CLASSICAL INTEGRABLE SYSTEMS AND GAUGE FIELD THEORIES

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In this review we consider the Hitchin integrable systems and their relations with the self-duality equations and twisted supersymmetric Yang–Mills theory in four dimensions. We define the symplectic Hecke correspondence between different integrable systems. As an example, we consider elliptic Calogero–Moser system and integrable Euler-Arnold top on coadjoint orbits of the group $GL(N, \mathbb{C})$ and explain the symplectic Hecke correspondence for these systems.

Дается обзор интегрируемых систем Хитчина и их связей с самодуальными уравнениями и твистованными суперсимметричными теориями Янга–Миллса в четырех измерениях. Определяется симплектическое соответствие Гекке между различными интегрируемыми системами. В качестве примера рассматриваются эллиптическая система Калоджеро–Мозера и интегрируемый волчок Эйлера–Арнольда на коприсоединенных орбитах группы $GL(N, \mathbb{C})$ и поясняется симплектическое соответствие Гекке для этих систем.

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INTRODUCTION

Some interrelations between classical integrable systems and field theories in dimensions 3 and 4 were proposed by N. Hitchin twenty years ago [1, 2]. This approach to integrable systems has some advantages. It immediately leads to the Lax representation with a spectral parameter, allows one to prove in some cases the algebraic integrability and to find separated variables [3,4]. It was found later that some well-known integrable systems can be derived in this way [5–12].

It was demonstrated in [13] that there exists an integrable regime in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory in four dimensions, which is described by Sieberg and Witten [14]. A general picture of interrelations between integrable models and gauge theories in dimensions 4, 5, and 6 was presented in review [15].

Some new aspects of interrelations between integrable systems and gauge theories were found recently in the framework of four-dimensional reformulation

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of the geometric Langlands program [16–18]. This review takes into account this approach, but also is based on the papers and reviews [1, 2, 11, 12, 19–22].

The derivation of integrable systems from field theories is based on the symplectic or the Poisson reduction. This construction is familiar in gauge field theories. The physical degrees of freedom in gauge theories are defined upon imposing the first- and the second-class constraints. The first-class constraints are analogs of the Gauss law generating the gauge transformations. A combination of the Gauss law and constraints coming from the gauge fixing yields the second-class constraints.

We start with gauge theories that have some important properties. First, they have at least a finite number of independent conserved quantities. After the reduction they will play the role of integrals of motion. Next, we assume that after a gauge fixing and solving the constraints, the reduced phase space becomes a finite-dimensional manifold and its dimension is twice of the number of integrals. The latter property provides the complete integrability. It is, for example, the theory of the Higgs bundles describing the Hitchin integrable systems [1]. This theory corresponds to a gauge theory in dimension three. On the other hand, the similar type of constraints arises in reduction of the self-duality equations in the four-dimensional Yang–Mills theory [1], and in the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [16] after reducing them to a space of dimension two.

We also analyze the problem of the classification of integrable systems. Roughly speaking, two integrable systems are called equivalent if the original field theories are gauged equivalent. We extend the gauge transformations by allowing singular gauge transformations of a special kind. On the field theory side, these transformations correspond to monopole configurations, or, equivalently, to the including of the 't Hooft operators [23, 24]. For some particular examples we establish in this way an equivalence of integrable systems of particles (the Calogero–Moser systems) and integrable Euler–Arnold tops. It turns out that this equivalence is the same as equivalence of two types of R matrices: of dynamical and vertex type [25, 26].

Before considering concrete cases we remind the main definitions of completely integrable systems [20, 21, 27].

1. CLASSICAL INTEGRABLE SYSTEMS

Consider a smooth symplectic manifold $\mathcal{R}$ of $\dim(\mathcal{R}) = 2l$. It means that there exists a closed nondegenerate two-form ω, and the inverse bivector π ($\omega_{a,b}\pi^{bc} = \delta_a^c$), such that the space $C^\infty(\mathcal{R})$ becomes a Lie algebra (the Poisson algebra) with respect to the Poisson brackets,

$$\{F, G\} = \langle dF|\pi|dG \rangle = \partial_a F \pi^{ab} \partial_b G.$$

Any $H \in C^\infty(\mathcal{R})$ defines a Hamiltonian vector field on $\mathcal{R}$

$$H \to \langle dH | \pi = \partial_a H \pi^{ab} \partial_b = \{H, .\}.$$

A Hamiltonian system is a triple $(\mathcal{R}, \pi, H)$ with the Hamiltonian flow

$$\partial_t x^a = \{H, x^a\} = \partial_b H \pi^{ba}.$$

A Hamiltonian system is called *completely integrable* if it satisfies the following conditions:

• There exists l Poisson commuting Hamiltonians on $\mathcal{R}$ (*integrals of motion*) $I_1, \ldots, I_l$;

• Since the integrals commute, the set $T_c = \{I_j = c_j\}$ is invariant with respect to the Hamiltonian flows $\{I_j, .\}$. Then being restricted on T_c, $I_j(x)$ are functionally independent almost for all $x \in T_c$, i.e., $\det(\partial_a I_b)(x) \neq 0$.

In this way we come to the hierarchy of commuting flows on $\mathcal{R}$

$$\partial_{t_j} \mathbf{x} = \{I_j(\mathbf{x}), \mathbf{x}\}. \tag{1.1}$$

T_c is a submanifold $T_c \subset \mathcal{R}$. It is a Lagrangian submanifold, i.e., ω vanishes on T_c. If T_c is compact and connected, then it is diffeomorphic to an l-dimensional torus. The torus T_c is called *the Liouville torus*. In a neighborhood of T_c there is a projection

$$p : \mathcal{R} \to B, \tag{1.2}$$

where the Liouville tori are generic fibers, and the base of fibration B is parameterized by the values of integrals. The coordinates on a Liouville torus («the angle» variables) along with dual variables on B («the action» variables) describe a linearized motion on the torus. Globally, the picture can be more complicated. For some values of c_j, T_c ceases to be a submanifold. In this way the action-angle variables are local.

Here we consider a complex analog of this picture. We assume that $\mathcal{R}$ is a complex algebraic manifold, and the symplectic form ω is a $(2,0)$ form, i.e., locally in the coordinates $(z^1, \bar{z}^1, \ldots, z^l, \bar{z}^l)$ the form is represented as $\omega = \omega_{a,b} dz^a \wedge dz^b$. General fibers of (1.2) are *Abelian subvarieties* of $\mathcal{R}$, i.e., they are complex tori $\mathbb{C}^l / \Lambda$, where the lattice Λ satisfies the Riemann conditions. Integrable systems in this situation are called *algebraically integrable systems*.

Let two integrable systems be described by two isomorphic sets of the action-angle variables. In this case, the integrable systems can be considered as equivalent. Establishing equivalence in terms of angle-action variables is troublesome. There exists a more direct way based on *the Lax representation*. The Lax representation is one of the commonly accepted methods of construction and investigation of integrable systems. Let $L(x, z)$, $M_1(x, z), \ldots, M_l(x, z)$ be a set of $l + 1$

matrices depending on $x \in \mathcal{R}$ with a meromorphic dependence on the *spectral parameter* $z \in \Sigma$, where Σ is a Riemann surface*. It is called *a basic spectral curve*. Assume that the commuting flows (1.1) can be rewritten in the matrix form

$$\partial_{t_j} L(\mathbf{x}, z) = [L(\mathbf{x}, z), M_j(\mathbf{x}, z)]. \tag{1.3}$$

Let f be a nondegenerate matrix of the same order as L and M. The transformation

$$L' = f^{-1} L f, \quad M'_j = f^{-1} \partial_{t_j} f + f^{-1} M_j f \tag{1.4}$$

is called the gauge transformation because it preserves the Lax form (1.3). The flows (1.3) can be considered as special gauge transformations

$$L(t_1, \ldots, t_l) = f^{-1}(t_1, \ldots, t_l) L_0 f(t_1, \ldots, t_l),$$

where L_0 is independent of time and defines an initial data, and $M_j = f^{-1} \partial_{t_j} f$. Moreover, it follows from this representation that the quantities $\mathrm{tr}\,(L(\mathbf{x}, z))^j$ are preserved by the flows and thereby can produce, in principle, the integrals of motion. As we mentioned above, it is reasonable to consider two integrable systems to be equivalent if their Lax matrices are related by nondegenerate gauge transformation.

We relax the definition of the gauge transformations and assume that $\det f$ can have poles and zeroes on the basic spectral curve Σ with some additional restrictions on f. This equivalence is called *the symplectic Hecke correspondence*. This extension of equivalence will be considered in this review in detail. The following systems are equivalent in this sense:

EXAMPLES

1. Elliptic Calogero–Moser system $\Leftrightarrow$ Elliptic $GL(N, \mathbb{C})$ Top [11];
2. Calogero–Moser field theory $\Leftrightarrow$ Landau–Lifshitz equation [10, 11];
3. Painlevé VI $\Leftrightarrow$ Zhukovsky–Volterra gyrostat [12].

The first example will be considered in Sec. 3.

The gauge invariance of the Lax matrices allows one to define the spectral curve

$$\mathcal{C} = \{(\lambda \in \mathbb{C}, z \in \Sigma) \,|\, \det(\lambda - L(x, z)) = 0\}. \tag{1.5}$$

The Jacobian of $\mathcal{C}$ is an Abelian variety of dimension g, where g is the genus of $\mathcal{C}$. If $g = l = 1/2 \dim \mathcal{R}$, then $\mathcal{J}$ plays the role of the Liouville torus and the system is algebraically integrable. In generic cases $g > l$, and to prove the algebraic integrability one should find additional reductions of the Jacobians, leading to Abelian spaces of dimension l.

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*It will be explained below that L and M are sections of some vector bundles over Σ.

Finally we formulate two goals of this review:
- derivation of the Lax equation and the Lax matrices from a gauge theory;
- explanation of the equivalence between integrable models by inserting 't Hooft operators in a gauge theory.

2. 1D FIELD THEORY

The simplest integrable models such as the rational Calogero–Moser system, the Sutherland model, the open Toda model can be derived from matrix models of a finite order. Here we consider a particular case — the rational Calogero–Moser system (RCMS) [32, 33].

2.1. Rational Calogero–Moser System (RCMS). The phase space of the RCMS is

$$\mathcal{R}^{\mathrm{RCM}} = \mathbb{C}^{2N} = \{(\mathbf{v}, \mathbf{u})\}, \quad \mathbf{v} = (v_1, \dots v_N), \quad \mathbf{u} = (u_1, \dots u_N)$$

with the canonical symplectic form

$$\omega^{\mathrm{RCM}} = \sum_{j=1}^{N} dv_j \wedge du_j, \quad \{v_j, u_k\} = \delta_{jk}. \tag{2.1}$$

The Hamiltonian describes interacting particles with complex coordinates $\mathbf{u} = (u_1, \dots u_N)$ and complex momenta $\mathbf{v} = (v_1, \dots v_N)$

$$H^{\mathrm{RCM}} = \frac{1}{2} \sum_{j=1}^{N} v_j^2 + \nu^2 \sum_{j>k} \frac{1}{(u_j - u_k)^2} \, .$$

The Hamiltonian leads to the equations of motion

$$\partial_t u_j = v_j, \tag{2.2}$$

$$\partial_t v_j = -\nu^2 \sum_{j>k} \frac{1}{(u_j - u_k)^3} \, . \tag{2.3}$$

The equations of motion can be put in the Lax form

$$\partial_t L(\mathbf{v}, \mathbf{u}) = [L(\mathbf{v}, \mathbf{u}), M(\mathbf{v}, \mathbf{u})]. \tag{2.4}$$

Here L, M are the $N \times N$ matricies of the form

$$L = P + X, \quad M = D + Y,$$

$$P = \mathrm{diag}\,(v_1, \dots v_N), \quad X_{jk} = \nu(u_j - u_k)^{-1}, \tag{2.5}$$

$$Y_{jk} = -\nu(u_j - u_k)^{-2}, \quad D = \operatorname{diag}(d_1, \ldots d_N), \tag{2.6}$$

$$d_j = \nu \sum_{k \neq j} (u_j - u_k)^{-2}.$$

The diagonal part of the Lax equation (2.4) implies $\partial_t P = [X, Y]_{\text{diag}}$. It coincides with (2.3). The nondiagonal part has the form

$$\partial_t X = [P, Y] + ([X, Y]_{\text{nondiag}} - [X, D]).$$

It can be found that $[X, Y]_{\text{nondiag}} = [X, D]$ and the equation $\partial_t X = [P, Y]$ coincides with (2.2).

The Lax equations produce the integrals of motion

$$I_m = \frac{1}{m} \operatorname{tr}(L^m), \quad \partial_t \operatorname{tr}(L^m) = 0, \quad m = 1, 2, \ldots N. \tag{2.7}$$

It will be proved later that they are in involution $\{I_m, I_n\} = 0$. In particular, $I_2 = H^{\text{RCM}}$. Eventually, we come to the RCMSS hierarchy

$$\partial_j f(\mathbf{v}, \mathbf{u}) = \{I_j, f(\mathbf{v}, \mathbf{u})\}. \tag{2.8}$$

2.2. Matrix Mechanics and the RCMS. This construction was proposed in [35, 36]. Consider a matrix model with the phase space $\mathcal{R} = gl(N, \mathbb{C}) \oplus gl(N, \mathbb{C})$

$$\mathcal{R} = (\Phi, \bar{A}), \quad \Phi, \bar{A} \in gl(N, \mathbb{C}), ^* \quad \dim \mathcal{R} = 2N^2.$$

The symplectic form on $\mathcal{R}$ is

$$\omega = \operatorname{tr}(d\Phi \wedge d\bar{A}) = \sum_{j,k} d\Phi_{jk} \wedge d\bar{A}_{kj}. \tag{2.9}$$

The corresponding Poisson brackets have the form

$$\{\Phi_{jk}, \bar{A}_{il}\} = \delta_{ki}\delta_{jl}.$$

Choose N commuting integrals

$$I_m = \frac{1}{m} \operatorname{tr}(\Phi^m), \quad \{I_m, I_n\} = 0, \quad m = 1, \ldots N.$$

Take as a Hamiltonian $H = I_2$. Then we come to the free motion on $\mathcal{R}$

$$\partial_t \Phi = \{H, \Phi\} = 0, \tag{2.10}$$

$$\partial_t \bar{A} = \{H, \bar{A}\} = \Phi. \tag{2.11}$$

Generally, we have a free matrix hierarchy

$$\partial_j \Phi = 0, \quad \partial_j \bar{A} = \Phi^{j-1} \quad (\partial_j = \{I_j, \}). \tag{2.12}$$

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*These notations will be justified in the next Section.

2.2.1. Hamiltonian Reduction for RCMS. The form ω and the integrals I_m are invariant with respect to the action of the gauge group

$$\mathcal{G} = GL(N, \mathbb{C}),$$

$$\Phi \to f^{-1}\Phi f, \quad \bar{A} \to f^{-1}\bar{A}f, \quad f \in GL(N, \mathbb{C}).$$

The action of gauge Lie algebra $\mathrm{Lie}\,(\mathcal{G}) = gl\,(N, \mathbb{C})$ is represented by the vector fields

$$V_\epsilon \Phi = [\Phi, \epsilon], \quad V_\epsilon \bar{A} = [\bar{A}, \epsilon]. \tag{2.13}$$

Let $\imath_\epsilon$ be the contraction operator with respect to the vector field V_ϵ $\left(\imath_\epsilon = \sum_{j,k} (V_\epsilon)_{jk} \dfrac{\partial}{\partial_{jk}} \right)$, and $\mathcal{L}_\epsilon = d\imath_\epsilon + \imath_\epsilon d$ be the corresponding Lie derivative. The invariance of the symplectic form and the integrals means that

$$\mathcal{L}_\epsilon \omega = 0, \quad \mathcal{L}_\epsilon I_m = 0.$$

Since the symplectic form is closed $d\omega = 0$, we have $d\imath_\epsilon \omega = 0$. Then, on the affine space $\mathcal{R}$ the one-form $\imath_\epsilon \omega$ is exact

$$\imath_\epsilon \omega = dF(\Phi, \bar{A}, \epsilon). \tag{2.14}$$

The function $F(\Phi, \bar{A}, \epsilon)$ is called *the momentum Hamiltonian*. The Poisson brackets with the momentum Hamiltonian generate the gauge transformations:

$$\{F, f(\Phi, \bar{A}\} = \mathcal{L}_\epsilon f(\Phi, \bar{A}).$$

The explicit form of the momentum Hamiltonian is

$$F(\Phi, \bar{A}, \epsilon) = \mathrm{tr}\,(\epsilon[\Phi, \bar{A}]).$$

Define *the moment map*

$$\mu : \mathcal{R} \to \mathrm{Lie}^*(\text{gauge group}) \sim gl\,(N, \mathbb{C}),$$

$$\mu(\Phi, \bar{A}) = [\Phi, \bar{A}], \quad (\Phi, \bar{A}) \mapsto [\Phi, \bar{A}]. \tag{2.15}$$

Let us fix its value as

$$\mu = [\Phi, \bar{A}] = \nu J, \tag{2.16}$$

$$J = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}. \tag{2.17}$$

It follows from the definition of the moment map that (2.16) is the first-class constraint. In particular, $\{F(\Phi, \bar{A}, \epsilon), F(\Phi, \bar{A}, \epsilon')\} = F(\Phi, \bar{A}, [\epsilon, \epsilon'])$. Note, that the matrix J is degenerate and is conjugated to the diagonal matrix $\operatorname{diag}(N - 1, -1, \ldots, -1)$. Let $\mathcal{G}_0$ be a subgroup of the gauge group preserving the moment value

$$\mathcal{G}_0 = \{f \in \mathcal{G} \mid f^{-1}Jf = J\} \quad (\dim(\mathcal{G}_0) = (N-1)^2 + 1).$$

In other words, $\mathcal{G}_0$ preserves the surface in $\mathcal{R}$

$$F^{-1}(\nu J) = \{[\Phi, \bar{A}] = \nu J\}. \tag{2.18}$$

Let us fix a gauge on this surface with respect to the $\mathcal{G}_0$ action. It can be proved that generic matrices $\bar{A}$ can be diagonalized by $\mathcal{G}_0$

$$f^{-1}\bar{A}f = \mathbf{u} = \operatorname{diag}(u_1, \ldots u_n), \quad f \in \mathcal{G}_0. \tag{2.19}$$

In other words, we have two conditions — the first-class constraints (2.16) and the gauge fixing (2.19). The reduced phase space $\mathcal{R}^{\text{red}}$ is the result of a putting both the types of constraints

$$\mathcal{R}^{\text{red}} = \mathcal{R}//\mathcal{G} = F^{-1}(\nu J)/\mathcal{G}_0.$$

It has dimension

$$\dim(\mathcal{R}^{\text{red}}) = \dim(\mathcal{R}) - \dim(\mathcal{G}) - \dim(\mathcal{G}_0),$$

$$2N - 2 = 2N^2 - N^2 - (N-1)^2 + 1).$$

Let us prove that $\mathcal{R}^{\text{red}} = \mathcal{R}^{\text{RCM}}$ and that the hierarchy (2.12) being restricted on $\mathcal{R}^{\text{RCM}}$ coincides with the RCMS hierarchy (2.8). Let $f \in \mathcal{G}_0$ diagonalize $\bar{A}$ in (2.19). Define

$$L = f^{-1}\Phi f. \tag{2.20}$$

Then it follows from (2.10) that L satisfies the Lax equation

$$\partial_t \Phi = 0 \Rightarrow \partial_t L = [L, M] \quad (M = -f^{-1}\partial_t f).$$

The moment constraint (2.18) allows one to find the off-diagonal part of L. Evidently, it coincides with X (2.5). The diagonal elements of L are free parameters. In a similar way the off-diagonal part Y (2.6) of M can be derived from the equation of motion for $\bar{A}$ (2.11). Thereby, we come to the Lax form of the equations of motion for RCMS. Since $\Phi \to L$ and $\bar{A} \to \mathbf{u}$, the symplectic form ω (2.9) coincides on $\mathcal{R}^{\text{RCM}}$ with ω^{RCM} (2.1). It follows from (2.20) that the integrals (2.7) Poisson commute. Therefore, we obtain the RCMS hierarchy.

The same system can be derived starting with the matrix mechanics based on $SL(N, \mathbb{C})$. In this case $I_1 = \operatorname{tr} \Phi = 0$ and thereby in the reduced system $\sum v_j = 0$.

2.2.2. Hamiltonian Reduction for 4d Yang–Mills Theory. In this subsection we make a step aside to illustrate the Hamiltonian reduction in terms of the familiar phase space of the Yang–Mills theory. For this purpose consider 4d YM theory with a group G in the Hamiltonian formalism [37, 38]. The phase space is generated by the space components on $\mathbb{R}^3 = (x_1, x_2, x_3)$ of the vector potential and the electric field

$$\mathcal{R} = \{ \mathbf{A} = (A_1, A_2, A_3), \quad \mathbf{E} = (E_1, E_2, E_3) \},$$

where

$$E_j = F_{0j} = \partial_0 A_j - \partial_j A_0 + [A_0, A_j].$$

Here we suppressed the Lie algebra indices. It is a symplectic space with the canonical form

$$\omega = \int_{\mathbb{R}^3} \langle d\mathbf{E} \wedge d\mathbf{A} \rangle = \sum_{j=1}^{3} \int_{\mathbb{R}^3} \operatorname{tr} (dE_j \wedge dA_j).$$

The Hamiltonian is quadratic in fields and has the form

$$H = \frac{1}{2} \int_{\mathbb{R}^3} \langle \mathbf{E}^2 \rangle + \langle \mathbf{B}^2 \rangle,$$

where $\mathbf{B} = (B_1, B_2, B_3)$ is the magnetic field

$$B_j = \epsilon_{jkl} F_{kl} = \varepsilon_{jkl} (\partial_k A_j - \partial_j A_k + [A_k, A_j]).$$

We assume that the fields are smooth and vanish on infinity such that the Hamiltonian and the symplectic form are well defined. The Hamiltonian defines the classical equations of motion

$$\partial_t A_j = E_j, \quad \partial_t E_j = \sum_k [\partial_k + A_k, F_{kj}].$$

The Hamiltonian and the form are invariant with respect to the gauge transformations

$$\mathbf{A} \to \mathbf{A}^f = f^{-1} df + f^{-1} \mathbf{A} f, \quad \mathbf{e} \to \mathbf{E}^f = f^{-1} \mathbf{E} f,$$

where $d = (\partial_1, \partial_2, \partial_3)$. We assume that f is a smooth map $f \in \mathcal{G} = C^\infty(\mathbb{R}^3 \to G)$, vanishing on infinity and at some marked points

$$f(\mathbf{y}^a) = 0, \quad \mathbf{y}^a = (y_1^a, y_2^a, y_3^a) \quad (a = 1, \dots, n).$$

Infinitesimal gauge transformations define the vector field on the phase space

$$V_\epsilon \mathbf{E} = [\mathbf{E}, \epsilon], \quad V_\epsilon \mathbf{A} = d\epsilon + [\mathbf{A}, \epsilon], \quad \epsilon \in \mathrm{Lie}\,(\mathcal{G}).$$

The corresponding momentum Hamiltonian is

$$F(\mathbf{E}, \mathbf{A}, \epsilon) = \int_{\mathbb{R}^3} \left\langle \epsilon \left(\sum_{j=1}^3 \partial_j E_j + [A_j, E_j] \right) \right\rangle$$

(compare with (2.14)). Therefore the moment takes the form

$$\mu(\mathbf{E}, \mathbf{A}) = \sum_{j=1}^3 \partial_j E_j + [A_j, E_j].$$

It is an element of the gauge co-algebra $\mathrm{Lie}^*(\mathcal{G})$. In other words, the moment belongs to the map of the phase space $\mathcal{R}$ to the distributions on $\mathbb{R}^3$ with values in $\mathrm{Lie}^*(G)$.

Let us fix it as

$$\sum_{j=1}^3 \partial_j E_j + [A_j, E_j] = \sum_a \delta(\mathbf{x} - \mathbf{y}^a) \rho_a, \tag{2.21}$$

where $\rho_a \in \mathrm{Lie}^*(\mathcal{G})$. The moment constraint (2.21) is none other than the Gauss law, and ρ_a are the electric charges.

To come to the reduced phase space $\mathcal{R}^{\mathrm{red}} = \mathcal{R}//\mathcal{G}$ we should add to the Gauss law a gauge fixing condition. Note that the gauge transformations vanish at the points $\mathbf{y}^a$, and in this way they preserve the right-hand side of (2.21). Starting with six fields (A_j, E_j) defining $\mathcal{R}$, we put two types of constraints — the Gauss law and the gauge fixing. Roughly speaking, they kill two fields and the reduced phase space describes the «transversal degrees of freedom».

3. 3D FIELD THEORY

3.1. Hitchin Systems. *3.1.1. Fields.* Define a field theory on $(2+1)$-dimensional space-time of the form $\mathbb{R} \times \Sigma_{g,n}$, where $\Sigma_{g,n}$ is a Riemann surface of genus g with a divisor $D = (x_1, \dots, x_n)$ of n marked points.

$(x_1, \ldots, x_4)$ − marked points
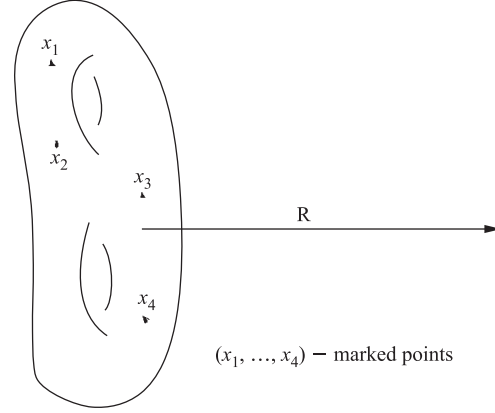
The phase space of the theory is defined by the following field content:

1) Consider a vector bundle E of rank N over $\Sigma_{g,n}$ equipped with the connection $d'' = \nabla_{\bar{z}} \otimes d\bar{z}$. It acts on the sections $s^T = (s_1, \ldots, s_N)$ of E as $d'' s = \bar{\partial}s + \bar{A}s$. *The vector fields $\bar{A}(z, \bar{z})$ are C^∞ maps $\Sigma_{g,n} \to gl\,(N, \mathbb{C})$.*

2) The scalar fields (*the Higgs fields*) $\Phi(z, \bar{z}) \otimes dz$, $\Phi : \Sigma_{g,n} \to gl\,(N, \mathbb{C})$. The Higgs field is a section of the bundle $\Omega^{(1,0)}(\Sigma_{g,n}, \operatorname{End} E)$. It means that Φ acts on the sections $s_j \to \Phi_{kj} s_j \otimes dz$. We assume that Φ has holomorphic poles at the marked points $\Phi \sim \dfrac{\Phi^a}{z - x_a} + \ldots$ Let $(\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g)$ be a set of fundamental cycles of $\Sigma_{g,n}$, $\left(\prod_j \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = 1 \right)$. The bundle E is defined by the monodromy matrices (Q_j, Λ_j)

$$\alpha_j : s \to Q_j^{-1} s, \quad \beta_j : \Lambda_j^{-1} s.$$

Similarly, for $\bar{A}$ and Φ we have

$$\begin{aligned}
\alpha_j &: \bar{A} \to Q_j \bar{\partial} Q_j^{-1} + Q_j \bar{A} Q_j^{-1}, \quad &\beta_j &: \bar{A} \to \Lambda_j \bar{\partial} \Lambda_j^{-1} + \Lambda_j \bar{A} \Lambda_j^{-1}, \\
\alpha_j &: \Phi \to Q_j \Phi Q_j^{-1}, \quad &\beta_j &: \Phi \to \Lambda_j \Phi \Lambda_j^{-1}.
\end{aligned} \tag{3.1}$$

3) *The spin variables* are attributed to the marked points $S^a \in gl\,(N, \mathbb{C})$, $a = 1, \ldots, n$, $S^a = g^{-1} S^a(0) g$, where $S^a(0)$ is a fixed element of $gl\,(N, \mathbb{C})$. In other words, S^a belong to coadjoint orbits $\mathcal{O}^a$ of $GL(N, \mathbb{C})$. They play the role of non-Abelian charges located at the marked points.

Let $\{T_\alpha\}$ $(\alpha = 1, \ldots, N^2)$ be a basis in the Lie algebra $gl\,(N, \mathbb{C})$, $[T_\alpha, T_\beta] = C_{\alpha,\beta}^\gamma T_\gamma$. Define the Poisson structure on the space of fields:

1) The Darboux brackets for the fields (A, Φ):

$$\bar{A}(z, \bar{z}) = \sum_{\alpha} \bar{A}_{\alpha}(z, \bar{z})T_{\alpha}, \quad \Phi(w, \bar{w}) = \sum_{\beta} \Phi_{\beta}(w, \bar{w})T_{\beta},$$

$$\{\Phi_{\alpha}, (w, \bar{w}), \bar{A}_{\beta}(z, \bar{z})\} = \langle T_{\alpha}T_{\beta}\rangle\delta(z - w, w - \bar{w}) \quad (\langle \ \rangle = \text{trace in ad}).$$

2) Linear Lie brackets for the spin variables:
$$\mathbf{S}^a = \sum_{\alpha} S_{\alpha}^a T_{\alpha}$$

$$\{S_{\alpha}^a, S_{\beta}^b\} = \delta^{a,b} C_{\alpha,\beta}^{\gamma} S_{\gamma}^a.$$

In this way we have defined the phase space

$$\mathcal{R} = (\bar{A}, \Phi, \mathbf{S}^a). \tag{3.2}$$

The Poisson brackets are nondegenerate and the space $\mathcal{R}$ is symplectic with the form

$$\omega = \omega^0 - \sum_{a=1}^{n} \int_{\Sigma_{g,n}} \omega^a \delta(z - x_a, \bar{z} - \bar{x}_a), \tag{3.3}$$

$$\omega^0 = \int_{\Sigma_{g,n}} \langle D\Phi \wedge D\bar{A}\rangle, \tag{3.4}$$

$$\omega^a = \langle D(S^a g^{-1}) \wedge Dg\rangle. \tag{3.5}$$

The last form is the Kirillov–Kostant form on the coadjoint orbits. The fields $(\Phi, \bar{A})$ are holomorphic coordinates on $\mathcal{R}$ and the form ω^0 is the $(2, 0)$ form in this complex structure on $\mathcal{R}$. Similarly, $(S^a g^{-1}, g)$ are the holomorphic coordinates on the orbit $\mathcal{O}^a$, and ω^a is also $(2, 0)$ form.

3.1.2. Hamiltonians. The traces $\langle \Phi^j\rangle$ $(j = 1, \dots, N)$ of the Higgs field are periodic $(j, 0)$ forms $\Omega^{(j,0)}(\Sigma_{g,n})$ with holomorphic poles of order j at the marked points. To construct integrals from $\langle \Phi^j\rangle$ one should integrate them over $\Sigma_{g,n}$ and to this end prepare $(1, 1)$ forms from the $(j, 0)$ forms. For this purpose consider the space of smooth $(1 - j, 1)$ differentials $\Omega^{(1-j,1)}(\Sigma_{g,n} \setminus D)$ vanishing at the marked points. Locally, they are represented as $\mu_j = \mu_j(z, \bar{z}) \left(\dfrac{\partial}{\partial z}\right)^{j-1} \otimes d\bar{z}$.

In other words, μ_j are $(0, 1)$ forms taking values in degrees of vector fields $\mathcal{T}$ on $\Sigma_{g,n} \setminus D$. For example, μ_2 is the Beltrami differential.

The product $\langle \Phi^j\rangle\mu_j$ can be integrated over the surface. We explain below that μ_j can be chosen as elements of basis in the cohomology space $H^1(\Sigma_{g,n} \setminus D, \mathcal{T}^{\otimes j-1})$. This space has dimension

$$n_j = \dim H^1(\Sigma_{g,n}, \mathcal{T}^{\otimes(j-1)}) = \begin{cases} (2j - 1)(g - 1) + jn, & j > 1, \\ g, & j = 1. \end{cases} \tag{3.6}$$

Let $\mu_{j,k}$ be a basis in $H^1(\Sigma_{g,n}, \mathcal{T}^{\otimes(|-\infty)})$ $(k = 1, \ldots, n_j)$. The product $\mu_{j,k}\langle\Phi^j\rangle$ can be integrated to define the Hamiltonians

$$I_{j,k} = \frac{1}{j} \int\limits_{\Sigma_{g,n}} \mu_{j,k}\langle\Phi^j\rangle, \quad j = 1, \ldots N. \tag{3.7}$$

It follows from (3.6) that the number of the independent integrals $\sum n_j$ for $GL(N, \mathbb{C})$ is

$$d_{N,g,n} = \sum_{j=1}^{N} n_j = (g-1)N^2 + 1 + n\frac{N(N-1)}{2}. \tag{3.8}$$

Since $\langle\Phi\rangle = 0$ for $SL(N, \mathbb{C})$ the number of the independent integrals is

$$d_{N,g,n} = \sum_{j=2}^{N} n_j = (g-1)(N^2-1) + n\frac{N(N-1)}{2}. \tag{3.9}$$

The integrals $I_{(j,k)}$ are independent and Poisson commute

$$\{I_{(j_1,k_1)}, I_{(j_2,k_2)}\} = 0. \tag{3.10}$$

Thus, we come to $d_{N,g,n}$ commuting flows on the phase space $\mathcal{R}(\bar{A}, \Phi, \mathbf{S}^a)$

$$\frac{\partial}{\partial t_{j,k}}\Phi = \{\nabla I_{j,k}, \Phi\} = 0, \tag{3.11}$$

$$\frac{\partial}{\partial t_{j,k}}\bar{A} = \mu_{j,k}\Phi^{j-1}, \tag{3.12}$$

$$\frac{\partial}{\partial t_{j,k}}\mathbf{S}^a = 0. \tag{3.13}$$

3.1.3. Action and Gauge Symmetries. The same theory can be described by the action

$$S = \sum_{j=2}^{N}\sum_{k=1}^{n_j} \int\limits_{\mathbb{R}_{j,k}}\int\limits_{\Sigma_{g,n}} \times$$

$$\times \left(\langle\Phi\partial_{j,k}\bar{A}\rangle + \sum_{a=1}^{n}\delta(z - x_a, \bar{z} - \bar{x}_a)\langle\mathbf{S}^a g_a^{-1}\partial_{j,k}g_a\rangle - I_{j,k}\right)dt_{j,k},$$

where the time-like Wilson lines at the marked points are included.

The action is gauge-invariant with respect to the gauge group

$$\mathcal{G}_{\mathbb{C}} = \{\text{smooth maps} : \Sigma_{g,n} \to GL(N, \mathbb{C})\}.$$

The elements $f \in \mathcal{G}_{\mathbb{C}}$ are smooth and have the same monodromies as the Higgs field (3.1).

The action is invariant with respect to the gauge transformations

$$\bar{A} \to f^{-1}\bar{\partial}f + f^{-1}\bar{A}f, \quad \Phi \to f^{-1}\Phi f,$$

$$g_a \to g_a f^a, \quad \mathbf{S}^a \to (f^a)^{-1}\mathbf{S}^a f^a, \quad f^a = f(z, \bar{z})|_{z=x_a}.$$

Consider the infinitesimal gauge transformations

$$V_\varepsilon \bar{A} = \bar{\partial}\varepsilon + [\bar{A}, \varepsilon], \quad V_\varepsilon \Phi = [\Phi, \varepsilon],$$

$$V_\varepsilon g_a = g_a \varepsilon(x_a), \quad V_\varepsilon \mathbf{S}^a = [\mathbf{S}^a, \varepsilon(x_a)], \quad \varepsilon \in \text{Lie}\,(\mathcal{G}_{\mathbb{C}}).$$

The Hamiltonian F generating the gauge vector fields $\imath_\varepsilon \omega = DF$ has the form

$$F = \int_{\Sigma_{g,n}} \left\langle \varepsilon(\bar{\partial}\Phi + [\bar{A}, \Phi] - \sum_{a=1}^{n} \mathbf{S}^a \delta(z - x_a, \bar{z} - \bar{x}_a)) \right\rangle.$$

The moment map

$$\mu : \mathcal{R}(\bar{A}, \Phi, \mathbf{S}^a) \to \text{Lie}^*(\mathcal{G}_{\mathbb{C}}), \ \mu = \bar{\partial}\Phi + [\bar{A}, \Phi] - \sum_{a=1}^{n} \mathbf{S}^a \delta(z - x_a, \bar{z} - \bar{x}_a).$$

The Gauss law (the moment constraints) takes the form

$$\bar{\partial}\Phi + [\bar{A}, \Phi] = \sum_{a=1}^{n} \mathbf{S}^a \delta(z - x_a, \bar{z} - \bar{x}_a). \tag{3.14}$$

Upon imposing these constraints, the residues of the Higgs fields become equal to the spin variables $\text{Res}\,\Phi_{z=x_a} = S^a$ by analogy with the Yang–Mills theory, where the Higgs field corresponds to the electric field and $\mathbf{S}^a$ are analogs of the electric charges.

The reduced phase space

$$\mathcal{R}^{\text{red}} = \mathcal{R}(\bar{A}, \Phi, \mathbf{S}^a)/(\text{Gauss law}) + (\text{gauge fixing})$$

defines the physical degrees of freedom, and the reduced phase space is the symplectic quotient

$$\mathcal{R}^{\text{red}} = \mathcal{R}(\bar{A}, \Phi, \mathbf{S}^a)//\mathcal{G}_{\mathbb{C}}. \tag{3.15}$$

3.1.4. Algebra-Geometric Approach. The operator d'' acting on sections defines a holomorphic structure on the bundle E. A section s is holomorphic if

$$(\bar\partial + \bar A)s = 0.$$

The moment constraint (3.14) means that the space of sections of the Higgs field over $\Sigma_g \setminus D$ is holomorphic.

Consider the set of holomorphic structures $\mathcal{L} = \{d_{\bar A}\}$ on E. Two holomorphic structures are called equivalent if the corresponding connections are gauge-equivalent. The moduli space of holomorphic structures is the quotient $\mathcal{L}/\mathcal{G}_{\mathbb{C}}$. Generically the quotient has very singular structure. To have a reasonable topology one should consider the so-called stable bundles. The stable bundles are generic and we consider the space of connection $\mathcal{L}^{\text{stable}}$ corresponding to the stable bundles. The quotient is called *the moduli space of stable holomorphic bundles*

$$\mathcal{M}(N, g, n) = \mathcal{L}^{\text{stable}}/\mathcal{G}.$$

It is a finite-dimensional manifold. The tangent space to $\mathcal{M}(N, g, n)$ is isomorphic to $H^1(\Sigma_{g,n}, \operatorname{End} E)$. Its dimension can be extracted from the Riemann–Roch theorem and for curves without marked points $(n = 0)$

$$\dim H^0(\Sigma, \operatorname{End} E) - \dim H^1(\Sigma, \operatorname{End} E) = (1 - g) \dim G.$$

For stable bundles and $g > 1$ $\dim(H^0(\Sigma, \operatorname{End} E)) = 1$ and

$$\dim \mathcal{M}(N, g, 0) = (g - 1)N^2 + 1$$

for $GL(N, \mathbb{C})$, and

$$\dim \mathcal{M}(N, g, 0) = (g - 1)(N^2 - 1)$$

for $SL(N, \mathbb{C})$.

Thus, in the absence of the marked points we should consider bundles over curves of genus $g \geqslant 2$. But the curves of genus $g = 0$ and 1 are important for applications to integrable systems. Including the marked points improves the situation.

We extend the moduli space by adding an additional data at the marked points. Consider an N-dimensional vector space V and choose a flag Fl$= (V_1 \subset V_2 \subset \ldots V_N = V)$. Note that flag is a point in a homogeneous space called the flag variety $Fl \in GL(N, \mathbb{C})/B$, where B is a Borel subgroup. If $(e_1, \ldots, e_N)$ is a basis in V and Fl is a flag

$$\text{Fl} = \{V_1 = \{a_{11}e_1\}, V_2 = \{a_{21}e_1 + a_{22}e_2\}, \ldots, V_N = V\},$$

then B is the subgroup of lower triangular matrices. The flag variety has dimension $\frac{1}{2}N(N-1)$. The moduli space $\mathcal{M}(N, g, n)$ is the moduli space $\mathcal{M}(N, g, 0)$ equipped with maps $g_a \in GL(N, \mathbb{C})$ of V to the fibers over the marked points $V \to E|_{x_a}$, preserving Fl in V. In other words, g_a are defined up to the right multiplication of B and therefore we supply the moduli space $\mathcal{M}(N, g, 0)$ with the structure of the flag variety $GL(N, \mathbb{C})/B$ at the marked points. We have a natural «forgetting» projection $\pi : \mathcal{M}(N, g, n) \to \mathcal{M}(N, g, 0)$. The fiber of this projection is the product of n copies of the flag varieties. The bundles with this structure are called *the quasi-parabolic bundles*. The dimension of the moduli space of quasi-parabolic holomorphic bundles is

$$\dim \mathcal{M}(N, g, n) = \dim \mathcal{M}(N, g, 0) + \frac{1}{2}nN(N-1).$$

For curves of genus $g > 1$, $\dim(\mathcal{M}(N, g, n))$ is independent of degree of the bundles $d = \deg(E) = c_1(\det E)$. In fact, we have a disjoint union of components labeled by the corresponding degrees of the bundles $\mathcal{M} = \bigsqcup \mathcal{M}^{(d)}$. For elliptic curves ($g = 1$) one has

$$\dim H^1(\Sigma, \operatorname{End} E) = \dim H^0(\Sigma, \operatorname{End} E),$$

and $\dim H^0(\Sigma, \operatorname{End} E)$ does depend on $\deg(E)$. Namely,

$$\dim(\mathcal{M}(N, 1, 0, d)) = \text{g.c.d.}(N, d). \tag{3.16}$$

In this case the structures of the moduli space for the trivial bundles (i.e., with $\deg(E) = 0$) and, for example, for bundles with $\deg(E) = 1$ are different.

Now consider the Higgs field Φ. As we already mentioned, Φ defines an endomorphism of the bundle E

$$\Phi : \Omega^{(0)}(\Sigma_{g,n}, E) \to \Omega^{(1,0)}(\Sigma_{g,n}, E), \quad s \to \Phi s \otimes dz.$$

Similarly, they can be described as sections of $\Omega^0_{C^\infty}(\Sigma_{g,n}, \operatorname{End} E \otimes K_D)$. Here K_D is the canonical class on $\Sigma \setminus D$ that locally apart from D is represented as dz. Remind that Φ has poles at D. On the other hand, as it follows from the definition of the symplectic structure (3.4) on the set of pairs $(\Phi, \bar{A})$, the Higgs field plays the role of a « covector» with respect to the vector $\bar{A}$. In this way the Higgs field Φ is a section of the cotangent bundle $T^*\mathcal{L}^{\text{stable}}$.

The pair of the holomorphic vector bundle and the Higgs field (E, Φ) is called *the Higgs bundle*. The reduced phase space (3.15) is *the moduli space of the quasi-parabolic Higgs bundles*. It is the cotangent bundle

$$\mathcal{R}^{\text{red}} = T^*\mathcal{M}(N, g, n, d). \tag{3.17}$$

Due to the Gauss law (3.14) the Higgs fields are holomorphic on $\Sigma \setminus D$. Then, on the reduced space $\mathcal{R}^{\text{red}}$

$$\Phi \in H^0(\Sigma_{g,n}, \text{End}^* E \otimes K_D). \tag{3.18}$$

A part of $T^*\mathcal{M}(N, g, n, d)$ comes from the cotangent bundle to the flag varieties $T^*(G/B)_a$ located at the marked points. Without the null section, $T^*(G/B)_a$ is isomorphic to a unipotent coadjoint orbit, while the null section is the trivial orbit. Generic coadjoint orbits passing through a semisimple element of $gl(N, \mathbb{C})$ is an affine space over $T^*(G/B)_a$. In this way we come to the moduli space of the quasi-parabolic Higgs bundles [29]. It has dimension

$$\dim \mathcal{R}^{\text{red}} = 2N^2(g - 1) + 2 + N(N - 1)n. \tag{3.19}$$

This formula is universal and valid also for $g = 0, 1$ and does not depend on $\deg(E)$. At the first glance, for $g = 1$ this formula contradicts to (3.16). In fact, we have a residual gauge symmetry generated by subgroup of the Cartan group of $GL(N, \mathbb{C})$. The symplectic reduction with respect to this symmetry kills these degrees of freedom and we come to $\dim \mathcal{R}^{\text{red}} = 2 + N(N - 1)n$ (see (3.6)). We explain this mechanism on a particular example in Subsubsec. 3.2.2. Formula (3.6) suggests that the phase spaces corresponding to bundles of different degrees may be symplectomorphic. We will see soon that it is the case.

It follows from (3.18) that $\langle \Phi^j \rangle \in H^0(\Sigma_{g,n}, K_D^j)$. In other words, $\langle \Phi^j \rangle$ are meromorphic forms on the curve with the poles of order j at the divisor D. Let ς^{jk} be a basis of $H^0(\Sigma_{g,n}, K_D^j)$. Then

$$\frac{1}{j}\langle \Phi^j \rangle = \sum_{k=1}^{n_j} I_{jk} \varsigma^{jk}. \tag{3.20}$$

The above-introduced basis μ_{jk} in $H^1(\Sigma_{g,n} \setminus D, \mathcal{T}^{\otimes j-1})$ is dual to the basis ς^{jk}

$$\int_{\Sigma_{g,n}} \mu_{jk} \varsigma^{lm} = \delta_j^l \delta_k^m.$$

Then the coefficients of the expansion (3.20) coincide with the integrals (3.7). The dimensions n_j (3.6) can be calculated as $\dim H^0(\Sigma_{g,n}, K_D^j)$.

The symplectic reduction preserves the involutivity (3.10) of the integrals (3.7). Since

$$\boxed{\frac{1}{2} \dim T^*\mathcal{M}(N, g, n) = \text{number of integrals}}$$

(see (3.8), (3.9)), we come to integrable systems on the moduli space of the quasi-parabolic Higgs bundles $\mathcal{R}^{\text{red}}$.

For $GL(N, \mathbb{C})$ the Liouville torus is the Jacobian of the spectral curve $\mathcal{C}$ (1.5). Consider bundles with the structure group replaced by a reductive group G. The algebraic integrability for $g > 1$ and G as a classical simple group was proved in [1]. The case of exceptional groups was considered in [30, 31].

3.1.5. Equations of Motion on the Reduced Phase Space. Let us fix a gauge $\bar{A} = \bar{A}_0$. For an arbitrary connection $\bar{A}$ define a gauge transform

$$f[\bar{A}] : \bar{A} \to \bar{A}_0, \quad \bar{A}_0 = (f^{-1}\bar{\partial}f)[\bar{A}] + f^{-1}[\bar{A}]\bar{A}f[\bar{A}].$$

Then $f[\bar{A}]$ is an element of the coset space $\mathcal{G}_{\mathbb{C}}/\mathcal{G}_0$, where the subgroup $\mathcal{G}_0$ preserves the gauge fixing

$$\mathcal{G}_0 = \{ f \,|\, \bar{\partial}f + [\bar{A}_0, f] = 0 \}.$$

The same gauge transformation brings the Higgs field to the form

$$L = f^{-1}[\bar{A}]\Phi f[\bar{A}].$$

The equations of motion for Φ (3.11) in terms of L take the form of the Lax equation

$$\boxed{\partial_{j,k} L = [L, M_{j,k}]}, \tag{3.21}$$

where $M_{j,k} = f^{-1}[\bar{A}]\partial_{j,k}f[\bar{A}]$. Therefore, the Higgs field becomes after reduction the Lax matrix. Equation (3.21) describes *the Hitchin integrable hierarchy*.

The matrix $M_{j,k}$ can be extracted from the second equation (3.12)

$$\bar{\partial}M_{j,k} - [M_{j,k}, \bar{A}_0] = \partial_{j,k}\bar{A}_0 - L^{j-1}\mu_{j,k}. \tag{3.22}$$

The Gauss law restricted on $\mathcal{R}^{\text{red}}$ takes the form

$$\bar{\partial}L + [\bar{A}_0, L] = \sum_{a=1}^{n} \mathbf{S}^a \delta(x_a, \bar{x}_a). \tag{3.23}$$

Thus, the Lax matrix is the matrix Green function of the operator $\bar{\partial} + \bar{A}_0$ on $\Sigma_{g,n}$ acting in the space $\Omega^{(1,0)}(\Sigma_{g,n}, \operatorname{End} E)$.

The linear system corresponding to the integrable hierarchy takes the following form. Consider a section ψ of the vector bundle E. The section is called the Baiker–Akhiezer function if it is a solution of the linear system for

$$\begin{cases} 1. & (\bar{\partial} + \bar{A}_0)\psi = 0, \\ 2. & (\lambda - L)\psi = 0, \\ 3. & (\partial_{j,k} + M_{j,k})\psi = 0. \end{cases}$$

The first equation means that ψ is a holomorphic section. Compatibility of the first equation and the second equation is the Gauss law (3.23), and the first equation and the last equation are the Lax equations (3.21).

In term of the Lax matrix the integrals of motion I_{jk} are expressed by the integrals (3.7)

$$I_{jk} = \frac{1}{j} \int\limits_{\Sigma_{g,n}} \mu_{jk} \operatorname{tr}\left(L(x,z)\right)^j, \tag{3.24}$$

or by the expansion (3.20)

$$\frac{1}{j}\langle L^j \rangle = \sum_{k=1}^{n_j} I_{jk}\varsigma^{jk}. \tag{3.25}$$

The moduli space of the Higgs bundles (3.17) is parameterized by the pairs $(\mathbf{A}_0, L)$. The projection (1.2)

$$T^*\mathcal{M}(N, g, n) \to B = \sum_{j=1}^{N} H^0(\Sigma_{g,n} \setminus D, K_D^j)$$

is called *the Hitchin fibration*.

An illustrative example of the Hitchin construction are the Higgs bundles over elliptic curves. These cases will be described explicitly in the next subsections.

3.2. N-Body Elliptic Calogero–Moser System (ECMS). *3.2.1. Description of the System.* Let C_τ be an elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, $(\operatorname{Im}\tau > 0)$. The phase space $\mathcal{R}^{\mathrm{ECM}}$ of ECMS is described by N complex coordinates and their momenta

$$\begin{aligned}
\mathbf{u} &= (u_1, \ldots, u_N), &&(u_j \in C_\tau) &&- \text{ coordinates of particles,} \\
\mathbf{v} &= (v_1, \ldots, v_N), &&(v_j \in \mathbb{C}) &&- \text{ momentum vector}
\end{aligned}$$

with the Poisson brackets $\{v_j, u_k\} = \delta_{jk}$.
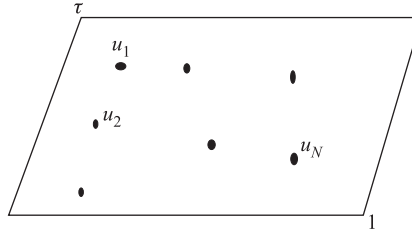
The Hamiltonian takes the form

$$H^{\mathrm{CM}} = \frac{1}{2}|\mathbf{v}|^2 + \nu^2 \sum_{j<k} \wp(u_j - u_k). \tag{3.26}$$

Here ν^2 is a coupling constant and $\wp(z)$ is the Weierschtrass function. It is a double periodic meromorphic function $\wp(z+1) = \wp(z+\tau) = \wp(z)$, with the second order pole $\wp(z) \sim z^{-2}$, $z \to 0$.

The system has the Lax representation [39] with the Lax matrix

$$L^{\text{CM}} = iV + X, \quad V = \text{diag}\,(v_1, \ldots, v_N), \tag{3.27}$$

$$X_{jk} = \nu \mathbf{e}\left(\frac{z - \bar{z}}{\tau - \bar{\tau}}(u_j - u_k)\right) \phi(u_j - u_k, z), \quad \mathbf{e}(x) = \exp\,(2\pi i x), \tag{3.28}$$

where

$$\phi(u, z) = \frac{\theta(u+z)\theta'(0)}{\theta(u)\theta(z)}, \tag{3.29}$$

and

$$\theta(z) = q^{1/8} \sum_{n \in \mathbf{Z}} (-1)^n \exp\left(2\pi\imath\left(\frac{1}{2}n(n+1)\tau + nz\right)\right), \tag{3.30}$$

$$q = \exp\,(2\pi i \tau)$$

is the standard theta-function with a simple zero at $z = 0$ and the monodromies

$$\theta(z+1) = -\theta(z), \quad \theta(z+\tau) = -q^{-1/2}\,\mathrm{e}^{-2\pi i z}\,\theta(z). \tag{3.31}$$

Then from (3.31) we have that

$$\phi(u, z+1) = \phi(u, z), \quad \phi(u, z+\tau) = \mathbf{e}(-u)\phi(u, z), \tag{3.32}$$

and $\phi(u, z)$ has a simple pole at $z = 0$

$$\text{Res}\,\phi(u, z)|_{z=0} = 1. \tag{3.33}$$

3.2.2. ECMS and the Higgs Bundles [6, 7]. To describe the ECMS as the Hitchin system consider a vector bundle E of rank N and degree 0 over an elliptic curve $\Sigma_{1,1}$ with one marked point. We assume that the curve is isomorphic to $C_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The quasi-parabolic Higgs bundle T^*E has coordinates

$$\mathcal{R}^0 = \{\Phi(z, \bar{z}), \bar{A}(z, \bar{z}), S\}, \quad \Phi, \bar{A} \in gl\,(N, \mathbb{C}), \quad S \in \mathcal{O},$$

where $\mathcal{O}$ is a degenerate orbit at the marked point $z = 0$

$$\mathcal{O} = \{S = g^{-1}S^0 g \,|\, g \in GL(N, \mathbb{C}), S^0 = \nu J\},$$

and J is the matrix (2.17). The orbit has dimension $\dim\,(\mathcal{O}) = 2N - 2$.

For degree zero bundles the monodromies around the two fundamental cycles can be chosen as $Q_1 = Id$ and $\Lambda_1 = \mathbf{e}(\mathbf{u})$, where $\mathbf{e}(\mathbf{u}) = \mathrm{diag}\,(\exp\,(2\pi i u_1, \ldots, 2\pi i u_N))$. A section with this monodromies is

$$s^T = (s_1, \ldots, s_N), \quad s_j = \phi(u_j, z). \tag{3.34}$$

where $\phi(u_j, z)$ is (3.29). It follows from (3.32) that the section has the prescribed monodromies.

For the fields and the gauge group we have the same monodromies

$$\bar{A}(z + 1) = \bar{A}(z), \quad \Phi(z + 1) = \Phi(z),$$

$$\bar{A}(z + \tau) = \mathbf{e}(\mathbf{u})\bar{A}(z)\,\mathbf{e}(-\mathbf{u}), \quad \Phi(z + \tau) = \mathbf{e}(\mathbf{u})\Phi(z)\,\mathbf{e}(-\mathbf{u}),$$

$$f(z + 1, \bar{z} + 1) = f(z, \bar{z}), \quad f(z + \tau, \bar{z} + \bar{\tau}) = \mathbf{e}(\mathbf{u})f(z, \bar{z})\,\mathbf{e}(-\mathbf{u}).$$

It can be proved that for bundles of zero degree, generic connection is trivial $\bar{A} = -\bar{\partial}f f^{-1}$ and therefore

$$\bar{A} \to \bar{A}_0 = 0. \tag{3.35}$$

It means that stable bundles E of rank N are decomposed into the direct sum of line bundles

$$E = \oplus_{j=1}^{N}\mathcal{L}_j,$$

with the sections (3.34). The elements u_j are the points of the Jacobian $\mathrm{Jac}\,(\Sigma_\tau)$. They play the role of the coordinates, and thereby, $C_\tau \sim \mathrm{Jac}\,(\Sigma_\tau)$.

This gauge fixing is invariant with respect to the constant diagonal subgroup D_0. It acts on the spin variables $S \in \mathcal{O}$. This action is Hamiltonian. The moment equation of this action is $\mathrm{diag}\,(\mathcal{O}) = 0$. This condition dictates the form of $S^0 = J$. The gauge fixing allows one to kill the degrees of freedom related to the spin variables, because $\dim\,(\mathcal{O}) = 2(N-1)$ and $\dim\,(D_0) = N - 1$. Thus, the symplectic quotient is a point $(\dim\,(\mathcal{O}//D_0) = 0)$.

Remark 3.1. *One can choose an arbitrary orbit $\mathcal{O}$. In this case we come to the symplectic quotient $\mathcal{O}//D_0$. It has dimension $\dim\,(\mathcal{O}) - 2(N-1)$.*

Now consider solutions of the moment equation (3.23) with the prescribed monodromies and prove that Φ becomes the Lax matrix $\Phi \to L^{\mathrm{CM}} = V + + X$ (3.27). Since $\bar{A}_0 = 0$, V does not contribute in (3.23) and its elements are free parameters. We identify them with momenta of the particles $V = \mathrm{diag}\,(v_1, \ldots, v_N)$. Due to the term with the delta-function in (3.23) the off-diagonal part should have a simple pole with the residue νJ and the prescribed monodromies. It follows from (3.32) and (3.33) that X_{jk} satisfies these conditions. They uniquely fix its matrix elements.

The reduced space is described by the variables $\mathbf{v}$ and $\mathbf{u}$. The symplectic form on the reduced space

$$\int\limits_{\Sigma_{1,1}} \langle L^{\mathrm{CM}}, \bar{A}_0 \rangle = \sum Dv_j \wedge Du_j$$

leads to the brackets $\{v_j, u_k\} = \delta_{jk}$.

From the general construction the integrals of motion come from the expansion of $\mathrm{tr}\,(L^{\mathrm{CM}})^j(\mathbf{v}, \mathbf{u}, z)$. They are double-periodic meromorphic functions with poles at $z = 0$. It is finite-dimensional space generated by a basis of derivative of the Weierschtrass functions. They are elements of the basis ς^{jk} in (3.25).

$$\frac{1}{j}\,\mathrm{tr}\,(L^{\mathrm{CM}})^j(\mathbf{v}, \mathbf{u}, z) = I_{0,j}^{\mathrm{CM}} + I_{2,j}^{\mathrm{CM}}\wp(z) + \ldots + I_{j,j}^{\mathrm{CM}}\wp^{(j)}(z). \tag{3.36}$$

There are $\dfrac{N(N+1)}{2} - 1$ integrals. Due to a special choice of the orbit only $N - 1$ integrals are independent. In particular,

$$\frac{1}{2}\,\mathrm{tr}\,(L^{\mathrm{CM}})^2(\mathbf{v}, \mathbf{u}, z) = -H^{\mathrm{CM}} + \nu^2\wp(z).$$

For generic orbits (see Remark 4.1) the Hamiltonian takes the form

$$H^{\mathrm{CM}} = \frac{1}{2}|\mathbf{v}|^2 + \sum_{j<k} S_{jk}S_{kj}\wp(u_j - u_k).$$

It is the ECMS with spin [34]. Note, that $I_{j,j}$ are the Casimir functions defining a generic orbit $\mathcal{O}$. Therefore we have $\dfrac{N(N+1)}{2} - 1 - (N-1) = \dfrac{N(N-1)}{2}$ commuting integrals of motion. The number of independent commuting integrals is always equal to $1/2\dim(\mathcal{O})$.

3.3. Elliptic Top (ET) on $GL(N,\mathbb{C})$. *3.3.1. Description of the System.* The elliptic top is an example of Euler–Arnold top related to the group $GL(N, \mathbb{C})$. Its phase space is a coadjoint orbit of $GL(N, \mathbb{C})$. The Hamiltonian is a quadratic form on the coalgebra $\mathfrak{g}^* = gl\,(N, \mathbb{C})^*$. The ET is an integrable Euler–Arnold top. Before to define the Hamiltonian, let us introduce a special basis in the Lie algebra $gl\,(N, \mathbb{C})$. Define the finite set

$$\mathbb{Z}_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}), \quad \tilde{\mathbb{Z}}_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}) \setminus (0,0)$$

and let $\mathbf{e}_N(x) = \exp\left(\dfrac{2\pi i}{N}x\right)$. Then a basis is generated by $N^2 - 1$ matrices

$$T_\alpha = \frac{N}{2\pi i}\,\mathbf{e}_N\left(\frac{\alpha_1\alpha_2}{2}\right)Q^{\alpha_1}\Lambda^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2) \in \tilde{\mathbb{Z}}_N^{(2)},$$

where

$$Q = \operatorname{diag}(1, \mathbf{e}_N(1), \ldots, \mathbf{e}_N(N-1)), \tag{3.37}$$

$$\Lambda = \sum_{j=1,N, \ (\text{mod } N)} E_{j,j+1}. \tag{3.38}$$

The commutation relations in this basis have a simple form

$$[T_\alpha, T_\beta] = \frac{N}{\pi} \sin \frac{\pi}{N}(\alpha \times \beta) T_{\alpha+\beta}.$$

Let $\mathbf{S} = \sum_{\alpha \in \mathbb{Z}_N^{(2)} \backslash (0,0)} S_\alpha T_\alpha \in \mathfrak{g}^*$. The Poisson brackets for the linear functions S_α come from the Lie brackets

$$\{S_\alpha, S_\beta\} = \frac{N}{\pi} \sin \frac{\pi}{N}(\alpha \times \beta) S_{\alpha+\beta}.$$

The phase space $\mathcal{R}^{\text{ET}}$ of the ET is a coadjoint orbit

$$\mathcal{R}^{\text{ET}} \sim \mathcal{O} = \{\mathbf{S} \in \mathfrak{g}^* \,|\, \mathbf{S} = g\mathbf{S}_0 g^{-1}, \ g \in GL(N, \mathbb{C})\}.$$

A particular orbit passes through $\mathbf{S}_0 = \nu J$, as for the spinless ECMS.

The Euler–Arnold Hamiltonian is defined by the quadratic form

$$H^{\text{ET}} = -\frac{1}{2} \operatorname{tr}(\mathbf{S} \cdot \mathbf{J}(\mathbf{S})),$$

where $\mathbf{J}$ is diagonal in the basis T_α

$$\mathbf{J}(\mathbf{S}) : S_\alpha \to \wp_\alpha S_\alpha, \quad \wp_\alpha = \wp\left(\frac{\alpha_1 + \alpha_2 \tau}{N}\right), \quad \alpha \in \tilde{\mathbb{Z}}_N^{(2)}.$$

The equations of motion corresponding to this Hamiltonian take the form

$$\partial_t \mathbf{S} = \{H^{\text{ET}}, \mathbf{S}\} = [\mathbf{J}(\mathbf{S}), \mathbf{S}],$$

$$\partial_t S_\alpha = \frac{N}{\pi} \sum_{\gamma \in \tilde{\mathbb{Z}}_N^{(2)}} S_\gamma S_{\alpha-\gamma} \wp_\gamma \sin \frac{\pi}{N}(\alpha \times \gamma).$$

3.3.2. Field Theory and the Higgs Bundles. The curve $\Sigma_{1,1}$ is the same as for the Calogero–Moser system. Consider a vector bundle E of a rank N and degree one over $\Sigma_{1,1}$. It is described by its sections $s = (s_1(z, \bar{z}), \ldots, s_N(z, \bar{z}))$ with monodromies

$$s^T(z+1, \bar{z}+1) = Q^{-1} s^T(z, \bar{z}), \quad s^T(z+\tau, \bar{z}+\bar{\tau}) = \tilde{\Lambda}^{-1} s^T(z, \bar{z}), \tag{3.39}$$

where Q is (3.37), $\tilde{\Lambda} = \mathbf{e}_N^{-(z+\tau/2)}\Lambda$, and Λ is (3.38). Since $\det Q = \pm 1$ and $\det \tilde{\Lambda} = \pm \mathbf{e}_1^{-(z+\tau/2)}$, the determinants of the transition matrices have the same quasi-periods as the Jacobi theta-functions. The theta-functions have a simple pole in $\Sigma_{1,1}$. Thereby, the vector bundle E_N has degree one.

The Higgs bundle has the same field content as the ECMS

$$\mathcal{R} = \{\bar{A}, \Phi, \mathbf{S}\}, \quad \bar{A}, \Phi \in gl\,(N, \mathbb{C}), \quad \mathbf{S} \in \mathcal{O}.$$

The orbit

$$\mathcal{O} = \{\mathbf{S} = g^{-1}\mathbf{S}^0 g, \quad g \in GL(N, \mathbb{C})\}$$

is located at the marked point $z = 0$.

It follows from (3.39) that the fields Φ, $\bar{A}$ have the monodromies

$$\bar{A}(z+1) = Q\bar{A}(z)Q^{-1}, \quad \Phi(z+1) = Q\Phi(z)Q^{-1},$$
$$\bar{A}(z+\tau) = \Lambda\bar{A}(z)\Lambda^{-1}, \quad \Phi(z+\tau) = \Lambda\Phi(z)\Lambda^{-1}.$$

The group of the automorphisms $\mathcal{G}_{\mathbb{C}} = \{f\}$ of E should have the same monodromies

$$f(z+1) = Qf(z)Q^{-1}, \quad f(z+\tau) = \Lambda f(z)\Lambda^{-1}.$$

Due to the monodromy conditions the generic field $\bar{A}$ is gauge equivalent to the trivial $f^{-1}\bar{A}f + f^{-1}\bar{\partial}f = 0$. Therefore

$$\bar{A} = -\bar{\partial}f[\bar{A}]f^{-1}[\bar{A}]. \tag{3.40}$$

It allows us to choose $\bar{A} = 0$ as an appropriate gauge. It means that there are no moduli of holomorphic vector bundles. More precisely, the holomorphic moduli are related only to the quasi-parabolic structure of E related to the spin variables $\mathbf{S}$. The monodromies of the gauge matrices prevent to have nontrivial residual gauge symmetries. Let $f[\bar{A}](z, \bar{z})$ be a solution of (3.40). Consider the transformation of Φ by solutions of (3.40)

$$L^{\mathrm{ET}}[\bar{A}, g](z, \bar{z}) = f[\bar{A}](z, \bar{z})\Phi(z, \bar{z})f^{-1}[\bar{A}](z, \bar{z}). \tag{3.41}$$

The moment constraints (3.14) take the form

$$\bar{\partial}L^{\mathrm{ET}} = \delta(z, \bar{z})\mathbf{S}.$$

The solution takes the form

$$\boxed{L^{\mathrm{ET}} = \sum_{\alpha \in \mathbb{Z}_N^{(2)}\setminus(0,0)} S_\alpha \varphi_\alpha(z)T_\alpha,}$$

where $\varphi_\alpha(z) = \mathbf{e}_N(\alpha_2 z)\phi\left(\dfrac{\alpha_1 + \alpha_2\tau}{N}, z\right)$. The Lax matrix was found in [40] using another approach. It is the Lax matrix of the vertex spin chain. The Lax matrix is meromorphic on $\Sigma_{1,1}$ with a simple pole with Res $L^{\mathrm{ET}}|_{z=0} = \mathbf{S}$. The monodromies of $\varphi_\alpha(z)$ are read off from (3.32)

$$\varphi_\alpha(z+1) = \mathbf{e}_N(\alpha_2)\varphi_\alpha(z), \quad \varphi_\alpha(z+\tau) = \mathbf{e}_N(-\alpha_1)\varphi_\alpha(z).$$

Then L^{ET} has the prescribed monodromies. The reduced phase space $\mathcal{R}^{\mathrm{ET}}$ is the coadjoint orbit:

$$\mathcal{R}^{\mathrm{ET}} = \{\mathcal{O} = \mathbf{S} = g\mathbf{S}_0 g^{-1}\},$$

$\mathbf{S} = \displaystyle\sum_{\alpha \in \mathbb{Z}_N^{(2)}\backslash(0,0)} S_\alpha T_\alpha \in \mathfrak{g}^*$. The symplectic form on $\mathcal{R}^{\mathrm{ET}}$ is the Kirillov–Kostant form (3.5).

For a particular choice of the orbit passing through J (ref J) its dimension coincides with the dimension of the phase of the spinless ECMS

$$\dim \mathcal{R}^{\mathrm{ET}} = \dim \mathcal{R}^{\mathrm{CMS}} = 2N - 2.$$

It is not occasional and we prove below that $\mathcal{R}^{\mathrm{CM}}$ is symplectomorphic to $\mathcal{R}^{\mathrm{ET}}$.

Since the traces $\mathrm{tr}\,(L^{\mathrm{ET}})^j$ are double periodic and have poles at $z = 0$, the integrals of motion come from the expansion (see (3.36))

$$\mathrm{tr}\,(L^{\mathrm{ET}}(z))^k = I_{0,k} + I_{2,k}\wp(z) + \ldots + I_{k,k}\wp^{(k-2)}(z).$$

In particular,

$$\mathrm{tr}\,(L^{\mathrm{ET}})^2 = H^{\mathrm{ET}} + C^2\wp(z).$$

The coefficients $I_{s,k}$ are in involution

$$\{I_{s,k}, I_{m,j}\} = 0.$$

In particular, all functions $I_{s,k}$ Poisson commute with the Hamiltonian H^{ET}. Therefore, they play the role of conservation laws of elliptic rotator hierarchy on $GL(N, \mathbb{C})$. We have a tower of $\dfrac{N(N+1)}{2}$ independent integrals of motion

$$
\begin{array}{ccccc}
I_{0,2} & I_{2,2} & & & \\
I_{0,3} & I_{2,3} & I_{3,3} & & \\
\cdots & \cdots & \cdots & \cdots & \\
I_{0,n} & I_{2,N} & \cdots & \cdots & I_{N,N}.
\end{array}
$$

There are no integrals $I_{1,k}$ because there are no double periodic meromorphic functions with one simple pole. The integrals $I_{k,k}$, $k = 0, 2, 3 \ldots, N$ are the Casimir functions corresponding to the coadjoint orbit

$$\mathcal{R}^{\mathrm{ET}} = \{\mathbf{S} \in gl\,(N, \mathbb{C}), \;\; \mathbf{S} = g^{-1}\mathbf{S}^{(0)}g\}.$$

The conservation laws $I_{s,k}$ generate commuting flows on $\mathcal{R}^{\text{rot}}$

$$\partial_{s,k}\mathbf{S} = \{I_{s,k}, \mathbf{S}\}_1 \quad (\partial_{s,k} := \partial_{t_{s,k}}).$$

3.4. Symplectic Hecke Correspondence. Let E and $\tilde{E}$ be two bundles over Σ of the same rank. Assume that there is a map $\Xi^+ : E \to \tilde{E}$ (more precisely a map of the space of sections $\Gamma(E) \to \Gamma(\tilde{E})$) such that it is an isomorphism on the complement to z_0 and it has one-dimensional cokernel at $x \in \Sigma$:

$$0 \to E \xrightarrow{\Xi^+} \tilde{E} \to \mathbb{C}|_{z_0} \to 0.$$

The map Ξ^+ is called *upper modification* of the bundle E at the point z_0. Let $w = z - z_0$ be a local coordinate in a neighborhood of z_0. We represent locally E as a sum of line bundles $E = \oplus_{j=1}^N \mathcal{L}_|$ with holomorphic sections

$$s = (s_1, s_2, \ldots, s_N). \tag{3.42}$$

After the modification we come to the bundle $\tilde{E} = \oplus_{j=1}^N \mathcal{L}_j \otimes \mathcal{O}(z_0)$. The sections of $\tilde{E}$ are represented locally as $\tilde{s} = (g_1(w)s_1, \ldots, w^{-1}g_N(w)s_N)$, where $g_j(0) \neq 0$. In this basis the upper modification at the point z_0 is represented by the matrix

$$\Xi^+ = \begin{pmatrix} \mathrm{Id}_{N-1} & 0 \\ 0 & w \end{pmatrix}.$$

It is a modification of order 1, since it increases the degree of E

$$\deg(\tilde{E}) = \deg(E) + \deg(\mathcal{O}(\ddagger_\prime)) = \deg(\mathcal{E}) + \infty. \tag{3.43}$$

On the complement to the point z_0 consider the map

$$E \xleftarrow{\Xi^-} \tilde{E},$$

such that $\Xi^- \Xi^+ = \mathrm{Id}$. It defines *the lower modification* at the point z_0. The upper modification Ξ^+ is represented by the vector $(0, \ldots, 1)$; and Ξ^-, by $(0, \ldots, -1)$.

For the Higgs bundles the modification acts as

$$(E, \Phi) \xrightarrow{\Xi} (\tilde{E}, \tilde{\Phi}),$$

$$\Xi\Phi = \tilde{\Phi}\Xi, \quad \Xi\tilde{\bar{A}} = \bar{\partial}\Xi + \bar{A}\Xi. \tag{3.44}$$

The Higgs fields Φ and $\tilde{\Phi}$ should be holomorphic with prescribed simple poles at the marked points. The holomorphity of the Higgs field puts restrictions on its form. Consider the upper modification $\Xi^+ \sim (0, \ldots, 1)$ and assume that Φ in the defined above basis takes the form

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a is a matrix of order $N - 1$. Then

$$\Xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & bw^{-1} \\ cw & d \end{pmatrix} \Xi.$$

We see that a generic Higgs field acquires the first order pole after the modification. To escape it we assume that there exists an eigenvector $\Phi\xi = \lambda\xi$ such that it belongs to the Ker Φ. Let $\xi = (0, 0, \ldots, 1)$ and

$$\Phi = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

Then the Higgs field $\tilde{\Phi}$ does not have a pole

$$\tilde{\Phi} = \begin{pmatrix} a & 0 \\ cw & d \end{pmatrix}.$$

In other words, the matrix elements $(\Phi)_{jN}$ should have first order null.

In this way the upper modification is lifted from E to the Higgs bundle. After the reduction we come to the map (see (3.17))

$$T^*\mathcal{M}(N, g, n, d) \rightarrow T^*\mathcal{M}(N, g, n, d+1).$$

We call it the upper *Symplectic Hecke Correspondence* (SHC).

Generically the modified bundle $\tilde{E}$ is represented locally as a sum of line bundles $\tilde{E} = \oplus_{j=1}^{N}(\mathcal{L}_j \otimes \mathcal{O}(z_0)_j^m$ $(m_j \in \mathbb{Z})$ with holomorphic sections

$$\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_N) = (w^{-m_1}g_1 s_1, w^{-m_2}g_2 s_2, \ldots, w^{-m_N}g_N s_N). \qquad (3.45)$$

It has degree

$$\deg(\tilde{E}) = \deg(E) + \sum_{j=1}^{N} m_j.$$

This modification is represented by the vector $(m_1, \ldots, m_n)$.

Remind that the Higgs field is an endomorphism of E $s \rightarrow \Phi s$ and near z_0 it acts as

$$\Phi \cdot s_j = (\Phi)_j^k s_k.$$

Similarly the modified Higgs field acts on sections of the modified bundle $\tilde{E}\tilde{s} \rightarrow \tilde{\Phi}\tilde{s}$. Then it follows from (3.45) that

$$\tilde{\Phi} \cdot \tilde{s}_j = \tilde{\Phi}_j^k \tilde{s}_k, \quad \tilde{\Phi}_j^k = w^{m_k - m_j} g_k(w) g_j^{-1}(w) \Phi_j^k.$$

Since $\tilde{\Phi}$ is holomorphic and $g_j(0) \neq 0$, $\Phi_j^k(z - z_0)^{m_k - m_j}$ must be regular at $z = z_0$. If we order $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_N$, then the number of parameters of the endomorphisms is $\sum\limits_{j<k}(m_j - m_k)$. In general case

$$T^*\mathcal{M}(N, g, n, d) \to T^*\mathcal{M}\left(N, g, n, d + \sum_{j=1}^{N} m_j\right).$$

If $\sum\limits_{j=1}^{N} m_j = 0$, the SHC does not change the topological type of the bundle. Therefore, such SHC defines a Bäcklund transformation of integrable hierarchy.

3.5. Symplectic Hecke Correspondence $\mathcal{R}^{\mathbf{CM}} \to \mathcal{R}^{\mathbf{ET}}$ [11]. We work directly with the Lax matrices

$$L^{\mathrm{ET}} \times \Xi = \Xi \times L^{\mathrm{CM}}.$$

The modification matrix should intertwine the multipliers corresponding to the fundamental cycles

$$\Xi(z + 1, \tau) = Q \times \Xi(z, \tau), \tag{3.46}$$

$$\Xi(z + \tau, \tau) = \tilde{\Lambda}(z, \tau) \times \Xi(z, \tau) \times \mathrm{diag}\,(\mathbf{e}(u_j)). \tag{3.47}$$

Consider the modification at $z = 0$. The Lax matrix of the CMS has the first order pole

$$L^{\mathrm{CM}} \sim \frac{1}{z}\nu J.$$

Its residue has an eigenvector $\xi^t = (1, \ldots, 1)$ with the eigenvalue $N - 1$. The matrix Ξ satisfying (3.46) and (3.47) that annihilate the vector ξ has the form

$$\Xi(z) = \tilde{\Xi}(z) \times \mathrm{diag}\left((-1)^l \prod_{j<k;j,k\neq l} \theta(u_k - u_j, \tau)\right),$$

$$\tilde{\Xi}_{ij}(z, u_1, \ldots, u_N; \tau) = \theta\left[\begin{array}{c} \dfrac{i}{N} - \dfrac{1}{2} \\ \dfrac{N}{2} \end{array}\right](z - Nu_j, N\tau).$$

Here $\theta\left[\begin{array}{c} \dfrac{i}{N} - \dfrac{1}{2} \\ \dfrac{N}{2} \end{array}\right](z - Nu_j, N\tau)$ is the theta-function with a characteristic. The determinant of Ξ can be calculated explicitly

$$\mathrm{det}\left[\frac{\tilde{\Xi}_{ij}(z, u_1, \ldots, u_N; \tau)}{i\eta(\tau)}\right] = \frac{\theta(z)}{i\eta(\tau)} \prod_{1 \leqslant k < l \leqslant N} \frac{\theta(u_l - u_k)}{i\eta(\tau)},$$

where $\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n)$ is the Dedekind function. It has a simple pole at $z = 0$ and therefore Ξ is degenerate.

We use the modification to write down the interrelations between the coordinates and momenta of the Calogero–Moser particles and the orbit variables of the elliptic top in the $SL(2, \mathbb{C})$ case

$$S_1 = -v\frac{\theta_{10}(0)}{\theta'(0)}\frac{\theta_{10}(2u)}{\theta(2u)} - \nu\frac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)}\frac{\theta_{00}(2u)\theta_{01}(2u)}{\theta^2(2u)},$$

$$S_2 = -v\frac{\theta_{00}(0)}{i\theta'(0)}\frac{\theta_{00}(2u)}{\theta(2u)} - \nu\frac{\theta_{00}^2(0)}{i\theta_{10}(0)\theta_{01}(0)}\frac{\theta_{10}(2u)\theta_{01}(2u)}{\theta^2(2u)}, \qquad (3.48)$$

$$S_3 = -v\frac{\theta_{01}(0)}{\theta'(0)}\frac{\theta_{01}(2u)}{\theta(2u)} - \nu\frac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)}\frac{\theta_{00}(2u)\theta_{10}(2u)}{\theta^2(2u)}.$$

Here $\theta_{1,0} = \sum_{n \in \mathbb{Z}} q^{1/2n^2} \exp \pi(2n-1)z$, $\theta_{0,0} = \sum_{n \in \mathbb{Z}} q^{1/2(n-1/2)^2} \exp 2\pi nz$, $\theta_{0,1} = \sum_{n \in \mathbb{Z}} (-1)^n q^{1/2(n-1/2)^2} \exp 2\pi nz$. These relations describe the Darboux coordinates $(v, u) \in \mathbb{C}^2$ of the coadjoint $SL(2, \mathbb{C})$-orbit $\sum S_\alpha^2 = \nu^2$.

It turns out that this modification is equivalent to the twist of R matrices. Namely, it describes the passage from the dynamical R matrix of the IRF models to the vertex R matrix [25, 26]. We don't discuss this aspect of SHC here.

4. 4D THEORIES

4.1. Self-Dual YM Equations and Hitchin Equations. *4.1.1. 2D Self-Dual Equations.* Consider a rank N complex vector bundle E over $\mathbb{R}^4$ with coordinates $\mathbf{x} = (x_0, x_1, x_2, x_3)$. Assume that the space of sections is equipped with a nondegenerate Hermitian metric h, $(h^+ = h)$. It satisfies the following condition $dh(x, y) = h(\nabla x, y) + h(x, \nabla y)$, where ∇ is a connection on E. If $dh(x, y) = 0$ for vectors in fibers $y \in V$, $x \in \bar{V}^t$, then there exist connections $\nabla_j = \partial_{x_j} + A_j$ such that

$$\mathbf{A}^+ = -h^{-1}dh - h^{-1}\mathbf{A}h \qquad \left(\mathbf{A} = \sum_{j=0}^3 A_j dx_j\right).$$

In this situation the transition functions are reduced to the unitary group $SU(N) \subset GL(N, \mathbb{C})$.

Let $F(\mathbf{A}) \in \Omega^{(2)}(\mathbb{R}^4, su(N))$ be the curvature $F_{ij} = [\nabla_i, \nabla_j]$ or $F(\mathbf{A}) = d\mathbf{A} + \mathbf{A}^2$. Here $su(N) = \{x \,|\, x^+ = -h^{-1}xh\}$.

The self-duality equation

$$F = \star F,$$

where $\star$ is the Hodge operator in $\mathbb{R}^4$, takes the form

$$F_{01} = F_{23}, \quad F_{02} = F_{31}, \quad F_{03} = F_{12}. \tag{4.1}$$

Assume that A_j depend only on (x_1, x_2). It means that the fields are invariant under the shifts in directions x_0, x_3. Then (A_0, A_3) become adjoint-valued scalar fields which we denote as (ϕ_1, ϕ_2). They are called the Higgs fields. In fact, they will be associated below with the Higgs field Φ. In this way we come to the self-dual equations on the plane $\mathbb{R}^2 = (x_1, x_2)$

$$F_{12} = [\phi_1, \phi_2], \tag{4.2}$$

$$[\nabla_1, \phi_1] = [\phi_2, \nabla_2], \tag{4.3}$$

$$[\nabla_1, \phi_2] = [\nabla_2, \phi_1]. \tag{4.4}$$

Introduce complex coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ and let $d' = \nabla_z$, $d'' = \nabla_{\bar{z}}$. Consider the fields, taking values in the Lie algebra $sl(N, \mathbb{C})$

$$\Phi_z = \frac{1}{2}(\phi_1 - i\phi_2)\, dz \in \Omega^{(1,0)}(\mathbb{R}^2, \mathrm{ad}\, E),$$

$$\Phi_{\bar{z}} = \frac{1}{2}(\phi_1 + i\phi_2)\, d\bar{z} \in \Omega^{(0,1)}(\mathbb{R}^2, \mathrm{ad}\, E).$$

They are not independent since the Hermitian conjugation acts as

$$\Phi_{\bar{z}}^+ = -h^{-1}\Phi_z h. \tag{4.5}$$

Similarly,

$$A_z = \frac{1}{2}(A_1 - iA_2), \ A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2),$$
$$A_{\bar{z}}^+ = -h^{-1}dh - h^{-1}A_z h. \tag{4.6}$$

In terms of fields

$$\mathcal{W} = (A, A_{\bar{z}}, \Phi_z, \Phi_{\bar{z}}), \tag{4.7}$$

(4.2)–(4.4) can be rewritten in the coordinate invariant way:

$$\begin{cases} 1. & F + [\Phi_z, \Phi_{\bar{z}}] = 0, \\ 2. & d''\Phi_z = 0, \\ 3. & d'\Phi_{\bar{z}} = 0, \end{cases} \tag{4.8}$$

where $[\Phi_z, \Phi_{\bar{z}}] = \Phi_z \Phi_{\bar{z}} + \Phi_{\bar{z}} \Phi_z$. Due to (4.5) and (4.6) the third equation is not independent. Thus, we have two equations with the left side of type $(1,1)$ for two complex valued fields $(\Phi_z, A_{\bar{z}})$ and the Hermitian matrix h.

Equations (4.8) are conformal invariant and thereby can be defined on a complex curve Σ_g. In this case

$$\Phi_z \in \Omega^{(1,0)}(\Sigma_g, su(N)), \quad \Phi_{\bar{z}} \in \Omega^{(0,1)}(\Sigma_g, su(N)),$$

$$d'' : \Omega^{(j,k)}(\Sigma_g, su(N)) \to \Omega^{(j,k+1)}(\Sigma_g, su(N)).$$

The self-duality equations (4.9) on Σ_g are called *the Hitchin equations*.

In fact, instead of (4.8) we will consider further a modified system

$$\begin{cases} 1. & F - [\Phi_z, \Phi_{\bar{z}}] = 0, \\ 2. & d'' \Phi_z = 0, \\ 3. & d' \Phi_{\bar{z}} = 0. \end{cases} \tag{4.9}$$

It comes from the self-duality on $\mathbb{R}^4$ with a metric of signature $(2,2)$. Consider the gauge group action on solutions of (4.9)

$$\mathcal{G} = \{ f \in \Omega^0(\Sigma_g, SU(N)) \}, \tag{4.10}$$

$$\Phi_z \to f^{-1} \Phi_z f, \quad \Phi_{\bar{z}} \to f^{-1} \Phi_{\bar{z}} f, \tag{4.11}$$

$$d'' \to f^{-1} d'' f. \tag{4.12}$$

If $(A, A_{\bar{z}}, \Phi_z, \Phi_{\bar{z}})$ are solutions of (4.9), then the transformed fields are also solutions. If f takes values in $GL(N, \mathbb{C})$, then it again transforms solutions to solutions. As above, we denote this gauge group as $\mathcal{G}_{\mathbb{C}}$.

Define the moduli space of solutions of (4.9) as a quotient under the gauge group action

$$\mathcal{M}_H(\Sigma_g) = \text{solutions of } (4.9)/\mathcal{G}. \tag{4.13}$$

Now look at the second equation in (4.9). It is the moment constraint equation for the Higgs bundles in the absence of marked points (3.14). The gauge group $\mathcal{G}_{\mathbb{C}}$ transforms solutions of (4.9) to other solutions but breaks (4.5), (4.6). Now restrict ourself with the second equation in (4.9). Dividing the space of its solution on the gauge group $\mathcal{G}_{\mathbb{C}}$ we come to the moduli space of the Higgs bundles $T^* \mathcal{M}(N, g, 0, d)$ (3.17). There exists a dense subset of moduli space of stable Higgs bundles $(T^* \mathcal{M}(N, g, 0, d))^{\text{stable}} \subset T^* \mathcal{M}(N, g, 0, d)$. The moduli space of stable Higgs bundles parameterizes the smooth part of $\mathcal{M}_H(\Sigma_g)$ (4.13) [2].

Consider a Higgs bundle with data $(\Phi, \bar{A})$ satisfying Eq. 2 in (4.9) and reconstruct from it solutions $(A_z, \Phi_z, A_{\bar{z}}, \Phi_{\bar{z}})$ of (4.9). Define them as

$$\Phi_{\bar{z}} = -h^{-1} \Phi^+ h, \quad \Phi_z = \Phi,$$
$$A_{\bar{z}} = \bar{A}, \quad\quad\quad A_z = -h^{-1} \bar{\partial} h - h^{-1} \bar{A}^+ h.$$

Then $(\Phi_{\bar{z}}, A_z)$ satisfy Eq. 3 in (4.9). Eq. 1 in (4.9) takes the form

$$\bar{\partial}(h^{-1}\bar{\partial}h + h^{-1}\bar{A}^+h) - \partial\bar{A} + [\bar{A}, (h^{-1}\bar{\partial}h + h^{-1}\bar{A}^+h)] - [\Phi, h^{-1}\Phi^+h] = 0.$$

For almost all $(\Phi, \bar{A})$ there exists a solution h of this equation (see Appendix of Donaldson in [2]). In this way we pass from the holomorphic data to solutions of system (4.9).

Summarizing, to define $\mathcal{M}_H(\Sigma_g)$ one can act in two ways:
1. Divide the space of solutions of (4.9) on the $SU(N)$-valued gauge group $\mathcal{G}$.
2. Consider the moduli space of stable Higgs bundles.

4.1.2. Hyper-Kähler Reduction. In this section we explain how to derive the moduli space $\mathcal{M}_H(\Sigma_g)$ (4.13) via an analog of the symplectic reduction. It is the so-called hyper-Kähler reduction [41]. We prove that infinite-dimensional space $\mathcal{W}$ (4.7) is a hyper-Kähler manifold, and $\mathcal{M}_H$ is its hyper-Kähler quotient, where (4.9) play the role of the moment equations.

To define a hyper-Kähler manifold we need three complex structures and a metric satisfying certain axioms. Define a flat metric on $\mathcal{W}$ depending on the complex structure on Σ

$$ds^2 = -\frac{1}{4\pi}\int_\Sigma \text{Tr}\left(\delta A_z \otimes \delta A_{\bar{z}} + \delta A_{\bar{z}} \otimes \delta A_z + \delta\Phi_z \otimes \delta\Phi_{\bar{z}} + \delta\Phi_{\bar{z}} \otimes \delta\Phi_z\right). \quad (4.14)$$

Introduce three complex structures I, J, K on $\mathcal{W}$. The corresponding operators act on the tangent bundle $T\mathcal{W}$, such that they obey the imaginary quaternion relations $I^2 = J^2 = K^2 = -\text{Id}$, $IJ = K, \ldots$ The complex structures are integrable because $\mathcal{W}$ is flat. Introduce a basis of the space of one-forms in $T^*\mathcal{W}$

$$V = (\delta A_{\bar{z}}, \delta\Phi_z, \delta A_z, \delta\Phi_{\bar{z}}).$$

Then the action of the conjugated operators on $T^*\mathcal{W}$ in this basis takes the form

$$I^T = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad J^T = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$K^T = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

Linear functions on $\mathcal{W}$ are holomorphic with respect to a complex structure if they are transformed under the action of the corresponding operator with eigenvalue $+i$. Thus A_z, Φ_z are holomorphic in the complex structure I, $A_{\bar{z}} + i\Phi_{\bar{z}}$, $A_z + i\Phi_z$ are holomorphic in the complex structure J, and $A_{\bar{z}} - \Phi_{\bar{z}}$, $A_z + \Phi_z$ are holomorphic in the complex structure K.

To be hyper-Kähler on $\mathcal{W}$, the metric ds^2 should be of type $(1,1)$ in each complex structure. It means that $ds^2 \sim (I^T \otimes I^T)\,ds^2 = (J^T \otimes J^T)\,ds^2 = (K^T \otimes K^T)\,ds^2$. In this way we have described a flat hyper-Kähler metric on $\mathcal{W}$. A linear combination of the complex structures produces a family of complex structures, parameterized by $\mathbb{CP}^1$.

We define three symplectic structures associated with the complex structures on $\mathcal{W}$ as $\omega_I = (I^T \otimes \mathrm{Id})\,ds^2$, $\omega_J = (J^T \otimes \mathrm{Id})\,ds^2$, $\omega_K = (K^T \otimes \mathrm{Id})\,ds^2$:

$$\omega_I = -\frac{i}{2\pi} \int_{\Sigma_g} \mathrm{tr}\,(DA_{\bar{z}} \wedge DA_z - D\Phi_z \wedge D\Phi_{\bar{z}}),$$

$$\omega_J = \frac{1}{2\pi} \int_{\Sigma_g} \mathrm{tr}\,(D\Phi_{\bar{z}} \wedge DA_z + D\Phi_z \wedge \delta A_{\bar{z}}), \qquad (4.15)$$

$$\omega_K = \frac{i}{2\pi} \int_{\Sigma_g} \mathrm{tr}\,(D\Phi_{\bar{z}} \wedge DA_z - D\Phi_z \wedge DA_{\bar{z}}).$$

These forms are closed and of type $(1,1)$ with respect to the corresponding complex structures.

Now consider the gauge transformations (4.10) of the fields (4.11), (4.12). Since the gauge transform takes values in $SU(N)$, the forms (4.15) are gauge-invariant. Therefore we can proceed as in the case of the standard symplectic reduction (2.14). But now we obtain three generating momentum Hamiltonians with respect to the three symplectic forms

$$F_I = \frac{i}{2\pi} \int_{\Sigma_g} \mathrm{tr}\,(\epsilon(F_{z,\bar{z}} - [\Phi_z, \Phi_{\bar{z}}])) \quad (\epsilon \in \mathrm{Lie}\,(\mathcal{G})),$$

$$F_J = -\frac{1}{2\pi} \int_{\Sigma_g} \mathrm{tr}\,(\epsilon(d'\Phi_{\bar{z}} + d''\Phi_z)),$$

$$F_K = -\frac{i}{2\pi} \int_{\Sigma_g} \mathrm{tr}\,(\epsilon(d'\Phi_{\bar{z}} - d''\Phi_z))$$

and the three moment maps $\mathcal{W} \to \mathrm{Lie}^*(\mathcal{G})$

$$\mu_I = iF - i[\Phi_z, \Phi_{\bar{z}}], \quad \mu_J = d'\Phi_{\bar{z}} + d''\Phi_z, \quad \mu_K = i(d'\Phi_{\bar{z}} - d''\Phi_z).$$

The zero-valued moments coincide with the Hitchin systems. The hyper-Kahler quotient $\mathcal{W}///\mathcal{G}$ is defined as

$$\mathcal{W}///\mathcal{G} = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)/\mathcal{G}.$$

To come to the system (4.9) consider the linear combination

$$\nu_I = \mu_J + i\mu_K = d''\Phi_{\bar{z}}. \tag{4.16}$$

This moment map is derived from the symplectic form

$$\Omega_I = \omega_J + i\omega_K = \frac{1}{\pi} \int_\Sigma \mathrm{tr}\,(D\Phi_z \wedge DA_{\bar{z}}).$$

It is a $(2,0)$ form in the complex structure I. Thus we have the holomorphic moment map ν_I in the complex structure I. Vanishing of the holomorphic moment map ν_I and the real moment map μ_I is equivalent to the Hitchin equations. Dividing their solutions on the gauge group $\mathcal{G}$ we come to the moduli space $\mathcal{M}_H(\Sigma_g)$ (4.13).

Now consider an analog of (4.16) corresponding to the complex structure J

$$\nu_J = \mu_K + i\mu_I = \mathcal{F}_{z,\bar{z}}, \qquad \mathcal{F}_{z,\bar{z}} = \mathcal{F}(\mathcal{A}_z, \mathcal{A}_{\bar{z}}),$$

$\mathcal{A}_z = A_z + i\Phi_z$, $\mathcal{A}_{\bar{z}} = A_{\bar{z}} + i\Phi_{\bar{z}}$. This moment map comes from the symplectic form

$$\Omega_J = \frac{1}{2\pi} \int_{\Sigma_g} \mathrm{tr}\,(D\mathcal{A} \wedge D\mathcal{A}).$$

It is $(2,0)$ form in the complex structure J. Putting $\nu_J = 0$ we come to flatness condition of the bundle E. Dividing the set of solutions $\mathcal{F}_{z,\bar{z}} = 0$ on the $GL(N,\mathbb{C})$ valued gauge transformations $\mathcal{G}_{\mathbb{C}}$ we come to the space

$$\mathcal{Y} = (\mathcal{F}_{z,\bar{z}} = 0)/\mathcal{G}_{\mathbb{C}} \tag{4.17}$$

of homomorphisms $\pi_1(\Sigma_g) \to GL(N,\mathbb{C})$ defined up to conjugations. According to [42] and Donaldson (the Appendix in [2]), generic flat bundles parameterize $\mathcal{M}_H(\Sigma_g)$ (4.13) in the complex structure J. This space is a phase space of non-autonomous Hamiltonian systems leading to monodromy preserving equations (see Conclusion). Thus, the space $\mathcal{M}_H(\Sigma_g)$ describes phase spaces of integrable systems $\mathcal{R}^{\mathrm{red}}$ (3.17) in the complex structure I and phase spaces of monodromy preserving equations $\mathcal{Y}$ (4.17) in the complex structure J.

4.2. $\mathcal{N} = 4$ SUSY Yang–Mills in Four Dimensions and Hitchin Equations.
Here we consider a twisted version of $\mathcal{N} = 4$ super Yang–Mills theory in four
dimensions. This theory was analyzed in detail in [16–18] to develop a field-
theoretical approach to the Geometric Langlands Program. The quantum Hitchin
systems are a one side of this construction and we use here only a minor part
of [16]. The twisted theory is a topological theory that contains a generalization
of the Hitchin equations (4.9) as a condition of the BRST invariance. Our goal is
to describe the Hecke transformations in terms of the theory. In Sec. 3 we have
defined the Hecke transformations as an instant singular gauge transformation.
The four-dimensional theory allows one to consider gauge transformations varying
along a space coordinate x_3. They become singular at some point, say $x_3 = 0$,
where a singular 't Hooft operator is located. It gives a natural description of
the symplectic Hecke correspondence in terms of a monopole configuration in the
twisted theory.

4.2.1. Twisting of $\mathcal{N} = 4$ SUSY $SU(N)$ Yang–Mills Theory. $\mathcal{N} = 4$ SUSY
$SU(N)$ Yang–Mills action in four dimensions can be derived from the $\mathcal{N} = 1$
SUSY $SU(N)$ Yang–Mills action in ten dimensions by the dimensional reduction.
We need only the bosonic part of the reduced theory.

The bosonic fields of the 4d Yang–Mills theory are four-dimensional gauge
potential

$$\mathbf{A} = (A_0, A_1, A_2, A_3),$$

and six scalar fields coming from six extra dimensions

$$\phi = (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5).$$

The bosonic part of the action has the form

$$I = \frac{1}{e^2} \int d^4x \, \mathrm{tr} \left(\frac{1}{2} \sum_{\mu,\nu=0}^{3} F_{\mu\nu} F^{\mu\nu} + \sum_{\mu=0}^{3} \sum_{i=1}^{6} D_\mu \phi_i D^\mu \phi_i + \frac{1}{2} \sum_{i,j=1}^{6} [\phi_i \phi_j]^2 \right).$$

The symmetry of the action is $\mathrm{Spin}\,(4) \times \mathrm{Spin}\,(6)$ (or $\mathrm{Spin}\,(1,3) \times \mathrm{Spin}\,(6)$ in
the Lorentz signature). The sixteen generators of the 4d supersymmetry are
transformed under $\mathrm{Spin}\,(1,3) \times \mathrm{Spin}\,(6) \sim SL(2) \times SL(2) \times \mathrm{Spin}\,(6)$ as $(2, 1, \bar{4}) \oplus
(1, 2, 4)$:

$$\{\bar{Q}_{AX}\} \oplus \{Q_{\dot{A}}^{Y}\} \quad (A = 1, 2;\ X = 1, \ldots, 4),\ (\dot{A} = 1, 2;\ Y = 1, \ldots, 4).$$

They satisfy the supersymmetry algebra

$$\{\bar{Q}_{AX}, Q_{\dot{A}}^{Y}\} = \delta_X^Y \sum_{\mu=0}^{3} \Gamma_{A\dot{A}}^{\mu} P_\mu, \quad \{Q, Q\} = 0, \quad \{\bar{Q}, \bar{Q}\} = 0. \tag{4.18}$$

The action of Q on the field X takes the form

$$\delta X = [Q, X\}.$$

Let κ be a map $\mathrm{Spin}\,(4) \to \mathrm{Spin}\,(6)$ and set

$$\mathrm{Spin}'(4) = (\mathrm{Id} \times \kappa)\,\mathrm{Spin}\,(4).$$

Define κ in such a way that the action of $\mathrm{Spin}'(4)$ on the chiral spinor $\mathcal{S}^+$ has an invariant vector. Let Q be the corresponding supersymmetry. It follows from (4.18) that it obeys $Q^2 = 0$. The twisted theory is defined by the physical observables from the cohomology groups $H^\bullet(Q)$. The twisted four scalar fields $\phi = (\phi_0, \dots, \phi_3)$ are reinterpreted as adjoint-valued one-forms on $\mathbb{R}^4$, while untwisted $\sigma, \bar{\sigma} = \phi_4 \pm \imath \phi_5$ remain adjoint-valued scalars.

In fact, there is a family of topological theories parameterized by $t \in \mathbb{CP}^1$. The bosonic fields to be invariant under Q should satisfy the equations

$$\begin{aligned}
&1) \quad (F - \phi \wedge \phi + tD\phi)^+ = 0, \\
&2) \quad (F - \phi \wedge \phi - t^{-1}D\phi)^- = 0, \\
&3) \quad \star D \star \phi = 0,
\end{aligned} \qquad (4.19)$$

where $+$ and $-$ denote the self-dual and the anti-self-dual parts for four-dimensional two-forms, $D = d + [\mathbf{A},\ ]$ and $\star$ is the Hodge operator in four dimensions. We are interested in solutions of this system up to gauge transformations.

This theory defined on flat $\mathbb{R}^4$ can be extended on any four-manifold M in such a way that it preserves the Q symmetry and contributions of metric come only from Q-exact terms. The bosonic part of the theory is described by connections $\mathbf{A} = (A_0, A_1, A_2, A_3)$ in a bundle E over M in a presence of the adjoint-valued one-forms $\phi = (\phi_0, \phi_1, \phi_2, \phi_3)$ satisfying (4.19).

The important for integrable systems case is $M = \mathbb{R}^2 \times \Sigma_g$, where $\mathbb{R}^2 = (\text{time} = x_0) \times \{x_3 = y\}$ and Σ_g will play the role of the basic spectral curve. $\mathbb{R}^2$ is not involved in the twisting and the fields (ϕ_0, ϕ_3) remain scalars, while ϕ_1, ϕ_2 become one-forms on Σ_g. It turns out that after the reduction the system (4.19) becomes equivalent to the Hitchin equations (4.9).

4.2.2. Hecke Correspondence and Monopoles. The system (4.19) for $t = 1$ can be replaced by

$$F - \phi \wedge \phi + \star D\phi = 0, \qquad (4.20)$$

$$\star D \star \phi = 0. \qquad (4.21)$$

Assume that the fields are time-independent and consider the system on the three-dimensional manifold $W = I(x_3) \times \Sigma_g$, where $-\infty \leqslant x_3 \leqslant \infty$. In terms of the

three-dimensional fields $\tilde{A}$ and $\tilde{\Phi}$ ($\mathbf{A} = (A_0, \tilde{A})$, $\phi = (\Phi_0 dx_0, \tilde{\Phi})$), the equations take the form [16]

$$\tilde{F} - \tilde{\Phi} \wedge \tilde{\Phi} = \star(D\Phi_0 - [A_0, \tilde{\Phi}]),$$

$$\star D\tilde{\Phi} = [\Phi_0, \tilde{\Phi}] + DA_0, \quad \star D \star \tilde{\Phi} + [A_0, \Phi_0] = 0. \tag{4.22}$$

Here the Hodge operator $\star$ is taken in the three-dimensional sense. Replace the coordinates $\mathbf{x} = (x_1, x_2, x_3)$ on $x_3 \to y$ and $(x_2, x_3) \to (z, \bar{z})$, where $(z, \bar{z})$ are local coordinates on Σ_g. Let $g(z, \bar{z})|dz|^2$ be a metric on Σ_g. The metric on W is $ds^2 = g|dz|^2 + dy^2$. Then the Hodge operator takes the form

$$\star dy = \frac{1}{2} ig dz \wedge d\bar{z}, \quad \star dz = -idz \wedge dy, \quad \star d\bar{z} = id\bar{z} \wedge dy.$$

It is argued in [16] that $\Phi_y = 0$ and $A_0 = 0$ are solutions of the system. Then we come to the equations

$$
\boxed{
\begin{aligned}
&1. \quad F(A_z, A_{\bar{z}}) - [\Phi_z, \Phi_{\bar{z}}] = \frac{1}{2} ig D_{A_y} \Phi_0, \\
&2. \quad D_{A_{\bar{z}}} \Phi_z = 0, \\
&3. \quad F(A_y, A_{\bar{z}}) = i D_{A_{\bar{z}}} \Phi_0, \\
&4. \quad D_{A_y} \Phi_z = -i[\Phi_z \Phi_0],
\end{aligned}
}
\tag{4.23}
$$

where as before $\Phi_z^+ = -h^{-1} \Phi_{\bar{z}} h$. The system is simplified in the gauge $A_y = 0$. In particular, for $\Phi_0 = 0$ the system (4.23) becomes essentially two-dimensional and coincides with the Hitchin equations (4.9).

Let Σ_g be an elliptic curve ($g = 1$). This case is important to application to integrable systems. The nonlinear system (4.23) can be rewritten as a compatibility condition for the linear system depending on the spectral parameter $\lambda \in \mathbb{C}$

$$(\partial_z + \lambda^{-1} a \partial_y + A_z + i\lambda^2 \Phi_z + i\lambda^{-1} a \Phi_0)\psi = 0, \tag{4.24}$$

$$(\partial_{\bar{z}} + \lambda a \partial_y + A_{\bar{z}} + i\lambda^{-2} \Phi_{\bar{z}} - i\lambda a \Phi_0)\psi = 0. \tag{4.25}$$

Here $a^2 = \dfrac{1}{\bar{\tau} - \tau}$. This linear system allows one to apply the methods of the inverse scattering problem or the Whitham approximation to find solutions of (4.23).

Define the complex connection $\mathcal{A}_z = A_z + i\Phi_z$. In terms of $(\mathcal{A}_z, \mathcal{A}_{\bar{z}}, A_y)$ the systems (4.23) assume the form of the Bogomolny equation

$$F = \star D\Phi_0. \tag{4.26}$$

Consider a monopole solution of this equation. Let $\tilde{W} = (W \setminus \mathbf{x}^0 = (y = 0, z = z_0))$. The Bianchi identity $DF = 0$ in the space $\tilde{W}$ implies that Φ_0 is the Green function for the operator $\star D \star D$

$$\star D \star D \Phi_0 = M\delta(\mathbf{x} - \mathbf{x}^0),$$

where $M \in gl(N, \mathbb{C})$.

Consider first the Abelian case $G = U(1)$. Then $F(A_z, A_{\bar{z}})$ is a curvature of a line bundle $\mathcal{L}$. Locally near $\mathbf{x}_0 = (y = 0, z = z_0, \bar{z} = \bar{z}_0)$ Φ_0 has a singularity

$$\Phi_0 \sim \frac{im}{2|\mathbf{x} - \mathbf{x}^0|}, \tag{4.27}$$

where m is a magnetic charge. Due to Eq. 1 in (4.23), F takes the form

$$F(A_z, A_{\bar{z}}) = \frac{1}{2}g\partial_y \Phi_0,$$

$$F(A_z, A_{\bar{z}}) \sim \frac{1}{2}mg(z, \bar{z})\frac{y}{|\mathbf{x} - \mathbf{x}^0|^{3/2}}.$$

Consider a small sphere S^2 enclosing $\mathbf{x}^0$. Due to (4.26) and (4.27)

$$\int_{S^2} F = m.$$

This solution describes the Dirac monopole of charge m corresponding to a line bundle over S^2 of degree m.

Let $\Sigma_g^{\pm} = \Sigma_g \times (\pm\infty)$ and $\mathcal{L}^{\pm}$ be the line bundles over $\Sigma_g^{\pm}$. The two-dimensional cycle C describing the boundary $C = \partial((W = I \times \Sigma_g) \setminus \mathbf{x}_0)$ is $\Sigma_g^+ - \Sigma_g^- - S^2$. Taking the integral over C we find that

$$\int_C F = 0.$$

In other words, for the Chern classes of the bundles $c(\mathcal{L}) = \deg(\mathcal{L})$ we have

$$\deg(\mathcal{L}^+) = \deg(\mathcal{L}^-) + m$$

or $\mathcal{L}^+ = \mathcal{L}^{\otimes}\mathcal{O}(z_0)^m$. Here $\mathcal{O}(z_0)^m$ is a line bundle whose holomorphic sections are holomorphic functions away from z_0 with a possible single pole of degree m at z_0. The line bundles over Σ_g are topologically equivalent for $y < 0$ or $y > 0$. The gauge transformation Φ_0 is smooth away from $\mathbf{x}_0$. The singularity changes the degree of the bundle.
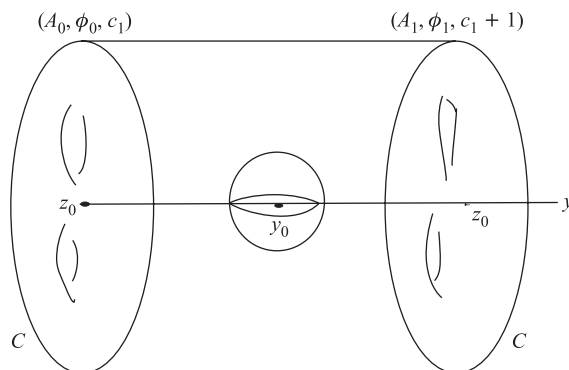
The monopole increases the Chern class: $c_1 \rightarrow c_1 + m$

In the four-dimensional Abelian theory we have the Dirac monopole singular along the time-like line $L = (x_0, \mathbf{x}^0)$. It corresponds to including *the 't Hooft operator* in the theory saying that the connections have the monopole singularity along the line L.

A generic vector bundles E near $\mathbf{x}_0$ split $E_y \sim \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_N$. Consider the gauge transformation

$$\Phi_0 \sim \frac{i}{2|\mathbf{x} - \mathbf{x}_0|} \, \mathrm{diag}\,(m_1, \ldots, m_N). \tag{4.28}$$

It causes the transformation $\mathcal{L}_j \rightarrow \mathcal{L}_j \otimes \mathcal{O}(z_0)^{m_j}$. The degree of the bundles E changes after crossing $y = 0$ by $\sum m_j$, as it was described for bundles over Σ in Subsec. 3.4.

To be more precise we specify the boundary conditions of solutions on the ends $y = -\infty$ and $y = +\infty$. Since $\Phi_0 \rightarrow 0$ for $y \rightarrow \pm\infty$ the system (4.23) coincides with the Hitchin system (4.9). If $\mathcal{M}_H(N, g, n, m^{\pm})$ is the moduli space of solutions on the boundaries $y = \pm\infty$, the gauge transformation with the monopole singularity stands that $m^{+} = m^{-} + \sum m_j$. It defines the SHC between two integrable systems related to $\mathcal{M}_H(N, g, n, m^{\pm})$. In particular, we have described it at the point $y = 0$ for $\mathcal{M}_H(N, 1, n, 0)$ and $\mathcal{M}_H(N, 1, n, 1)$.

CONCLUSION

Here we shortly discuss some related issues not included in the review.

1. Solutions of the Hitchin equations (4.9) corresponding to quasi-parabolic Higgs bundles were analyzed in [17]. In the three-dimensional gauge theory considered in Subsec. 3.3 we have the Wilson lines located at the marked points.

In the four-dimensional Yang–Mills theory they correspond to singular operators along two-dimensional surfaces. Locally on a punctured disc around a marked point, the Hitchin system (4.9) assumes the form of the Nahm equations [43]. It was proved in [46] that the space of its solutions after dividing on a special gauge group is symplectomorphic to a coadjoint orbit of $SL(N,\mathbb{C})$. A hyper-Kähler structure on the space of solutions induces a hyper-Kähler structure on the orbits. It establishes the interrelations between the Hitchin equations and the Higgs bundles with the marked points (the quasi-parabolic Higgs bundles).

2. There exists a generalization of this approach to Higgs bundles of infinite rank. In other words, the structure group $G = GL(N,\mathbb{C})$ or $SL(N,\mathbb{C})$ of the bundles is replaced by an infinite-rank group. One way is to consider the central extended loop group $S^1 \to G$. Then the Higgs field depends on additional variable $x \in S^1$ and instead of the Lax equation we come to the Zakharov–Shabat equation

$$\partial_j L - \partial_x M_j + [M_j, L] = 0.$$

This equation describes an infinite-dimensional integrable hierarchy like the KdV hierarchy. The two-dimensional version of the ECMS was constructed in [10,11]. In particular, the SHC establishes an equivalence of the two-particle ($N = 2$) elliptic Calogero–Moser field theory with the Landau–Lifshitz equation [44,45]. The latter system is the two-dimensional version of the $SL(2,\mathbb{C})$ elliptic top. The relations (3.48) are working in the two-dimensional case.

Another way is to consider $GL(\infty)$ bundles. In [48], the ECMS for infinite number of particles $N \to \infty$ was analyzed. The elliptic top on the group of the noncommutative torus was considered in [47]. It is a subgroup of $GL(\infty)$. This construction describes an integrable modification of the hydrodynamics of the ideal fluid on a noncommutative two-dimensional torus.

3. Consider dynamical systems where the role of times is played by parameters of complex structures of curves $\Sigma_{g,n}$. In this case we come to monodromy preserving equations, like the Schlesinger system or the Painlevé equations. They can be constructed in the similar fashion as the integrable Hitchin systems [8]. To this purpose, one should replace the Higgs bundles by the flat bundles and afterwards use the same symplectic reduction (see (4.17)). In this situation the Lax equations take the form

$$\partial_j L - \partial_z M_j + [M_j, L] = 0.$$

An analysis of this system is more complicated in comparison with the standard Lax equations due to the presence of derivative with respect to the spectral parameter. Note that M_j corresponds only to the quadratic Hamiltonians, since they are responsible for the deformations of complex structures. Concrete examples of this construction were given in [8,52,53]. Interrelations with Higgs bundles were analyzed in [8,49]. It is remarkable that the symplectic Hecke correspondence is

working in this case. It establishes an equivalence of the Painlevé VI equation and a nonautonomous Zhukovski–Volterra gyrostat [12].

4. A modification of the Higgs bundles allows one to construct relativistic integrable systems [50]. The role of Higgs field is played by a group element $g = \exp\left(cK^{-1}\Phi\right)$ where K is a canonical class on Σ and c is the relativistic parameter. This construction is working only for curves of genus $g \leqslant 1$. This approach was realized in [28] to derive the elliptic Rujesenaars system and in [51,53] to derive the elliptic classical r matrix of Belavin–Drinfeld [54] and a quadratic Poisson algebra of the Sklyanin–Feigin–Odesski type [55,56].

Including the relativistic systems allows one to define a duality in integrable systems [57,58] (see [59] for recent developments). This type of dualities has a natural description for the corresponding quantum integrable systems in terms of Hecke algebras [60]. It is called there the Fourier transform and takes the form of S duality. Another form of duality in the classical Hitchin system is considered in [61–63]. It is related to Langlands duality and is similar to T duality of fibers in the Hitchin fibration.

5. There exists useful description of the moduli space of holomorphic vector bundles closely related to the modification described in Subsec. 3.4. It is the so-called Tyurin parameterization [64]. This construction was applied to describe Higgs bundles and integrable systems related to curve of arbitrary genus in [10, 52,65]. Using this approach, classical r matrices with a spectral parameter living on curves of arbitrary genus were constructed in [67].

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