ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2012. Т. 43. ВЫП. 5

HAMILTONIAN REDUCTION AND UNFOLDING OF DYNAMICAL SYSTEMS WITH GAUGE SYMMETRIES

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We investigate the reduction and unfolding of dynamical systems with gauge symmetries. An application is provided by a nonrelativistic point charge in the field of a Dirac monopole. The corresponding dynamical system possessing a Kepler-type symmetry is associated with the Taub-NUT metric using a reduction procedure of symplectic manifolds with symmetries. The reverse of the reduction procedure is done by stages performing the unfolding of the gauge transformation followed by the Eisenhart lift in connection with scalar potentials.

PACS: 04.62.+v

INTRODUCTION

In the case of a symplectic manifold on which a group of symmetries acts symplectically, it is possible to reduce the original phase space to another symplectic manifold in which the symmetries are divided out. Such a situation arises when one has a particle moving in an electromagnetic field [1].

On the other hand the reverse of the reduction procedure can be used to investigate complicated systems. It is possible to use a sort of unfolding of the initial dynamics by imbedding it in a larger one which is easier to integrate [2]. Sometimes the equations of motion in a higher dimensional space are quite transparent, e.g., geodesic motions, but the equations of motion of the reduced system appear more complicated [3].

As an illustration of the reduction of a symplectic manifold with symmetries and the opposite procedure of oxidation of a dynamical system, we shall consider the principal bundle $\pi : \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$ with structure group $U(1)$. The Hamiltonian function on the cotangent bundle $T^\star(\mathbb{R}^4 - \{0\})$ is invariant under the $U(1)$ action and the reduced Hamiltonian system proves to describe the three-dimensional Kepler problem in the presence of a centrifugal potential and Dirac's monopole field.

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Concerning the unfolding of the reduced Hamiltonian system, we shall perform it by stages. In the first stage of unfolding we use an opposite procedure to the reduction by an $U(1) \simeq S^1$ action to a four-dimensional generalized Kepler problem. Finally we resort to the method introduced by Eisenhart [4] who added one or two extra dimensions to configuration space to represent trajectories by geodesics.

1. HAMILTONIAN REDUCTION

Let us start to consider the principal fiber bundle $\pi : \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$ with structure group $U(1)$. The $U(1)$ action is lifted to a symplectic action on $T^\star(\mathbb{R}^4 - \{0\})$ equipped with the standard symplectic form $d\Theta$.

Let $\Psi : T^\star(\mathbb{R}^4 - \{0\}) \to \mathbb{R}$ be the moment map associated with the $U(1)$ action

$$\Psi(x, y) = \frac{1}{2}(-x_2 y_1 + x_1 y_2 - x_4 y_3 + x_3 y_4), \tag{1}$$

where $(x, y) \in (\mathbb{R}^4 - \{0\}) \times \mathbb{R}^4$.

The reduced phase-space P_μ is defined through

$$\pi_\mu : \Psi^{-1}(\mu) \to P_\mu := \Psi^{-1}(\mu)/U(1), \tag{2}$$

which is diffeomorphic with $T^\star(\mathbb{R}^3 - \{0\}) \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$. The coordinates $(q_k, p_k) \in (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ are given by the Kustaanheimo–Stiefel transformation [5].

Let $\iota_\mu : \Psi^{-1}(\mu) \to T^\star(\mathbb{R}^4 - \{0\})$ be the inclusion map. The reduced symplectic form ω_μ is determined on P_μ by

$$\pi_\mu^\star \omega_\mu = \iota_\mu^\star d\Theta, \tag{3}$$

namely

$$\omega_\mu = \sum_{k=1}^{3} dp_k \wedge dq_k - \frac{\mu}{r^3}(q_1\, dq_2 \wedge dq_3 + q_2\, dq_3 \wedge dq_1 + q_3\, dq_1 \wedge dq_2); \tag{4}$$

ω_μ consists of the standard symplectic form on $T^\star(\mathbb{R}^3 - \{0\})$ and in addition a term corresponding to Dirac's monopole field $\mathbf{B} = -\mu(\mathbf{q}/r^3)$ of strength $-\mu$. The reduced Hamiltonian is determined by

$$H \circ \iota_\mu = H_\mu \circ \pi_\mu. \tag{5}$$

For the purpose of the present paper, we shall be concerned with the reduction of the dynamical system associated with the geodesic flows of the generalized

Taub-NUT metric on $\mathbb{R}^4 - \{0\}$. This metric is relevant for (conformal) Coulomb problem [6], MIC-Zwanziger system [7, 8], Euclidean Taub-NUT [9–11] and its extensions [12, 13], etc. The generalized Taub-NUT metric is

$$ds_4^2 = f(r)(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)) + g(r)(d\psi + \cos\theta\, d\phi)^2, \tag{6}$$

written in curvilinear coordinates and $r = \sum\limits_1^4 x_j^2 = \sqrt{\sum\limits_1^3 q_k^2}$.

We consider the Hamiltonian on the cotangent bundle $T^\star(\mathbb{R}^4 - \{0\})$

$$H = \frac{1}{2f(r)}p_r^2 + \frac{1}{2r^2 f(r)}p_\theta^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{2r^2 f(r)\sin^2\theta} + \frac{p_\psi^2}{2g(r)} + V(r). \tag{7}$$

The Hamiltonian function is invariant under the $U(1)$ action with the infinitesimal generator $\partial/\partial\psi$ so that the conserved momentum is $\mu = p_\psi$.

The reduced Hamiltonian (5) has the form

$$H_\mu = \frac{1}{2f(r)}\sum_{k=1}^{3} p_k^2 + \frac{\mu^2}{2g(r)} + V(r), \tag{8}$$

and the reduced symplectic form is

$$d\Theta_\mu = dp_r \wedge dr + dp_\theta \wedge d\theta + dp_\phi \wedge d\phi. \tag{9}$$

2. UNFOLDING

It is interesting to analyze the reverse of the reduction procedure which can be used to investigate difficult problems [2]. For example, the equations of motion for the dynamical system (4), (8) look quite complicated. Using a sort of *unfolding* of the 3-dimensional dynamics imbedding it in a higher dimensional space, the conserved quantities are related to the symmetries of this manifold.

2.1. Unfolding of the Gauge Symmetry. To exemplify let us start with the reduced Hamiltonian, and at each point of $T^\star(\mathbb{R}^3 - \{0\})$ we define the fiber S^1, the group space of the gauge group $U(1)$. On the fiber we consider the motion whose equation is

$$\frac{d\psi}{dt} = \frac{\mu}{g(r)} - \frac{\cos\theta}{r^2 f(r)\sin^2\theta}(p_\phi - \mu\cos\theta). \tag{10}$$

The metric on $\mathbb{R}^4$ defines horizontal spaces orthogonal to the orbits of the circle — this is a connection on the principal bundle [14]. Using the above trivialization, we have the coordinates (r, θ, ϕ, ψ) with the horizontal spaces annihilated by the connection $d\psi + \cos\theta\, d\phi$.

The metric on $\mathbb{R}^4$, which admits a circle action leaving invariant the symplectic form (9), can be written in the form

$$ds_4^2 = \sum_{i,j=1}^{4} g_{ij}\, dq^i\, dq^j = f(r)(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)) + h(r)(d\psi + \cos\,\theta\, d\phi)^2. \tag{11}$$

Considering the geodesic flow of ds_4^2 and taking into account that ψ is a cycle variable, $p_\psi = h(r)(\dot{\psi} + \cos\,\theta\dot{\phi})$ is a conserved quantity. To make contact with the Hamiltonian dynamics on $T^\star(\mathbb{R}^3 - \{0\})$ we must identify [15]

$$h(r) = g(r). \tag{12}$$

2.2. Eisenhart Lift. In many concrete problems, after the unfolding of the gauge symmetry, one ends up with a dynamical system on an extended phase space and an Hamiltonian with a «residual» scalar potential.

In the final stage of the unfolding of the dynamical system we shall apply the Eisenhart lift, or oxidation [4]. In order to simplify the problem, we shall assume that the constraints of the dynamical system and the potential V do not involve time. In this simplified case, it is adequately to consider a Riemannian space with $n+1$ (in our particular case $4+1$) dimensions with the metric

$$ds_5^2 = \sum_{i,j=1}^{4} g_{ij}\, dq^i\, dq^j + A du^2, \tag{13}$$

where it is assumed that A does not involve u. The lifted system is equivalent to geodesic motion on the enlarged spacetime (13), the coordinate u being related to the action by

$$u = -2\int T\, dt + 2(E+b)t, \tag{14}$$

where $E = T + V$ is the energy of the system, b is a constant and

$$\frac{1}{2A} = V + b. \tag{15}$$

Acknowledgements. This work is supported in part by a joint Romanian-LIT, JINR, Dubna Research Project, theme No. 05-6-1060-2005/2013.

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