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JORDAN–BRANS–DICKE THEORY AS GENERALIZATION OF EINSTEIN THEORY OF GRAVITATION

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Different representations of JBD theory arising at conformal transformations of metrics are considered. Propositions establishing mathematical equivalence of these representations, which allows one to generate new solutions in other representations based on known exact solutions in one of the representations, are formulated. It is shown, in particular, how one can obtain new solutions in GR based on known solutions in JBD theory of gravitation, and vice versa.

INRODUCTION

Einstein's relativistic ideas and the idea of Minkowski that physical events occur in the four-dimensional space-time enabled unification of the electric and magnetic fields of the classical electrodynamics. On creation of the general theory of relativity, the quest for unification of the then known two types of interactions, electromagnetic and gravitational, was natural. Kaluza [1] inferred that the expression for the electromagnetic field tensor

$$F_{\alpha\beta} = \left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right)$$

(the Greek indices run the values 0, 1, 2, 3; A is the 4-potential of the electromagnetic field) will be contained in the definition of the Christoffel symbols

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right),$$

if, following Minkowski, an additional fifth dimension is introduced in the theory. In such a five-dimensional manifold with the metric form

$$ds^2 = G_{AB}dx^A dx^B, \qquad A, B = \mu, 5,$$

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where

$$G_{AB} = \begin{pmatrix} g_{\mu\nu} & g_{\mu 5} \\ g_{5\nu} & g_{55} \end{pmatrix},$$

and under the assumption that

$$g_{55} = \text{const}, \quad \text{and} \quad \frac{\partial(\dots)}{\partial x_5} = 0.$$

Kaluza found

1. for the electromagnetic field tensor:

$$F_{\alpha\gamma} = \Gamma_{\alpha 5\gamma} = \left(\frac{\partial A_\gamma}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\gamma}\right), \quad \text{if} \quad A_\alpha = \frac{1}{2} g_{\alpha 5},$$

2. for the action of the theory:

$$W = \int \sqrt{-^{(5)}g}\left(-\frac{^{(5)}R}{16\pi G} + L_m\right) d^5x,$$

where

$$^{(5)}R = {}^{(4)}R + GF^{\mu\nu}F_{\mu\nu}$$

and, as a result of vanishing the variations W with respect to $g_{\mu\nu}$ and A_μ, ten field equations of the Einstein theory of gravitation and the system of Maxwell equations.

Considering the transformation properties of the five-dimensional Kaluza–Klein theory [1, 2] with respect to the unified group of gauge and general coordinate transformations

$$K \cup G_4 \longrightarrow G_5$$

Jordan (see, e. g., [3]) concluded that under such transformations g_{55} turns out to be not a constant, as Kaluza thought, but rather a scalar

$$\acute{g}_{55}(\acute{x}) = g_{55}(x) \equiv y(x),$$

$$G_{AB} = \begin{pmatrix} g_{\mu\nu} & g_{\mu 5} \\ g_{5\nu} & y(x^\alpha) \end{pmatrix}.$$

In realization of Dirac hypothesis («decrepited» gravitation) [4, 5] Jordan assumes that in each space-time point, the scalar $y(x^\alpha)$ substitutes gravitation constant, whereas $y \sim 1/G$, and formulates a theory of gravitation, which is different from GR:

$$W = \int \sqrt{-g}\left[-\frac{y}{16\pi}\left({}^{(4)r}R - \zeta\frac{y^\mu y_\mu}{y^2}\right) + L_m\right] d^4x, \tag{1}$$

$$G^{\mu}_{\nu} = \frac{8\pi}{y} T^{\mu}_{\nu} + \frac{\nabla_{\nu} y^{\mu}}{y} + \zeta \frac{y_{\nu} y^{\mu}}{y^2} - \delta^{\mu}_{\nu} \left(\frac{\nabla_{\alpha} y^{\alpha}}{y} + \frac{\zeta}{2} \frac{y_{\alpha} y^{\alpha}}{y^2} \right), \tag{2}$$

$$\nabla_{\alpha} y^{\alpha} = \frac{8\pi T}{3 + 2\zeta}. \tag{3}$$

Here ζ is a dimensionless coupling constant of theory. In 1961 Dicke and Brans [6, 7] utilized the heuristic idea of Mach on the influence of remote masses on the origin of inertia and formulated a theory with the same field equations. The course of their argumentation is based of Sciama [8] relation, which easily results from simple estimation reasonings: On the one hand, a particle being at a distance r from the centre of the body of mass m falls on it acceleration $a = Gm/r^2$; on the other hand, assumming, following the Mach principle, that acceleration depends not only on the gravitational effect of fixed bodies, but also on the distribution of matter in the vicinity of a probe particle, which gives $a \sim mc^2 R/Mr^2$, where M is the mass of the Universe observable part and R is its radius, we obtain after the comparison of these expressions

$$\frac{GM}{c^2 R} \sim 1.$$

The latter can be rewritten in an equivalent form

$$\frac{1}{G(r)} = \sum \frac{m_i}{c^2 (r - r_i)}.$$

Thus, in Jordan–Brans–Dicke theory (1)–(3) of gravity, scalar is created by matter and nongravitational fields, and more precisely is obeying a type of wave equation with a source as a trace of energy-momentum of the matter and nongravitational fields. The influence of scalar field on the movement of particles is manifested not through direct interaction, but thanks to change of metric tensor caused by that field. In this case one of achievements of GR is saved: it follows from field equations that the covariant divergency of energy-momentum tensor of matter and nongravitational fields becomes zero,

$$\nabla_{\mu} T^{\mu}_{nu} = 0,$$

which ensures compliance with the requirements of week principle of equivalence — test uncharged, spinless particles and rays of light move along geodesic lines. The Jordan–Brans–Dicke theory satisfied the following condition, it is necessary, according to Dicke, for a gravitational theory to be true:

1. The theory has to be complete, i. e., the analysis of the results of any trustworthy experiments, can be conducted on the basis of «first principles».

2. The theory has to be self-agreed, i. e., prediction of the results of any experiment, received through different means shall coincide.

3. The theory has to be relativistic, i. e., nongravitational laws of physics shall be reduced to relations of Special Theory of Relativity.

4. The theory has to be true in Newtonian limit.

In recent years, the number of papers related to the JBD theory considerably increased due to a renewed interest in gravitation theories with a scalar field. We think, there are several reasons for that:

1. Renormalization of gauge theories with spontaneous symmetry breakdown is ensured only if a scalar (Higgs) field is introduced.

2. Supersymmetrical theories reveal the existence of scalar fields with whole (0) and half-integer (3/2) spins.

3. In Friedman cosmological model, expansion is described by scale factor $a(t)$ which characterizes the change of the distance between cosmic objects, depending on time, while $a \sim t^{1/2}$ or $a \sim t^{2/3}$. In other words, the cosmic scale factor increases slow enough. However, if one extrapolates it for the past, then too large values $a(t)$ correspond to small values of t from the point of view of modern observations, that is why the classical Friedman model cannot be extrapolated for the early stage of evolution of Universe. On the other hand, Friedman model survived the test on correspondence with observation data, that is why there is no necessity to modify the model itself, but it is necessary to limit its applications with reasonable values of t from the low end: Friedman model is in compliance with observations, if one takes into account, that the Universe evolution takes place in accordance with prescriptions of that model starting from $t \sim 10^{-34}$ s. New model, which works in a gap between Planck $t_{\rm pl} \sim 5 \cdot 10^{-43}$ s until that moment, shall give the initial conditions of standard Friedman model, whereas the expansion regime has to be different (power or exponential).

4. Power or exponential («inflational») growth of scale factor permits the fields which imitate the matter, the state of which is simulated by the equation $P = -\varepsilon$ (P — pressure, ε — density of energy). Indeed, in this case the source of gravitational field is not the energy, but also the pressure, which brings to gravitational repulsion and, thus, ensures inflational regime. In this case the energy density remains unchanged, because the work of the forces of pressure compensated the loss of energy in the process of exponential expansion.

5. At his time Einstein introduced the so-called cosmological constant Λ into the equations of his theory of gravitation. However, after Friedman's work, and, especially after Hubble's discovery, he considered the introduction of Λ as a big mistake. At present many researchers are tended to the necessity to reanimate this parameter, because it is not difficult to prove, that the cosmological constant in Einstein equations mimics the equation of state for matter with $P = -\varepsilon$. Indeed, if $P = \Lambda c^4/8\pi G$ and $P = -\varepsilon$, then $T_{\mu\nu} = (\varepsilon + P)u_\mu u_\nu - P g_{\mu\nu} = \Lambda g_{\mu\nu}$. Thus, introduction of Λ allows inflationary stage of the evolution of universe.

1. NEWTONIAN LIMIT OF JBD THEORY

For slow motion $v^2/c^2 \ll 1$ and weak gravitational fields $Gm/c^2r \ll 1$

$$g_{00} \approx 1 + \frac{2\varphi}{c^2} = 1 - \frac{2Gm}{c^2 r}, \quad g_{0i} = 0, \quad g_{ik} = \mathrm{diag}\,(-1,-1,-1).$$

If one looks for solution of JBD theory equation in the form

$$y \approx y_0(1+\sigma), \qquad g_{00} \approx 1 + g,$$

then

$$\triangle\sigma = 4\pi\left(-\frac{2}{y_0(3+2\zeta)}\right)\rho, \quad \triangle g = 4\pi\left(-\frac{4(2+\zeta)}{y_0(3+2\zeta)}\right)\rho.$$

At the condition $\sigma, g \sim 0(1/r)$, the solution of these equations is:

$$\sigma(\mathbf{r},t) = \frac{2}{y_0(3+2\zeta)} I, \quad g(\mathbf{r},t) = -\frac{4(2+\zeta)}{y_0(3+2\zeta)} I.$$

Comparing it with the solution of Newtonian potential

$$\varphi(\mathbf{r},t) = -GI, \quad I = \int \frac{\rho(\acute{\mathbf{r}},t)d\acute{V}}{\mid \mathbf{r} - \acute{\mathbf{r}} \mid}$$

we'll find

$$y_0 = \frac{2(2+\zeta)}{G(3+2\zeta)}, \qquad \sigma = -\varphi/(2+\zeta), \quad g = -\frac{2Gm}{c^2 r}.$$

2. CONFORMAL CORRESPONDENCE OF THE JBD AND GR THEORIES

Let two conformly connecting spaces V_4 and $\bar{V}_4$ are given in the same manifold, which have Riemann structure

$$d\bar{s}^2 = \sigma^2(x) ds^2 = \sigma^2(x) g_{\mu\nu} dx^\mu dx^\nu, \qquad \bar{g}_{\mu\nu} = \sigma^2(x) g_{\mu\nu}. \tag{4}$$

One can connect local conform transformations with utilization of different systems of units of measurement of physical parameters [9–11]. It is natural to assume, that

$$\bar{c} = c, \qquad \bar{\hbar} = \hbar. \tag{5}$$

Let us assume also that

$$A_\mu = \bar{A}_\mu \quad (\text{but } \bar{A}^\mu = \bar{g}^{\mu\nu}\bar{A}_\nu = (\sigma)^{-2} A^\mu).$$

On the other hand,

$$u^\mu = \frac{dx^\mu}{ds} = \frac{dx^\mu}{\sigma^{-1}d\bar{s}} = \sigma \bar{u}^\mu, \qquad u_\mu = g_{\mu\nu}u^\nu = \sigma^{-1}\bar{u}_\mu.$$

For the units of measurement of:

distances — $\bar{l} = \sigma l$,
time — $\bar{t} = \sigma t$,
mass — $\bar{m} = \sigma^{-1}m$,
energy density — $\bar{\varepsilon} = \sigma^{-4}\varepsilon$, etc.

Let us assume, that metric tensor of space V_4 obeys the equations of tensor-scalar theory of gravitation, which are obtained from variation of the action

$$W = \int \sqrt{-g}\Big[-F(\phi)R + \frac{1}{2}\Phi(\phi)g^{\mu\nu}\phi_\mu\phi_\nu + L_m\Big]d^4x. \tag{6}$$

Let us move into conformly connected space $\bar{V}_4$ in accordance with

$$\bar{g}_{\mu\nu} = \frac{F(\phi)}{F_0}g_{\mu\nu}, \qquad F_0 = \text{const},$$

then

$$\bar{W} = \int \sqrt{-\bar{g}}\Big[-F_0\bar{R} + \frac{1}{2}\bar{g}^{\mu\nu}\psi_\mu\psi_\nu + \bar{L}_m\Big]d^4x, \tag{7}$$

where

$$\psi_\alpha = \phi_\alpha\sqrt{3F_0\frac{\acute{F}^2}{F^2} + F_0\frac{\Phi}{F}}, \qquad \acute{F} = \frac{\partial F}{\partial \phi}.$$

Corresponding equations have the following form:

$$\bar{G}_{\alpha\beta} = \frac{1}{2F_0}(\bar{T}^m_{\alpha\beta} + \bar{T}^s_{\alpha\beta}), \qquad \bar{g}^{\alpha\beta}\nabla_\alpha\psi_\beta = 0,$$

$$\bar{T}^s_{\alpha\beta} = \psi_\alpha\psi_\beta - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}\psi_\mu\psi_\nu.$$

Let us select $F_0 = 1/2\kappa_0 = c^3/16\pi G$, now we can formulate

Statement 1. *Tensor-scalar theories of gravitation (6) in conformly connected space with a metric tensor*

$$\bar{g}_{\mu\nu} = (F(\phi)/F_0)g_{\mu\nu},$$

are conformly equivalent to GR in a form of minimally linked scalar field.

Let now $g_{\mu\nu}(x)$ of the space V_4 is obeying to the equations of JBD theory

$$G^\mu_\nu = \frac{8\pi}{y}T^\mu_\nu + \frac{\nabla_\nu y^\mu}{y} + \zeta\frac{y_\nu y^\mu}{y^2} - \delta^\mu_\nu\left(\frac{\nabla_\alpha y^\alpha}{y} + \frac{\zeta}{2}\frac{y_\alpha y^\alpha}{y^2}\right), \tag{8}$$

$$\nabla_\alpha y^\alpha = \frac{8\pi T}{3+2\zeta} \tag{9}$$

or in equivalent form

$$R^\mu_\nu = \frac{8\pi}{y}\left[T^\mu_\nu - \delta^\mu_\nu\frac{1+\zeta}{3+2\zeta}T\right] + \frac{\nabla_\nu y^\mu}{y} + \zeta\frac{y_\nu y^\mu}{y^2},$$

$$R = -\frac{16\pi}{y}\frac{\zeta}{3+2\zeta}T + \zeta\frac{y_\alpha y^\alpha}{y^2}.$$

Here ζ is the dimensionless coupling constant of JBD theory.

Equations (8) and (9) can be obtained through variation of action

$$W = \int\sqrt{-g}\left[-\frac{y}{16\pi}\left(R - \zeta\frac{y^\mu y_\mu}{y^2}\right) + L_m\right]d^4x \tag{10}$$

by $g_{\mu\nu}$ and $y(x)$. Let us note, that action (10) is a special case of (6), if one puts $F = y/16\pi$, ($F_0 = y_0/16\pi \equiv 1/2\kappa$), $\Phi = \zeta y/8\pi$, $\phi_\mu = y_\mu/y$ and is transformed into Gilbert–Einstein action at $y = y_0$ and $\zeta \to \infty$.

Let us move into conformly corresponding space $\tilde{V}_4$, the metric tensor of which

$$\tilde{g}_{\mu\nu} = \left(\frac{y}{y_0}\right)^k g_{\mu\nu}. \tag{11}$$

In this case

$$\tilde{W} = \int\sqrt{-\tilde{g}}\left[-\frac{1}{2\kappa}\left(\frac{y}{y_0}\right)^{1-k}\left(\tilde{R} - \frac{A}{2}\tilde{g}^{\alpha\beta}\frac{y_\alpha y_\beta}{y^2}\right) + \tilde{L}_m\right]d^4x. \tag{12}$$

Here a notation is introduced

$$A = (3+2\zeta) - 3(1-k)^2, \qquad k = 1 - \sqrt{\frac{3+2\zeta-A}{3}}. \tag{13}$$

Variation of (12) by $\tilde{g}_{\alpha\beta}$ and y brings to the equations

$$\tilde{G}^\alpha_\beta = \kappa\left(\frac{y}{y_0}\right)^{k-1}\tilde{T}^\alpha_\beta + \left(\frac{A}{2} - k(1-k)\right)\frac{y_\beta y^{\tilde{\alpha}}}{y^2} + \\ + \delta^\alpha_\beta(k(1-k) - A/4)\frac{y_\mu y^{\tilde{\mu}}}{y^2} + \frac{1-k}{y}[\tilde{\nabla}_\beta y^{\tilde{\alpha}} - \delta^\alpha_\beta\tilde{\nabla}_\mu y^{\tilde{\mu}}], \tag{14}$$

$$(1-k)\tilde{R} - A(1+k)\frac{y_\alpha y^{\tilde{\alpha}}}{2y^2} + A\frac{\tilde{\nabla}_\alpha y^{\tilde{\alpha}}}{y} = 0. \tag{15}$$

Combining the convolution of equation (14)

$$-\tilde{R} = \kappa\Big(\frac{y}{y_0}\Big)^{k-1}\tilde{T} - \left(\frac{A}{2} - 3k(1-k)\right)\frac{y_\alpha y^{\tilde{\alpha}}}{y^2} - 3(1-k)\frac{\tilde{\nabla}_\alpha y^{\tilde{\alpha}}}{y} \tag{16}$$

with (15) we obtain equation, which defines gravitational scalar

$$\frac{\tilde{\nabla}_\alpha y^{\tilde{\alpha}}}{y} - k\frac{y^{\tilde{\alpha}} y_\alpha}{y^2} = \kappa(1-k)\Big(\frac{y}{y_0}\Big)^{k-1}\frac{\tilde{T}}{3+2\zeta}. \tag{17}$$

Thus, as a result of conform transformations (11) field equations (8) and (9) in the space $\tilde{V}_4$ are being transformed into (14) and (17).

Comment 1. Let us redefine the gravitational scalar of JBD theory in a way, that

$$\frac{\tilde{y}}{y_0} = \Big(\frac{y}{y_0}\Big)^{1-k} \tag{18}$$

and let us introduce a dimensionless coupling constant

$$\tilde{\zeta} = \frac{A}{2(1-k)^2} = -\frac{3}{2} + \frac{3+2\zeta}{2(1-k)^2}. \tag{19}$$

In new notations action (10) obtains the form

$$\tilde{W} = \int \sqrt{-\tilde{g}}\Big[-\frac{\tilde{y}}{16\pi}\left(\tilde{R} - \tilde{\zeta}\frac{\tilde{y}^\mu \tilde{y}_\mu}{\tilde{y}^2}\right) + \Big(\frac{\tilde{y}}{y_0}\Big)^{1-k^2} L_m\Big]d^4x. \tag{20}$$

Content of this comment proves,

Statement 2. *JBD theory equations are invariant with respect to conform transformations for any $k \neq 1$ if one redefines the gravitational scalar and redenote coupling constant in accordance with (18) and (19) (at $k = 2$, $\tilde{\zeta} = \zeta$).*

Einstein representation of JBD theory. Let us consider action (12) for the case, when the exponent of the power of conform factor in (11) $k = 1,\ A = 3+2\zeta$. Let us introduce

$$\phi_\alpha = \frac{y_\alpha}{y}\sqrt{\frac{(3+2\zeta)y_0}{16\pi}} \tag{21}$$

and let us rewrite action (12) in the form

$$\tilde{W} = \int \sqrt{-\tilde{g}}\Big[-\frac{\tilde{R}}{2\kappa} + \frac{1}{2}\tilde{g}^{\alpha\beta}\phi_\alpha\phi_\beta + \tilde{L}_m\Big]d^4x. \tag{22}$$

Here

$$2\kappa = \frac{16\pi}{y_0} = \frac{8\pi G(3+2\zeta)}{2+\zeta}. \tag{23}$$

The following equation corresponds to action (22)

$$\tilde{G}_{\alpha\beta} = \kappa\Big(\tilde{T}^{m}_{\alpha\beta} + \phi_\alpha\phi_\beta - \frac{1}{2}g_{\alpha\beta}\tilde{g}^{\mu\nu}\phi_\mu\phi_\nu\Big), \tag{24}$$

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\phi_\beta = 0. \tag{25}$$

Let us note, that

- The equation (25) arises as a consequence of covariant constancy of $G_{\mu\nu}$.
- Einstein gravitational constant is renormalized according to (23). Thus, one can consider, that the following was proved:

Statement 3. *JBD theory equations are transformed into GR equations as a result of conform transformation (11) with* $k = 1$, *redenoted Einstein gravitational constant and a source in a form of a sum of energy-momentum tensors of the matter, nongravitational fields and minimally coupled scalar field.*

In other words, *the conform transformations transfer JBD theory from its own representation into Einstein representation. Whereas, if in its own representation the gravitational constant G changes from point to point proportionally to gravitational scalar, but universal constants c, $\hbar$ and the mass of the particles remain unchanged, then in Einstein representation G, c and $\hbar$ are constant, but the masses in different space-time points are different* $\tilde{m} = (y/y_0)^{-1/2}m$.

Let us move into other space $\breve{V}_4$, according to the following relations:

$$\breve{g}_{\mu\nu} = \frac{1}{4}z^{(n+1)/n}[1+z^{-n}]^2 g_{\mu\nu}, \tag{26}$$

$$\psi = \frac{6}{\tilde{\kappa}}\frac{z^n-1}{z^n+1}, \qquad n = \sqrt{\frac{3+2\zeta}{3}}, \qquad z = (y/y_0)^n. \tag{27}$$

In this case (10) is transformed into

$$\breve{W} = \int\sqrt{-\breve{g}}\Big[-\frac{1}{2\kappa}\breve{R}\Big(1 - \frac{1}{6}\kappa\psi^2\Big) + \frac{1}{2}\breve{g}^{\alpha\beta}\psi_\alpha\psi_\beta + \breve{L}_m\Big]d^4x, \tag{28}$$

which proves the following

Statement 4. *The JBD equations are transformed into GR equations by conform transformation (26), (27) with a source in a form of nongravitational* *fields and conformly coupled massless scalar field ψ, which satisfies to Penrose–Chernikov–Tagirov equation:*

$$\breve{g}^{\alpha\beta}\breve{\nabla}_\alpha\psi_\beta - \frac{1}{6}\breve{R}\psi = 0. \tag{29}$$

Comment 2. Let us neglect the term $(-\breve{R}/2\kappa)$ in action (28) and let us add Higgs potential $(\lambda\psi^4/12)$. Let us transform conformly the resulting expression

$$\breve{W} = \int \sqrt{-\breve{g}}\Big[\frac{1}{12}\breve{R}\psi^2 + \frac{1}{2}\breve{g}^{\alpha\beta}\psi_\alpha\psi_\beta - \frac{1}{12}\lambda\psi^4 + \breve{L}_m\Big]d^4x, \tag{30}$$

selecting conform factor in a way, that, as a result, the potential of scalar field would be transformed into constant and «kinetic» term would be absorbed by additions, conditioned by conformal factor. The following transformations satisfy to those conditions:

$$\hat{\psi} = \frac{\psi}{\chi} = \epsilon = \text{const}, \qquad \psi = \epsilon\chi, \qquad \hat{g}_{\alpha\beta} = \chi^2\breve{g}_{\alpha\beta}. \tag{31}$$

Let us introduce also

$$\hat{\kappa} = -\frac{6}{\epsilon^2} \quad \text{and} \quad \Lambda = \frac{1}{2}\lambda\epsilon^2,$$

then

$$\hat{W} = \int \sqrt{-\hat{g}}\Big[-\frac{1}{2\hat{\kappa}}(\hat{R} + 2\Lambda) + \hat{L}_m\Big]d^4x. \tag{32}$$

This result (see also [12]) allows the following interpretation: *violation of conformal symmetry may bring to the induction of gravitational field by conform scalar field.*

3. ELECTROVACUUM SOLUTION OF GR WITH MINIMAL COUPLED SCALAR FIELD

We use elecrtovacuum spherically-symmetrical solution of JBD equations [14]

$$ds^2 = \frac{1}{F^2}f^{1/\eta}dt^2 - \\ - F^2 f^{(a-1/\eta)}\left[du^2 + \frac{u^2-k^2}{1-v^2}dv^2 + (u^2-k^2)(1-v^2)d\varphi^2\right].$$

Here

$$f = \frac{u-k}{u+k}, \qquad k = \eta\sqrt{m^2 - Q^2}, \qquad 2F = q + (2-q)f^{(2-a)/2\eta},$$

$$q = \sqrt{1+\tilde{Q}^2}, \qquad \tilde{Q}^2 = \frac{2\eta Q}{k(2-a)},$$

m is the mass; Q is the charge of the source of gravitational field. Gravitational scalar

$$y = y_0 f^{-a/2\eta},$$

and the only nonvanishing component of the tension of the electromagnetic field

$$E = \frac{Q}{u^2}\frac{f^{(a-2)/2\eta}}{F^2(1-k^2/u^2)}.$$

The constants η and a are connected through relation

$$\eta^2 = (a-1)^2 + a + \frac{1}{2}\zeta a^2$$

and are parametrized by central density (or pressure) of each of the member of the family of self-graviting bodies. This solution can be rewritten in uniform coordinates R or in modified coordinates of curvature x with substitution

$$u = x - k = R(1 + k^2/4R^2)$$

(let us note, that $dv^2/(1-v^2) + (1-v^2)d\varphi^2 = d\theta^2 + \sin^2\theta d\varphi^2$). Conform transformation $\bar{g}_{\mu\nu} = (y/y_0)g_{\mu\nu}$ transforms the JBD theory equation into GR equation with a source in a form of minimally coupled scalar field, and the solution of these equations can be obtained through conform transformation from the solutions of the problem in JBD theory and it has the following form:

$$d\bar{s}^2 = \frac{f^n}{F^2}dt^2 - F^2 f^{-n}[du^2 + (u^2-k^2)d\Omega^2],$$

$$\varphi = -\sqrt{\frac{1-n^2}{2}}\ln f, \qquad E = \frac{Q}{u^2}\frac{f^{-n}}{F^2(1-k^2/u^2)}.$$

Here

$$k = \sqrt{m^2 - Q^2}, \qquad n^2 \le 1, \qquad \tilde{Q} = \frac{Q}{kn},$$

the remaining notations are the previous ones. Substitution

$$u = R\Big(1 - \frac{k^2}{4R^2}\Big) = r - k$$

enables to rewrite this solution in homogeneous coordinates R or in curvature coordinates r. If one introduces a new coordinate z in such a way, that

$$f = \frac{u-k}{u+k} = \mathrm{e}^{-z}, \qquad \mathrm{e}^{-z} = \mathrm{e}^{-\varphi\sqrt{2/(1-n^2)}},$$

then

$$d\bar{s}^2 = \frac{\mathrm{e}^{-nz}}{F^2}dt^2 - \frac{k^2}{\mathrm{sh}^2\, z/2}F^2\,\mathrm{e}^{nz}\left[\frac{dz^2}{\mathrm{sh}^2\, z/2} + d\Omega^2\right].$$

In special case of absence of electric field ($F = 1$) this solution coincides with the one obtained by Stanjukovich and Melnikov [13]. Let us introduce $x = z/2$, $\tau = 2kt$, $\alpha = -2n$, $H = 2kF$, then

$$d\bar{s}^2 = \frac{\mathrm{e}^{\alpha x}}{H^2}d\tau^2 - \frac{H^2\mathrm{e}^{-\alpha x}}{4\,\mathrm{sh}^2\, x}\left(\frac{4dx^2}{\mathrm{sh}^2\, x} + d\Omega^2\right), \qquad \varphi = x\sqrt{\frac{4-\alpha^2}{2}}.$$

4. STATIONARY GRAVITATIONAL FIELD WITH AXIAL SYMMETRY

The field equations of the Einstein and the Jordan–Brans–Dicke gravitation theories are essentially nonlinear, which makes their solution difficult. Nevertheless, the number of the known solutions is rather large, though only some of them go well together with the true physical problems, i. e., can be interpreted physically. (Rather a complete and consistent set of exact solutions GR is given in [15].) From a point of view of physical applications of most interest is the study of such gravitational fields which possess certain symmetry.

Space-time, corresponding to the stationary gravitational field with axial symmetry, admits a group of motions with two linearly independent and commuting Killing vectors ξ, η. In accord with the problem symmetry we choose coordinates so that

$$x^\mu = (t, x^i, \varphi), \qquad \mu, \nu ... = 0, i, 3, \qquad i, k ... = 1, 2.$$

Then,

$$\xi = \xi^\mu\partial/\partial x^\mu, \quad \xi^\mu = \delta^\mu_0, \qquad \eta = \eta^\nu\partial/\partial x^\nu, \quad \eta^\nu = \delta^\nu_3.$$

In a general case, after rejecting four components of the metric tensor, whose equality to zero follows from invariance of the metric with respect to individual time inversions ($t \mapsto -t$) and the azimuthal angle ($\varphi \mapsto -\varphi$), we write down the metric of the stationary, axial-symmetric space-time in the form

$$ds^2 = ds^2_{\mathrm{I}} - ds^2_{\mathrm{II}}, \; ds^2_{\mathrm{I}} = \mathrm{e}^{2\alpha}\left(dt - q d\varphi\right)^2 - \mathrm{e}^{2\gamma}d\varphi^2, \; ds^2_{\mathrm{II}} = \mathrm{e}^{2\beta}\left(dx^1\right)^2 + \mathrm{e}^{2\mu}\left(dx^2\right)^2.$$

Such a metric form is invariant under

• the transformation

$$x^a = \alpha^a_b \acute{x}^b, \qquad \alpha^a_b = \text{const}, \qquad a, b, = 0, 3$$

in particular including a simultaneous inversion of time and azimuth $(t, \varphi) \mapsto (-t, -\varphi)$;

• general transformations

$$x^i = x^i(\acute{x}^k),$$

on the two-dimensional surface $x^1 x^2$.

Invariance with respect to specular reflection $(t, \varphi) \mapsto (-t, -\varphi)$ physically means that *a motion of a source of a gravitational field is a mere rotation around the symmetry axis, i. e., space-time corresponding to this metric is related to the rotating body.*

Let us specialize the coordinates in such a way, that $\beta = \mu$. One of the combinations of field equations appears to be the Laplace equation. Let us select that one and harmonic function conjugated to it as coordinates

$$\acute{x}^1 = z(x^1, x^2), \qquad \acute{x}^2 = \rho(x^1, x^2),$$

and let us perform the metric conform transformation (see [16])

$$\bar{g}_{\mu\nu} = y g_{\mu\nu},$$

then

$$d\bar{s}^2 = y ds^2 = \psi(dt - q d\varphi)^2 - \rho^2 d\varphi^2 - \frac{\Phi^2}{\psi}(dz^2 + d\rho^2),$$

$$\psi = y\,\mathrm{e}^{2\alpha}, \qquad \Phi = y\,\mathrm{e}^{\alpha+\beta}.$$

Let us introduce

$$G = \mathrm{e}^{\nu - p\sigma} \begin{pmatrix} 1 & -q \\ -q & q^2 - \rho^2 \mathrm{e}^{-2\nu} \end{pmatrix}, \qquad p = \sqrt{3 + 2\zeta}.$$

The field equation of the block with indices (0,3)

$$\nabla(g^{-1} \nabla g) = 0.$$

The remaining equations have the following form

$$\ln{(\bar{g}_{ik})}_{,z} = \frac{1}{2} \rho \,\mathrm{Sp}\,(f_{,z} f_{,\rho}), \quad \ln{(\bar{g}_{ik})}_{,\rho} = -\frac{1}{\rho} + \frac{\rho}{4}\,\mathrm{Sp}\,(f^2_{,\rho} - f^2_{,z}),$$

$$f_{,i} = g^{-1} g_{,i}.$$

Thus, the system of field equations of JBD theory in Weil canonical coordinates and after conform transformations has such a form, as GR equations of analogous problem, which allows one to formulate the following

Statement 5. *If the set*

$$y, \mathrm{e}^{2\alpha}, \mathrm{e}^{2\beta}, q, A, B$$

is electrovacuum solution of stationary axially symmetric problem of JBD theory in Weil canonical coordinates, then the analogous GR problem has the following solution

$$y\,\mathrm{e}^{2\alpha}, y\,\mathrm{e}^{2\beta}/\Phi_y^{2(3+2\zeta)}, q, A, B.$$

And vice versa: one can find by known solution of GR the solution of analogous problem of JBD theory, if y and Φ_y are known.

Example: Axially symmetric solution of GR with charge. Let us use statement 5 for transformation of the electrovacuum static solution of the JBD theroy [14] (see point 4) to the corresponding GR solution. After simple calculations we obtain an axially symmetric electrovacuum new solution of the Einstein equations:

$$ds^2 = \frac{k^2}{F^2}\left(\frac{x-1}{x+1}\right)^n dt^2 - F^2\left(\frac{x+1}{x-1}\right)^n \Big\{(x^2-1)(1-y^2)d\varphi^2 + \\ + \left(\frac{x^2-1}{x^2-y^2}\right)^{n^2}(x^2-y^2)\left[\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right]\Big\},$$

$$F = \frac{m+2r_0}{2n}\left[1-\frac{m-2r_0}{m+2r_0}\left(\frac{x-1}{x+1}\right)^n\right], \quad k = \frac{2r_0}{n}, \quad 2r_0 = \sqrt{m^2-Q^2},$$

$$E = \frac{Q}{F^2(x^2-1)}\left(\frac{x-1}{x+1}\right)^n\left(\frac{x^2-1}{x^2-y^2}\right)^{(1-n^2)/2},$$

$$g_{00} \approx 1 - \frac{2m}{r} + \frac{DP_2(\cos\theta)}{r^3} + ..., \quad D = \frac{2}{3}\frac{n^2-1}{n^2}m(m^2-Q^2).$$

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