FORMATION OF THE $SU(3)$-POLARIZATION STATES IN ATOM-QUANTUM ELECTROMAGNETIC FIELD SYSTEM UNDER CONDITION OF THE BOSE–EINSTEIN CONDENSATE EXISTENCE

A. P. Alodjants[1], *A. V. Prokhorov*[1], *S. M. Arakelian*[2]

Vladimir State University, Vladimir, Russia

We consider the problem of multipartite entanglement of two-level atom system in Bose–Einstein condensate (BEC) and quantized electromagnetic field. The main accent is made on polarization properties of such a system. The $SU(3)$-polarization states are introduced for three-mode quantum system satisfying both the Gell-Mann symmetry and the isopolarization property for atoms in condensate state. Superstructure of revivals and collapses of quantum state in the system is described by degree of polarization changing in time.

INTRODUCTION

Quantum states of ultracold atoms in Bose–Einstein condensate (BEC) are under intensive study in modern quantum and atomic optics [1]. Such a mesoscopic system is of great interest for both basic and applied research. In the latter case, the problem of quantum computing is considered in many papers (e. g., [2]). Entangled multipartite states of condensate atoms and external electromagnetic (EM) field become principal for the case [3]. The effect is very well known for classical optical field when two hyperfine levels of atoms are coupled by the field and the Rabi oscillations for population imbalance and phase difference of two-level system take place and demonstrate the nonlinear behaviour [4]. Another case is the Josephson effect resulting in nonclassical statistics and squeezing in atomic variables for junctions of two conductors at low temperature [5].

In our previous paper [6], we considered the problem of interaction of two-level atoms with quantized EM field under the Jaynes–Cummings (or Dicke) model. In the frameswork of the model, nonclassical effects of collapse and revivals for quantum atomic system and also the squeezing effects in terms of the $SU(9)$ observables (i. e., the spin operators) have been predicted. For some conditions a quantum steady-state excitation can appear. In experiments [7] with ultracold sodium atoms, the light velocity reduction has been demonstrated for passing EM radiation, and such a behaviour can be explained by the effect of electromagnetic field induced transparency for «bright» and «dark» polaritons (i. e., the spin-wave excitations) arising in three-level atomic system [7, 8]. Mathematically, the excitations represent an

[1]e-mail: laser@vpti.vladimir.su
[2]e-mail: arak@vpti.vladimir.su

example of phenomena under formalism of so-called $SU(2)$-algebra deformation (see, e. g., [9]).

On the other hand, complete description of the three-component bosonic system should be presented by using the $SU(3)$-algebra symmetry, being the mathematical apparatus in both quantum chromodynamics and physics of elementary particles [10]. Although such an approach is well known for describing the atom–field interaction [11, 12], the $SU(3)$-polarization problem is still open now.

In the present paper we develop the $SU(3)$-polarization approach for the two-level atomic and EM field system interaction.

1. QUANTUM DESCRIPTION OF THE $SU(3)$-POLARIZATION PROBLEM

Let us describe the $SU(3)$ symmetry of Bose system in the Schwinger representation by Hermitian Gell-Mann operators λ_j $(j = 0, 1, \ldots, 8)$ (cf. [10]):

$$\lambda_0 = a_1^+ a_1 + a_2^+ a_2 + a_3^+ a_3, \tag{1}$$

$$\lambda_1 = a_1^+ a_2 + a_2^+ a_1, \quad \lambda_2 = i\left(a_2^+ a_1 - a_1^+ a_2\right), \quad \lambda_3 = a_1^+ a_1 - a_2^+ a_2, \tag{2}$$

$$\lambda_4 = a_1^+ a_3 + a_3^+ a_1, \quad \lambda_5 = i\left(a_3^+ a_1 - a_1^+ a_3\right), \tag{3}$$

$$\lambda_6 = a_2^+ a_3 + a_3^+ a_2, \quad \lambda_7 = i\left(a_3^+ a_2 - a_2^+ a_3\right), \tag{4}$$

$$\lambda_8 = \frac{1}{\sqrt{3}}\left(a_1^+ a_1 + a_2^+ a_2 - 2a_3^+ a_3\right), \tag{5}$$

where $a_j(a_j^+)$, $j = 1, 2, 3$ are the bosonic annihilation (creation) operators, respectively, under commutation relation:

$$\left[a_i; a_j^+\right] = \delta_{ij}, \quad [a_i; a_j] = 0. \tag{6}$$

Using Eqs. (1), (6), we have the $SU(3)$-algebra commutation relations for λ-operators:

$$[\lambda_j; \lambda_k] = i\varepsilon_{jkm}\lambda_m, \quad j, k, m = 1, 2, 3, \tag{7}$$

$$[\lambda_4; \lambda_5] = i\left(\lambda_3 + \sqrt{3}\lambda_8\right), \quad [\lambda_6; \lambda_7] = i\left(\sqrt{3}\lambda_8 - \lambda_3\right), \tag{8}$$

$$[\lambda_0; \lambda_i] = 0, \quad i = 1, \ldots, 8, \tag{9}$$

where structural coefficients ε_{jkm} are the completely antisymmetric with the values (cf. [12])

$$\varepsilon_{123} = 2, \quad \varepsilon_{584} = \varepsilon_{678} = \sqrt{3}, \quad \varepsilon_{147} = \varepsilon_{246} = \varepsilon_{257} = \varepsilon_{345} = \varepsilon_{516} = \varepsilon_{637} = 1. \tag{10}$$

According to definition (1), the operator λ_0 describes a total number of particles in Bose system. The operators $\lambda_{1,2,3}$ can be considered as an isospin of $SU(6)$ subgroup in $SU(3)$ algebra under Gell-Mann symmetry [10]. The operators $\lambda_{1,2,3}$ characterize a two-level atomic system for Bose condensate. Another set of the operators λ_j $(j = 4, \ldots, 8)$ determines the interaction of atoms and mode of quantum field a_3.

Now consider the $SU(3)$-polarization problem for three-mode system by analogy with the case discussed above (see, e. g., [13]).

Let us introduce a unit «vector of polarization» $\mathbf{e}$:

$$\mathbf{e}a = \mathbf{e}_1 a_1 + \mathbf{e}_2 a_2 + \mathbf{e}_3 a_3, \tag{11}$$

where a is the annihilation operator for a total three-mode system; $\mathbf{e}_j (j = 1, 2, 3)$ are the orthogonal basic vectors under the condition

$$\sum_{j=1}^{3} |\mathbf{e}_j| = 1. \tag{12}$$

Expression (11) can be represented in the form

$$a = e_1^* a_1 + e_2^* a_2 + e_3^* a_3, \tag{13}$$

where $e_j^* = \mathbf{e}^* \mathbf{e}_j$.

According to the $SU(3)$ symmetry (see, e. g., [12]), we can rewrite Eq. (13) in terms of the four phase parameters Θ, ϕ, ψ_1, ψ_2:

$$\begin{gathered} e_1 = \mathrm{e}^{i\psi_1} \sin\Theta \sin\phi, \quad e_2 = \mathrm{e}^{i\psi_2} \sin\Theta \cos\phi, \\ e_3 = \cos\Theta, \quad 0 < \Theta, \quad \phi \leq \frac{\pi}{2}, \quad 0 \leq \psi_{1,2} < 2\pi. \end{gathered} \tag{14}$$

A simple geometric interpretation of introduced quantities is presented in Fig. 1. In fact, the phase parameters Θ, ϕ, ψ_1, ψ_2 as well as the operators a_j determine completely the

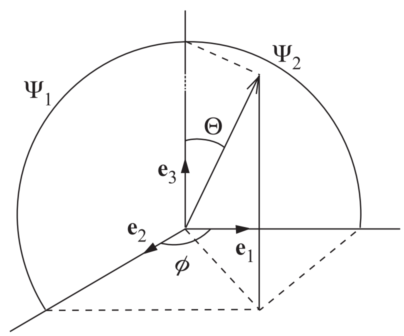

Fig. 1. 3D geometric presentation of the $SU(3)$ polarization for three-level bosonic system. The angles ϕ, Θ and relative phase $\psi_{1,2}$ determine the parameterization of polarization state of the system

«polarization» states of the $SU(3)$-symmetry system in a Hilbert space. The polarization states of atomic system can be associated with isopolarization states of the $SU(6)$ subgroup. Thus, the degree of isopolarization looks like (cf. [13])

$$P_{\mathrm{IP}} = \frac{\left(\langle\lambda_1\rangle^2 + \langle\lambda_2\rangle^2 + \langle\lambda_3\rangle^2\right)^{1/2}}{\langle a_1^+ a_1\rangle + \langle a_2^+ a_2\rangle}, \tag{15}$$

and we can introduce the parameter for total system

$$P = \frac{\sqrt{3}}{2} \frac{\left(\sum\limits_{j=1}^{8} \langle \lambda_j \rangle^2 \right)^{1/2}}{\langle \lambda_0 \rangle}, \tag{16}$$

being independent of the $P_{\rm IP}$ parameter.

The numerical factor $\sqrt{3}/2$ in Eq. (16) arises due to normalization condition for the P quantity. For absolutely polarized state of the system, we have

$$P = 1, \quad gP_{\rm IP} = 1. \tag{17}$$

2. QUANTUM POLARIZATION IN THE BEC ENTANGLED STATE

For the system including the two-level ultracold atoms and the quantum EM field (described by the operators a_3, a_3^+), we have the Hamiltonian [6]

$$H = H_f + H_{\rm BEC} + H_{\rm OPT}, \tag{18}$$

where

$$H_f = \hbar \omega_f a_3^+ a_3, \tag{19}$$

$$H_{\rm BEC} = \hbar \left(\omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \frac{1}{2} \gamma_1 a_1^{+2} a_1^2 + \frac{1}{2} \gamma_2 a_2^{+2} a_2^2 \right), \tag{20}$$

$$H_{\rm OPT} = \hbar k \left(a_3^+ a_2^+ a_1 + a_1^+ a_2 a_3 \right) \tag{21}$$

and rotating wave approximation and lossless cavity taken into account.

The operators $a_1(a_1^+)$, $a_2(a_2^+)$ describe an annihilation (creation) of the atoms in condensate states being induced by two hyperfine levels «1» and «2», respectively.

For «small» number of particles ($N \approx 200$, see, e. g., [5]) that is exactly the case, we can neglect the motion of the atoms.

The parameters $\gamma_{1,2}$ are proportional to the scattering length, and characterize the atomic collisions in the Born approximation. The Hamiltonian $H_{\rm OPT}$ (see (21)) describes a direct coupling between the atom levels arising due to quantum EM modes. The Hamiltonian H_f (see (19)) corresponds to the energy of the free EM field.

Let us characterize the entangled state determined by a single-photon state and atoms and described by the following ansatz of the state vector of the system in Schrödinger representation:

$$|\Psi(t)\rangle = \sum_{p=0}^{N-1} \{ P_p(t) \, |p\rangle_1 \, |N-p\rangle_2 \, |1\rangle_3 + D_p(t) \, |p+1\rangle_1 \, |N-p-1\rangle_2 \, |0\rangle_3 \}, \tag{22}$$

where $P_p(t)$ and $D_p(t)$ are the unknown coefficients being subject of calculating by variational method. Physically, the $P_p(t)$ amplitudes determine the probability to find the number $(N-p)$ and p atoms in the lower and upper level of the condensate, correspondingly.

The coefficient $D_p(t)$ characterizes the probability of a single-photon state absorption in the condensate.

The following normalization condition takes place:

$$\langle \Psi(t) \mid \Psi(t) \rangle = \sum_{p=0}^{N} \left(|P_p(t)|^2 + |D_p(t)|^2 \right) = 1. \tag{23}$$

For in-+itial state, we have

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2^N N!}} \left(a^+ + b^+\right)^N |0\rangle_{12} |1\rangle_3 \tag{24}$$

in the limit of a large number of atoms ($N \gg 1$) (cf. [3]).

Using the variational method to obtain the coefficients $P_p(t)$ and $D_p(t)$, we have the equations

$$\begin{aligned} \frac{dP_p(t)}{dt} &= -i\left(\chi_0 P_p(t) + \theta D_p(t)\right), \\ \frac{dD_p(t)}{dt} &= -i\left(\chi_D D_p(t) + \theta P_p(t)\right), \end{aligned} \tag{25}$$

where the parameters χ_0, χ_D, θ characterize the quantum atomic modes in the condensate state:

$$\begin{gathered} \chi_0 = \omega_2(N-p) + \omega_1 p + \frac{1}{2}\gamma_2(N-p)(N-p-1) + \frac{1}{2}\gamma_1 p(p-1) + \omega_3, \\ \chi_D = \omega_2(N-p-1) + \omega_1(p+1) + \frac{1}{2}\gamma_2(N-p-1)(N-p-2) + \frac{1}{2}\gamma_1 p(p+1), \\ \theta = k\sqrt{(N-p)(p+1)}. \end{gathered} \tag{26}$$

Stationary solutions of the equations (25) reduce to the form

$$P_p(t) = Q_p \,\mathrm{e}^{-i\omega_S t} U_P, \quad D_p(t) = Q_p \,\mathrm{e}^{-i\omega_S t} U_D, \tag{27}$$

where ω_S is the frequency for the evolution behaviour in time; $Q_p = \dfrac{\sqrt{N!}}{\sqrt{2^N (N-p)!p!}}$.

Coefficients U_P, U_D do not depend on time t and characterize the initial distribution of the atoms in condensate. In particular, we have

$$U_P^2 = \frac{1}{2}\left(1 \pm \sqrt{\frac{\delta^2}{4\theta^2 + \delta^2}}\right), \quad U_D^2 = \frac{1}{2}\left(1 \mp \sqrt{\frac{\delta^2}{4\theta^2 + \delta^2}}\right), \tag{28}$$

where $\delta \equiv \chi_D - \chi_0 = \Delta\omega - \gamma_2(N-p-1) + \gamma_1 p$ is the phase difference describing the «self-interaction» effects for the atom/field modes; $\Delta\omega = \omega_1 - \omega_2 - \omega_f$ is the phase retardation.

The dispersion relation associated with solutions (27) determines the two frequency branches $\Omega_{1,2}$ for quantum excitations (i. e., spin waves) in such a system:

$$\Omega_{1,2} = \frac{\chi_D + \chi_0 \pm \sqrt{\delta^2 + 4\theta^2}}{2}. \tag{29}$$

In optical region, the coupling for two atomic levels «1» and «2» by frequencies $\Omega_{1,2}$ results in a «bright» and a «dark» condensate polaritons, correspondingly. The frequencies $\Omega_{1,2}$ depend on the quantum number p characterizing the population of atomic levels in condensate state (see Fig. 2).

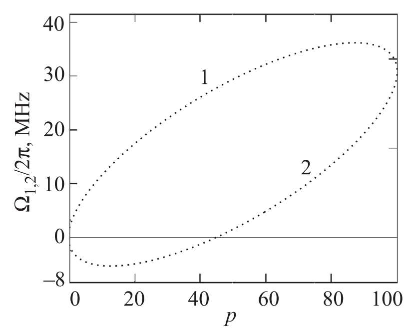

Fig. 2. The frequency for «bright» (1) and «dark» (2) excitations as a function of particle number p for $N = 100$, $\Delta\omega = 0$, $\gamma_1 = \gamma_2 = 2.1 \cdot 10^5\ \mathrm{s}^{-1}$, $k = 1.7 \cdot 10^9\ \mathrm{s}^{-1}$. The magnitude $\omega_{12}/2\pi = 5.5$ MHz determines an initial and final energy gap between the two types of excitations

Existence of two excitations, i. e., positive ($\Omega_j > 0$) and negative ($\Omega_j < 0$), associated with two quasiparticles propagating with supersonic velocities results in instabilities for the condensate, and macroscopic effect of formation of shock waves takes place [7]. The parameter $\omega_{12} = \sqrt{\delta^2 + 4\theta^2}$ determines an energy gap for the two branches of excitations which have a minimum for $p = 0$ and $p = N - 1$.

For «dark» polariton in the case when $\Omega_2 = 0$ (Fig. 2), the condition $\chi_0\chi_D = \theta^2$ (see Eq. (29)) determines the frequency of the EM field driving the atomic system (p is fixed).

For the state (22), the expressions (15), (12) reduce to the form

$$P_{\mathrm{IP}}^2 = 1 - \frac{4\left(\left\langle a_1^+ a_1\right\rangle \left\langle a_2^+ a_2\right\rangle - \left\langle a_1^+ a_2\right\rangle \left\langle a_2^+ a_1\right\rangle\right)}{N^2}, \tag{30}$$

$$P^2 = 1 - \frac{3\left(\left\langle a_1^+ a_1\right\rangle \left\langle a_2^+ a_2\right\rangle + N\left\langle a_3^+ a_3\right\rangle - \left\langle a_1^+ a_2\right\rangle \left\langle a_2^+ a_1\right\rangle\right)}{\left(N + \left\langle a_3^+ a_3\right\rangle\right)^2}. \tag{31}$$

In the case described by Eqs. (24) and (21), we have

$$P_{\mathrm{IP}}^2 = 1, \quad P^2 = 1 - \frac{3N}{(N+1)^2}. \tag{32}$$

The first expression in (32) corresponds to polarization of own coherent state for atomic system, but depolarization of total system is determined by a single-photon state. The time evolution behaviour for P, P_{IP} under resonance condition $\Delta\omega = 0$ is shown in Fig. 3.

The phenomena described above determine the principal dynamics behaviour of the polarization states for the system, and the collapse and revival effects (see Fig. 3 and cf. [6]) arise. At the same time the depolarization occurs in the system.

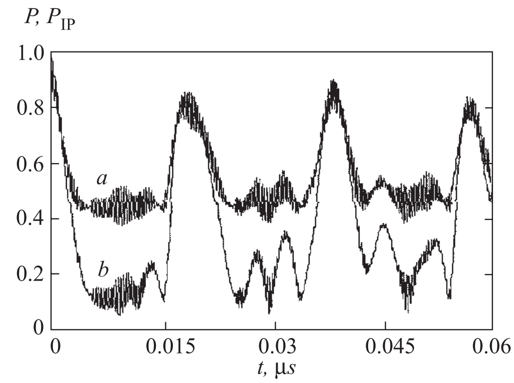

Fig. 3. Time dependence of the degree of polarization P (*a*) and isopolarization P_{IP} (*b*) for nonstationary regimes. The parameters of the atom–field interaction: $N = 15''$, $\Delta\omega = 0$, $\gamma_1 = \gamma_2 = 2.1 \cdot 10^5\ \mathrm{s}^{-1}$, $k = 1.7 \cdot 10^9\ \mathrm{s}^{-1}$

The difference between two parameters, i. e., the degree of isopolarization for atoms in condensate state and the degree of polarization for total system, can be explained by the influence of EM field. In fact, the P behaviour demonstrates the superstructure of the revivals as a result of the atom–field interaction [6].

CONCLUSION

Nonclassical effects of collapse and revival for degree of polarization are obtained in the atom–EM-field system in condensate state. For experimental observation of predicted effects, the interferometric methods are necessary [14], but special analysis should be carried out for the $SU(3)$ interferometer to measure simultaneously all the Gell-Mann observables with some ultimate accuracy.

Finally, the problem of the Elliott $SU(3)$ symmetry for atomic system [10, 11], which we do not consider in the present paper, demands that the angular momentum be taken into account as well (cf. [15]).

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