*ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА*
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# RELATIONS BETWEEN $SU(2)$- AND $SU(3)$-LECs IN CHIRAL PERTURBATION THEORY

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Chiral perturbation theory in the two-flavour sector allows one to analyze Green functions in QCD in the limit where the strange quark mass is considered to be large in comparison to the external momenta and to the light quark masses $m_u$ and $m_d$. In this framework, the low-energy constants of $SU(2)_R \times SU(2)_L$ depend on the value of the heavy quark masses. For the coupling constants which occur at order $p^2$ and $p^4$ in the chiral expansion, we worked out in [1] the dependence on the strange quark mass at two-loop accuracy, and provided in [2] analogous relations for some of the couplings $c_i$ which are relevant at order $p^6$. This talk comments on the methods used, and illustrates implications of the results obtained.

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## INTRODUCTION

At low energies and small quark masses, the Green functions of quark currents can be analyzed in the framework of chiral perturbation theory ($\chi$PT) [3–5]. The method allows one to work out the momentum and quark mass dependence of the quantities of interest in a systematic and coherent manner. It is customary to perform the quark mass expansion either around $m_u = m_d = 0$, with the strange quark mass held fixed at its physical value ($\chi\text{PT}_2$), or to consider an expansion in all three quark masses around $m_u = m_d = m_s = 0$ ($\chi\text{PT}_3$). The corresponding effective Lagrangians contain low-energy constants (LECs) that parametrize the degrees of freedom which are integrated out. The two expansions are not independent: in a particular limit specified below, $\chi\text{PT}_3$ reduces to $\chi\text{PT}_2$. As a result of this, one can express the LECs in the two-flavour case through the ones in $\chi\text{PT}_3$, in a perturbative manner. The relations amount to a series expansion in the strange quark mass. Generically,

$$k^{\rm r} = \sum_{m \geqslant -m_0} d_m z^m, \quad z = \frac{m_s B_0}{(4\pi F_0)^2}. \tag{1}$$

Here, $k^{\mathrm{r}}$ stands for any of the renormalized $\chi$PT$_2$ LECs $F$, $B$, $\ell_i^{\mathrm{r}}$, $c_i^{\mathrm{r}} \ldots$,while $F_0, B_0$ are the LECs at order $p^2$ in $\chi$PT$_3$. The coefficients $d_m$ (whose dependence on the chosen $k^{\mathrm{r}}$ is suppressed in the notation) contain renormalized LECs from $\chi$PT$_3$, and powers of the logarithm $\ln(m_s B_0/\mu^2)$, where $\mu$ denotes the standard renormalization scale. For $k^{\mathrm{r}}$ at order $p^{2(N+1)}$, one has $m_0 = N$, and the corresponding leading term $d_{m_0} z^{-m_0}$ is generated by three graphs in $\chi$PT$_3$. The next-to-leading order term requires a one-loop calculation, etc. In the following, we refer to the relations (1) as *matching relations*, obtained by *matching* $\chi$PT$_2$ to $\chi$PT$_3$ in the specific limit mentioned. The matching relations are useful, because they provide i) additional information on the LECs in $\chi$PT$_3$, and ii) internal consistency checks.

For the LECs at order $p^2$ and $p^4$, the matching was performed to one loop (to two loops) in [5] ([1]), and for a subclass of LECs at order $p^6$ to two loops in [2].

We comment on related work which is available in the literature.

i) The strange quark mass expansion of the $\chi$PT$_2$ LEC $B$ ($F^2 B$) was provided at two-loop accuracy in [6] ([7]).

ii) Matching of the order $p^6$ LECs in the parity-odd sector was performed recently in [8].

iii) Analogous work was done in the baryon sector in [9, 10], and for electromagnetic interactions in [11–14].

## 1. MATCHING OF GENERATING FUNCTIONALS IN $\chi$PT$_2$ AND $\chi$PT$_3$

We have developed in [1] a generic method for the matching, which is based on the path integral formulation of $\chi$PT. The idea of this method is not to compare matrix elements that can be obtained in both formulations, but rather to restrict the three-flavour theory such that it only describes the same physics as the two-flavour formulation. Then, one compares their generating functionals containing all the Green functions and reads off the matching of the LECs.

The LECs do not depend on the light quark masses $m_u$ and $m_d$. Since both theories are expansions around vanishing quark masses, we may set $m_u = m_d = 0$ for the purpose of the matching.

The comparison of the generating functionals is in fact a comparison of all possible Green functions, which depend on the external fields. Obviously, they can only be compared with each other if they depend on the *same* external fields. Therefore, the external fields of $\chi$PT$_3$ need to be restricted to those of $\chi$PT$_2$. We also have to assure that the heavy mesons $K$ or $\eta$ running in the loops do not have the possibility to go on-shell. Therefore, we consider in addition the case where all external momenta are small compared to the kaon mass. The physics of $SU(3)_R \times SU(3)_L$ then reduces to the one of $SU(2)_R \times SU(2)_L$. We refer to this limit as the *two-flavour limit*.

The LECs are the coefficients of local chiral operators in the effective Lagrangian. Once one evaluates the generating functional with the effective Lagrangian, besides the local terms also many nonlocal contributions are generated, both in $\chi$PT$_2$ and in $\chi$PT$_3$. However, the nonlocal contributions, appearing in $\chi$PT$_3$ as the result of low-energy expansion, will be exactly canceled by $\chi$PT$_2$ counterparts once the matching is performed. Therefore, to obtain the matching relations, it is sufficient to restrict oneself to the local parts in the evaluation of the generating functional of $\chi$PT$_3$.

## 2. LECs AT ORDER $p^2$ AND $p^4$

All the relations may be put in the form of Eq. (1). To render the formulae more compact, we found it convenient to slightly reorder the expansions, such that they become a series in the quantity $\bar{M}_K^2$, which stands for the one-loop expression of the (kaonmass)$^2$ in the limit $m_u = m_d = 0$, see, e.g., [5]. The result is

$$
\begin{aligned}
&Y = Y_0[1 + a_Y\, x + b_Y\, x^2 + \mathcal{O}(x^3)], \quad Y = F, \Sigma, \\
&l_i^{\mathrm{r}} = a_i + x\, b_i + \mathcal{O}(x^2), \quad (i \neq 7), \quad l_7 = \frac{F_0^2}{8B_0 m_s} + a_7 + x\, b_7 + \mathcal{O}(x^2), \\
&x = \frac{\bar{M}_K^2}{NF_0^2}, \quad N = 16\pi^2, \quad \Sigma = F^2 B, \quad \Sigma_0 = F_0^2 B_0.
\end{aligned} \tag{2}
$$

We denote the contributions proportional to $a_i$ $(b_i)$ as NLO (NNLO) terms, generated by one-loop (two-loop) graphs in $\chi$PT$_3$. Note that $l_7$ receives a contribution at leading order (LO) as well, proportional to $m_s^{-1}$, in agreement with the remarks made in the introduction. The LO and NLO terms were determined in [5] more than 25 years ago, whereas the NNLO terms $b_i$ were only recently worked out [1]. They have the following structure:

$$
b = p_0 + p_1\, \ell_K + p_2\, \ell_K^2 \ , \tag{3}
$$

where $\ell_K = \ln(\bar{M}_K^2/\mu^2)$ is the chiral logarithm, and where we have dropped for simplicity the index $i$. The polynomials $p_j$ are independent of the strange quark mass, and their scale dependence is such that in combination with the logarithms it adds up to the scale independent quantity $b$. In other words, the scale dependence of $l_i$ is exclusively generated by the one-loop contribution $a_i$. The explicit results for the polynomials $p_j$ are displayed in Tables 2–4 of [1].

## 3. LECs AT ORDER $p^6$

The evaluation of all matching relations at two-loop order for the LECs at order $p^6$ is very complicated. To ease the calculations, we did not deal with the full framework in [2], but rather switched off the sources $s$ and $p$ (while retaining $m_s$). This yields the following simplifications:

i) the solution of the classical EOM for the eta-field is trivial, $\eta = 0$;

ii) there is no mixing between the $\eta$ and the $\pi^0$ fields.

Point i) greatly simplifies the transition from the $\chi\text{PT}_3$ building blocks of the monomials to those of two flavours, as it suppresses any effects from the eta, whereas point ii) eliminates many possible graphs and hence considerably reduces the requested labour. For example, in this restricted framework, the one-particle reducible graphs (two one-loop diagrams linked by a single propagator) do not contribute to the matching, see [2].

Aiming for the $\mathcal{L}_6$-monomials in the generating functional requires the evaluation of many graphs with sunset-like topology. In the two-flavour limit, where one is interested in the local contributions only, one can simplify the loop calculations by using a short distance expansion for the massive propagators. This simplifies drastically the involved loop integrals; however, the contributions from individual graphs are not chirally invariant. Collecting terms stemming from different graphs to obtain a manifestly chirally invariant result is rather cumbersome. Since we are interested in the local terms only, we use a shortcut which is based on gauge invariance*: one may choose a gauge such that at some fixed space-time point $x_0$, the totally symmetric combination of up to three derivatives acting on the chiral connection vanishes. Up to four ordinary derivatives are then indistinguishable from the fully symmetric combinations of covariant derivatives. This allows us to write even intermediate results in a manifestly chirally invariant manner.

As already stated in [15], the monomial $P_{27}$ of the $p^6$ Lagrangian for $\chi\text{PT}_2$ can be discarded as it is linearly dependent on the other monomials $P_i$. Therefore, the matching relations will certainly be a combination of some $c_i^{\text{r}}$ and $c_{27}^{\text{r}}$. Due to the restricted framework, only relations for LECs not involving monomials dependent on the sources $s$ or $p$ are nontrivial. In the restricted framework, there is an additional relation among the remaining $SU(2)$ monomials. Because the EOM is different in the full framework, this relation is no longer valid there. We used this relation to exclude the monomial $P_1$ from our consideration. As a result, we give the matching for the 27 combinations of $c_i^{\text{r}}$. In the full framework, an additional matching relation (apart from the ones for the monomials involving the sources $s$ and $p$) for $c_1^{\text{r}}$ could be worked out, yielding the only missing piece in the matching for the 28 LECs worked out.

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The final result may be written in the form

$$x_i = p_i^{(0)} + p_i^{(1)}\ell_K + p_i^{(2)}\ell_K^2 + O(m_s)\ , \tag{4}$$

where $x_i$ denotes one of the 27 linear combinations of the $c_i^{\rm r}$. The explicit expressions for the polynomials $p_i^{(n)}$ in the $\chi$PT$_3$–LECs are displayed in Tables 2 and 3 of our article [2].

## SUMMARY

In this talk, we have discussed a general procedure [1, 2] to work out the matching relations between the LECs in $\chi$PT$_2$ and $\chi$PT$_3$ in a perturbative manner. For the LECs at order $p^2$ and $p^4$, and for a subset of those at order $p^6$, the relations are now available at two-loop order. The method could be used with only moderate adaption to work out more general matching relations, like the ones for $\chi$PT$_{N-1}$ to $\chi$PT$_N$. To obtain the matching relations of the remaining LECs at order $p^6$ to the same accuracy would require, however, a very big amount of work.

We refer the interested reader to our articles [1, 2] for the matching relations found, and for further details on the method used.

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