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GENERAL CONSTRAINTS ON SPIN OBSERVABLES; APPLICATIONS TO $\bar{p} + p \to \bar{\Lambda} + \Lambda$ AND TO POLARIZED QUARK DISTRIBUTIONS

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The density matrix formalism and the positivity conditions for general multiple-spin asymmetries are reviewed, taking as an example the case $\bar{p} + p \to \bar{\Lambda} + \Lambda$ in which one, two or three spins are analyzed. Some aspects related to quantum information and entangled states are discussed. Some positivity domains for pairs and triplets of spin parameters are displayed, together with the experimental points. The case of inclusive reaction is also treated, taking as an example the spin- and transverse momentum-dependent quark distributions.

Обсуждается формализм матрицы плотности и условия положительности для общих многоспиновых асимметрий с использованием в качестве примера одно-, двух- и трехспиновых асимметрий в процессе $\bar{p} + p \to \bar{\Lambda} + \Lambda$. Рассмотрены отдельные аспекты, связанные с квантовой информацией и перепутанными состояниями. Представлены некоторые области положительной определенности в сравнении с экспериментальными данными. Также обсуждаются инклюзивные реакции на примере кварковых распределений, зависящих от спина и поперечного импульса.

INTRODUCTION

The single- or multiple-spin asymmetries, which can be measured using polarized beams, polarized targets or analyzing the final-particle polarizations, provide important information about the elementary processes in particle physics. These spin observables may be related by *equalities* coming from the symmetries of the processes. Besides, they satisfy *inequalities* expressing the positivity of a *Grand Density Matrix*, R, which describes all possible polarized cross sections. The positivity of R insures that the cross section is positive for any initial and final spin

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states, including entangled ones. The resulting inequalities provide consistency checks of experimental data, constrain any parametrization of polarized structure functions and, in the future, may be applied to Monte-Carlo event generators with spins.

The positivity conditions are also interesting from the point of view of the *quantum information* carried by spins. The information about the scattering amplitudes is maximal when all the independent spin observables are measured. There are inequalities that define the allowed domain and are saturated in this case. Conversely, there is a loss of information, in other words an increase of entropy, when some particles are not analyzed or in the case of inclusive reactions. In this case, most of the inequalities may be nonsaturated.

There is an abundant literature on positivity conditions. Some key papers of the 60's are still relevant. See, for instance, Ref. 1 for a survey and some references. The subject has been revisited in recent months due to the results on the reaction

$$\bar{p} + p \to \bar{\Lambda} + \Lambda, \tag{1}$$

measured by the PS185 collaboration at CERN [2]. The polarization of the outgoing hyperon or antihyperon is revealed by its weak decay. Since a polarized target was used in the last runs, observables up to rank 3 can be accessed.

This contribution is a short introduction to the density matrix formalism and derivation of the positivity conditions. In Secs. 1–4 we will consider the case of *exclusive* reactions, illustrating it by reaction (1). We will also present in Sec. 5 some results of an alternative «empirical» method by which the positivity domains of a subset of observables can be very easily discovered.

Finally, we will consider the case of *inclusive* reactions and obtain the inequalities which must be satisfied by the *spin-dependent quark distribution*, considered as the probability of the elementary splitting process

$$\text{nucleon} \to \text{quark} + X \tag{2}$$

for a given longitudinal momentum ratio $x = p_L(\text{quark})/p_L(\text{nucleon})$.

1. THE SPIN OBSERVABLES

The fully polarized differential cross section of (1), more generally $A+B \to C+D$, where A, B, C, and D have spin $1/2$, can be expressed as

$$\frac{d\sigma}{d\Omega}\,(\mathbf{S}_A, \mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_D) = \frac{1}{4}\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} C_{\lambda\mu\nu\tau}\; S_A^\lambda\, S_B^\mu\, S_C^\nu\, S_D^\tau. \tag{3}$$

The $\mathbf{S}$'s are the polarization vectors of pure spin states ($|\mathbf{S}| = 1$). In the right-hand side they are promoted to 4-vectors with $S^0 = 1$. The indices λ, μ, ν, τ, run from 0 to 3, whereas latin indices i, j, k, l, take the values 1, 2, 3, or x, y, z. A summation is understood over each repeated index. S^x, S^y, S^z are measured in a triad of unit vectors $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ which may differ from one particle to the other. $C_{\lambda\mu\nu\tau}$ are the *correlation parameters*. For example, $C_{0000} \equiv 1$, $C_{xy00} \equiv A_{xy}$ is an initial double-spin asymmetry, C_{000y} is the spontaneous polarization of particle D along $\hat{\mathbf{y}}$, $C_{0y0y} \equiv D_{yy}$ is a spin transmission coefficient from B to D and $C_{00xy} \equiv C_{xy}$ is a final spin correlation.

Equation (3) also applies to the case of incomplete initial polarizations, replacing the unit vector $\mathbf{S}_A$ by $\mathbf{P}_A$ with $|\mathbf{P}_A| \leqslant 1$ and the same for B. The final polarizations generally depend on the initial ones, e.g.,

$$\mathbf{P}_C \equiv \langle \mathbf{S}_C \rangle = \left(\frac{1}{4}\frac{d\sigma}{d\Omega}\right)^{-1}_{\text{unpol}} \nabla_{\mathbf{S}_C} \frac{d\sigma}{d\Omega}(\mathbf{P}_A, \mathbf{P}_B, \mathbf{S}_C; \mathbf{S}_D = 0). \tag{4}$$

2. THE DENSITY MATRIX FORMALISM

Let $\{|\alpha\rangle\}$, with $\alpha = \pm 1/2$, be the basic spin states of particle A, $\{|\beta\rangle\}$ those of B, etc. The quantification axis $\hat{\mathbf{z}}$ may differ from one particle to the other. It can be the *helicity* axis $\mathbf{p}/|\mathbf{p}|$ or the *transversity* axis, $\hat{\mathbf{n}} = \mathbf{p}_A \times \mathbf{p}_C \,/\, |\mathbf{p}_A \times \mathbf{p}_C|$. We write the spin-dependent amplitude of (1) as

$$(\langle \bar{\Lambda}, \gamma| \otimes \langle \Lambda, \delta|)\ M\ (|\bar{p}, \alpha\rangle \otimes |p, \beta\rangle) \equiv \langle \gamma\delta | M | \alpha\beta \rangle. \tag{5}$$

For each spin-$1/2$ particle we have the single-spin density matrix

$$\rho(\mathbf{P}) \equiv \frac{1}{2}(1 + \mathbf{P} \cdot \sigma) \tag{6}$$

For *partial* polarizations of A and B, but definite spins of C and D, cross section (3) becomes

$$\frac{d\sigma}{d\Omega}(\mathbf{P}_A, \mathbf{P}_B, \mathbf{S}_C, \mathbf{S}_D) = \text{trace}\,\{M\rho(\mathbf{P}_A) \otimes \rho(\mathbf{P}_B) M^\dagger \rho(\mathbf{S}_C) \otimes \rho(\mathbf{S}_D)\}. \tag{7}$$

The final two-spin density matrix is

$$\rho_{C,D} = \frac{M\rho(\mathbf{P}_A) \otimes \rho(\mathbf{P}_B) M^\dagger}{\text{trace}\,\{MM^\dagger\}}. \tag{8}$$

It describes the individual polarizations of C and D and their spin correlations. The polarization of the C is obtained by taking the partial trace over the D spin variable:

$$\rho_C = \text{trace}_D\ \rho_{C,D}, \quad \text{i.e.,} \quad \langle \gamma | \rho_C | \gamma' \rangle = \sum_\delta \langle \gamma\delta | \rho_{C,D} | \gamma'\delta \rangle. \tag{9}$$

The lack of information about a system can be measured by various estimators, among which the entropy $S = -\mathrm{trace}\{\rho \log \rho\}$, and the rank of ρ. Pure states (maximum information) have zero entropy and unit rank. The entropy (resp. rank) of the initial state is the sum (resp. product) of the single-particle entropies (resp. ranks). The rank of the final density matrix (8) is less than or equal to the initial one. Therefore, complete initial polarizations ($|\mathbf{P}(p)| = |\mathbf{P}(\bar{p})| = 1$) lead to a final pure state. It does not imply that the individual polarization of the $\bar{\Lambda}$, obtained from (9), is complete, because the $\bar{\Lambda}\Lambda$ state may be *entangled* [3].

Let us now generalize the density matrix in order to describe in an unified way the spin *correlations* inside the final state and the *transmission* of polarizations (i.e., of spin information) between the initial and the final particles. For this purpose we consider the fictitious *crossed* reaction of (1),

$$|\mathrm{vacuum}\rangle \to p + \bar{p} + \bar{\Lambda} + \Lambda. \tag{10}$$

We restrict this crossing to *spin* and *flavor* variables (we do not consider the momenta). An initial particle *ket* becomes a *final anti-particle bra* of *opposite spin*, for instance,

$$|p, \beta\rangle \quad \to \quad \langle \bar{p}, -\beta| \equiv \langle \mathrm{p}, \beta| CPT, \tag{11}$$

where C, P, and T are the charge conjugation, parity and time-reversal operators (it does not matter if reaction (10) does not conserve energy-momentum). Accordingly, we can rewrite the spin-dependent amplitude (5) as

$$\langle \gamma, \delta | M | \alpha, \beta \rangle = \langle -\alpha, -\beta, \gamma, \delta |\, M^{\mathrm{crossed}} | \mathrm{vacuum}\rangle \equiv \langle -\alpha, -\beta, \gamma, \delta | \Psi \rangle. \tag{12}$$

Thus we produce a one-to-one correspondence between the 2-particle transition *operator* M and a 4-particle *state vector* $|\Psi\rangle$, which we will call the *Grand Wave Function*. To shorten the equations, we will introduce the notation $\bar{\alpha} \equiv -\alpha$, $\bar{\beta} \equiv -\beta$, etc. For explicit values of α, we will use the notations u and d (for «up» and «down», like for quark isospin states) instead of $+1/2$ and $-1/2$. Therefore (11) will be written as

$$|u\rangle \to \langle \bar{u}|, \quad |d\rangle \to \langle \bar{d}|. \tag{13}$$

The *Grand Density Matrix*, R, which describes all possible spin correlations in reaction (1), is defined by

$$\langle \bar{\alpha}\bar{\beta}\gamma\delta | R | \bar{\alpha}'\bar{\beta}'\gamma'\delta' \rangle \equiv \langle \gamma\delta | M | \alpha\beta \rangle \langle \alpha'\beta' | M^\dagger | \gamma'\delta' \rangle = \langle \bar{\alpha}\bar{\beta}\gamma\delta | \Psi \rangle \langle \Psi | \bar{\alpha}'\bar{\beta}'\gamma'\delta' \rangle. \tag{14}$$

Like ordinary density matrices, it is hermitian and semipositive. Its trace is given by

$$\mathrm{trace}\,(R) = \langle \Psi | \Psi \rangle = \mathrm{trace}\,(M M^\dagger), \tag{15}$$

which is 2^2 times the unpolarized cross section. Dividing by (15), we can rescale R to unit trace, as a standard density matrix. As can be seen from (14), R describes a pure state: $R = |\Psi\rangle\langle\Psi|$, and is therefore of rank one. This is a particular property of *exclusive* reactions.

Expression (8) for the final density matrix can be rewritten as

$$\rho(\bar{\Lambda}, \Lambda) = \frac{\mathrm{trace}_{\bar{\alpha},\bar{\beta}}\{R[\rho^t(p) \otimes \rho^t(\bar{p})]\}}{\mathrm{trace}\,(R)}, \tag{16}$$

where $\rho^t(p)$ is the transpose of $\rho(p)$. This transposition is explained in the Appendix.

The Grand Density Matrix can be expressed in terms of the correlation parameters and vice versa through

$$R = 2^{-4}\ C_{\lambda\mu\nu\tau}\ \sigma^t_\lambda(A) \otimes \sigma^t_\mu(B) \otimes \sigma_\nu(C) \otimes\ \sigma_\tau(D), \tag{17}$$

$$C_{\lambda\mu\nu\tau} = \mathrm{trace}\left\{R\left[\sigma^t_\lambda(A) \otimes \sigma^t_\mu(B) \otimes \sigma_\nu(C) \otimes\ \sigma_\tau(D)\right]\right\}, \tag{18}$$

where σ_0 is the unit 2×2 matrix.

3. REDUCTION OF THE DENSITY MATRIX

It is difficult to have polarized antiprotons. Therefore the practical spin observables in reaction (1) concern only p, Λ, and $\bar{\Lambda}$. They are encoded in the subdensity matrix $R(p, \bar{\Lambda}, \Lambda) = \mathrm{trace}_{\bar{\alpha}}\{\rho^t(\bar{p})R(\bar{p}, p, \bar{\Lambda}, \Lambda)\}$ with $\rho(\bar{p}) = \dfrac{1}{2}I$, more explicitly

$$\langle\bar{\beta}\gamma\delta|R(p, \bar{\Lambda}, \Lambda)|\bar{\beta}'\gamma'\delta'\rangle = \sum_{\bar{\alpha}}\langle\bar{\alpha}\bar{\beta}\gamma\delta|R(\bar{p}, p, \bar{\Lambda}, \Lambda)|\bar{\alpha}\bar{\beta}'\gamma'\delta'\rangle. \tag{19}$$

This density matrix has dimension 8×8, which is still rather large to write down the positivity conditions (the original one was 16×16). It has a nonzero entropy, brought by the $\bar{p}$, and rank 2 because $\bar{\alpha}$ takes two values in (19).

An important simplification occurs due to the symmetry of reaction (1) under the reflection Π about the scattering plane, which reverses the spin components parallel to this plane (it is the «B-symmetry» mentioned in [4]). Using the transversity basis, where $\hat{\mathbf{z}} \equiv \hat{\mathbf{n}}$, the scattering matrix M is even under $\sigma_x \to -\sigma_x$, $\sigma_y \to -\sigma_y$, and amplitude (5) vanishes when an odd number of transversities are negative. Accordingly, the Grand Wave Function Ψ has no components like $|\bar{u}\bar{u}ud\rangle$ and the original 4-particle density matrix R is reduced to 8×8.

For the same reason, the density matrices restricted to fewer particles like $R(p, \bar{\Lambda}, \Lambda)$, $\rho(\bar{\Lambda}, \Lambda)$ or $\rho(\Lambda)$ do not mix states with even and odd numbers of d's.

Thus $R(p, \bar{\Lambda}, \Lambda)$ is block-diagonal, being the direct sum of two 4×4 matrices, one corresponding to $\bar{\alpha} = \bar{u}$, the other to $\bar{\alpha} = \bar{d}$ in Eq. (19). Each of these submatrices is of rank one. Similarly, $\rho(\bar{\Lambda}, \Lambda)$ is block-diagonal in two 2×2 submatrices of rank 2, one corresponding to $(\bar{\alpha}, \bar{\beta}) = (\bar{u}, \bar{u})$ or $(\bar{d}, \bar{d})$, the other to $(\bar{\alpha}, \bar{\beta}) = (\bar{u}, \bar{d})$ or $(\bar{d}, \bar{u})$ in Eq. (16). Finally, $\rho(\Lambda)$ is diagonal, which means that the Λ polarization is normal to the scattering plane.

4. THE POSITIVITY CONDITIONS

The positivity of the Grand Density Matrix comes from the very general, but nontrivial requirement that the probability of any process is positive. It is *not sufficient* to require that cross section (3) is positive for any set of polarizations $\{\mathbf{S}_A, \mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_D\}$. Let us suppose, for instance, that (3) possesses the factor $(1 + \mathbf{S}_C \cdot \mathbf{S}_D)$. This factor is positive or null for any $\mathbf{S}_C$ and $\mathbf{S}_D$. However, it corresponds to a final density matrix of the form $\rho_{C,D} = [1 + \sigma_C^i \otimes \sigma_D^i]/4$ which is nonpositive. For example, for the *singlet* spin state, we have $\sigma_C^i \otimes \sigma_D^i = -3$. The probability that reaction (1) produces a $(\bar{\Lambda}, \Lambda)$ system in the singlet state would be negative! Note that the singlet state is *entangled*. This shows that positivity has to be tested with nonentangled *and* entangled states.

Similarly, a factor $(1 - \mathbf{S}_A \cdot \mathbf{S}_C)$, which leads to the complete spin reversal $\mathbf{S}_C = -\mathbf{S}_A$ according to (4), gives a nonpositive R and is therefore forbidden. As an example, let us consider the splitting $\pi \to q + \bar{q}$ followed by a quark–hadron scattering $q + h \to q' + h'$ where the $\bar{q}$ is spectator. The intermediate spin correlation is in $(1 - \mathbf{S}_q \cdot \mathbf{S}_{\bar{q}})$. If there were a complete spin reversal $\mathbf{S}_q = -\mathbf{S}_{q'}$ in the quark–hadron scattering, it would lead to a final correlation in $(1 + \mathbf{S}_{q'} \cdot \mathbf{S}_{\bar{q}})$, which is forbidden as explained before.

The general positivity conditions are as follows: a $N \times N$ hermitian matrix ρ is positive (respectively semipositive) if all its eigenvalues r_i are positive (resp. positive or null). Let us consider the symmetric functions of the eigenvalues

$$\Sigma_1 = \sum_i r_i, \quad \Sigma_2 = \sum_{i<j} r_i r_j, \quad \Sigma_3 = \sum_{i<j<k} r_i r_j r_k, \ldots, \Sigma_N = r_1 r_2 \cdots r_n. \tag{20}$$

Σ_n is the sum of the *on-diagonal subdeterminants* of order n (when a submatrix has its diagonal on the diagonal of ρ, we call it «on-diagonal»). A necessary and sufficient condition of positivity, or semipositivity with N_0 vanishing eigenvalues, is

$$\Sigma_n > 0 \quad \text{for} \quad n \leqslant N - N_0, \qquad \Sigma_n = 0 \quad \text{for} \quad N - N_0 < n \leqslant N. \tag{21}$$

If ρ is (semi-)positive, *each* of its on-diagonal subdeterminants is (null or) positive. This may provide inequalities simpler than, but redundant with (21), in the same manner as $|x^2| < 1$ is redundant with $|x^2| + |y^2| < 1$.

The matrix ρ depends on N^2 real parameters. They can be $\mathrm{Re}\,(\rho_{ii'})$ for $i \leqslant i'$ and $\mathrm{Im}(\rho_{ii'})$ for $i < i'$, or the correlation parameters, which are linear combinations of them. In the N^2-dimensional parameter space, the domain of positivity of ρ is a convex half-cone. Its intersection with the hyperplane $\Sigma_1 \equiv \mathrm{trace}\,(\rho) = 1$ is a finite convex domain $\mathcal{D}$. The boundary of $\mathcal{D}$ is a sheet of the hypersurface $\Sigma_N \equiv \det\,(\rho) = 0$. It is a (N^2-2)-dimensional manifold of degree N. On this boundary, ρ is only semipositive. The other conditions, $\Sigma_n \geqslant 0$ for $n = 2, \ldots, N-1$ define domains which *include* $\mathcal{D}$. These «auxiliary» conditions serve to eliminate the other sheets of the hypersurface $\Sigma_N = 0$. The hypersurface where Σ_n, or any on-diagonal subdeterminant, vanishes is externally tangent to $\mathcal{D}$.

As we have seen, for an exclusive reaction the Grand Density Matrix R is of rank one. Therefore all the Σ_n's are vanishing for $n \geqslant 2$ and all the positivity constraints are saturated. It can be shown that R is on a «corner» of $\mathcal{D}$. On the contrary, when much information is lost through nondetected or nonanalyzed particles, R is «deep inside» $\mathcal{D}$.

5. EMPIRICAL APPROACH

The search for inequalities is straightforward using the density matrix method, but does not reveal at once the shape of the allowed domains. Also, when one writes the conditions on the density matrix, one gets in general a combination of several spin observables, and thus one has to make appropriate combinations of inequalities to obtain constraints on two or three given observables of interest.

To circumvent this difficulty, the following method was used in Ref. 5. The real and imaginary parts of the amplitudes were chosen randomly, and the spin observables were computed using their explicit expression in terms of amplitudes. This detects which pairs or triplets of observables fulfill inequalities, and then these inequalities can be derived by straightforward calculus. The case of pairs of observables is extensively discussed in Ref. 5, and preliminary results on triplets presented at the LEAP2003 conference [6]. A sample of the results is displayed in Figs. 1 and 2.

Notice that only a few types of inequalities are encountered. For pairs of observables, say X and Y, each being typically restricted to $[-1, +1]$, one gets the following possibilities:

- nothing: X and Y might reach any point of the square $[-1, +1]^2$,
- the disk $X^2 + Y^2 \leqslant 1$,
- a triangle $4Y^2 \leqslant (1 + X)^2$.

For triplets of observables, say X, Y, and Z, the following situations are obtained:

- nothing, any point of cube $[-1, +1]^3$ is allowed,

- a sphere $X^2 + Y^2 + Y^2 \leqslant 1$,
- a cone $(1 + Z)^2 \geqslant 4X^2 + 4Y^2$,
- a cubic of the type $X^2 + Y^2 + Z^2 \pm XYZ \leqslant 1$.

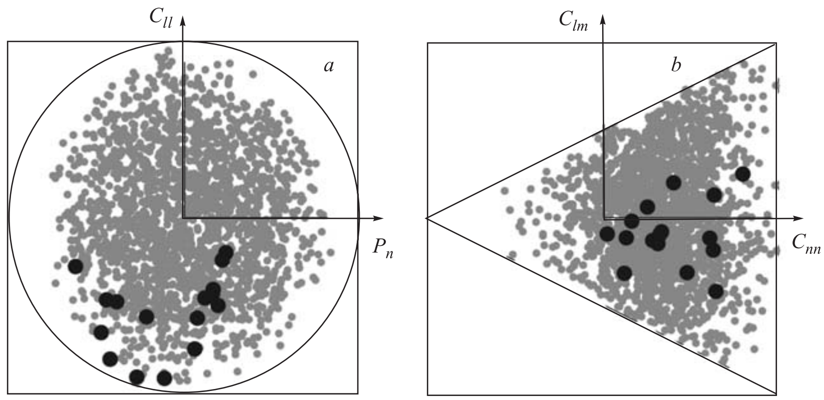

Fig. 1. Pair of observables restricted to the unit disk (a — here polarization and C_{ll} are shown) or to a triangle (b — here C_{nn} and C_{lm} are shown). The small grey dots correspond to hypothetical, randomly generated, amplitudes; and the larger dots, to actual data

The latter case is the most interesting, since the domain is restricted in space of three observables, but each projection covers the whole square, i.e., there is no restriction for any pair of observables within (X, Y, Z). The border has the shape of a twisted cushion.

6. INCLUSIVE CASE: THE SPIN-DEPENDENT PARTON DENSITIES

As an example of *inclusive* reaction, let us consider now the elementary process (2), which we rewrite for fixed momenta and spin vectors as

$$N(\mathbf{p}, \mathbf{S}_N) \to q(\mathbf{k}, \mathbf{S}_q) + X, \tag{22}$$

with $\mathbf{k} = x\mathbf{p} + \mathbf{k}_T$. The probability of (22) is the *spin- and* $\mathbf{k}_T$*-dependent* quark density in the nucleon. All what we will say below also applies to the quark fragmentation $q \to \mathrm{baryon} + X$, only commuting q and N, or to any inclusive reaction of the type

$$A\uparrow + B \to C\uparrow + X \tag{23}$$

already treated by Doncel and Méndez [4]. Thus the problem was solved long ago. One can also relate the inequalities in (23) to those of the crossed reaction [7]

$$A_1 \uparrow + A_2 \uparrow \rightarrow A_3 + X \tag{24}$$

by the correspondence $\mathbf{S}(A_2) \leftrightarrow -\mathbf{S}(C)$.

For a given spectator state X, the matrix element in spin Hilbert space can be written as $\langle\beta|M_X|\alpha\rangle$. Like in (10), (11), we consider the fictitious crossed process

$$\bar{X} \rightarrow \bar{N}(\mathbf{p}, -\mathbf{S}_N) + q(\mathbf{k}, \mathbf{S}_q), \tag{25}$$

where $|\bar{X}\rangle \equiv CPT|X\rangle$. Note that we have also moved the spectator system X to the *initial* state. The Grand Wave Function and Grand Density Matrix are then defined by

$$\langle\bar{\alpha}\beta|\Psi_X\rangle = \langle\beta|M_X|\alpha\rangle, \tag{26}$$

$$R = \sum_X |\Psi_X\rangle\langle\Psi_X|. \tag{27}$$

Thus R corresponds to a statistical mixture. Its rank r is the dimension of the subspace spanned by the vectors $|\Psi_X\rangle$ in the $(\bar{N}q)$ spin Hilbert space. It cannot exceed the number of possible quantum states of the spectator system. In general $r > 1$, which means that some information is lost, taken away by the spectator partons.

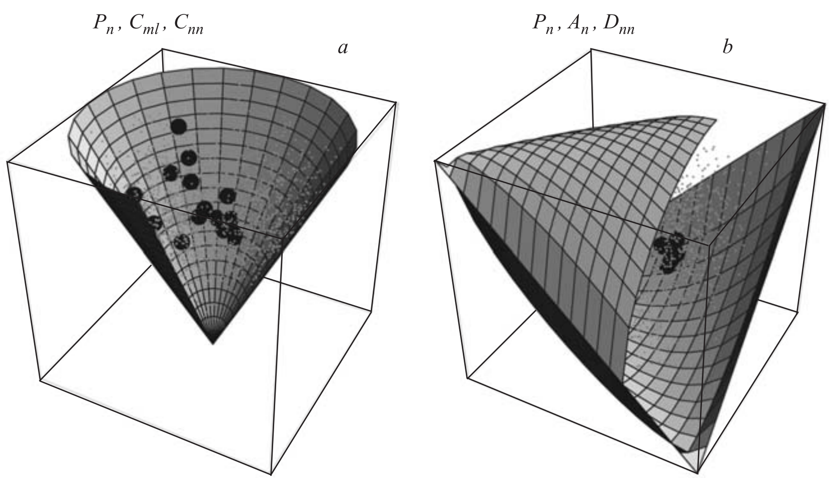

Fig. 2. Triplet of observables restricted to the inner volume of a cone (*a*) or of a cubic (*b*). The small dots correspond to randomly generated amplitudes; the larger ones (partly hidden), to actual data

In our case R has dimension 4×4 and depends on *a piori* 16 correlation parameters $C_{\mu\nu}$ through the analog of (17). However, like in the $2 \to 2$ reaction (1), the plane defined by $\mathbf{p}$ and $\mathbf{k}_T$ is a symmetry plane. It is therefore convenient to use the *transversity* basis with $\hat{\mathbf{z}}$ normal to this plane, instead of the helicity basis (unless one integrates over $\mathbf{k}_T$). In this basis R is even under $\sigma_x \to -\sigma_x$, $\sigma_y \to -\sigma_y$ and the only nonvanishing coefficients are $C_{00} \equiv 1$, C_{0z}, C_{z0}, C_{zz}, C_{xx}, C_{xy}, C_{yx}, and C_{yy}. If we take $\hat{\mathbf{x}}$ along $\mathbf{p}$ these parameters are respectively proportional to f_1, $-h_1^{\perp}$, $f_{1T}^{\perp}$, $h_1 - h_{1T}^{\perp}$, g_1, $h_{1L}^{\perp}$, g_{1T} and $h_1 + h_{1T}^{\perp}$ of Ref. 8, all kinematical factors in p_T/M or $p_T^2/(2M^2)$ removed. However the following inequalities are independent of the choice of the x and y axes in the production plane. Let us introduce

$$\frac{1 \pm C_{zz}}{2} \equiv D_{nn}^{\pm}, \; \frac{C_{0z} \pm C_{z0}}{2} \equiv A_n^{\pm}, \; \frac{C_{xx} \pm C_{yy}}{2} \equiv U^{\pm}, \; \frac{C_{xy} \pm C_{yx}}{2} \equiv V^{\pm}. \tag{28}$$

Putting the $|\bar{N}q\rangle$ basic spin states in the order $\{|\bar{u}u\rangle, |\bar{u}d\rangle, |\bar{d}u\rangle, |\bar{d}d\rangle\}$, one has

$$R = \frac{1}{2} \begin{pmatrix} D_{nn}^+ + A_n^+ & 0 & 0 & U^+ - iV^+ \\ 0 & D_{nn}^- - A_n^- & U^- + iV^+ & 0 \\ 0 & U^- - iV^+ & D_{nn}^- + A_n^- & 0 \\ U^+ + iV^- & 0 & 0 & D_{nn}^+ - A_n^+ \end{pmatrix}. \tag{29}$$

As expected from the symmetry about the (x, y) plane, R is block-diagonal in two rank-2 submatrices, which obey the separate positivity conditions:

$$(D_{nn}^{\pm})^2 \geqslant (A_n^{\pm})^2 + (U^{\pm})^2 + (V^{\mp})^2 \tag{30}$$

and

$$D_{nn}^{\pm} \geqslant 0, \quad \text{i.e.,} \quad |C_{zz}| \equiv |D_{nn}| \leqslant 1, \tag{31}$$

which agrees with the results of Bacchetta et al. [8] and of Ref. 4.

If we integrate over $\mathbf{k}_T$, the only surviving parameters are $C_{00} \equiv 1$, $C_{xx} \equiv \Delta q(x) q(x)$, and $C_{yy} = C_{zz} \equiv \delta q(x) q(x)$, where $q(x)$, $\Delta q(x)$, and $\delta q(x)$ are the quark *number*, quark *helicity* and quark *transversity* [9, 10] distributions. One obtains the Soffer inequality [11]:

$$2\delta q(x) \leqslant q(x) + \Delta q(x). \tag{32}$$

In a simple model of quark distribution where X just consists in a scalar diquark, all inequalities (30), (31) of the $\mathbf{k}_T$-dependent case are saturated. Indeed, such an object has no spin to carry quantum information away; a fully polarized nucleon delivers a fully polarized quark [10]. This is no more the case if we integrate over the degree of freedom $\mathbf{k}_T$. Nevertheless, the Soffer bound keeps saturated.

7. CONCLUSIONS AND OUTLOOK

The formalism developed in the 60's remains extremely powerful to analyze the consistency of spin observables. However, it needs some freshening, and new methods are needed to quickly derive the inequalities within a subset of accessible observables. We hope to have worked in this direction.

One of the basic tools, already used in [4], is a fictitious crossing which gathers all the particles on the same side. It is expressed as a partial matrix transposition. Usual crossing also links different physical reactions, just reversing the polarization vectors.

Particle spin physics also touches the more general theory of quantum information, in particular with the concept of entanglement. The fact that the particles considered here have definite momenta is not essential. The inequalities obtained in particle physics could also apply to «gates» between other kinds of quantum information channels like optical fibers.

In a forthcoming article, we shall provide more details about the positivity conditions and their physical interpretation. In particular, we will show explicitly that the method of the Grand Density Matrix gives the same constraints on observables that the empirical approach based on randomly generated amplitudes.

Appendix
EFFECT OF CROSSING ON THE OPERATORS ACTING ON AN INITIAL PARTICLE

Together with (11), we have $\langle\alpha'| \to |\bar{\alpha}'\rangle$,

$$|\alpha\rangle\langle\alpha'| \to |\bar{\alpha}'\rangle\langle\bar{\alpha}|, \tag{A.1}$$

and for a linear combination of such elementary operators

$$\sum_{\alpha,\alpha'} |\alpha\rangle A_{\alpha\alpha'} \langle\alpha'| \quad \to \quad \sum_{\alpha,\alpha'} |\bar{\alpha}'\rangle A_{\alpha\alpha'} \langle\bar{\alpha}|. \tag{A.2}$$

Here we have assumed that crossing acts *linearly* on operators. Indeed (11) is the product of two antilinear operations: (i) applying CPT, (ii) changing a ket into a bra. Equation (A.2) amounts to the matrix transposition $A \to A^t$, provided we choose the same ordering for the crossed basis vectors $\{|\bar{u}\rangle,\ |\bar{d}\rangle\}$ as in the initial basis vectors $\{|u\rangle,\ |d\rangle\}$ (the ordering in *magnetic* number s_z is reversed: it becomes $\{|-1/2\rangle,\ |+1/2\rangle\}$). For a single-spin matrix density, the transformation is

$$\rho = \frac{1}{2}(1 + \mathbf{P}\cdot\sigma) \quad \to \quad \rho^t = \rho^\dagger = \frac{1}{2}(1 + \bar{\mathbf{P}}\cdot\underline{\sigma}), \tag{A.3}$$

where $\bar{\mathbf{P}} = -\mathbf{P}$ due to spin reversal, and $\underline{\sigma_i} = -\sigma_i^t$ are the Pauli matrices for the $\{\bar{2}\}$ representation of $SU(2)$ (which is not often used, due to the equivalence $\{2\} \leftrightarrow \{\bar{2}\}$).

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