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ADVECTION OF PASSIVE MAGNETIC FIELD BY THE GAUSSIAN VELOCITY FIELD WITH FINITE CORRELATIONS IN TIME AND SPATIAL PARITY VIOLATION

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By using the field-theoretic renormalization group technique, the model of weak magnetic field passively advected by an incompressible isotropic helical turbulent flow is investigated up to the second order of the perturbation theory (two-loop approximation) in the framework of an extended Kazantsev–Kraichnan model of kinematic magnetohydrodynamics. Statistical fluctuations of the velocity field are taken in the form of a Gaussian distribution with zero mean and defined noise with finite correlations in time. The two-loop analysis of all possible scaling regimes is done, and the influence of helicity on the stability of scaling regimes is discussed and shown in the plane of exponents $\varepsilon - \eta$, where ε characterizes the energy spectrum of the velocity field in the inertial range $E \propto k^{1-2\varepsilon}$ and η is related to the correlation time at the wave number k which is scaled as $k^{-2+\eta}$. It is shown that in nonhelical case the scaling regimes of the present vector model are completely identical and have also the same properties as those obtained in the corresponding model of passively advected scalar field. Besides, it is also shown that when the turbulent environment under consideration is helical, then the properties of the scaling regimes in models of passively advected scalar and vector (magnetic) fields are essentially different. The results demonstrate the importance of the presence of a symmetry breaking in a given turbulent environment for investigation of the influence of an internal tensor structure of the advected field on the inertial range scaling properties of the model under consideration and will be used in the analysis of the influence of helicity on the anomalous scaling of correlation functions of passively advected magnetic field.

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INTRODUCTION

The main conclusion of the phenomenological Kolmogorov–Obukhov (KO) theory [1, 2] is the statement that the statistical properties of random fields deep inside in the inertial interval $l \ll r \ll L$ of fully developed turbulent system are independent of the integral scale L (a typical scale on which the energy is pumped into the system) as well as the viscous scale l (a typical scale on which

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the energy starts to dissipate). This behavior was formulated in the form of the well-known Kolmogorov hypotheses. The assumption of validity of these hypotheses, together with simple dimensional analysis, then leads to the scaling behavior of correlation functions with definite exponents.

On the other hand, it is also well known that both experimental and theoretical studies show the existence of deviations from the predictions of the KO theory. Actually, the dependence of the correlation functions on the integral scale L is detected in contradiction with the first Kolmogorov hypothesis. Such deviations, referred to as anomalous or nondimensional scaling, manifest themselves in a singular dependence of the correlation functions on the distances and the integral scale L and are usually explained by the existence of strong developed fluctuations of the dissipative rate (intermittency) [1–4].

During the last two decades this problem was intensively studied within the scope of models of passive scalar and vector fields (concentration of an admixture, temperature, or weak magnetic field are examples) advected by a «synthetic» velocity field with prescribed Gaussian statistics. The reason is twofold. In the first place, the deviation from the classical theory is even more strongly noticeable for a passively advected field than for the velocity field itself, see, e.g., [3–5], and secondly, the problem of passive advection is considerably easier for theoretical investigation. Moreover, it reproduces many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. Thus, the theoretical study of the models of a passive scalar or vector advection can be treated as the first step on the long way of the investigation of intermittency and anomalous scaling in fully developed turbulence. In this respect, during a long period the crucial role in the theoretical investigations of anomalous scaling was played by the simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space, the so-called Kraichnan rapid-change model [6]. Namely, in the framework of the rapid-change model, for the first time, the anomalous scaling was established on the basis of a microscopic model and corresponding anomalous exponents were calculated within controlled approximation in the framework of the so-called zero-mode approach (see, e.g., [4] and references cited therein).

A considerable progress in the understanding of the anomalous scaling in turbulence was also done by the renormalization group (RG) technique which represents an effective method for investigation of self-similar scaling behavior [7–9]. In [10, 11], the field-theoretic RG and the operator-product expansion (OPE) were used in the systematic investigation of the anomalous scaling in Kraichnan's rapid-change model. It was shown that in the framework of the field-theoretic RG approach the anomalous scaling is related to the existence in the model of *dangerous* composite operators with negative critical dimensions in the OPE (see, e.g., [9, 12] for details). Thereafter, the field-theoretic RG technique was widely used for investigation of the anomalous behavior of various descendants of the Kraichnan model, e.g., models with inclusion of small-scale anisotropy, compressibility, models with the finite correlation time of the velocity field, and spatial parity violation (helicity) (see, e.g., [5, 13–15] and references cited therein). Besides, advection of the passive vector field by the Gaussian self-similar velocity field (with and without large and small-scale anisotropy, pressure, compressibility, and finite correlation time) has also been investigated, all possible asymptotic scaling regimes and cross-over among them have been classified, and anomalous scaling was analyzed [16–18]. A general conclusion of all these investigations is that the anomalous scaling remains valid for all generalized models.

The general solution of the problem of anomalous scaling in the framework of the field-theoretic approach [9, 12] is divided into two main stages. In the first stage, the multiplicative renormalizability of the corresponding field-theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behavior of the latter on their ultraviolet argument (r/l) for $r \gg l$ and any fixed (r/L) is given by infrared stable fixed points of those equations. It involves some «scaling functions» of the infrared argument (r/L), whose form is not determined by the RG equations. In the second stage, the behavior of scaling functions at $r \ll L$ is found from the OPE within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite operators, which give rise to an infinite family of independent aforementioned scaling exponents (and hence to multiscaling).

However, unlike the investigations of the anomalous scaling of passive scalar admixture in the framework of the Kraichnan model, generalized Kraichnan model [5], as well as in the model with advection by the Navier–Stokes velocity field [19], which were done up to the second-order (two-loop) approximation (in the case of the Kraichnan model, three-loop analysis of the anomalous exponents has also been done [11]), the complete field-theoretic RG analysis of the passively advected vector field, even within the simplest model, the so-called Kazantsev–Kraichnan kinematic magnetohydrodynamic (MHD) turbulence, is known only to the first order of approximation. Only quite recently [20,21], brief RG discussions of this problem have been done in two-loop approximation. At the same time, the calculation and deeper analysis of the two-loop corrections to the anomalous exponents in the framework of the Kazantsev–Kraichnan model, as well as in the framework of its various generalizations, are important from theoretical as well as experimental point of view. First of all, it is well known that at one-loop level of approximation the scaling regimes and the critical exponents of the most important composite operators that determine the anomalous scaling of the single-time correlation or structure functions of advected scalar or vector fields are the same in the corresponding models of passive advection (see, e.g., [10,16,18,22]). It means that at one-loop approximation it is impossible to identify and study possible influence of the internal tensor structure of the advected field on its scaling properties.

On the other hand, it is also well known that in the framework of the simplest models of the passive scalar or vector advection, namely, in the framework of the Kraichnan model and the corresponding Kazantsev–Kraichnan model of kinematic MHD, by using the field-theoretic methods it is impossible to study the behavior of the model under the influence of more realistic properties of the turbulent environment. For example, it is well known that in the framework of the Kraichnan model it is impossible to study the influence of the presence of spatial parity violation (helicity) on the properties of the diffusion processes of passively advected scalar field. At the same time, as was shown in [15], the importance of the presence of helicity for diffusion processes of a scalar quantity in turbulent environment can be rather significant. Besides, due to the structure of the Feynman diagrams, the effects of helicity on the diffusion processes can be studied only starting from the two-loop approximation.

In the present paper, we shall investigate the generalized Kazantsev–Kraichnan model of passively advected magnetic field by the Gaussian velocity field with finite time correlations and with the presence of helicity. The aim is twofold. First of all, we shall analyze the structure of all possible scaling regimes of the model in two-loop approximation, and the results will be compared to those obtained in the framework of the corresponding model of passively advected scalar field [15]. It will be shown that, in the case where the turbulent environment is incompressible, isotropic, and nonhelical, the structure of the scaling regimes as well as the corresponding coordinates of the IR stable fixed points for the model of scalar advection and vector advection are completely the same; i.e., it is shown that the internal tensor structure of the advected field in the framework of the present model is not important for the properties of diffusion processes in fully symmetric turbulent environments. The second aim is to investigate the influence of the presence of the spatial parity violation on the scaling regimes of the model and to compare the results to those obtained in the corresponding scalar problem [15]. As we shall see, the presence of helicity in the studied turbulent systems leads to the different diffusion behavior of the scalar field in comparison with the behavior of the vector field.

Here, we consider only the first stage of the solution of the problem of anomalous scaling in the framework of the field-theoretic approach; i.e., we shall only establish the possible scaling regimes of the model. The next step will be to use the obtained results for the investigation of the properties of the scaling functions of the correlation functions of the advected magnetic field in the framework of the OPE to determine the critical dimensions of the most important composite operators that lead to the anomalous scaling. However, the problem of anomalous scaling will be studied elsewhere.

The paper is organized as follows. In Sec. 1, the generalized Kazantsev–Kraichnan model of the passively advected vector (magnetic) field with presence of helicity is introduced and its field-theoretic formulation is given in Sec. 2. In Sec. 3, the RG analysis of the model is done, and the possible scaling regimes and their IR stability under the influence of helicity are given in Sec. 4. In the Conclusion, the discussion of results is presented.

1. DESCRIPTION OF THE MODEL

In what follows, we shall consider the advection of a solenoidal passive magnetic field $\mathbf{b} \equiv \mathbf{b}(x)$ $(x \equiv (t, \mathbf{x}))$ by an incompressible velocity field $\mathbf{v} \equiv \mathbf{v}(x)$, which is described by the following advection-diffusion equation:

$$\partial_t \mathbf{b} = \nu_0 \triangle \mathbf{b} - (\mathbf{v} \cdot \partial)\mathbf{b} + (\mathbf{b} \cdot \partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \tag{1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator; $\nu_0 = c^2/(4\pi\sigma)$ represents the magnetic diffusivity (in what follows, a subscript 0 will denote bare parameters of the unrenormalized theory); c is the speed of light, and σ is the conductivity. Thus, both $\mathbf{v}$ and $\mathbf{b}$ are divergence-free vector fields: $\partial \cdot \mathbf{v} = \partial \cdot \mathbf{b} = 0$.

The magnetic energy pumping given by a transverse Gaussian random noise $\mathbf{f}^{\mathbf{b}} = \mathbf{f}^{\mathbf{b}}(x)$ with zero mean and the correlation function

$$D_{ij}^b(x; 0) \equiv \langle f_i^b(x) f_j^b(0) \rangle = \delta(t)\, C_{ij}(|\mathbf{x}|/L) \tag{2}$$

represents the source of the fluctuations of the magnetic field $\mathbf{b}$ and maintains the steady state of the system. Here, L is an integral scale related to the corresponding stirring, and C_{ij} is a function finite in the limit $L \to \infty$. In what follows, the detailed form of the function C_{ij} is unimportant; the only condition which must be satisfied is that C_{ij} decreases rapidly for $|\mathbf{x}| \gg L$. If C_{ij} depends on the direction of the vector $\mathbf{x}$ and not only on its modulus $r = |\mathbf{x}|$, then it can be considered as a source of the large-scale anisotropy (see, e.g., [16]).

In real problems it is usually supposed that the velocity field $\mathbf{v}(x)$ satisfies the stochastic Navier–Stokes equation. In spite of this fact, in what follows, we shall suppose that the statistics of the velocity field is given in the form of Gaussian distribution with zero mean and correlation function [18, 22, 23]

$$\langle v_i(x) v_j(x') \rangle \equiv D_{ij}^v(x; x') = \int \frac{d\omega d^d\mathbf{k}}{(2\pi)^{d+1}} R_{ij}(\mathbf{k}) \tilde{D}^v(\omega, k) \times \\ \times \exp\left[-i\omega(t - t') + i\mathbf{k}(\mathbf{x} - \mathbf{x}')\right], \tag{3}$$

with

$$\tilde{D}^v(\omega, k) = \frac{g_0 \nu_0^3\, k^{4-d-2\varepsilon-\eta}}{(i\omega + u_0 \nu_0\, k^{2-\eta})(-i\omega + u_0 \nu_0 k^{2-\eta})}, \tag{4}$$

where $k = |\mathbf{k}|$ is the wave number; ω is frequency; d is the dimensionality of the $\mathbf{x}$ space (of course, when one investigates system with helicity the dimension of the $\mathbf{x}$ space must be strictly equal to three; nevertheless, in what follows, we shall remain the d-dimensionality of all results which are not related to helicity to be also able to study d-dependence of nonhelical case of the model). The geometric properties of the velocity correlator are given by the form of the transverse (due to incompressibility of the fluid) projector $R_{ij}(\mathbf{k})$. In the simplest isotropic nonhelical case, it has the form of the standard transverse projector $R_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) \equiv \delta_{ij} - k_i k_j / k^2$. On the other hand, the transition to a helical fluid corresponds to the giving up of conservation of spatial parity. Technically, this is expressed by the fact that the correlation function is specified in the form of mixture of a true tensor and a pseudotensor. In our approach, it is represented by two parts of transverse projector

$$R_{ij} = P_{ij}(\mathbf{k}) + H_{ij}(\mathbf{k}), \tag{5}$$

which consists of nonhelical standard transverse projector $P_{ij}(\mathbf{k})$ as it is given above and $H_{ij}(\mathbf{k}) = i\rho\varepsilon_{ijl} k_l / k$ which represents the presence of helicity in the flow. Here, ε_{ijl} is Levi-Civita's completely antisymmetric tensor of rank 3 (it is equal to 1 or -1 according to whether (i, j, l) is an even or odd permutation of $(1, 2, 3)$ and zero otherwise), and the real parameter of helicity, ρ, characterizes the amount of helicity. Due to the requirement of positive definiteness of the correlation function, the absolute value of ρ must be in the interval $|\rho| \in [0, 1]$. Physically, nonzero helical part (proportional to ρ) expresses existence of nonzero correlations $\langle \mathbf{v} \cdot \mathrm{rot}\, \mathbf{v} \rangle$.

The correlator (4) is directly related to the energy spectrum via the frequency integral [22, 24, 25]

$$E(k) \simeq k^{d-1} \int d\omega \tilde{D}^v(\omega, k) \simeq \frac{g_0 \nu_0^2}{u_0} k^{1-2\varepsilon}. \tag{6}$$

Therefore, the coupling constant g_0 and the exponent ε describe the equal-time velocity correlator or, equivalently, energy spectrum. On the other hand, the constant u_0 and the second exponent η are related to the frequency $\omega \simeq u_0 \nu_0 k^{2-\eta}$ which characterizes the mode k [22, 24–27]. Thus, in our notation, the value $\varepsilon = 4/3$ corresponds to the well-known Kolmogorov «five-thirds law» for the spatial statistics of velocity field, and $\eta = 4/3$ corresponds to the Kolmogorov frequency. Simple dimensional analysis shows that the parameters (charges) g_0 and u_0 are related to the characteristic ultraviolet (UV) momentum scale Λ (of the order of inverse Kolmogorov length) as follows:

$$g_0 \simeq \Lambda^{2\varepsilon+\eta}, \quad u_0 \simeq \Lambda^{\eta}. \tag{7}$$

In [23], it was shown that the Gaussian model (3), (4) is not Galilean invariant and, as a consequence, it does not take into account the self-advection of turbulent eddies. As a result of these so-called «sweeping effects», the different time correlations of the Eulerian velocity are not self-similar and depend strongly on the integral scale; see, e.g., [28]. But, on the other hand, the results presented in [23] show that the Gaussian model gives reasonable description of the passive advection in the appropriate frame, where the mean velocity field vanishes. One more argument to justify the model based on statistics of the velocity field given in Eqs. (3) and (4) is that, in what follows, we shall be interested in the equal-time, Galilean invariant quantities (structure or correlation functions), which are not affected by the sweeping, and, therefore, as we expect (see, e.g., [22, 29, 30]), their absence in the Gaussian model (3), (4) is not essential.

Model (3), (4) contains two special cases that are interesting themselves. One of them is the so-called rapid-change model limit (in our context, one comes to the so-called Kazantsev–Kraichnan model of kinematic MHD), where $u_0 \to \infty$ and $g_0' \equiv g_0/u_0^2 = \text{const}$,

$$\tilde{D}^v(\omega, k) \to g_0' \nu_0 \, k^{-d-2\varepsilon+\eta}, \tag{8}$$

and the other one is the so-called quenched (time-independent or frozen) velocity field limit which is defined by $u_0 \to 0$ and $g_0'' \equiv g_0/u_0 = \text{const}$,

$$\tilde{D}^v(\omega, k) \to g_0'' \nu_0^2 \pi \delta(\omega) k^{-d+2-2\varepsilon}, \tag{9}$$

which is similar to the well-known models of the random walks in random environment with long-range correlations; see, e.g., [31, 32].

2. FIELD-THEORETIC FORMULATION OF THE MODEL

According to the well-known theorem (see, e.g., [9] and references cited therein), the stochastic problem (1)–(4) is equivalent to the field-theoretic model of the set of three fields $\Phi \equiv \{\mathbf{b}, \mathbf{b}', \mathbf{v}\}$ with action functional

$$\begin{aligned} S(\Phi) = -\frac{1}{2} \int dt_1 \, d^d\mathbf{x_1} \, dt_2 \, d^d\mathbf{x_2} \, v_i(t_1, \mathbf{x_1}) [D^v_{ij}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2})]^{-1} \times \\ \times v_j(t_2, \mathbf{x_2}) + \frac{1}{2} \int dt_1 \, d^d\mathbf{x_1} \, dt_2 \, d^d\mathbf{x_2} \, b'_i(t_1, \mathbf{x_1}) D^b_{ij}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2}) b'_j(t_2, \mathbf{x_2}) + \\ + \int dt \, d^d\mathbf{x} \; \mathbf{b}'[-\partial_t \mathbf{b} + \nu_0 \triangle \mathbf{b} - (\mathbf{v} \cdot \partial)\mathbf{b} + (\mathbf{b} \cdot \partial)\mathbf{v}], \end{aligned} \tag{10}$$

where $\mathbf{b}'$ is an auxiliary vector field which has the same properties as the field $\mathbf{b}$, and D^b_{ij} and D^v_{ij} are correlators (2) and (4), respectively. In the action (10) all the required integrations over $x = (t, \mathbf{x})$ and summations over the vector indices are understood. The second and the third integrals in Eq. (10) represent the Dominicis–Jansen-type action for the stochastic problem (1), (2) at fixed $\mathbf{v}$, and the first integral represents the Gaussian averaging over $\mathbf{v}$.

Model (10) corresponds to a standard Feynman diagrammatic perturbation theory with bare propagators (in frequency-momentum representation)

$$\langle b_i(\omega, \mathbf{k}) b'_j(-\omega, -\mathbf{k})\rangle_0 = \frac{P_{ij}(\mathbf{k})}{-i\omega + \nu_0 k^2}, \tag{11}$$

$$\langle b'_i(\omega, \mathbf{k}) b_j(-\omega, -\mathbf{k})\rangle_0 = \langle b_i(\omega, \mathbf{k}) b'_j(-\omega, -\mathbf{k})\rangle_0^*, \tag{12}$$

$$\langle b_i(\omega, \mathbf{k}) b_j(-\omega, -\mathbf{k})\rangle_0 = \frac{C_{ij}(\mathbf{k})}{|-i\omega + \nu_0 k^2|^2}, \tag{13}$$

$$\langle b'_i(\omega, \mathbf{k}) b'_j(-\omega, -\mathbf{k})\rangle_0 = 0, \tag{14}$$

and the bare propagator $\langle v_i v_j\rangle_0$ for the velocity field is given directly in Eqs. (3) and (4). $C_{ij}(\mathbf{k})$ in Eq. (13) is the Fourier transform of the function $C_{ij}(|\mathbf{x}|/L)$ from Eq. (2). The graphical representation of nonzero propagators is presented in Fig. 1 (the end with a slash in the propagator $\langle b_i b'_j\rangle_0$ corresponds to the field $\mathbf{b}'$ and the end without a slash corresponds to the field $\mathbf{b}$). The triple (interaction) vertex

$$b'_i[-v_j\partial_j b_i + b_j \partial_j v_i] = b'_i V_{ijl} b_j v_l, \tag{15}$$

with the vertex factor (in frequency-momentum representation)

$$V_{ijl} = i(k_l \delta_{ij} - k_j \delta_{il}), \tag{16}$$

is shown in Fig. 2, where the momentum $\mathbf{k}$ is flowing into the vertex via the auxiliary field $\mathbf{b}'$.

Let us remind that the formulation of the stochastic problem (1)–(4) through the action functional (10) replaces the statistical averages of random quantities with equivalent functional averages with weight $\exp S(\Phi)$.

Fig. 1. Graphical representation of the propagators of the model

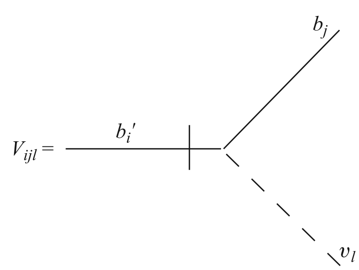

Fig. 2. The interaction vertex of the model

3. RENORMALIZATION GROUP ANALYSIS

The information about possible UV divergences in a field-theoretic model can be found by the standard analysis of canonical dimensions [7, 8]. The field-theoretic model defined by the action functional (10) belongs among the so-called two-scale models [9, 12] for which the total canonical dimension d_Q of some quantity Q (which plays the same role in the renormalization theory of our dynamical model as the simple momentum dimension does in static models) is defined by two numbers, namely, the momentum dimension d_Q^k and the frequency dimension d_Q^ω with the standard normalization conditions $d_k^k = -d_x^k = 1, d_\omega^\omega = -d_t^\omega = 1, d_k^\omega = d_x^\omega = d_\omega^k = d_t^k = 0$. In the present model the total canonical dimension is given as $d_Q = d_Q^k + 2d_Q^\omega$.

The canonical dimensions of the model under consideration are presented in table, where the canonical dimensions of the renormalized parameters are also shown (see below). The model is logarithmic for $\varepsilon = \eta = 0$ (the coupling constants g_0 and u_0 are dimensionless); therefore, the UV divergences in the correlation functions have the form of the poles in ε, η, and their linear combinations. It is also important to stress that, like in the model of real MHD turbulence described by the stochastic MHD equations with the presence of helicity (see, e.g., [33] and references cited therein), in the present vector model with helicity the linear divergences can appear. Their correct treatment in genuine MHD turbulence leads to the appearance of the homogeneous large-scale magnetic field (turbulent dynamo effect) generated by a kind of spontaneous symmetry-breaking mechanism. However, this mechanism needs the presence of the Lorentz force term in the stochastic Navier–Stokes equation in the model. Because such kind of term is not present in the model under consideration, where the velocity field has a Gaussian statistics, we shall leave the problem of the linear divergences untouched in the present paper and we shall concentrate only on the problem of the existence and stability of the IR scaling regimes which can be studied without considering the linear divergences. At the same time, we are aware of the fact that the full problem with the presence of helicity can be solved only in the framework of the genuine MHD turbulence described by the stochastic MHD equations. The corresponding analysis will be given elsewhere. Thus, in what follows, we shall consider and work only with the logarithmic divergences.

Canonical dimensions of the fields and parameters of the model under consideration

Q	$\mathbf{v}$	$\mathbf{b}$	$\mathbf{b}'$	m, Λ, μ	ν_0, ν	g_0	u_0	g, u
d_Q^k	-1	0	d	1	-2	$2\varepsilon + \eta$	η	0
d_Q^ω	1	0	0	0	1	0	0	0
d_Q	1	0	d	1	0	$2\varepsilon + \eta$	η	0

Detailed analysis of the possible divergences of the present model was done in [18]; therefore, it is not necessary to repeat it here. The final conclusion of the analysis is that the only superficially divergent function of the model is the one-irreducible Green's function $\langle b'_i b_j \rangle_{1-\mathrm{ir}}$ and can be removed multiplicatively by the only counterterm $b'_i \triangle b_j$. It can be explicitly expressed in the multiplicative renormalization of the bare parameters g_0, u_0, and ν_0 in the following form:

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon+\eta} Z_g, \quad u_0 = u\mu^\eta Z_u. \tag{17}$$

Here the dimensionless parameters g, u, and ν are the renormalized counterparts of the corresponding bare ones; μ is the renormalization mass (a scale-setting parameter) in the minimal subtraction (MS) scheme; and $Z_i = Z_i(g, u)$ are renormalization constants.

On the other hand, the renormalized action functional has the following form:

$$\begin{aligned} S^R(\Phi) = -\frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2}\, v_i(t_1, \mathbf{x_1}) \times \\ \times [D^v_{ij}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2})]^{-1}\, v_j(t_2, \mathbf{x_2}) + \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \times \\ \times b'_i(t_1, \mathbf{x_1}) D^b_{ij}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2}) b'_j(t_2, \mathbf{x_2}) + \\ + \int dt\, d^d\mathbf{x}\, \mathbf{b}'[-\partial_t \mathbf{b} + \nu Z_1 \triangle \mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v}], \end{aligned} \tag{18}$$

with the only renormalization constant Z_1 related to the renormalization constants defined in Eq. (17) as follows (the terms with correlators D^v_{ij} and D^b_{ij}, as well as the fields, are not renormalized):

$$Z_\nu = Z_1, \quad Z_g = Z_\nu^{-3}, \quad Z_u = Z_\nu^{-1}. \tag{19}$$

The second and third relations are consequence of the absence of the renormalization of the term with D^v in renormalized action (18), i.e.,

$$g_0 \nu_0^3 = g\nu^3 \mu^{2\epsilon+\eta}, \quad u_0\nu_0 = u\nu\mu^\eta. \tag{20}$$

In our case, the only independent renormalization constant Z_1, in general, contains poles of linear combinations of ε and η, i.e., $Z_1 = Z_1(g, u, d, \rho; \varepsilon, \eta)$. However, as detailed analysis shows, to obtain all important quantities as the γ functions, β functions, coordinates of fixed points, and the critical dimensions, the knowledge of the renormalization constants for the special choice $\eta = 0$ is sufficient up to two-loop approximation (see, e.g., [22] for details). It is important here that the parameter ε alone provides the UV regularization for the theory, hence the renormalization constant Z_1 remains finite at $\eta = 0$.

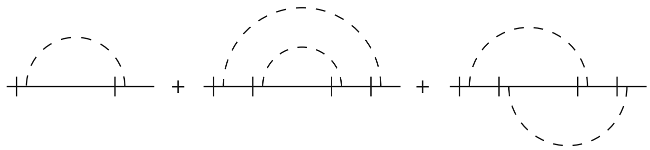

Fig. 3. The one- and two-loop contributions to the self-energy operator $\Sigma^{b'b}$

The renormalization constant Z_1 can be determined by the requirement that the one-irreducible Green's function $\langle b_i' b_j \rangle_{1-\text{ir}}$ must be UV finite when is written in the renormalized variables; i.e., it has no singularities in the limit $\varepsilon \to 0$ (as was briefly discussed above, one can put $\eta = 0$ in calculations). By using this condition, the renormalization constant Z_1 is determined up to a UV finite contribution which is fixed by the choice of the renormalization scheme. Up to the second order of the perturbation theory in the standard MS scheme, used in the present paper, the Z_1 has the following form:

$$Z_1(g,u,d,\rho;\varepsilon) = 1 + \sum_{n=1}^{2} g^n \sum_{j=1}^{n} \frac{z_{nj}(d)}{\varepsilon^j}. \tag{21}$$

Thus, to determine the renormalization constant Z_1 to the second order of the perturbation theory, i.e., to the order in which we are working here, it is necessary to find the coefficients z_{11}, z_{21} and z_{22} of the series (21). On the other hand, one-irreducible Green's function $\langle b_i' b_j \rangle_{1-\text{ir}}$ is related (through the Dyson equation [8,9]) to the self-energy operator $\Sigma^{b'b}$, which is expressed via the corresponding set of Feynman diagrams shown in Fig. 3.

By explicit calculations it can be shown that the singular parts of diagrams in Fig. 3 have the following analytical form:

$$A = -\frac{S_d}{(2\pi)^d} \frac{g\nu p^2 \delta_{ij}}{4u(1+u)} \frac{d-1}{d} \left(\frac{\mu}{m}\right)^{2\varepsilon} \frac{1}{\varepsilon}, \tag{22}$$

$$\begin{aligned} B_1 = {} & \frac{S_d^2}{(2\pi)^{2d}} \frac{g^2 \nu p^2 \delta_{ij}}{16u^2(1+u)^3} \frac{(d-1)^2}{d^2} \left(\frac{\mu}{m}\right)^{4\varepsilon} \times \\ & \times \frac{1}{\varepsilon} \left[\frac{1}{2\varepsilon} + \frac{{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right)}{(d+2)(1+u)^2} - \right. \\ & \left. - \rho^2 \frac{2(d-2)\pi\, {}_2F_1\left(-\frac{1}{2},\frac{3}{2};1+\frac{d}{2};\frac{1}{(1+u)^2}\right)}{(d-1)^2} \right], \end{aligned} \tag{23}$$

$$B_2 = \frac{S_d^2}{(2\pi)^{2d}} \frac{g^2 \nu p^2 \delta_{ij}}{16u^2(1+u)^3} \frac{(d-1)}{d^2} \left(\frac{\mu}{m}\right)^{4\varepsilon} \frac{1}{\varepsilon} \times \\ \times \left[\frac{{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right)}{(d+2)(1+u)} - \right. \\ \left. - \rho^2 \frac{(d-2)(d-5)\pi\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};1+\frac{d}{2};\frac{1}{(1+u)^2}\right)}{2(d-1)} \right], \quad (24)$$

where A corresponds to one-loop contribution (the first diagram in Fig. 3), B_1 is related to the second diagram in Fig. 3, and B_2 is the result for the third diagram. Here, $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the d-dimensional sphere and ${}_2F_1(a,b,c,z) = 1 + \frac{a\,b}{c \cdot 1} z + \frac{a(a+1)b(b+1)}{c(c+1)\cdot 1 \cdot 2} z^2 + \ldots$ represents the corresponding hypergeometric function. In further investigations the helical terms with ρ^2 in B_1 and B_2 has to be taken with $d = 3$, but for completeness we have remained the d-dependence of these parts in B_1 and B_2 in Eqs. (23) and (24).

Finally, the renormalization constant $Z_1 = Z_\nu$ is given as follows:

$$Z_\nu = 1 - \frac{\bar{g}}{\varepsilon} \frac{d-1}{d} \frac{1}{4u(1+u)} + \frac{\bar{g}^2}{\varepsilon^2} \frac{(d-1)^2}{d^2} \frac{1}{32u^2(1+u)^3} + \\ + \frac{\bar{g}^2}{\varepsilon} \frac{(d-1)(d+u)}{d^2(d+2)} \frac{1}{16u^2(1+u)^5}\, {}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right) - \\ - \frac{\bar{g}^2}{\varepsilon} \frac{\rho^2 \pi \delta_{3d}}{72u^2(1+u)^3} \left[{}_2F_1\left(-\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right) - \right. \\ \left. - \frac{1}{2}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right) \right], \quad (25)$$

where $\bar{g} = gS_d/(2\pi)^d$. In the helical part (the last two lines), we have already put $d = 3$ and we have also introduced the Konecker symbol δ_{3d} to show explicitly that this term has sense only for $d = 3$.

The basic RG differential equation, for example, for the renormalized connected correlation functions $W^R = \langle \Phi \cdots \Phi \rangle^R$ (the counterparts of the bare connected correlation functions $W = \langle \Phi \cdots \Phi \rangle$) are obtained from the relation $S(\Phi, e_0) = S^R(\Phi, e, \mu)$, where e_0 stands for the complete set of bare parameters and e stands for the renormalized one, together with the fact that fields $\mathbf{v}, \mathbf{b}$, and $\mathbf{b}'$ are not renormalized. It leads to the relation

$$W^R(e, \mu, \ldots) = W(e_0, \ldots), \quad (26)$$

where the dots stand for other arguments which are untouched by renormalization, e.g., coordinates and times. Further, using the fact that unrenormalized correlation functions are independent of the scale-setting parameter μ, one can apply the differential operator $\mu\partial_\mu$ at fixed unrenormalized parameters on both sides in Eq. (26) which leads to the basic RG equation

$$\mathcal{D}_{\mathrm{RG}} W^R(A, e, \mu) = 0, \tag{27}$$

where operator $\mathcal{D}_{\mathrm{RG}}$ has the following explicit form:

$$\mathcal{D}_{\mathrm{RG}} = \mu\partial_\mu + \beta_g(g,u)\partial_g + \beta_u(g,u)\partial_u - \gamma_\nu(g,u)\mathcal{D}_\nu, \tag{28}$$

where we denote $\mathcal{D}_\nu \equiv \nu\partial_\nu$ and the RG functions (the β and γ functions) are given by well-known definitions and in our case, using relations (19) for renormalization constants, they have the following form:

$$\gamma_\nu \equiv \mu\partial_\mu \ln Z_\nu, \tag{29}$$

$$\beta_g \equiv \mu\partial_\mu g = g(-2\varepsilon - \eta + 3\gamma_\nu), \tag{30}$$

$$\beta_u \equiv \mu\partial_\mu u = u(-\eta + \gamma_\nu). \tag{31}$$

Now using the definition of the anomalous dimension γ_ν in Eq. (29), together with the explicit expression for Z_ν as it is given in Eq. (25), one comes to the following result:

$$\gamma_\nu = -2(\bar{g}\mathcal{A} + 2\bar{g}^2\mathcal{B}), \tag{32}$$

where

$$\mathcal{A} = -\frac{d-1}{d}\frac{1}{4u(1+u)} \tag{33}$$

is the one-loop contribution to anomalous dimension γ_ν and the two-loop one is

$$\begin{aligned}\mathcal{B} = \frac{(d-1)(d+u)}{16d^2(d+2)u^2(1+u)^5}\, {}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right) - \\ - \frac{\pi\rho^2\delta_{3d}}{72u^2(1+u)^3}\Bigg[{}_2F_1\left(-\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right) - \\ - \frac{1}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right)\Bigg].\end{aligned} \tag{34}$$

Finally, the possible asymptotic scaling behavior of the correlation functions of the model (the possible scaling regimes of the model deep inside in the inertial interval) is given by the IR stable fixed points of the RG equations. On the other hand, the coordinates of possible fixed points g_* and u_* are determined by the requirement of vanishing of the β functions (30) and (31), namely,

$$\beta_g(g_*, u_*) \equiv g_*(-2\varepsilon - \eta + 3\gamma_\nu^*) = 0, \tag{35}$$

$$\beta_u(g_*, u_*) \equiv u_*(-\eta + \gamma_\nu^*) = 0, \tag{36}$$

where γ_ν^* denotes the function (32) taken at the fixed points g_*, u_*.

All possible fixed points and the corresponding scaling regimes will be classified and regions of their IR stability will be studied in the next section. But, first of all, let us briefly discuss the consequences of the very existence of an IR scaling regime on the behavior of important statistical characteristics of the system.

Existence of the stable IR fixed point means that the correlation functions of the model exhibit scaling behavior with given critical dimensions in the IR range. The issue of interest are especially multiplicatively renormalizable equal-time two-point quantities $G(r)$ (see below). The IR scaling behavior of a function $G(r)$ (for $r/l \gg 1$ and any fixed r/L), namely,

$$G(r) \simeq \nu_0^{d_G^\omega} l^{-d_G} (r/l)^{-\Delta_G} R(r/L) \tag{37}$$

is related to the existence of IR stable fixed point of the RG equations (27). In Eq. (37), d_G^ω and d_G are the corresponding canonical dimensions of the function G (the canonical dimensions of the model are given in table), $l = 1/\Lambda$, $L = 1/m$, $R(r/L)$ is a scaling function, which cannot be determined by the RG equations (see, e.g., [9]), and Δ_G is the critical dimension defined as

$$\Delta_G = d_G^k + \Delta_\omega d_G^\omega + \gamma_G^*. \tag{38}$$

Here, γ_G^* is the fixed point value of the anomalous dimension $\gamma_G \equiv \mu\partial_\mu \ln Z_G$, where Z_G is the renormalization constant of the multiplicatively renormalizable quantity G, i.e., $G = Z_G G^R$ [9], and $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of the frequency with γ_ν^* which is defined in (32) taken at the corresponding fixed point. However, from Eqs. (35) and (36) one can immediately find the exact values of the γ_ν^* for the corresponding scaling regimes. They are exact one-loop results; i.e., no higher-loop corrections to the γ_ν^* exist. It also means that the critical dimension of frequency Δ_ω for the corresponding scaling regime is also known exactly, as well as the critical dimensions of the fields. In the next section, we shall present them explicitly for all possible IR stable fixed points, i.e., for all possible scaling regimes of the model.

An example of the equal-time quantities built of the magnetic field $\mathbf{b}$ that are usually studied in the literature are the equal-time two-point correlation functions

$$B_{N-m,m}(r) \equiv \langle b_r^{N-m}(t, \mathbf{x}) b_r^m(t, \mathbf{x}') \rangle, \quad r = |\mathbf{x} - \mathbf{x}'|, \tag{39}$$

which are studied deep inside in the inertial range $l \ll r \ll L$, where b_r denotes the component of the magnetic field directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ [16, 18].

However, as was already mentioned in the Introduction, in the present paper we shall concentrate only on the analysis of the possible asymptotic scaling regimes of the present model and the analysis of the so-called anomalous scaling of the correlation functions of the model will be given elsewhere.

4. FIXED POINTS AND THE SCALING REGIMES

As was already mentioned in the previous section, possible scaling regimes of a renormalized model are directly given by the infrared (IR) stable fixed points of the corresponding system of the RG equations [8,9]. The fixed point of the RG equations is defined by β functions, namely, by requirement of their vanishing. In our model the coordinates g_*, u_* of the fixed points are found from the system of two equations

$$\beta_g(g_*, u_*) = \beta_u(g_*, u_*) = 0, \tag{40}$$

which are explicitly shown in Eqs. (35) and (36). To investigate the IR stability of a fixed point, it is enough to analyze the eigenvalues of the corresponding matrix of the first derivatives Ω:

$$\Omega_{ij} = \begin{pmatrix} \dfrac{\partial \beta_g}{\partial g} & \dfrac{\partial \beta_g}{\partial u} \\ \dfrac{\partial \beta_u}{\partial g} & \dfrac{\partial \beta_u}{\partial u} \end{pmatrix}. \tag{41}$$

Possible IR asymptotic behaviors are governed by the IR stable fixed points, i.e., those for which both eigenvalues are positive.

The possible scaling regimes of the model in the framework of the one-loop approximation were investigated in [18]. The aim of the present paper is to analyze the problem in two-loop approximation, as well as to analyze the influence of the spatial parity violation (helicity) on the scaling regimes and to compare the results to the corresponding results obtained in the problem of passive scalar advection studied in [15].

First of all, we shall study the rapid-change limit which describes the Kazantsev–Kraichnan model of kinematic MHD: $u \to \infty$. For this aim it is convenient to make the transformation to new variables, namely, $w \equiv 1/u$, and $g' \equiv g/u^2$ [18, 22], with the corresponding changes in the β functions:

$$\beta_{g'} = g'(\eta - 2\varepsilon + \gamma_\nu), \tag{42}$$

$$\beta_w = w(\eta - \gamma_\nu). \tag{43}$$

In this notation the anomalous dimension γ_ν acquires the following form:

$$\gamma_\nu = -2(\bar{g}'\mathcal{A}' + 2\bar{g}'^2\mathcal{B}'), \tag{44}$$

where again $\bar{g}' = g' S_d/(2\pi)^d$. The one-loop contribution $\mathcal{A}'$ acquires the form

$$\mathcal{A}' = -\frac{d-1}{d}\frac{1}{4(1+w)}, \tag{45}$$

and the two-loop correction $\mathcal{B}'$ is

$$\mathcal{B}' = \frac{(d-1)(dw+1)w^2}{16d^2(d+2)(1+w)^5}\,{}_2F_1\left(1,1;2+\frac{d}{2};\frac{w^2}{(1+w)^2}\right) - \\ - \frac{\pi\rho^2\delta_{3d}w}{72(1+w)^3}\left[{}_2F_1\left(-\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{w^2}{(1+w)^2}\right) - \right. \\ \left. - \frac{1}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{w^2}{(1+w)^2}\right)\right]. \tag{46}$$

Here, it is evident that in the rapid-change limit $w \to 0$ ($u \to \infty$) one comes to the known result that the two-loop contribution $\mathcal{B}'$ is equal to zero. It is related to the fact that in the rapid-change limit one obtains the well-known Kazantsev–Kraichnan kinematic MHD [16], where no higher-loop corrections to the self-energy operator exist and the anomalous dimension γ_ν is determined exactly at one-loop level of approximation and has the following form [16]:

$$\gamma_\nu = \lim_{w\to 0}\frac{(d-1)\bar{g}'}{2d(1+w)} = \frac{(d-1)\bar{g}'}{2d}. \tag{47}$$

In this limit two different fixed points exist. Let us denote them as FPI and FPII. The first fixed point is trivial, namely,

$$\mathrm{FPI}:\ w_* = g'_* = 0, \tag{48}$$

with $\gamma_\nu^* = 0$, and diagonal matrix Ω with eigenvalues (diagonal elements)

$$\lambda_1 = \eta, \quad \lambda_2 = \eta - 2\varepsilon. \tag{49}$$

The region of its stability is shown in Fig. 4. The second point is defined as

$$\mathrm{FPII}:\ w_* = 0, \quad \bar{g}'_* = \frac{2d}{d-1}(2\varepsilon - \eta), \tag{50}$$

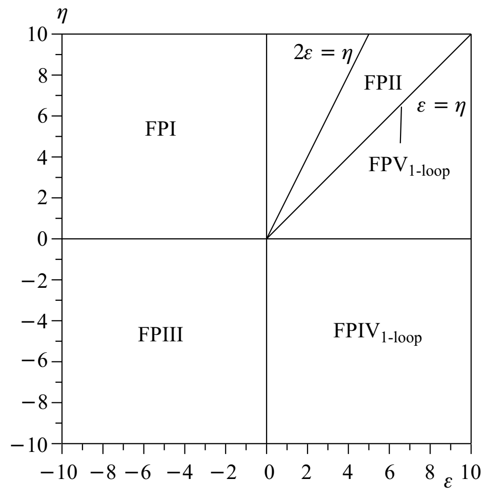

Fig. 4. Regions of the stability for the fixed points in one-loop approximation. The regions of stability for fixed points FPI, FPII, and FPIII are exact, i.e., are not influenced by loop corrections. The fixed point FPIV is shown in one-loop approximation. The d-dependence of the FPIV in two-loop approximation is shown in Fig. 5

with $\gamma_\nu^* = 2\varepsilon - \eta$. These are exact one-loop expressions as a result of non-existence of the higher-loop corrections. The corresponding matrix of the first derivatives is triangular with diagonal elements (eigenvalues):

$$\lambda_1 = 2(\eta - \varepsilon), \quad \lambda_2 = 2\varepsilon - \eta. \tag{51}$$

The region of stability of this fixed point is shown in Fig. 4.

The second limit of the present model corresponds to the so-called «frozen regime» with frozen velocity field. This regime is obtained in the limit $u \to 0$. To study this transition, it is appropriate to change the variable g to the new variable $g'' \equiv g/u$ [18, 22]. In this case, the β functions are transformed to the following ones:

$$\beta_{g''} = g''(-2\varepsilon + 2\gamma_\nu), \tag{52}$$

$$\beta_u = u(-\eta + \gamma_\nu). \tag{53}$$

In this notation the anomalous dimension γ_ν has the form

$$\gamma_\nu = -2(\bar{g}''\mathcal{A}'' + 2\bar{g}''^2\mathcal{B}''), \tag{54}$$

where $\bar{g}'' = g'' S_d/(2\pi)^d$. Here, the one-loop contribution $\mathcal{A}''$ is

$$\mathcal{A}'' = -\frac{d-1}{d}\frac{1}{4(1+u)}, \tag{55}$$

and the two-loop one, $\mathcal{B}''$, is given as follows:

$$\mathcal{B}'' = \frac{(d-1)(d+u)}{16d^2(d+2)(1+u)^5}\, {}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right) - \\ - \frac{\pi\rho^2\delta_{3d}}{72(1+u)^3}\left[{}_2F_1\left(-\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right) - \right. \\ \left. - \frac{1}{2}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right)\right]. \quad (56)$$

In the limit $u \to 0$ the functions $\mathcal{A}''$ and $\mathcal{B}''$ aquire the following form:

$$\mathcal{A}''_0 = -\frac{d-1}{4d}, \quad (57)$$

and

$$\mathcal{B}''_0 = \frac{(d-1)}{16d(d+2)}\, {}_2F_1\left(1,1;2+\frac{d}{2};1\right) - \\ - \frac{\pi\rho^2\delta_{3d}}{72}\left[{}_2F_1\left(-\frac{1}{2},\frac{3}{2};\frac{5}{2};1\right) - \frac{1}{2}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};1\right)\right]. \quad (58)$$

The system of β functions (52) and (53) exhibits two fixed points, denoted as FPIII and FPIV in [22], related to the corresponding two scaling regimes. One of them is again trivial, namely,

$$\text{FPIII}: \; u_* = g''_* = 0, \quad (59)$$

with $\gamma_\nu^* = 0$. The eigenvalues of the corresponding matrix Ω, which is diagonal in this case, are

$$\lambda_1 = -2\varepsilon, \quad \lambda_2 = -\eta. \quad (60)$$

Thus, this regime is IR stable only if both parameters ε and η are negative simultaneously, as can be seen in Fig.4. The second, nontrivial, point is

$$\text{FPIV}: \quad u_* = 0, \quad \bar{g}''_* = -\frac{\varepsilon}{2\mathcal{A}''_0} - \frac{\mathcal{B}''_0}{2\mathcal{A}''^3_0}\varepsilon^2, \quad (61)$$

where $\mathcal{A}''_0$ and $\mathcal{B}''_0$ are defined in Eqs. (57) and (58), respectively.

First of all, let us briefly discuss the influence of two-loop approximation on the IR stability of this scaling regime without helicity in general d-dimensional case. Let us denote the corresponding fixed point as FPIV_0. Its coordinates are

$$\text{FPIV}_0: \; u_* = 0, \quad \bar{g}''_* = \frac{2d}{d-1}\left(\varepsilon + \frac{1}{d-1}\varepsilon^2\right), \quad (62)$$

with anomalous dimension γ_ν defined as

$$\gamma_\nu^* = \frac{d-1}{2d}\left(\bar{g}_*'' - \frac{\bar{g}_*''^2}{2d}\right) = \varepsilon, \tag{63}$$

which is the exact one-loop result [22]. The eigenvalues of the matrix Ω (taken at the fixed point) are

$$\lambda_1 = 2\varepsilon\left(1 + \frac{1}{1-d}\varepsilon\right), \quad \lambda_2 = \varepsilon - \eta. \tag{64}$$

The conditions $\bar{g}_*'' > 0, \lambda_1 > 0$, and $\lambda_2 > 0$ for the IR stable fixed point lead to the following restrictions on the values of the parameters ε and η:

$$\varepsilon > 0, \quad \varepsilon > \eta, \quad \varepsilon < d-1. \tag{65}$$

The region of stability of the regime for different values of the spatial dimension d is shown in Fig. 5. The region of stability of this IR fixed point increases when the dimension of the coordinate space d increases. The result is completely the same as in the corresponding model of passively advected scalar field [15]; i.e., when given turbulent environment is completely symmetric, there is no difference between scaling regimes in the frozen limit in the models of passively advected scalar and vector (magnetic) field.

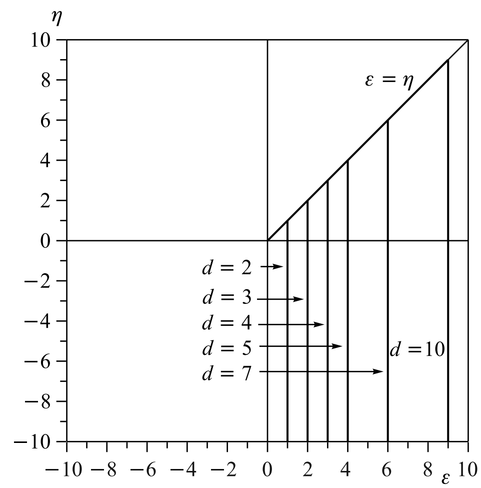

Fig. 5. Regions of the stability for the fixed point FPIV in two-loop approximation without helicity for different space dimensions d. The IR fixed point is stable in the region given by inequalities: $\varepsilon > 0$, $\varepsilon > \eta$, and $\varepsilon < 2(d-1)$

Now let us turn to the helical case. Here, the dimension of the space is $d = 3$. The fixed point FPIV is now given as

$$u_* = 0, \quad \bar{g}''_* = 3\varepsilon + \frac{3}{2}\varepsilon^2. \tag{66}$$

It is independent of the helicity unlike the corresponding fixed point obtained in the framework of the model of passive scalar advection [15], where the value of the two-loop fixed point depends explicitly on helicity parameter ρ. The nondependence of the fixed point on the helicity is given by the following identity:

$${}_2F_1\left(-\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; 1\right) - \frac{1}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; 1\right) = 0; \tag{67}$$

i.e., the helical part in Eq. (58) is simply equal to zero.

However, in principle, the presence of helicity in the system can have nontrivial impact on the region of stability of the fixed point. Nevertheless, the matrix of the first derivatives Ω, which is triangular, has the following diagonal elements (eigenvalues) taken at the fixed point:

$$\lambda_1 = 2\varepsilon - \varepsilon^2, \tag{68}$$

$$\lambda_2 = \varepsilon - \eta, \tag{69}$$

which are again independent of the helicity parameter ρ. It means that, unlike the advection of the passive scalar field by the frozen velocity field, where nontrivial dependence of the coordinate of the fixed point on the parameter of helicity exists, the frozen limit of the present model of the advection of the vector (weak magnetic) field does not feel the spatial parity violation of the system. Moreover, the stability of the scaling regime also does not depend on the presence of helicity in the system and the regime is IR stable for $\varepsilon > \eta$ and $\varepsilon < 2$.

In the end, let us turn to the most interesting scaling regime with finite value of the fixed point for the variable u. In this case, the system of equations (see also [18, 22])

$$\beta_g = g(-2\varepsilon - \eta + 3\gamma_\nu) = 0, \tag{70}$$

$$\beta_u = u(-\eta + \gamma_\nu) = 0 \tag{71}$$

can be fulfilled simultaneously for finite values of g, u only if $\varepsilon = \eta$. In this case, the function β_g is proportional to the function β_u. As a result, we have not one fixed point of this type but a curve of fixed points in the $g - u$ plane. The value of the fixed point for variable g in two-loop approximation is given as follows (we denote this fixed point as FPV):

$$\mathrm{FPV}: \; \bar{g}_* = -\frac{1}{2\mathcal{A}_*}\,\varepsilon - \frac{1}{2}\frac{\mathcal{B}_*}{\mathcal{A}_*^3}\,\varepsilon^2, \tag{72}$$

with exact one-loop result for $\gamma_\nu^* = \varepsilon = \eta$. Here $\mathcal{A}_*$ and $\mathcal{B}_*$ are expressions $\mathcal{A}$ and $\mathcal{B}$ in Eqs. (33) and (34) taken in the fixed point value u_* of the variable u. The possible values of the fixed point for the variable u can be restricted as we shall discuss below. The matrix Ω has the following eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = 3\bar{g}^* \left(\frac{\partial \gamma_\nu}{\partial \bar{g}}\right)_* + u^* \left(\frac{\partial \gamma_\nu}{\partial u}\right)_*, \tag{73}$$

where γ_ν is given in Eq. (32). The vanishing of the λ_1 is an exact result which is related to the degeneracy of the system of Eqs. (70) and (71) when nonzero solutions in respect to g and u are assumed, or, equivalently, it reflects the existence of a marginal direction in the $g - u$ plane along the line of the fixed points.

In the nonhelical case ($\rho = 0$), the coordinate g_* of the possible fixed point as function of the spatial dimension d and arbitrary fixed point value of parameter u_* is given as follows:

$$\bar{g}_* = \frac{2du_*(1+u_*)}{d-1}\varepsilon + \frac{2du_*(d+u_*)\,{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u_*)^2}\right)}{(d-1)^2(d+2)(1+u_*)^2}\varepsilon^2. \tag{74}$$

To have positive value of the fixed point for variables g and u, the condition $\varepsilon > 0$ must be fulfilled. However, possible restrictions on the IR fixed point value of the variable u can be found from condition $\lambda_2 > 0$. The explicit form of λ_2 is the same as in the model of passively advected scalar field [15], namely,

$$\begin{aligned}\lambda_2 = \frac{2+u_*}{1+u_*}\varepsilon + \frac{\varepsilon^2}{(d-1)(d+2)(d+4)(1+u_*)^6}\times \\ \times\Bigg[(1+u_*)^2(4+d)(2d(u_*-1)+(u_*-3)u_*)\times \\ \times\,{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u_*)^2}\right)+ \\ +4u_*(d+u_*)\,{}_2F_1\left(2,2;3+\frac{d}{2};\frac{1}{(1+u_*)^2}\right)\Bigg].\end{aligned} \tag{75}$$

In Fig. 6, the regions of stability for the fixed point FPV without helicity in the $u - \varepsilon$ plane for different space dimension d are shown. Thus, in two-loop approximation nontrivial d-dependence of IR stability of the fixed point appears in contrast to the one-loop approximation [18]. At the same time, again the result is the same as in the corresponding model of passively advected scalar field.

Now, let us turn to the helical case; i.e., let us investigate the influence of the presence of helicity on the value of the fixed point as well as on its IR stability.

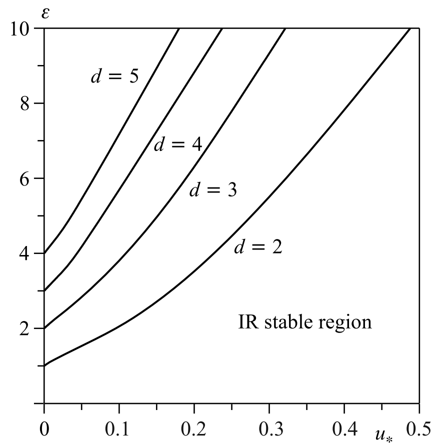

Fig. 6. Regions of the IR stability for the fixed point FPV in two-loop approximation without helicity. The d-dependence of the stability is shown. The restrictions are the same as in the corresponding model of advection of a passive scalar field

In helical case, one works directly in three-dimensional space and the coordinate g_* of the fixed point is given by the following equation:

$$\bar{g}_* = 3u_*(1+u_*)\varepsilon + \frac{3u_*\varepsilon^2}{20(1+u_*)^2}\Big\{2(3+u_*)\,{}_2F_1\times \times\left(1,1;\frac{7}{2};\frac{1}{(1+u_*)^2}\right) - 5\pi(1+u_*)^2\rho^2\Big[2\,{}_2F_1\left(-\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{1}{(1+u_*)^2}\right) - - {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u_*)^2}\right)\Big]\Big\}. \quad (76)$$

On the other hand, the explicit form of the eigenvalue λ_2 in Eq. (73) as function of the parameter of helicity ρ has the following form:

$$\lambda_2 = \frac{2+u_*}{1+u_*}\varepsilon + \frac{\varepsilon^2}{48(1+u_*)^3}\times \times\Big\{8(1+u_*)\big[-6+u_*(51+u_*(95+12u_*(5+u_*)))- - 3u_*\sqrt{u_*(2+u_*)}(21+u_*(33+4u_*(5+u_*)))\arccos(1+u_*)\big]+ + 6\pi u_*^2\rho^2\Big[3\sqrt{u_*(2+u_*)}(2-u_*(7+2u_*(5+2u_*)))+ + 3(1+u_*)^2(2+u_*(7+2u_*(5+2u_*)))\times \times\arccos(1+u_*) - 4(1+u_*)(2+u_*)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u_*)^2}\right)\Big]\Big\}. \quad (77)$$

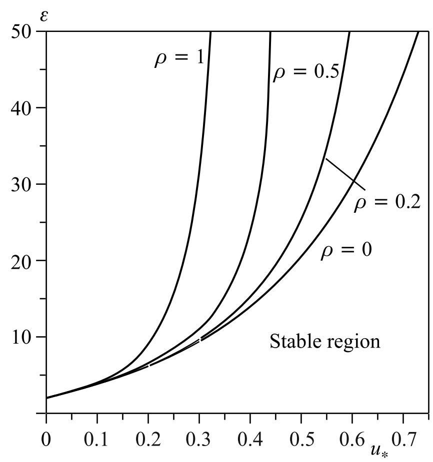

Fig. 7. Regions of the IR stability for the fixed point FPV in two-loop approximation with presence of helicity

In Fig. 7 the regions of stability in the plane $u - \varepsilon$ are shown for various values of the helicity parameters. It is evident that the presence of helicity in our vector model enlarges the region of values of parameters for which the IR stable scaling regime can exist.

The most important conclusion of our two-loop investigation of the scaling regimes in the present model is the fact that the possible restrictions on the regions of stability of the IR fixed points are «pressed» to the region with large values of the parameter ε, namely, $\varepsilon \geqslant 2$, and do not disturb the regions with important relatively small values of ε. For example, the Kolmogorov point ($\varepsilon = \eta = 4/3$) is not disturbed by the two-loop corrections, as well as by the presence of helicity in the system.

Now, we have all needed results to return to the basic analysis of the scaling behavior of the correlation functions in the scaling regimes given by the IR stable fixed points as discussed at the end of Sec. 3. As was shown in the present section, the fixed point value of the anomalous dimension γ_ν^* is exactly given already at the one-loop level of approximation. It means that the critical dimension of frequency $\Delta_\omega = 2 - \gamma_\nu^*$, as well as of fields $\Phi \equiv \{\mathbf{b}, \mathbf{b}', \mathbf{v}\}$, is also defined exactly at one-loop level approximation. Thus, one has

$$\Delta_\omega = \begin{cases} 2 - 2\varepsilon + \eta & \text{for FPII,} \\ 2 - \varepsilon & \text{for FPIV,} \\ 2 - \varepsilon = 2 - \eta & \text{for FPV} \end{cases} \tag{78}$$

and

$$\Delta_{\mathbf{v}} = 1 - \gamma_\nu^*, \quad \Delta_{\mathbf{b}} = 0, \quad \Delta_{\mathbf{b}'} = d. \tag{79}$$

By using all these results together with the explicit scaling representation given in Eq. (37) with the critical dimensions defined in Eq. (38), the scaling behavior of the most interesting equal-time two-point quantities (for example, usually studied equal-time correlation functions defined in Eq. (39)) can be studied except for the properties of the scaling function $R(r/L)$. As was briefly discussed in the Introduction, the scaling behavior of the scaling functions can be investigated by means of the OPE. However, this question is beyond the scope of the present paper and will be studied elsewhere.

CONCLUSION

In the present paper, we have investigated the advection of a weak magnetic field by a turbulent environment with spatial parity violation in the framework of extended Kazantsev–Kraichnan model of kinematic MHD, where turbulent flow is given by the Gaussian statistics of the velocity field with finite correlations in time. The complete analysis of all possible scaling regimes was done and the IR stability of the corresponding fixed points of the RG equations was analyzed in detail. It is shown that in the case when the turbulent environment is isotropic and nonhelical the scaling regimes of the model of passive advection of the vector (magnetic) field in the framework of extended Kazantsev–Kraichnan model of kinematic MHD have completely the same properties as in the model of passive advection of the scalar field in the framework of the corresponding extended Kraichnan model (see, e.g., [15]). On the other hand, it is also shown that when the turbulent environment exhibits the spatial parity violation, nontrivial differences between properties of the scaling regimes for the scalar and vector models appear. For example, within the so-called frozen limit the coordinate of the fixed point, as well as its IR stability, does not depend on the presence of helicity in the system in the framework of the extended Kazantsev–Kraichnan model of passively advected vector field unlike the extended Kraichnan model of passively advected scalar field, where the system feels the presence of helicity and the coordinate of the fixed point depends explicitly on the parameter that controls the amount of helicity in the turbulent environment. On the other hand, in the most interesting case with finite correlations in time of the velocity field, the coordinates of the fixed points, as well as their stability, for both models, namely, extended Kraichnan model of passively advected scalar field and extended Kazantsev–Kraichnan model of passively advected magnetic field, depend on helicity. However, the dependence is different for the scalar and vector model. These results demonstrate importance and necessity of inclusion of various symmetry breaking (e.g., helicity or small-scale anisotropy) into the turbulent models for analysis of the influence of the existence of internal tensor structure of advected fields on scaling properties of the corresponding models deep inside in the inertial interval.

In the present paper, we have analyzed only the first stage of the solution of the problem of the anomalous scaling in the framework of the field-theoretic approach; i.e., we have established all possible scaling regimes of the model and discussed their IR stability. The most important conclusion of our two-loop investigation of the scaling regimes is the fact that the fixed points remain stable under the influence of helicity for $\varepsilon \leqslant 2$; i.e., the IR scaling regimes are not changed for relatively small values of ε. For example, the Kolmogorov scaling regime that corresponds to $\varepsilon = \eta = 4/3$ is not disturbed by the two-loop corrections, as well as by the presence of helicity in the system.

The next step will be to use the obtained results for the investigation of the properties of the scaling functions of the correlation functions of the advected magnetic field (in this respect, the most interesting are the single-time two-point correlation functions defined in Eq. (39)) in the framework of the OPE to determine the critical dimensions of the most important composite operators that lead to the anomalous scaling. However, the problem of anomalous scaling will be studied elsewhere.

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