atoms of some metals (alkali, gold, etc.) can form bound systems (clusters) showing shell structure and magic numbers as in nuclei and atoms [117,118]. Thus, this area can be regarded as a new application field of nuclear-physics ideas and methods [119].

The usual WS potential was used in Ref. 117 in analysing the mass spectrum for sodium clusters in the pioneering work, where shell effects were observed. The same potential was imployed in [119] in parametrizing the effective radial electronic potential, that had been obtained on the basis of local density approximation, to examine shell and supershell effects. The parameters were: $V_0 = 6$ eV, $R = r_0 N^{1/3}$, $r_0 = 2.25$ Å and $a = 0.74$ Å.

In Refs. 120 the WS and SWS potentials were applied to describe the properties of metal (sodium) clusters. In those papers a complex expression for the potential radius R, that is a sort of generalization of the expression used for hypernuclei, was considered, and also another potential of the WS-type, namely, the «wine bottle» SWS given by

$$V_{\mathrm{WB}} = -V_0 \left(1 + \frac{wr^2}{R^2}\right) \frac{\sinh(R/a)}{\cosh(r/a) + \cosh(R/a)}, \qquad 0 \leq r < \infty \tag{5.56}$$

was discussed.

5.3.3. The SWS *Potential and the Determination of $\hbar\omega$ for Hypernuclei and Metal Clusters.* The determination of the HO energy level spacing $\hbar\omega$ and its variation with the particle number was discussed in Subsect. 5.1 for ordinary nuclei and metal clusters on the basis of the Fermi distribution for the particle density. Here we address the same problem for hypernuclei and metal clusters employing the single particle (s.p.) potential which is assumed to be of the (S)WS shape. Our brief discussion is based on Refs. 85 and 121 where pertinent details are given.

We consider $\hbar\omega_\Lambda$ the HO energy level spacing of a Λ in hypernuclei. We should firstly point out that one could use for our purpose other simpler two-parametric potentials [85, 110] which, however, are less realistic compared with the (S)WS one in describing the surface of the system. Secondly, the way in determining $\hbar\omega_\Lambda$ is not unique and therefore one encounters with various possibilities in such an endeavour.

Let us write the s. p. potential in the form

$$V(r) = -V_0 f(r), \qquad 0 \leq r < \infty \tag{5.57}$$

and the HO one as

$$V_{\mathrm{HO}} = -V_0 + V_0 \frac{r^2}{R_0^2}, \qquad 0 \leq r < \infty. \tag{5.58}$$

Then, given the Λ-nucleus potential $V(r)$, we require, in order to approximate it by the HO one in the nuclear interior and as far as possible in the nuclear surface, «V_{HO} to approximate best in the mean the $V(r)$ in the interval $0 \leq r \leq R_0$». Thus, we require the following condition to be satisfied

$$\int_0^{R_0} |V(r) - V_{\mathrm{HO}}(r)|^2 dr = \min. \tag{5.59}$$

The optimum value $R_0 = R_m$ should satisfy the conditions

$$\left[\frac{8}{15} + f^2(R_m)\right] R_m^3 = 4\int_0^{R_m} f(r) r^2 dr \tag{5.60}$$

and

$$f(R_m) \;<\; \left[\frac{2}{5} + \frac{3}{2} f^2(R_m)\right] + \frac{R_m}{4}\frac{df^2(R_0)}{dR_0}|_{R_0=R_m}. \tag{5.61}$$

In case of the (S)WS potential, neglecting terms which are expected to be small, one is led to [85]

$$R_m = \left(\frac{5}{2}\right)^{1/3} R\left[1 + \left(\frac{\pi a}{R}\right)^2\right]^{1/3}, \tag{5.62}$$

from which the expression of $\hbar\omega$ follows, since

$$\hbar\omega_\Lambda = \left[\frac{\hbar^2}{\mu} 2V_0\right]^{1/2} \frac{1}{R_m}. \tag{5.63}$$

It is seen that $\hbar\omega_\Lambda$ depends on the expression for R considered for the SWS potential. If the complex relation (5.46) is used, then $\hbar\omega_\Lambda$ is simplified within this treatment,

$$\hbar\omega_\Lambda = \left[\frac{\hbar^2}{\mu}\frac{2V_0}{r_0^2}\right]^{1/2} \left(\frac{2}{5}\right)^{1/3} \frac{1}{A_c^{1/3}}. \tag{5.64}$$

Expressions of this or other forms (see Ref. 85) can be helpful in miscellaneous problems of hypernuclear physics [122, 123] where $\hbar\omega_\Lambda$ occurs.

The corresponding result for metal clusters in this sort of treatment looks rather similar

$$\hbar\omega = \left[\frac{\hbar^2}{m_e}\frac{2V_0}{r_0^2}\right]^{1/2} \left(\frac{2}{5}\right)^{1/3} \frac{1}{(\beta + N)^{1/3}}, \tag{5.65}$$

cf. [121] where one can see that the origin of β is due to the respective formula for the potential radius R.

6. SUMMARY AND CONCLUSIONS

In this last section we give a summary of some basic topics discussed in this review, along with certain remarks.

A considerable part of the survey was devoted to the evaluation of the F and SF integrals (including the more general case, in which the limits of integration were from R_i to R_f, $R_i < c < R_f$) on the basis of the Sommerfeld approximation and pertinent expansions in powers of the diffuseness parameter a were given. Results can be presented in a unified way for the S and SF functions. Furthermore, the interesting problem of the evaluation of F-type integrals beyond the Sommerfeld approximation that is when the Sommerfeld lemma fails (particularly when the integrand in the Fermi-type integrals contains a rapidly oscillating function) was considered. A pertinent treatment based on Fourier transforms and the properties of the hypergeometric function, was described and the systematic evaluation of the correction terms of any order in $\exp(-c/a)$ was pointed out. Special attention was also paid to the Dingle representation for the Fermi function.

In the framework of the conventional model of the S matrix it has been shown how the elastic scattering amplitude can be divided into the diffractive and nondiffractive contributions. The former oscillate with a high frequency $L \sim kR \gg 1$ (the «edge» effect). The latter have smooth angular dependence and are due to the diffuseness of nuclear boundary (the «rim» effect).

The nondiffractive contributions discussed here can be classified in powers of an «effective» parameter $\exp(-L/\Delta)$. Although this parameter is exponentially small under the strong absorption condition $L/\Delta \gg 1$, their relative role increases with θ against the rapid decrease of the magnitude of the diffractive contributions. Of course, it is not compulsory that the magnitude of nondiffractive contributions to the diffraction scattering amplitude to be regulated by the exponent $\exp(-L/\Delta)$. However, a similar interplay between them and the corresponding diffractive contributions should be retained in any SAM with smooth cutoff (a black disk with a grey, partially transparent edge).

Among the applications of Fermi-type functions in studying the properties of nuclei, hypernuclei and metal clusters, the variation of the HO energy level spacing $\hbar\omega$ with the particle number was discussed in some detail. Correction terms to the standard formula for a nucleon in nuclei $\hbar\omega = \mathrm{const}\, A^{-1/3}$ were given first, on the basis of a Fermi function for the nuclear density (and an approximate treatment of the open shell nuclei). An analogues treatment for metal clusters (neutral sodium clusters) was also made by parametrizing through a Fermi function, Eckardt's electronic densities in his self-consistent jelium model and local density approximation calculations.

Furthermore, a comparative discussion of the Woods–Saxon-type potentials was made and a way in estimating the corresponding parameters was described. Attetion was also paid to the analytic determination of the ground (1s) state energy of a particle in the WS potential and application to hypernuclei was made allowing also for an expression for the potential radius R somewhat more general than the simple one $R = r_0 A_c^{1/3}$. Finally, the problem of the determination of $\hbar\omega$

was also discussed on the basis of a (S)WS potential and under the assumption of an integral constraint.

We conclude by emphasizing that the usefulness of the Fermi and symmetrized Fermi functions in describing surface effects in various problems of nuclear physics and related areas has stimulated theoretical work in treating them analytically, as far as possible. Here we have tried to show that such analytic treatments have considerable advantages.

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REFERENCES

1. *Wilson A.H.* The Theory of Metals. Cambridge: Cambridge University Press, 1960.
2. *Wasserman A. et al.* // J. Math. Phys. 1970. V.11. P.477.
3. *Hofstadter R.* Nuclear and Nucleon Structure. N.Y., 1963. P.315.
4. *Elton R.L.B.* Nuclear Sizes. Oxford University Press, 1960.
5. *Uberall H.* Electron Scattering from Complex Nuclei. Part A. N.Y.: Academic Press, 1971.
6. *Barett R., Jackson D.* Nuclear Sizes and Structure: Intern. Series of Monographs on Physics. Oxford: Clarendon Press, 1977.
7. *Petkov I.Zh., Lukyanov V.K., Pol Yu.S.* // Sov. J. Nucl. Phys. 1967. V.4. P.41.
8. *Bohr A., Mottelson B.* // Nuclear Structure: Advanced Book Program Reading. Benjamin; Massachusetts, 1969. V.1.
9. *Blair J.S.* // Phys. Rev. 1954. V.95. P.1218.
10. *McIntyre J.A., Wang K.H., Becker L.C.* // Phys. Rev. 1960. V.117. P.1337.
11. *Frahn W.E., Venter R.H.* // Ann. Phys. 1963. V.24. P.234.
12. *Inopin E.V.* // Sov. Phys. JETP. 1965. V.21. P.1090.
13. *Ericson T.E.O.* Preludes in Theoretical Physics / Ed. A. De-Shalit, H. Feshbach, L. Van Hove. Amsterdam, 1966. P.321.
14. *Shebeko A.V.* // Sov. J. Nucl. Phys. 1967. V.5. P.543.
15. *Eldyshev Yu.N., Lukyanov V.K., Pol Yu.S.* // Yad. Fiz. 1972. V.16. P.506.
16. *Burov V.V. et al.* JINR Preprint E4-8029. Dubna, 1972.
17. *Buck B., Pilt A.* // Nucl. Phys. A. 1977. V.280. P.133.
18. *Grypeos M.E. et al.* // J. Phys. G: Nucl. Part. Phys. 1991. V.17. P.1093.
19. *Lukyanov V.K.* // J. Phys. G: Nucl. Part. Phys. 1995. V.21. P.145.
20. *Landau L.D., Lifshitz E.M.* Course of Theoretical Physics, Statistical Physics. N.Y.: Pergamon, 1959. V.5.
21. *Levin S.N.* Quantum Physics of Electronics. N.Y.: Macmillan, 1965.

22. *Grypeos M. et al.* // J. Phys. G: Nucl. Part. Phys. 1998. V.24. P.1913.
23. *Sprung D.W.L., Martorell J.* // J. Phys. A: Math. Gen. 1997. V.30. P.6525; ibid. 1998. V.31. P.8973.
24. *Sommerfeld A.* // Z. Phys. 1928. V.47. P.1.
25. *Bateman H.* Higher Transcendental Functions / Ed. A. Erdelyi. London: McGraw-Hill, 1953. V.I.
26. *Gradshteyn I., Ryzhik I.* Tables of Series, Products and Integrals. 1987 (H. Deutsh) (Engl. transl.).
27. *Gel'fand I.M., Shilov G.E.* Generalized Functions. M.: Fizmatgiz, 1958. V.1.
28. *Bateman H.* Higher Transcendental Functions / Ed. A. Erdelyi. London: McGraw-Hill, 1953. V.II.
29. *Khin Maung Maung. Deutehman P.A., Royalts W.D.* // Can. J. Phys. 1989. V.67. P.95.
30. *Treiner J., Krivine H., Bohigas O.* // Nucl. Phys. A. 1981. V.371. P.253.
31. *Krivine H., Treiner J.* // J. Math. Phys. 1981. V.22. P.2484.
32. *Velicia F.* // J. Phys. A: Math. Gen. 1987. V.20. P.2293.
33. *Shebeko A.V.* // Yad. Fiz. 1977. V.25. P.747.
34. *Kotlyar V.V., Shebeko A.V.* // Yad. Fiz. 1981. V.34. P.370.
35. *Dingle R.* // Appl. Sci. Res. B. 1956. V.6. P.225.
36. *Glauber R.J.* Lectures in Theor. Phys. N.Y., 1959. P.315.
37. *Sitenko A.G.* // Ukrainian Phys. J. 1959. V.4. P.152.
38. *Lavrent'ev M.A., Shabat B.V.* Methods in Theory of Functions of a Complex Variable. M., 1958.
39. *Lukyanov V.K., Pol Yu.S.* // Sov. J. Part. and Nucl. 1975. V.5. P.385.
40. *Burov V.V. et al.* // Phys. of At. Nucl. 1998. V.61. P.525.
41. *Rowley N., Marty C.* // Nucl. Phys. A. 1976. V.266. P.494.
42. *Anni R., Taffara L.* // Nuovo Cim. A. 1974. V.20. P.595.
43. *Choudhury D.C., Scura M.A.* // Phys. Rev. C. 1994. V.50.
44. *Akhiezer A.I., Berezhnoy Yu.A., Pilipenko V.V.* // Particles and Nuclei. 2000. V.31. P.458.
45. *Inopin E.V., Tischenko B.I.* // Sov. J. Nucl. Phys. 1968. V.7. P.618.
46. *Sommerfeld A.* Partial Differential Equations in Physics. N.Y.: Academic, 1949.
47. *Fuller R.C.* // Phys. Rev. C. 1975. V.12. P.1561.
48. *Fuller R.C., Moffa P.J.* // Phys. Rev. C. 1977. V.15. P.266.
49. *Anni R., Renna L., Taffara L.* // Nuovo Cim. A. 1980. V.55. P.456.
50. *Frahn W.E.* Diffractive Processes in Nuclear Physics. Oxford: Clarendon Press, 1985.
51. *Tischenko B.I., Shebeko A.V.* // JETP. 1966. V.50. P.1674.
52. *Inopin E.V., Shebeko A.V.* // Sov. J. Nucl. Phys. 1970. V.11. P.78.
53. *Evgraffov M.* Analytical Functions. M.: Nauka, 1965.
54. *Titsmarsh E.S.* Introduction to the Theory of Fourier Integrals. Oxford: Clarendon Press, 1948.
55. *Levy V., Keller M.* // Communs. Pure and Appl. Math. 1959. V.12. P.159.
56. *Shebeko A.V., Zemlyanaya E.V.* JINR Preprint P4-2000-115. Dubna, 2000.
57. *Frahn W.E.* // Ann. Phys. 1966. V.75. P.577.

58. *Shebeko A.V.* // Proc. of the 8th Inten. Conf. «On Nuclear Reaction Mechanisms» / Ed. E. Gadioli. Varenna, 1997. P.483.

59. *Kotlyar V.V., Shebeko A.V.* // Bull. Russ. Acad. Sci. 1999. V.63, No.1. P.188.

60. *Daskaloyannis C.B. et al.* // Phys. Lett. B. 1983. V.121. P.91.

61. *Moszkowski S.A.* // Handb. Phys. 1957. V.39. P.470.

62. *De Shalit A., Feshbach H.* Theoretical Nuclear Physics. N.Y., 1974. V.I. P.197.

63. *Hornyak W.F.* Nuclear Structure. N.Y.: Academic, 1975.

64. *Bethe H.* Private communication to J. Negele // Phys. Rev. C. 1970. V.1. P.1260.

65. *Chandra H., Sauer G.* // Phys. Rev. C. 1976. V.13. P.245.

66. *Shlomo S., Riska D.U.* // Nucl. Phys. A. 1975. V.254. P.281.

67. *De Vries H., de Jager C.W., de Vries C.* // At. Data Nucl. Data Tables. 1987. V.36. P.495.

68. *Brown B.A., Massen S.E., Hodgson P.E.* // J. Phys. 1979. V.5. P.1655.

69. *Grypeos M.E., Panos C.P.* // Int. J. Theor. Phys. 1987. V.26. P.583.

70. *Lalazissis G.A., Panos C.P.* // Z. Phys. A. 1992. V.344. P.17.

71. *Lalazissis G.A., Panos C.P.* // J. Phys. G: Nucl. Part. Phys. 1993. V.19. P.283.

72. *Lalazissis G.A., Panos C.P.* // Phys. Rev. C. 1995. V.51. P.1247.

73. *Lalazissis G.A., Panos C.P.* // Int. J. Mod. Phys. E. 1996. V.5. P.669.

74. *Gambhir Y.K., Patil S.H.* // Z. Phys. A. 1984. V.321. P.161.

75. *Gambhir Y.K., Patil S.H.* // Z. Phys. A. 1986. V.324. P.9.

76. *Angeli I. et al.* // J. Phys. G. 1980. V.6. P.303.

77. *Kotsos B.A., Grypeos M.E.* // Proc. of 9th Intern. Symposium «Small Particles and Inorganic Clusters»: Book of Abstr. Lausanne, 1998. P.219.

78. *Kotsos B.A., Grypeos M.E.* // Int. J. of Quant. Chem. (in press).

79. *Ekardt W.* // Phys. Rev. B. 1984. V.29. P.1558.

80. *Hodgson P.E.* Nuclear Reactions and Nuclear Structure. Oxford: Clarendon Press, 1971;
Irvine J.M. Nuclear Structure Theory. Oxford: Pergamon, 1972;
Soloviev V.G. Theory of Complex Nuclei. Oxford: Pergamon, 1976.

81. *Mang H.J.* // Proc. of Winter College on Fundamental Nuclear Physics, Trieste, 1984 / Eds. K. Dietricht, M. di Toro, H. J. Mang. Singapore, 1985. P.15.

82. *Buck B., Dover C.B., Vary J.P.* // Phys. Rev. C. 1975. V.11. P.1803.

83. *Carbonell J. et al.* // J. Phys. G: Nucl. Phys. 1985. V.11. P.325.

84. *Carbonell J.* These de 3eme cycle Universite de Grenoble. 1983.

85. *Grypeos M.E. et al.* // Nuovo Cim. A. 1989. V.102. P.445.

86. *Daskaloyannis C. et al.* // Lett. Nuovo Cim. 1985. V.42. P.257.

87. *Kodama T.* // Prog. Theor. Phys. 1971. V.45. P.1112.

88. *Elton L.R.B., Swift A.* // Nucl. Phys. A. 1967. V.94. P.52.

89. *Inopin E., Shebeko A.* // Ukr. Phys. J. 1964. V.9. P.1161.

90. *Sitenko A., Tartakovski V.* Lectures on Nuclear Theory. Oxford: Pergamon, 1975.

91. *Salcedo L.L. et al.* // Nucl. Phys. A. 1988. V.484. P.557; *Oset E., Salcedo L.L., Brockmann R.* // Phys. Rep. 1990. V.188. P.79.

92. *Satchler G.R.* Direct Nuclear Reactions. Oxford: Clarendon Press, 1983.
93. *Kolesnikov N.N., Chernov S.M.* // Sov. J. Nucl. Phys. 1976. V.23. P.505.
94. *Downs B.W., Kunz P.D.* // Proc. of Intern. Conf. on Hypernuclear Physics / Eds. A. R. Bodmer, C. G. Hyman. Argonne, 1969. V.II. P.796.
95. *Dalitz R.H.* // Proc. of Summer School of Physics, Les Houches / Eds. de Witt, C. Gillet. N.Y., 1969. P.703.
96. *Stepien-Rudzka W., Wycech S.* // Nucleonika. 1977. V.22. P.929.
97. *Volkov A.B.* // Nucl. Phys. 1965. V.74. P.33.
98. *Jacob G., Maris J.A.J.* // Rev. Mod. Phys. 1973. V.45. P.6.
99. *Frullani S., Mougey J.* // Adv. Nucl. Phys. 1984. V.14. P.1.
100. *Collard H.R. et al.* Nuclear Radii (Landolt Bornstein). Berlin: Springer, 1967.
101. *Löwdin P.O.* // Phys. Rev. 1955. V.97. P.1490; *Brink D.M.* // Proc. of School of Physics of Enrico Fermi. Course V.36 / Ed. C. Bloch. N.Y., 1966. P.247.
102. *Malaguti F. et al.* // Nucl. Phys. A. 1978. V.297. P.287; *Malaguti F. et al.* // Riv. Nuovo Cimento. 1982. 5 Ser.3 (1). P.23.
103. *Mahaux C. et al.* // Phys. Rep. 1985. V.120. P.1.
104. *Belostotskii S.L. et al.* // Sov. J. Nucl. Phys. 1985. V.41. P.903.
105. *Bhowmik B., Chand T., Chopra D.V.* // Nuovo Cim. A. 1967. V.52. P.1375.
106. *Gal A.* // Adv. Nucl. Phys. 1975. V.8. P.1.
107. *Flügge S.* Practical Quantum Mechanics. Part 1. Berlin, 1971. P.162.
108. *Deloff A.* // Nucl. Phys. B. 1971. V.27. P.149.
109. *Walecka D.* // Nuovo Cim. 1960. V.16. P.342.
110. *Daskaloyanniset C.B. et al.* // Phys. Lett. B. 1984. V.134. P.147.
111. *Goyal D.P.* // Nucl. Phys. 1966. V.83. P.639; *Pniewski J., Ziemiska D.* // Proc. of Seminar on Kaon-Nuclear Interaction and Hypernuclei. Zvenigorod, 1977. P.33.
112. *Lalazissis G.A., Grypeos M.E., Massen S.E.* // J. Phys. G: Nucl. Part. Phys. 1990. V.17. P.1093.
113. *Povh B.* // Ann. Rev. Nucl. Part. Sci. 1978. V.28. P.1; *Chrien R.E.* // Nucl. Phys. A. 1988. V.478. P.705c.
114. *Millener D.J., Dover C.B., Gal A.* // Phys. Rev. C. 1988. V.38. P.2700.
115. *Lalazissis G.A.* // J. Phys. G: Nucl. Part. Phys. 1993. V.19. P.695.
116. *Koutroulos C.* // Z. Naturforsch. 1989. V.44a. P.1234; ibid. 1990. V.45a. P.191; J. Phys. G: Nucl. Part. Phys. 1989. V.15. P.1659; Proc. of the 5th Workshop «Perspectives in Nuclear Physics at Intermediate Energies» / Eds. S. Boffi, C. Ciofi degli Atti, M. Giannini. Trieste, 1991. P.408.
117. *Knight W.D. et al.* // Phys. Rev. Lett. 1984. V.52. P.2141.
118. *Katakuse I. et al.* // Int. J. Mass Spectr. Ion Processes. 1986. V.67. P.229; ibid. 1986. V.69. P.109.
119. *Nishioka H., Hansen K., Mottelson B.R.* // Phys. Rev. B. 1990. V.42. P.9377.
120. *Kotsos B.A., Grypeos M.E.* // Proc. of Intern. Conf. «Atomic and Nuclear Clusters», Santorini, 1993 / Eds. G. Anagnostatos, W. Von Oerzen. Berlin; Heidelberg, 1995. P.242; *Kotsos B.A., Grypeos M.E.* // Physica B. 1997. V.229. P.173; see also Ref. 2 in [77].
121. *Grypeos M.E., Kotsos B.A.* // J. Phys. B: At. Mol. Opt. Phys. 1996. V.29. P.L473.
122. *Grypeos M.E. et al.* // J. Phys. G: Nucl. Part. Phys. 1990. V.16. P.1627.
123. *Karpeshin F.F., Koutroulos C.G., Grypeos M.E.* // Nucl. Phys. A. 1995. V.595. P.209.