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GEOMETRIZATION OF THE ELECTROMAGNETIC FIELD AND DARK MATTER

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Fundamental concepts, symmetries, and dynamic equations of the theory of dark matter are derived from the simple relation: everything in the concept of space and the concept of space in everything. It is shown that the electromagnetic field is the singlet state of the dark matter field and, hence, the last may be considered as a generalized electromagnetic field (shortly gef) and a simple solution is given to the old problem of connecting the electromagnetic field with geometric properties of the physical manifold itself. It is shown that gauge fixing renders the generalized electromagnetic field effectively massive while the Maxwell electromagnetic field remains massless. To learn more about interactions between matter and dark matter on the microscopic level (and to recognize the fundamental role of internal symmetry in this case), the general covariant Dirac equation is derived and its natural generalization is considered. The experiment is suggested to test the formulated theory.

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INTRODUCTION

The problem of invisible mass is acknowledged to be among the greatest puzzles of modern cosmology and field theory (see, for example, [1] and [2]). The most direct evidence for the existence of large quantities of dark matter in the Universe comes from the astronomical observation of the motion of visible matter in galaxies [3]. One neither knows the identity of the dark matter nor whether there is one or more types of its structure elements. The most commonly discussed theoretical elementary particle candidates are a massive neutrino, a sypersymmetric neutralino, and the axion. So, at the present time there is a good probability that the set of known fields is by no means limited to those fields. Moreover, we are free to look for deeper reasons for the existence of a new entity unusual in many respects. Of course, such reasoning is grounded on the point of view that there is a general and easily visible mathematical structure that

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stands behind all phenomena that we observe. We put forward an idea that a needed mathematical structure is defined by the fundamental relation: everything in the concept of manifold and the concept of manifold in everything. Here a field theory of the so-called dark matter is derived from the only first principle. According to the modern viewpoint, a fundamental physical theory is the one that possesses a mathematical representation whose elements are a smooth manifold and geometrical objects defined on this manifold. Most physicists nowadays consider a theory be fundamental only if it does make explicit use of this concept. This picture is generally accepted and it is based on such a long history of physical research that there is no reason to question it. The geometrical structure of the physical manifold (points, curves, congruences of curves, families of curves) determines a very restricted set of really geometrical quantities and along with that geometrical internal symmetry that makes these quantities variable and forms from them the fundamental physical fields [4]: the Riemann metric g_{ij}; the linear (affine) connection P^i_{jk} (the group of geometrical internal symmetry is a general linear group $GL(n, R)$); the scalar and covariant vector fields, and antisymmetric covariant tensor fields which are connected by the geometrical internal symmetry (spin symmetry) into the spinning field

$$\mathbf{A} = (a, a_i, a_{ij}, \ldots, a_{ijk\ldots l}), \quad i, j, k, l = 1, 2, \ldots, n.$$

(the group of geometrical internal symmetry is a general linear group $GL(2^n, R)$). It should be noted that the idea of geometrical internal symmetry was at first introduced by Weyl as the process of recalibration [5]. The concept of really geometrical quantity is tightly connected with the general concept of potential field defined below. The last concept can be considered as exact mathematical expression of the minimality principle of the gravitational interactions.

We connect the second of these fields with the problem of dark matter. Thus, the field that we put in correspondence with dark matter has a fundamental geometrical interpretation (parallel transport defines congruence of curves) and carries intrinsically inherent local symmetry that guarantees the uniqueness of the theory. The equations of the dark matter field are derived which are invariant with respect to the local transformations. It is shown that the electromagnetic field can be considered as the singlet state of the dark matter field. Thus, the dark matter field may be considered as a generalized electromagnetic field (shortly gef) and, at the same time, we get a simple solution of the old problem raised by Weyl, Einstein, and Eddington to connect the electromagnetic field with geometrical properties of the physical manifold itself. The idea is that the process of local symmetry breaking is an intrinsic property of the system itself which means that gauge fixing cannot be arbitrary. This approach is realized here in the framework of the concept of the gef ground state. It is interesting that the ground field belongs to the set of potential fields as well. It should be noted that gauge fixing renders the gef effectively massive while the Maxwell electromagnetic field remains massless (in this context a particle of dark matter is a heavy photon). To learn more about interactions between matter and dark matter on the microscopical level, we use the Dirac theory. The general covariant Dirac equation is derived in the Minkowski space–time and, in course of this, the fundamental role of internal symmetry is recognized. On this ground, the Dirac equation is derived which describes the interactions of the spinor field with the ground field. This leads to the general conclusion that interactions of the generalized electromagnetic field with the Dirac spinor field occur only via the above-introduced ground field. The general conclusion is that dark matter gravitates, but there is no actually direct interactions of this new form of matter with known physical fields that represent luminous matter. A rather simple and feasible experiment is proposed to verify this conclusion.

1. CONCEPT OF POTENTIAL FIELD

The concept of really geometrical quantity is tightly connected with the concept of potential field which will be introduced here in the most general form. If we take the components of the symmetrical covariant tensor field g_{ij} and form its derivatives $(\partial_i g_{jk})$, then these derivatives are neither the components of a tensor nor of any geometrical object. However, from g_{ij} and these partial derivatives one can form (with the help of algebraic operations only) a new geometrical object

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_j g_{kl} - \partial_l g_{jk}), \tag{1}$$

which is called the Christoffel connection, where g^{il} are contravariant components of g_{ij}. Now we can formalize this particular case and give general definition of the potential field.

If some geometrical object (or a geometrical quantity) is given and from the components of this object and its partial derivatives one can form (using the algebraic operations only) a new geometrical object (or geometrical quantity), then we deal with a new geometrical quantity that will be called a potential field. The potential field is characterized by the potential P and the strength H and in what follows it will be written in the form (P, H). The connection between the potential and the strength is then called a natural derivative and in a symbolic form can be written as follows: $H = \partial P$. If we go back to our starting point, g_{ij} is a potential and Γ^i_{jk} is a strength of the potential field (g, Γ) known after Einstein as the gravitational field.

Let us consider the Riemann tensor of the connection P^i_{jk}

$$B_{ijl}{}^k = \partial_i P^k_{jl} - \partial_j P^k_{il} + P^k_{im} P^m_{jl} - P^k_{jm} P^m_{il}. \tag{2}$$

Now we go back to the definition of the potential field and see that our really geometrical quantity defines a new potential field (P, B). The theory of this potential field is defined by the group of geometrical internal symmetry inherent in this entity (gef symmetry). Let S^i_j be components of a tensor field of type $(1, 1)$ (a field of linear operator), $\mathrm{Det}\,(S^i_j) \neq 0$. Of two tensor fields S^i_j and Q^i_j of type $(1, 1)$, a tensor field $P^i_j = S^i_k Q^k_j$ of type $(1, 1)$ can be constructed, called their product. With the operation of multiplication thus defined, the set of tensor fields of type $(1, 1)$ with a nonzero determinant forms the group $GL(n, R)$. At the given vector field E^i, any element of the group $GL(n, R)$ defines a bundle of vector fields which is defined as follows:

$$\bar{E}^i = S^i_j E^j, \quad \tilde{E}^i = T^i_j E^j, \quad \text{etc.},$$

where T^i_j are the components of the field S^{-1} inverse to S, $S^i_k T^k_j = \delta^i_j$. It is clear that the notion of the parallel transport is not applied to the bundle of the vector fields and the parallel transport of the bundle of the vector fields is defined by the bundle of the linear connections which is defined by the relation

$$\overline{P}^i_{jk} = S^i_m P^m_{jn} T^n_k + S^i_m \partial_j T^m_k.$$

It is easy to see from this formula that the tensor $P^i_{jk} - P^i_{kj}$ defines no representation of the group $GL(n, R)$.

Thus, we shall expand the diffeomorphism group to include into the consideration the group of local symmetry $GL(n, R)$ defined above. It can be shown that the diffeomorphism group is the group of external automorphisms of the group of local symmetry, i.e., the group $GL(n, R)$ is invariant under the transformations of the group $\mathrm{Diff}\,(M)$. Thus, we have a nontrivial unification of these symmetries and possibility to consider one more potential field with the nontrivial and most wide internal symmetry.

We conclude that the theory of the potential field (P, B) should be invariant not only with respect to the general transformations of the coordinates but with respect to the transformations of the local symmetry group $GL(n, R)$ as well. We put in correspondence to this field the so-called dark matter and develop theory of dark matter as the theory of this new potential field. For brevity, we use the matrix notation

$$\mathbf{S} = (S^k_l), \quad \mathbf{P}_i = (P^k_{il}), \quad \mathbf{E} = (\delta^k_l), \quad \mathbf{H}_{ij} = (H_{ijl}{}^k), \quad \mathrm{Tr}\,\mathbf{U} = U^k_k.$$

The transformations of gef symmetry take the form

$$\overline{\mathbf{P}}_i = \mathbf{S}\mathbf{P}_i\mathbf{S}^{-1} + \mathbf{S}\partial_i\mathbf{S}^{-1} = \mathbf{P}_i + \mathbf{S}D_i\mathbf{S}^{-1}, \tag{3}$$

where D_i is the natural differential operator associated with gef symmetry only

$$D_i\mathbf{S} = \partial_i\mathbf{S} + \mathbf{P}_i\mathbf{S} - \mathbf{U}\mathbf{P}_i = \partial_i\mathbf{S} + [\mathbf{P}_i, \mathbf{S}]$$

and which is especially convenient when one deals with local symmetry in question. In what follows, we shall meet many examples of this. Relation (3) is indeed the transformation of the connection, since $\mathbf{S}D_i\mathbf{S}^{-1}$ is a tensor field of type $(1,2)$ and, hence, $\overline{\mathbf{P}}_i$ is the connection with respect to the coordinate transformations. Since the connection between the potential and strength in matrix notation is given by the formula

$$\mathbf{B}_{ij} = \partial_i \mathbf{P}_j - \partial_j \mathbf{P}_i + [\mathbf{P}_i, \mathbf{P}_j],$$

from (3) it follows that under the transformations of the group $GL(n,R)$ the strength is transformed as follows:

$$\overline{\mathbf{B}}_{ij} = \mathbf{S}\mathbf{B}_{ij}\mathbf{S}^{-1}. \tag{4}$$

For $\mathbf{B}_{ij}$ we have

$$D_i\mathbf{B}_{jk} = \partial_i \mathbf{B}_{jk} + [\mathbf{P}_i, \mathbf{B}_{jk}],$$

and if $\overline{D}_i$ is defined by the potential $\overline{\mathbf{P}}_i$, then from (3) and (4) it follows that

$$\overline{D}_i\overline{\mathbf{B}}_{jk} = \mathbf{S}(D_i\mathbf{B}_{jk})\,\mathbf{S}^{-1}. \tag{5}$$

In the general case the operator D_i is not general covariant; however, the commutator $[D_i, D_j]$ is always general covariant and we get the important relation for the strength tensor

$$[D_i, D_j]\mathbf{B}_{kl} = [\mathbf{B}_{ij}, \mathbf{B}_{kl}]. \tag{6}$$

Thus, in our approach the theory of dark matter is tightly connected with the local symmetry, it is general covariant and has a profound geometrical interpretation.

2. MAIN EQUATIONS OF THE THEORY

The simplest general covariant and gauge invariant Lagrangian of the potential $\mathbf{P}_i$

$$\mathcal{L}_P = -\frac{1}{4}\mathrm{Tr}\,(\mathbf{B}_{ij}\mathbf{B}^{ij}),$$

was considered in [6] and [7], where the gauge theory of oriented media was formulated, and it was proven that the Euler equation $\delta L_P = 0$ has nontrivial solutions. We do not consider this Lagrangian here for the following reason.

The Riemann tensor of $\mathbf{P}_i$ is reducible with respect to the transformations (4) since

$$\mathbf{B}_{ij} = \left(\mathbf{B}_{ij} - \frac{1}{4}\mathrm{Tr}\,(\mathbf{B}_{ij})\,\mathbf{E}\right) + \frac{1}{4}\mathrm{Tr}\,(\mathbf{B}_{ij})\mathbf{E}.$$

Hence, the strength tensor of the generalized electromagnetic field is given by the formula

$$\mathbf{H}_{ij} = \mathbf{B}_{ij} - \frac{1}{4}\mathrm{Tr}\,(\mathbf{B}_{ij})\mathbf{E}, \quad \mathrm{Tr}\,(\mathbf{H}_{ij}) = 0,$$

and the singlet state of the gef defines the strength tensor of the electromagnetic field $F_{ij} = \mathrm{Tr}\,(\mathbf{B}_{ij})$ which should be considered independently of $\mathbf{H}_{ij}$. The ground state of gef is defined by the equation $\mathbf{B}_{ij} = 0$ which means that this state transfers a new form of energy. We give a general solution of this equation. Let four linear independent vector fields E^i_μ be given and one can construct purely algebraical components of the four covector fields E^μ_i, so that $E^i_\mu E^\mu_j = \delta^i_j$ holds valid. Setting $P^i_{jk} = L^i_{jk}$, where $L^i_{jk} = E^i_\mu \partial_j E^\mu_k$ is the canonical connection, we get the general solution of the equation in question. Let us introduce a tensor field $Q^i_{jk} = P^i_{jk} - L^i_{jk}$ and consider the irreducible deviation tensor

$$T^i_{jk} = Q^i_{jk} - \frac{1}{4}Q^l_{jl}\delta^i_k, \quad \mathbf{T}_j = \mathbf{Q}_j - \frac{1}{4}\mathrm{Tr}\,(\mathbf{Q}_j)\mathbf{E}, \quad \mathrm{Tr}\,\mathbf{T}_i = 0.$$

The Lagrangian of gef dynamics takes the form

$$\mathcal{L}_P = -\frac{1}{4}\mathrm{Tr}\,(\mathbf{H}_{ij}\mathbf{H}^{ij}) + \frac{\mu^2}{2}\mathrm{Tr}\,(\mathbf{T}_i\mathbf{T}^i), \tag{7}$$

where μ is a constant of dimension of cm^{-1},

$$\mathbf{H}^{ij} = g^{ik}g^{jl}\mathbf{H}_{kl}, \quad \mathbf{T}^i = g^{ik}\mathbf{T}_k.$$

Varying the Lagrangian $\mathcal{L}_P$ with respect to $\mathbf{P}_i$ the following equations of the generalized electromagnetic field hold valid:

$$\frac{1}{\sqrt{g}}D_i(\sqrt{g}\mathbf{H}^{ij}) + \mu^2\mathbf{T}^j = 0, \tag{8}$$

where $g = -\mathrm{Det}\,(g_{ij})$. From the properties of the operator D_i it is not difficult to see that equations (8) are invariant with respect to the local symmetry group in question. The tensor character of these equations can be seen from the identity

$$\left(\frac{1}{\sqrt{g}}D_i(\sqrt{g}\mathbf{H}^{ij})\right)^k_l = \overset{p}{\nabla}_i\,(\mathbf{H}^{ij})^k_l + \omega_i(\mathbf{H}^{ij})^k_l - \frac{1}{2}(P^j_{im} - P^j_{mi})(\mathbf{H}^{im})^k_l,$$

where $\overset{p}{\nabla}_i$ is the usual covariant derivative with respect to the connection $\mathbf{P}_i$ and $\omega_i = \partial_i \ln\sqrt{g} - P^k_{ki}$ are the components of the covector field. Thus, it is shown that the group of diffeomorphisms is the group of covariance of equations (8). Equations (8) form the first group of equations of gef. The second one is presented by the identity

$$D_i\mathbf{H}_{jk} + D_j\mathbf{H}_{ki} + D_k\mathbf{H}_{ij} = 0. \tag{9}$$

From the definition of the operator D_i it follows that the left-hand side of relation (9) is a tensor and hence it is general covariant.

We see that in some sense one can treat μ as the effective mass of the heavy photon. Since trace of $\mathbf{H}_{ij}$ equals zero, it is clear why we need to consider an irreducible tensor of deviation. In our case, the trace of $\mathbf{T}^i$ is trivial and the system of equations (8) is compatible. From (8), it follows that $\mathbf{T}^i$ has to satisfy the equation

$$\frac{1}{\sqrt{g}} D_i(\sqrt{g}\mathbf{T}^i) = 0, \tag{10}$$

in accordance with (6), $D_i D_j(\sqrt{g}\mathbf{H}^{ij}) = 0$. It is very important that the same equation appears under varying (7) with respect to E^i_μ. Equations (10) represent sixteen additional invariant constraints on the potential $\mathbf{P}_i$.

However, Eqs. (8), (9), and (10) are invariant with respect to the local transformations and, hence, we still have a problem of gauge fixing. It is interesting that there is only one plausible possibility to solve this problem which will be considered in what follows.

Varying the Lagrangian L_P with respect to g^{ij} we obtain the so-called metric tensor of energy–momentum of gef (dark matter field)

$$T_{ij} = -\mathrm{Tr}\,(\mathbf{H}_{ik}\mathbf{H}_j{}^k) - g_{ij}\mathcal{L}_P + \mu^2 \mathrm{Tr}\,(\mathbf{T}_i\mathbf{T}_j), \tag{11}$$

where $\mathbf{H}^k_j = \mathbf{H}_{jl}g^{kl}$. With Eqs. (8) and (9) one can show that the metric tensor of the energy–momentum satisfies the equations

$$\nabla_i T^{ij} = 0, \tag{12}$$

where ∇_i denotes, as usual, the covariant derivative with respect to the Christoffel connection (1) and $\nabla^i = g^{ik}\nabla_k$. It is evident that the metric tensor energy–momentum is invariant with respect to the group of gef symmetry. Now we can write down the full action for the fields g_{ij} and $\mathbf{P}_i$:

$$A = -\frac{l^{-2}}{2}\int R\sqrt{g}\,d^4x - \int \frac{1}{4}\mathrm{Tr}\,(\mathbf{H}_{ij}\mathbf{H}^{ij})\,\sqrt{g}\,d^4x + \int \frac{\mu^2}{2}\mathrm{Tr}\,(\mathbf{T}_i\mathbf{T}^i)\sqrt{g}\,d^4x,$$

where R is the scalar curvature and l is the constant of the dimension of length. From the geometrical interpretation of the field $\mathbf{P}_i$ it follows that it has the dimension cm^{-1}. As all coordinates can be considered to have the dimension cm, the action A is dimensionless.

Varying the full action A with respect to g^{ij} we derive the Einstein equations

$$R_{ij} - \frac{1}{2}g_{ij}R = l^2 T_{ij}, \tag{13}$$

where T_{ij} is the metric tensor of energy–momentum of gef. Thus, it is shown that the interactions of the generalized electromagnetic field with the gravitational field are characterized by some length l. Equations (8), (9), and (13) are compatible in view of (12). This system of equations describes a new form of matter which is known as dark matter.

Now consider a question concerning the potential of the electromagnetic field. Since tensor of the electromagnetic field is the singlet state of gef, we have

$$F_{ij} = \partial_i \mathrm{Tr}\,\mathbf{P}_j - \partial_j \mathrm{Tr}\,\mathbf{P}_i = \partial_i P^k_{jk} - \partial_j P^k_{ik}.$$

Taking trace of relation (3) we get

$$\overline{P}^k_{ik} = P^k_{ik} - \partial_i \ln |\triangle|,$$

where $\triangle = \mathrm{Det}\,(S^i_j)$. We put $\mathrm{Tr}\mathbf{P}_i = P^k_{ik} - L^k_{ik} + L^k_{ik}$. Since $L^k_{ik} = (1/p)\partial_j p$, $p = \mathrm{Det}\,(E^\mu_i)$, for a singlet state of gef, we have $F_{ij} = \partial_i A_j - \partial_j A_i$, where $A_i = P^k_{ik} - L^k_{ik}$ is the covector field. Thus, the question of the nature of the gauge transformations is completely solved and the geometrical origin of the electromagnetic field is recognized.

Now we have to solve the problem of the general covariant gauge fixing that is provided by the Cauchy problem for the field in question. The distinctive feature of the generalized electromagnetic field is that it is self-interacting: it is nonlinear even in the absence of other fields. Two potentials $\overline{\mathbf{P}}_i$ and $\mathbf{P}_i$ are physically equivalent if there is a local transformation which takes $\mathbf{P}_i$ into $\overline{\mathbf{P}}_i$, and clearly $\overline{\mathbf{P}}_i$ satisfies the field equations if and only if $\mathbf{P}_i$ does. In order to obtain a definite member of the equivalence class of potentials, one has to introduce general covariant gauge conditions. These conditions have to remove the sixteen degrees of freedom and lead to a unique solution for the potential components. To solve these problems, we suggest that gauge fixing is an internal property of the system in question connected with the notion of the ground state of gef.

3. EQUATIONS OF THE GROUND FIELD

The local symmetry will be broken if we introduce the quantity

$$U^i_{jk} = E^i_\mu(\partial_j E^\mu_k - \partial_k E^\mu_j). \tag{14}$$

From the definition it follows that U^i_{jk} is evidently a tensor field antisymmetric in covariant indices. On the other hand, from the definition it follows that this tensor is not a geometrical object with respect to the local symmetry group. The tensor U^i_{jk} defines no representation of the group $GL(n, R)$. Thus, it is convenient for our goal. Further, we shall establish a geometrically motivated Lagrangian that can be constructed for this ground field. It leads us to the investigation of the geometry of affine space which is characterized by the connection

$$L^i_{jk} = \Gamma^i_{jk} + U^i_{jk}, \tag{15}$$

where the first summand is given by expression (1). The physical meaning of this connection is to investigate two quite independent potential fields in the uniform geometrical framework. Consider the most important geometrical quantity defined by connection (15). For the Riemann tensor as a function of the potentials of gravity and ground state we have

$$B_{ijk}{}^{l} = R_{ijk}{}^{l} + \nabla_i U^l_{jk} - \nabla_j U^l_{ik} + U^l_{im} U^m_{jk} - U^l_{jm} U^m_{ik}, \tag{16}$$

where

$$R_{ijk}{}^{l} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik} \tag{17}$$

is the Riemann curvature tensor of metric g_{ij}, and ∇_i as earlier stands for the covariant derivative with respect to the Christoffel connection (1)

$$\nabla_i U^l_{jk} = \partial_i U^l_{jk} + \Gamma^l_{im} U^m_{jk} - \Gamma^m_{ij} U^l_{mk} - \Gamma^m_{ik} U^l_{jm}.$$

By contraction we get from (16) the tensor

$$B_{jk} = B_{ijk}{}^{i} = R_{jk} + \nabla_i U^i_{jk} - \nabla_j U^i_{ik} + U^i_{im} U^m_{jk} - U^i_{jm} U^m_{ik}, \tag{18}$$

where R_{jk} is the Ricci tensor. From (18) one can find by contraction with g^{jk} the following expression for the scalar:

$$B = g^{jk}\, B_{jk} = R + g^{jk} U^l_{jm} U^m_{kl} - \nabla_j U^j,$$

where R is the Ricci scalar curvature and $U^j = g^{jk} U_k = g^{jk} U^l_{lk}$. Hence, connection (15) uniquely determines the geometrical Lagrangian of the potential fields of the curvature and the ground state which is a natural generalization of the Einstein–Gilbert Lagrangian of the gravitational field. Thus, we shall derive equations describing the interactions of the gravitational field and the ground field from the action

$$A = \frac{l^{-2}}{2} \int B \sqrt{g}\, d^4x. \tag{19}$$

From (19) it follows that connection (15) uniquely determines the Lagrangian $\mathcal{L}_{gf\mathcal{L}_{gf}}$ of the ground field itself

$$\mathcal{L}_{gf} = \frac{1}{2} g^{jk}\, U^l_{jm}\, U^m_{kl}. \tag{20}$$

It is natural that the Lagrangian of the ground field like the dark matter Lagrangian contains no derivatives of the components of the gravitational potential since U^i_{jk} can be considered as a strength with respect to E^μ_i.

Varying action (20) with respect to g_{ij}, we get the Einstein equations

$$G_{ij} = T_{ij},$$

where

$$T_{ij} = g_{ij}\mathcal{L}_{gf} - U^k_{il}U^l_{jk} \tag{21}$$

is the metric tensor energy–momentum of the ground field. From (22) and (21) it follows that $g^{ij}T_{ij} = 2\mathcal{L}_{gf}$ and hence the equations of the ground field are not conformally invariant. It is yet another general property of gravity and ground state.

It is convenient to introduce the tensor

$$F^{ij}_k = g^{il}U^j_{lk} - g^{jl}U^i_{lk} = U^{ij}_k - U^{ji}_k$$

with inverse transformation

$$U^i_{jk} = \frac{1}{2}(g^{il}F^{mn}_l g_{jm}g_{kn} + g_{jl}F^{il}_k - g_{kl}F^{il}_j).$$

Now we make small variations in our field quantities E^l_μ, and the variational principle provides the following equations for the potential of the ground field:

$$E^\mu_k\nabla_j F^{jk}_l + F^{jk}_l\nabla_j E^\mu_k + F^{jk}_m E^\mu_k E^\nu_l \nabla_j E^m_\nu = 0.$$

It is possible to rewrite this equations in the simplest form setting

$$F^{i\mu}{}_\nu = F^{ik}_l E^l_\nu E^\mu_k.$$

As a result, the following equations hold valid:

$$\nabla_i F^{i\mu}{}_\nu = 0. \tag{22}$$

Like the equations of the gravitational field and the generalized electromagnetic field (dark matter field or gef), the equations of the ground field are essentially nonlinear. Now we shall consider the interactions of the generalized electromagnetic field with matter in the framework of the Dirac theory that is very important since nothing is known about the interactions of dark matter with luminous matter.

4. THE DIRAC EQUATION IN THE GENERAL COVARIANT FORM

The description of the interactions between the matter and dark matter will be provided in the framework of the Dirac equation which is the basis for the description of matter. It is one of the fundamental principles of modern geometry and theoretical physics that the laws of geometry and physics do not depend on the choice of coordinate systems. It is natural to write all equations in the coordinate basis since the problem to rewrite these equations in any other basis is a formal and hence trivial task. In our days, this statement is as canonical as the energy conservation. Let us show that the original Dirac equation is in full agreement with this fundamental statement and that it is defined by the internal symmetry. As is known, internal symmetries play a fundamental role in modern physical theories and hence it is very important to have clear understanding of the role of internal symmetries in the Dirac theory, which is the basis for all modern theories of elementary particles and their interactions; in particular, Dirac's Hamiltonian defines entirely the space-time sector of the standard model.

Let $\mathbf{C}^4$ be a linear space of columns of four complex scalar fields ψ_1, ψ_2, ψ_3, ψ_4. Linear transformations in this space can be represented by the complex matrices (4×4). The set of all invertible (4×4) complex matrices forms a group denoted by $GL(4, \mathbf{C})$. Dirac's γ^μ matrices belong to $GL(4, \mathbf{C})$ and obey the anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu},$$

where $\eta^{\mu\nu}$ is digital matrix originated by the fundamental quadratic form

$$\varphi = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

From γ^μ one can construct sixteen linear independent matrices that form a basis of the Lie algebra of $GL(4, \mathbf{C})$. This basis is especially important since the matrices $S_{\mu\nu} = (1/2)\gamma^\mu \gamma^\nu$ form the basis of the Lie algebra of the Lorentz group (subgroup of $GL(4, \mathbf{C})$). Thus, a spinor is an element of the space $\mathbf{C}^4$ that is equipped with the matrix $\eta_{\mu\nu}$ defined above. It should be noted that in the space $\mathbf{C}^3$ there are no matrices like γ^μ.

If one considers ψ_1, ψ_2, ψ_3, ψ_4 as a set of complex scalar fields on the space–time manifold, then a spinor field emerges on the manifold as a basis of irreducible representation of the group $GL(4, \mathbf{C})$. It is not difficult to understand that $GL(4, \mathbf{C})$ is a group of internal symmetry since its transformations involve only functions of the spinor field and do not affect the coordinates. In other words, spin symmetry is *internal symmetry*.

Now, on this ground we consider the general covariant formulation of the Dirac equation in the Minkowski space–time. We shall follow the fundamental physical principle that was mentioned above. With respect to an arbitrary curviliner system of coordinates the Minkowski space–time is characterized by the metric

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

of the Lorentz signature, which satisfies the equation $R_{ijk}{}^l = 0$. At given g_{ij}, the generators of the group of space–time symmetry can be represented as a set of linear independent solutions of general covariant system of equations (Killing's equations)

$$K^i \partial_i g_{jk} + g_{ik} \partial_j K^i + g_{ji} \partial_k K^i = 0$$

for a vector field K^i. In the case of the Minkowski metric, we have ten linear independent solutions of the Killing equations which are denoted by K^i_μ and $K^i_{\mu\nu} = -K^i_{\nu\mu}$ and, hence, the Greek indices enumerate vector fields and take the values $0, 1, 2, 3$, like coordinate Latin indices.

It is well known that the generators of the Poincaré group

$$\mathbf{P}_\mu = K^i_\mu \frac{\partial}{\partial x^i}, \quad \mathbf{M}_{\mu\nu} = K^i_{\mu\nu} \frac{\partial}{\partial x^i}$$

satisfy the following commutation relations:

$$[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0, \tag{23}$$

$$[\mathbf{P}_\mu, \mathbf{M}_{\nu\lambda}] = \eta_{\mu\nu}\mathbf{P}_\lambda - \eta_{\mu\lambda}\mathbf{P}_\nu. \tag{24}$$

It is evident that all these relations are general covariant and that the operators $\mathbf{P}_\mu = K^i_\mu(\partial/\partial x^i)$ transform a scalar field into the scalar one.

Now we shall show that the general covariant Dirac equation has the form

$$i\gamma^\mu \mathbf{P}_\mu \psi = \frac{mc}{\hbar}\psi, \tag{25}$$

where ψ is a column of four complex scalar fields in question and $\mathbf{P}_\mu$ are the generators of space–time translations. To be exact in all details, let us explain what it means that the Dirac equation is general covariant. Transformation φ of the local group of diffeomorphisms (group of general coordinate transformations) can be represented by the smooth functions

$$\varphi : x^i \Rightarrow \varphi^i(x), \quad \varphi^{-1} : x^i \Rightarrow f^i(x), \quad \varphi^i(f(x)) = x^i.$$

Induced transformation of the metric tensor is of the form

$$\tilde{g}_{ij}(x) = g_{kl}(f(x)) f^k_i(x) f^l_j(x),$$

where $f^k_i(x) = \partial_i f^k(x)$. For the scalar and vector fields we have

$$\tilde{\psi}(x) = \psi(f(x)), \quad \tilde{P}^i(x) = P^k(f(x))\varphi^i_k(f(x)),$$

where $\varphi^i_k(x) = \partial_k \varphi^i(x)$. It is not difficult to verify that if $K^i(x)$ is a solution of the Killing equations for the metric $g_{ij}(x)$, then $\tilde{K}^i(x)$ is a solution of the Killing equations for the metric $\tilde{g}_{ij}(x)$. Further, if $\psi(x)$ is a solution of the Dirac equation (25), then $\tilde{\psi}(x)$ will be a solution of equation (25) when $K^i_\mu(x)$ is substituted by $\tilde{K}^i_\mu(x)$. Besides, the transformations of the diffeomorphisms group conserve the form of the commutation relations of the Poincaré group. Dirac's equation is covariant with respect to the general coordinate transformations. It is known that in the Minkowski space–time, there is a preferred class of the coordinate systems. In the preferred system of coordinates the Dirac equation (25) has a customary form.

It is also clear that Eq. (25) is equivalent to the equation

$$i\tilde{\gamma}^{\mu}\mathbf{P}_{\mu}\psi = \frac{mc}{\hbar}\psi,$$

if $\tilde{\gamma}^{\mu} = S\gamma^{\mu}S^{-1}$, where $S \in GL(4, \mathbf{C})$ (the Dirac equation (25) is covariant with respect to the transformations of the group $GL(4, \mathbf{C})$).

Now we have found enough to provide some valuable insights into the connection between the space–time and internal transformations. Consider again the generators of the internal Lorentz group $S_{\mu\nu} = (1/4)(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu})$ and pay attention to the commutation relations

$$[\gamma_{\mu}, S_{\nu\lambda}] = \eta_{\mu\nu}\gamma_{\lambda} - \eta_{\mu\lambda}\gamma_{\nu}. \qquad (26)$$

Comparing (24) and (26) it is not difficult to verify that the operators

$$\mathbf{L}_{\mu\nu} = \mathbf{M}_{\mu\nu} + S_{\mu\nu}$$

commute with the Dirac operator $D = i\gamma^{\mu}\mathbf{P}_{\mu}$ and satisfy the commutation relations of the Poincaré group. Thus, in the Minkowski space–time, there is a relation between the internal symmetry group and the space–time symmetry group. The consequence is that Dirac's equation (25) is invariant with respect to the transformations of the Poincaré group. Thus, the geometrical and group-theoretical meaning of both the spinor and original Dirac equation is quite clear. We see that the structure of the Dirac equation is defined by the internal symmetry and the derivatives with respect to the given directions. In the considered case these derivatives coincide with the generators of the translation group. In this respect, the Dirac equation differs radically from the Einstein equations where internal symmetry has no role at all. The spinor enters into the world of tensors as a four-component complex scalar field being a carrier of nontrivial internal symmetry which, thus, was discovered together with the Dirac equation.

Consider now the possible natural generalizations of the general covariant Dirac equation. We will strive to realize the project when the diffeomorphism group is the group of invariance (not covariance) of the generalized theory and internal symmetry remains without change. There is only one natural way to do this and it will be the subject of our consideration in later sections.

5. GENERALIZATION OF THE DIRAC THEORY

In this chapter, it is shown that a spinor field can be represented as a natural origin of the ground field considered above.

We take that the canonical energy–momentum tensor plays a fundamental role in the theory of the spinor fields and, in accordance with this, the generalized Dirac Lagrangian has the form

$$\mathcal{L}_D = \frac{i}{2} E^i_\mu \left(\bar{\psi} \gamma^\mu D_i \psi - (D_i \bar{\psi}) \gamma^\mu \psi \right) - m \bar{\psi} \psi, \tag{27}$$

where E^i_μ are contravariant components of the potential of the ground field,

$$D_i \psi = (\partial_i - iqA_i)\psi, \quad D_i \bar{\psi} = (\partial_i + iqA_i)\bar{\psi}.$$

It is evident that varying (27) with respect to E^i_μ results in the canonical energy–momentum tensor of the spinor field. Lagrangian (27) is invariant with respect to the substitutions

$$\psi \Rightarrow \mathrm{e}^{i\varphi}\psi, \quad \bar{\psi} \Rightarrow \mathrm{e}^{-i\varphi}\bar{\psi}, \quad A_i \Rightarrow A_i + \partial_i \varphi$$

and, hence, it is general covariant and invariant with respect to the local transformation of the group $U(1)$. The action has the form

$$A = \int \mathcal{L}_D \, p \, d^4 x,$$

where $p = \mathrm{Det}\,(E^\mu_i)$. Since

$$E^i_\mu \partial_j E^\mu_i = \frac{1}{p} \partial_j p,$$

this action leads to the Dirac equations in the presence of the external ground field and the electromagnetic field

$$i E^i_\mu \gamma^\mu \left(D_i + \frac{1}{2} U_i \right) \psi = m\psi, \tag{28}$$

$$i E^i_\mu (D_i + \frac{1}{2} U_i) \bar{\psi} \gamma^\mu = -m\bar{\psi}, \tag{29}$$

where, as earlier, $U_i = U^k_{ik}$.

Setting

$$W^\mu_i = \frac{i}{2} (\bar{\psi} \gamma^\mu D_i \psi - (D_i \bar{\psi}) \gamma^\mu \psi),$$

we have $\mathcal{L}_D = E^i_\mu W^\mu_i - m\bar{\psi}\psi$. Hence, from the action

$$A = \int \mathcal{L}_D \, p \, d^4 x + \frac{l^{-2}}{2} \int \mathcal{L}_{\mathrm{gf}} \sqrt{g} d^4 x, \quad g = -\mathrm{Det}\,(g_{ij})$$

we verify (in accordance with (22)) the following equations for the potential of the ground field:

$$\nabla_j F^{j\mu}{}_\nu + l^2 W^\mu_\nu = 0, \tag{30}$$

where

$$W^\mu_\nu = \epsilon E^l_\nu W^\mu_l, \quad \epsilon = p/\sqrt{g}.$$

Equations (30) generalize Eqs. (22) and together with the Dirac equations (28) and (29) explain clearly how the ground field interacts with the spinor field. There is no direct interaction of the generalized electromagnetic field with the spinor field. From Eq. (30) an interesting relation can be derived. By summing over the indices μ and ν, we get that a trace of U^i_{jk} satisfies the following equation:

$$\nabla_i U^i = m\bar{\psi}\psi, \tag{31}$$

where $U^i = g^{ik}U_k$. We conclude that for $m = 0$, the interactions of the ground field and the spinor field are characterized by a new conserved quantity. Indeed, this fact simply means that the action is invariant under the mapping

$$E^\mu_i \to aE^\mu_i, \quad \psi \to a^{-(1/2)}\psi,$$

where a is dimensionless constant. Thus, the introduction of the ground field into the framework of the standard model may shed new light on the mechanism of the lepton mass generation.

CONCLUSION

Here we suggest an experiment to test the formulated theory. It is suggested to measure the gravitational acceleration of electrons and positrons in the Earth gravitational field. The motivation is as follows.

In 1967, Witteborn and Fairbank measured the net vertical component of gravitational force on electrons in vacuum enclosed by a copper tube [8]. This force was shown to be less than $0.09mg$, where m is the inertial mass of the electron and g is 980 cm/s^2. They concluded that this result supported the contention that gravity induced an electric field outside a metal surface of such magnitude and direction that the gravitational force on electrons was cancelled. If this is true, then the positrons will fall in this tube with the acceleration $a = 2g$. The conclusion from the theory presented here is that electrons and positrons do not interact with the gravitational field directly but only through the ground field and electromagnetic channel. And the result presented by the measurements may be considered as an estimation for the energy of the ground field generated by electron (and positron). Thus, the new measurements of the net vertical component of the force on positrons in vacuum enclosed by a copper tube will have the fundamental significance for understanding the conceptual basis of contemporary theoretical physics and for the understanding the nature of dark matter as well.

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