ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2011. Т. 42. ВЫП. 1

EXACT RESULTS IN QED

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This review is devoted to exact calculations of cross sections of QED processes in relativistic approach. We consider the case when the velocity of final heavy particles is not particularly close to the speed of light. Especially, the cases of muon radiative pair creation are considered. In the framework of QED with pion form factor, we estimate the pion radiative pair creation. All leading terms are included in the framework of structure function approach. The annihilation of electron–positron to the hadronic final state with one additional tagged photon is considered. The radiative corrections are calculated, and the numerical estimation is made. Target spin asymmetry and charge asymmetry are investigated for $e\mu$-scattering processes. Initial-state radiation (returning to resonance) mechanism is investigated including radiative corrections to initial and final states. Possible method of experimental extraction of the ratio of electric and magnetic form factors of a proton is suggested for different processes in lepton–proton scattering. Some useful algebraic relations and integrals are presented.

Рассмотрены вычисления различных процессов в рамках КЭД с учетом масс тяжелых конечных частиц. Отдельные разделы посвящены рождению мюонных и пионных пар вблизи порога. Показано, что лидирующие радиационные поправки могут быть учтены при помощи метода структурных функций. Рассмотрен механизм возвращения на резонанс с учетом радиационных поправок для образования пионных и мюонных пар. Показано, что сечение процесса может быть представлено в форме сечения процесса Дрелла–Яна, и приведены численные оценки нелидирующих вкладов. В отдельном разделе изучен альтернативный механизм экспериментального измерения отношения электрического и магнитного формфакторов протона в процессах лептон-адронного рассеяния. В разд. 3 приведены интегралы, часто используемые при вычислении сечений, а также полезные формулы для расчета следов.

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To the blessed memory of Teachers Alexandr Ilich Akhiezer, Sergei Semenovich Sannikov, and Vladimir Naumovich Gribov

INTRODUCTION

In this paper we obtain the differential cross sections in relativistic approach. We consider the case when the velocity of final particles is not particularly close to the speed of light. It gives us the possibility of applying our formulas to the process near the particle-creation threshold.

Annihilation channels with production of muon and pion pairs with additional photon are explicitly considered in Sec. 1. We investigate separately the effects of initial- and final-state emission. We calculate the explicit form of the third structure function, beyond Pauli and Dirac form factors, in the amplitude of the process of elastic electron–muon scattering, in the presence of two-photon exchange. We also consider the method of extracting the ratio of electric and magnetic proton form factors in elastic electron-polarized proton scattering as well as inelastic electron–proton and photon–proton scattering processes.

The heavy photon Compton tensor in the case of longitudinally polarized initial fermion and the double logarithmic corrections to the beam asymmetry in the process of elastic polarized electron–proton scattering are considered at the end of Sec. 1.

Processes of initial-state hard-photon emission in annihilation channel of electron–positron collisions are considered in Sec. 2. Particularly, we investigate in detail the processes of creation of muon and pion pairs with a point-like model of pion. The main attention is paid to the so-called mechanism of return to resonance in the channel of hadron production. The lowest order RC are taken into account. The results are generalized to higher orders of PT using the LSF approach.

Detailed consideration of Compton and double Compton scattering processes with taking into account RC is given. In Sec. 2 we demonstrate analytically the cancellation of the dependence on the auxiliary angular parameter separating the regions of collinear and noncollinear additional photon emission.

In Sec. 3 we give the tables for one-loop Feynman integrals of scalar, vector, and tensor types, with two, three, four, and five denominators. All formulae are presented with the accuracy up to the terms of the order of the ratio of the electron to the muon masses squared, and the kinematic invariants are assumed to be large compared to the electron mass squared.

Also we present some integral which could be used for phase volume calculation with finite mass of particles and put some tricks for trace conversion.

Throughout our paper we use the next designations: DIS — deep inelastic scattering; FD — Feynman diagram; LLA — leading logarithmic approximation; LSF — lepton structure functions; NL — next-to-leading; PD — photon detector; QCD — quantum chromodynamics; QED — quantum electrodynamics; RC — radiative corrections; SM — Standard Model.

1. EXACT RESULTS FOR $2 \to 3$ TYPE PROCESSES

1.1. The Cross Sections of the Charged Muon- and Pion-Pairs Production at Electron–Positron Annihilation near the Threshold. Exact evaluation of the hadron's contribution to the anomalous magnetic moment of muon implies the knowledge of the cross sections of muon- (tau) and pion-pairs production at the region where the total c.m.s. energy of pair does not exceed threshold value significantly. The lowest order RC and effects due to the Coulomb interaction in the final state become essential [25].

For the purpose of comparison with experimental data, the cases with hard additional photon must be calculated in the framework of PT. It is a weak point of approaches based on dimensional regularization methods where the separation of soft- and hard-photon emission cannot be arranged. Here we calculate these contributions in the framework of traditional QED approach with assigning to photon small mass and calculate the virtual-, soft-, and hard-photon contributions separately. In papers [26, 27], the spectra and total cross sections were obtained, but the calculation method of distribution on invariant mass of muon (tau) pair and the contribution to the total cross section were too complicated. The radiative corrections to the pion-pair production including intital- and final-state radiation was also considered in [99, 100]. The leptonic and hadronic cross sections with $\mathcal{O}(\alpha)$ corrections are required to improve the calculation precision of the vacuum polarization effects in photon propagator at low energies.

Below, using the invariant integration method, we obtain the hard-photon emission contribution to the spectral distribution on the muon pair effective mass and the corresponding contribution to the total cross section due to photon radiation by initial or final particles. We do not consider the interference of these amplitudes assuming the experimental setup to be charge-blind. In this case, the interference contribution to the total cross section is zero. Similar calculations of the charged pion-pair production (assuming pion to be point-like object) are done. Adding the known results for contribution of virtual and real soft-photon emission we obtain the corresponding total cross sections. These results are in agreement with ones obtained in previously published papers (see references below), but have the form more convenient for different applications. Whenever possible, the analytical results are used as a cross-check with ultrarelativistic limit.

1.1.1. Final-State Radiation (FSR) in Muon-Pair Production. As well as we are interested in muon effective mass spectrum let us put the cross section in the form:

$$d\sigma = \frac{1}{8s} \int \sum_{\text{spins}} |M|^2 \, d\Gamma.$$

The summed over spin states matrix element squared can be put in the form:

$$\begin{gathered}\sum |M|^2 = -(4\pi\alpha)^2 \frac{1}{s^2} L_{\mu\nu} T^{\mu\nu}, \quad s = (p_+ + p_-)^2, \\ L_{\mu\nu} = \mathrm{Tr}\,[\hat{p}_- \gamma_\mu \hat{p}_+ \gamma_\nu], \quad T_{\mu\nu} = \mathrm{Tr}\left[(\hat{q}_- + M) O_{\mu\eta} (\hat{q}_+ - M) \tilde{O}_{\nu\eta}\right],\end{gathered} \tag{1.1}$$

with

$$O_{\mu\nu} = \gamma_\nu \frac{\hat{q}_- + \hat{k} + M}{\chi_-}\gamma_\mu + \gamma_\mu \frac{-\hat{q}_+ - \hat{k} + M}{\chi_+}\gamma_\nu, \qquad \chi_\pm = 2kq_\pm,$$
$$p_- + p_+ = q = q_- + q_+ + k, \quad q_\pm^2 = M^2, \quad p_\pm^2 = m^2, \quad k^2 = 0. \tag{1.2}$$

Introducing the energy fractions of final particles we have:

$$\nu_\pm = \frac{2qq_\pm}{s}, \quad \nu = \frac{2qk}{s}, \quad \nu + \nu_+ + \nu_- = 2,$$
$$\int d\Gamma = \int \frac{1}{(2\pi)^5}\frac{d^3q_-}{2E_-}\frac{d^3q_+}{2E_+}\frac{d^3k}{2\omega}\delta^4(p_+ + p_- q_+ - q_- k) = \frac{s}{2^7\pi^3}\int\limits_\Delta^{\beta^2} d\nu \int\limits_{\nu_1}^{\nu_2} d\nu_+,$$
$$\nu_{1,2} = \frac{1}{2}(2-\nu) \pm \frac{\nu}{2}R(\nu), \qquad (1-\nu)(1-\nu_-)(1-\nu_+) > \sigma\nu^2,$$
$$R(\nu) = \sqrt{1 - \frac{4\sigma}{1-\nu}} = \sqrt{\frac{\beta^2-\nu}{1-\nu}}, \qquad \beta^2 = 1 - 4\sigma, \qquad \sigma = \frac{M^2}{s}.$$

Due to gauge invariance of tensor $T^{\mu\nu}$, we can write down the following:

$$\int d\Gamma T_{\mu\nu} = \frac{1}{3}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\int d\Gamma T_\eta^\eta.$$

Further simplification follows from gauge invariance of initial lepton tensor $L^{\mu\nu}q_\mu = 0$. Simple calculation gives

$$\sum T_\eta^\eta = 4\left[\frac{A}{(1-\nu_+)^2} + \frac{B}{1-\nu_+} + C + (\nu_+ \to \nu_-)\right],$$
$$A = -\frac{1}{2}(3-\beta^2)(1-\beta^2), \qquad C = -2,$$
$$B = \frac{1}{\nu}(3-\beta^2)(1+\beta^2) - 2(3-\beta^2) + 2\nu.$$

Integration on the muon-energy fraction can be performed using the expressions:

$$\int\limits_{\nu_1}^{\nu_2} d\nu_+ \left[\frac{1}{(1-\nu_+)^2};\ \frac{1}{1-\nu_+};\ 1\right] = \left[\frac{1-\nu}{\nu\sigma}R(\nu);\ \ln\frac{1+R(\nu)}{1-R(\nu)};\ \nu R(\nu)\right]. \tag{1.3}$$

Distribution on the invariant mass square of muons $q^2 = (q_+ + q_-)^2 = s(1-\nu)$ for the case when the energy of hard photon exceeds some value

$\omega > \sqrt{s}\Delta/2, \Delta \ll 1$ has the form

$$\frac{d\sigma^h_{\rm FSR}}{d\nu} = \frac{2\alpha^3}{3s}\left[\left[\frac{(1+\beta^2)(3-\beta^2)}{\nu} - 2(3-\beta^2) + 2\nu\right]\ln\frac{1+R(\nu)}{1-R(\nu)} - \right.$$
$$\left. -2\left[\frac{3-\beta^2}{\nu}(1-\nu)+\nu\right]R(\nu)\right]. \quad (1.4)$$

Contribution to the total cross section can be obtained performing the integration on invariant muon mass. We use the set of integrals:

$$\int\limits_\Delta^{\beta^2} R(\nu)\left[\frac{1}{\nu};\ 1;\ \nu\right]d\nu = \left[-L_\beta + \beta\ln\frac{4\beta^2}{(1-\beta^2)\Delta};\ \beta - \frac{1-\beta^2}{2}L_\beta;\right.$$
$$\left.\beta\frac{3-\beta^2}{4} - \frac{(3+\beta^2)(1-\beta^2)}{8}L_\beta\right] + \mathcal{O}(\Delta),$$
$$(1.5)$$
$$\int\limits_\Delta^{\beta^2} \ln\frac{1+R(\nu)}{1-R(\nu)}\left[\frac{1}{\nu};\ 1;\ \nu\right]d\nu = \left[L_\beta\ln\frac{1}{\Delta} + 2\Phi(\beta);\ \ -\beta + \frac{1}{2}(1+\beta^2)L_\beta;\right.$$
$$\left.\frac{1}{16}(3+2\beta^2+3\beta^4)L_\beta - \frac{3}{8}\beta(1+\beta^2)\right] + \mathcal{O}(\Delta),$$

with

$$L_\beta = \ln\frac{1+\beta}{1-\beta},$$
$$(1.6)$$
$$\Phi(\beta) = \mathrm{Li}_2(1-\beta) - \mathrm{Li}_2(1+\beta) - \mathrm{Li}_2\left(\frac{1-\beta}{2}\right) + \mathrm{Li}_2\left(\frac{1+\beta}{2}\right).$$

The result is

$$\sigma_h^{e^+e^-\to\mu^+\mu^-\gamma} = \frac{2\alpha}{\pi}\sigma_B(s)\left[\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right)\ln\frac{1}{\Delta} + \frac{7}{4} - \ln\frac{4\beta^2}{1-\beta^2} - \right.$$
$$\left. -\frac{3(1+\beta^2)}{8(3-\beta^2)} + \frac{9-2\beta^2+\beta^4}{16\beta(3-\beta^2)}L_\beta + \frac{1+\beta^2}{\beta}\Phi(\beta)\right], \quad (1.7)$$

where $\sigma_B(s) = 2\pi\alpha^2\beta(3-\beta^2)/(3s)$ is the cross section of muon-pair production in Born approximation. In the ultrarelativistic limit we have

$$\sigma_h^{e^+e^-\to\mu^+\mu^-\gamma}\Big|_{\beta\to 1} = \frac{4\pi\alpha^2}{3s}\frac{2\alpha}{\pi}\left[(l_\mu - 1)\ln\frac{1}{\Delta} - \frac{3}{4}l_\mu + \frac{11}{8} - \xi_2\right], \quad (1.8)$$

where $l_\mu = \ln(s/M^2)$, $\xi_2 = \pi^2/6$. The contribution from emission of soft real photons $\omega = \sqrt{k^2+\lambda^2} < \sqrt{s}\Delta/2$ by muons (λ is «photon mass») is:

$$\sigma^s_{\mathrm{FSR}} = \sigma_B(s)\left(-\frac{\alpha}{4\pi^2}\right)\int\frac{d^3k}{\omega}\left(\frac{q_-}{q_-k}-\frac{q_+}{q_+k}\right)^2,$$

performing the standard calculations it can be written in the form:

$$\begin{aligned}\sigma^s_{\mathrm{FSR}} = \frac{2\alpha}{\pi}\sigma_B(s)\Biggl[&\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right)\left(\ln\frac{M}{\lambda}+\ln\Delta\right) + \\ &+\frac{1+\beta^2}{2\beta}\left[\frac{1}{4}L_\beta^2 - \mathrm{Li}_2(\beta)+\mathrm{Li}_2(-\beta)-\mathrm{Li}_2\left(\frac{1-\beta}{2}\right) - \right.\\ &\left.-\ln\left(\frac{1+\beta}{2}\right)\ln(1-\beta)+\frac{1}{2}\ln^2(1+\beta)+\mathrm{Li}_2\left(\frac{1}{2}\right)+L_\beta\ln\frac{2}{1+\beta}\right] + \\ &+\ln\left(\frac{1+\beta}{2}\right)+\frac{1-\beta}{2\beta}L_\beta\Biggr]. \end{aligned} \tag{1.9}$$

Virtual photon emission correction includes the Dirac and Pauli form factors of muon [2, 3]. It has the form:

$$\begin{aligned}\sigma^v_{\mathrm{FSR}} = \frac{2\alpha}{\pi}\sigma_B(s)\Biggl[&\left(1-\frac{1+\beta^2}{2\beta}L_\beta\right)\ln\frac{M}{\lambda} - 1 + \left(\frac{1+\beta^2}{2\beta}-\frac{1}{4\beta}\right)L_\beta + \\ &+\frac{1+\beta^2}{2\beta}\left[2\xi_2 - \frac{1}{4}L_\beta^2 - L_\beta\ln\frac{2\beta}{1+\beta}+\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right] - \frac{3(1-\beta^2)}{4\beta(3-\beta^2)}L_\beta\Biggr]. \end{aligned} \tag{1.10}$$

The sum of the contributions from virtual and soft real photons reads to be:

$$\begin{aligned}\sigma^{v+s}_{\mathrm{FSR}} = \frac{2\alpha}{\pi}\sigma_B(s)\Biggl[&\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right)\ln\Delta - 1 + \ln\frac{1+\beta}{2} + \\ &+\left(\frac{3-2\beta+2\beta^2}{4\beta}-\frac{3(1-\beta^2)}{4\beta(3-\beta^2)}\right)L_\beta + \\ &+\frac{1+\beta^2}{2\beta}\left(-2\mathrm{Li}_2(\beta)+2\mathrm{Li}_2(-\beta)+\mathrm{Li}_2\left(\frac{1+\beta}{2}\right)-\mathrm{Li}_2\left(\frac{1-\beta}{2}\right)+3\xi_2\right)\Biggr]. \end{aligned} \tag{1.11}$$

In ultrarelativistic limit we have:

$$\sigma^{v+s}_{\mathrm{FSR}}\Big|_{\beta\to 1} = \frac{2\alpha}{\pi}\sigma_B(s)\left[(l_\mu - 1)\ln\Delta - 1 + \frac{3}{4}l_\mu + \xi_2\right]. \tag{1.12}$$

The total sum of contributions from virtual, soft, and hard real photons does not contain photon mass λ and the separation parameter Δ:

$$\sigma_{\rm FSR}^{e^+e^-\to\mu^+\mu^-\gamma} = \frac{2\alpha}{\pi}\sigma_B(s)\Delta_{\rm FSR}^{\mu^+\mu^-}(\beta), \tag{1.13}$$

where

$$\begin{aligned}\Delta_{\rm FSR}^{\mu^+\mu^-}(\beta) &= \frac{3(5-3\beta^2)}{8(3-\beta^2)} + \frac{(1-\beta)(33-39\beta-17\beta^2+7\beta^3)}{16\beta(3-\beta^2)}L_\beta + \\ &\qquad + 3\ln\left(\frac{1+\beta}{2}\right) - 2\ln\beta + \frac{1+\beta^2}{2\beta}F(\beta), \\ F(\beta) &= -2{\rm Li}_2(\beta) + 2{\rm Li}_2(-\beta) - 2{\rm Li}_2(1+\beta) + 2{\rm Li}_2(1-\beta) + \\ &\qquad + 3{\rm Li}_2\left(\frac{1+\beta}{2}\right) - 3{\rm Li}_2\left(\frac{1-\beta}{2}\right) + 3\xi_2.\end{aligned} \tag{1.14}$$

The quantity $\Delta_{\rm FSR}^{\mu^+\mu^-}(\beta)$ agrees with the result obtained in [30]. One can check this result, describing photon emission in final state, due to that in the ultrarelativistic limit it arrives to the value 3/8:

$$\sigma_{\rm FSR}^{e^+e^-\to\mu^+\mu^-\gamma}|_{\beta\to1} = \frac{4\pi\alpha^2}{3s}\frac{2\alpha}{\pi}\frac{3}{8} = \frac{\alpha^3}{s}.$$

Cancellation of «large» logarithms $l_\mu = \ln(s/M^2)$ is the consequence of Kinoshita–Lee–Nauenberg theorem [18].

1.1.2. Initial-State Radiation (ISR) in Muon-Pair Production. Matrix element of the process of muon-pair production with hard photon radiated from initial state has the form:

$$M_{\rm ISR} = \frac{(4\pi\alpha)^{3/2}}{s(1-\nu)}\bar v(p_+)\left[\hat Q\frac{\hat p_- - \hat k + m}{-2kp_-}\hat e(k) + \hat e(k)\frac{-\hat p_+ + \hat k + m}{-2kp_+}\hat Q\right]u(p_-),$$

with $Q_\eta = \bar u(q_-)\gamma_\eta v(q_+)$ being the muon current.

Using the gauge condition for muon current $q^\eta Q_\eta = 0$, $q = q_+ + q_- = p_+ + p_- - k$ we have

$$\begin{aligned}\sum\int Q_\mu(Q_\nu)^*\frac{d^3q_+}{2E_+}\frac{d^3q_-}{2E_-}\delta^4(q-q_+-q_-) &= D\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right), \\ D = -\frac{2\pi s}{3}\left[\frac{3-\beta^2}{2} - \nu\right]R(\nu), \qquad q^2 &= s(1-\nu),\end{aligned}$$

with notations given above. Using this relation, the calculation of the summed upon spin states of matrix element squared is straightforward. Performing the angular integrations by means of

$$\int\limits_{-1}^{1} dc \left[\frac{1}{1-\beta_e c}; \frac{4m^2}{s(1-\beta_e c)^2}; 1\right] = [l_e; 2; 2],$$
$$l_e = \ln\frac{s}{m^2}, \qquad \beta_e = \sqrt{1-\frac{4m^2}{s}}, \tag{1.15}$$

with m being the electron mass, we obtain the distribution on the muons invariant mass:

$$\frac{d\sigma^h_{\rm ISR}}{d\nu} = \frac{4\alpha^3}{3s\nu(1-\nu)^2}[1+(1-\nu)^2](l_e-1)\left(\frac{3-\beta^2}{2}-\nu\right)R(\nu). \tag{1.16}$$

Further integration on the photon-energy fraction ν can be performed using the set of integrals given above and two additional ones:

$$\int\limits_{0}^{\beta^2} R(\nu)\left[\frac{1}{(1-\nu)^2}; \frac{1}{1-\nu}\right] d\nu = \left[\frac{2\beta^3}{3(1-\beta^2)}; -2\beta + L_\beta\right].$$

As a result, we obtain for contribution of hard photon ISR to the total cross section:

$$\sigma^h_{\rm ISR} = \frac{2\alpha}{\pi}\sigma_B(s)(l_e-1)\left[\ln\frac{1}{\Delta} - \frac{1-3\beta+\beta^3}{\beta(3-\beta^2)}L_\beta - \frac{4}{3} + 2\ln\frac{2\beta}{1+\beta}\right]. \tag{1.17}$$

Taking into account the virtual and soft real photons to the initial state gives:

$$\sigma^{s+v}_{\rm ISR} = \frac{2\alpha}{\pi}\sigma_B(s)\left[(l_e-1)\ln\Delta + \frac{3}{4}l_e - 1 + \xi_2\right]. \tag{1.18}$$

The total sum is

$$\sigma^{s+v+h}_{\rm ISR} = \frac{2\alpha}{\pi}\sigma_B(s)\Delta^{\mu^+\mu^-}_{\rm ISR}(\beta),$$
$$\Delta^{\mu^+\mu^-}_{\rm ISR}(\beta) = (l_e-1)\left[-\frac{1-3\beta+\beta^3}{\beta(3-\beta^2)}L_\beta - \frac{4}{3} + 2\ln\frac{2\beta}{1+\beta}\right] + \frac{3}{4}l_e - 1 + \xi_2. \tag{1.19}$$

The total sum in ultrarelativistic limit has the form:

$$\sigma^{s+v+h}_{\rm ISR+FSR}|_{\beta\to 1} = \frac{8\alpha^3}{3s}\left[\frac{1}{2}l_e l_\mu - \frac{1}{2}l_\mu - \frac{7}{12}l_e + \xi_2 + \frac{17}{24}\right], \tag{1.20}$$

which is in agreement with [26, 27]. Leading term $\sim l_e l_\mu$ is in agreement with the result of [29].

1.1.3. Two Charged Pion Production in Electron–Positron Annihilation. It is worth to remind that the total cross section $\sigma(e^+e^- \to \pi^+\pi^-)$ with $\mathcal{O}(\alpha)$ corrections is required in many subjects of particle physics. Particularly, it is required to determine, with a better accuracy, the precision of the evaluation of vacuum polarization effects in photon propagator. Another well-known application is the calculation of the hadronic contribution to the anomalous magnetic moment of muon $a_\mu^{\rm hadr}$:

$$a_\mu^{\rm hadr} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_{4M_\pi^2}^{\infty} ds\, \frac{R(s)K(s)}{s}, \quad R(s) = \frac{\sigma^{e^+e^-\to{\rm hadr}(s)}}{\sigma^{e^+e^-\to\mu^+\mu^-(s)}} \tag{1.21}$$

with

$$K(s) = \int\limits_0^1 \frac{dx(1-x)x^2}{x^2+(1-x)\rho}, \quad \rho = \frac{s}{M_\pi^2}.$$

A contribution to this integral coming from high-energy region can be calculated within QCD framework, while for the low-energy range the experimental values $R(s)$ have to be taken as an input. A numerical evaluation of this integral in relative unities gives the value of ~ 70 ppm.

The goal of the new experiment at BNL (E969) is to measure the anomalous magnetic moment of muon with the relative accuracy of about ~ 0.25 ppm and to improve the previous result [24] by a factor of two. It follows that the value $a_\mu^{\rm hadr}$ should be calculated as precisely as possible. In this context, the required theoretical precision of the cross sections with RC as well as the calculation accuracy of the vacuum polarization effects should be not worse than $\sim 0.2\%$ as it follows from the estimation: 70 ppm $\times 0.2\% \sim 0.14$ ppm. This short observation shows why high precision calculations of the hadronic cross sections are extremely important.

1.1.4. Final-State Radiation in Pion-Pair Production. As well as it was done for the muons, the contributions with one-photon radiation in the final state can be divided into three separate parts: virtual, soft, and hard. The expression for the virtual photon emission from final state is given by

$$\sigma_v = \frac{\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\Bigg[2\ln\frac{M}{\lambda}\left(1-\frac{1+\beta^2}{2\beta}L_\beta\right) - 2 + \frac{1+\beta^2}{\beta}L_\beta + \\ + \frac{1+\beta^2}{\beta}\left(-\frac{1}{4}L_\beta^2 + L_\beta\ln\frac{1+\beta}{2\beta} + 2\xi_2 + {\rm Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right)\Bigg]. \tag{1.22}$$

Here L_β, λ, β were defined above, β is the pion velocity in c.m. frame; $F_\pi(s)$ — pion strong interaction form factor; $\sigma_B^{\pi^+\pi^-}(s) = (\pi\alpha^2\beta^3)/(3s)|F_\pi(s)|^2$ is the cross section of the charged pion-pair production in the Born approximation.

Additional real soft-photon emission contribution when photon energy does not exceed $\Delta\varepsilon$ in c.m. frame reads:

$$\sigma^s_{\mathrm{FSR}} = \frac{\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\Bigg[2\ln\left(\frac{2\Delta\varepsilon}{\lambda}\right)\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right) + \frac{1}{\beta}L_\beta + \\ + \frac{1+\beta^2}{\beta}\left(-\frac{1}{4}L_\beta^2 + L_\beta\ln\frac{1+\beta}{2\beta} - \xi_2 + \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right)\Bigg], \quad \Delta\varepsilon \ll \varepsilon = \frac{\sqrt{s}}{2}. \tag{1.23}$$

The sum of the contributions from virtual and real soft-photon emission can be presented in convenient way as

$$\sigma^{v+s}_{\mathrm{FSR}} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\left[\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right)\ln\Delta + b(s)\right], \tag{1.24}$$

where

$$b(s) = -1 + \frac{1-\beta}{2\beta}\rho + \frac{2+\beta^2}{\beta}\ln\frac{1+\beta}{2} + \\ + \frac{1+\beta^2}{2\beta}\left[\rho + \xi_2 + L_\beta\ln\frac{1+\beta}{2\beta^2} + 2\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right],$$

$$\rho = \ln\frac{4}{1-\beta^2}, \qquad \Delta = \frac{\Delta\varepsilon}{\varepsilon}.$$

Calculations similar to ones given above for FSR muon-pair production lead to pion invariant mass distribution $m^2_{\pi\pi} = s(1-\nu)$:

$$\frac{\sigma^h_{\mathrm{FSR}}}{d\nu} = \frac{2\alpha^3\beta^2}{3s}\left[\left(\frac{\nu}{\beta^2} - \frac{1-\nu}{\nu}\right)R(\nu) + \left(\frac{1+\beta^2}{2\nu} - 1\right)\ln\frac{1+R(\nu)}{1-R(\nu)}\right]|F_\pi(s)|^2, \\ R(\nu) = \sqrt{\frac{\beta^2-\nu}{1-\nu}}. \tag{1.25}$$

Contribution to the total cross section can be obtained performing the integration on invariant pion mass. It is in agreement with the results of papers [19,28]. The relevant contribution has the form:

$$\sigma^h_{\mathrm{FSR}} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\Bigg[\ln\frac{1}{\Delta}\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right) + \\ + 2 + \frac{3-\beta^2}{4\beta^2} - \frac{(3+\beta^2)(1-\beta^2)}{8\beta^3}L_\beta - \ln\frac{4\beta^2}{1-\beta^2} + \frac{1+\beta^2}{\beta}\Phi(\beta)\Bigg], \tag{1.26}$$

with $\Phi(\beta)$ defined above (1.6).

Now we can collect all the discussed terms and write down the complete expression for the total cross section including all corrections of order α and the parametric enhanced Coulomb factor. This expression is given by

$$\sigma_{\mathrm{FSR}}^{e^+e^-\to\pi^+\pi^-\gamma} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta), \tag{1.27}$$

and

$$\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta) = \frac{3(1+\beta^2)}{4\beta^2} - 2\ln\beta + 3\ln\frac{1+\beta}{2} + \\ + \frac{(1-\beta)(-3-3\beta+7\beta^2-5\beta^3)}{8\beta^3}L_\beta + \frac{1+\beta^2}{2\beta}F(\beta), \tag{1.28}$$

with the same expression for $F(\beta)$ as in muon case (1.14). The factor $\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}$ represents the correction to the Born cross section caused by final-state radiation and the Coulomb interaction. In ultrarelativistic limit we have $\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta\to 1) = 3/2$. One can see again, that all «large» logarithms cancel out in accordance with Kinoshita–Lee–Nauenberg theorem. In a low β limit $\Delta_{\mathrm{FSR}}^{\pi^+\pi^-} = \pi^2/(4\beta)$, which is the manifestation of the Coulomb interaction of pions.

Expression for $\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta)$ coincides with one obtained in [19, 28, 31]. It can be noticed that in these papers the quantity $\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta)$ was presented without separator Δ between soft and hard photons. But for some applications it can be useful to have these two parts of the cross sections separated. For instance, the differential cross section (1.25) as a function of the invariant pion's mass is a valuable tool to verify assumption that pions can be considered as the point-like objects and scalar QED can be applied.

To verify the validity of this assumption it is necessary to extract events with FSR from experimental data whereas events with ISR should be suppressed as much as possible. Numerically the cross section with ISR exceeds the cross section with FSR by a factor of ten. There are two ways how to isolate this type of events. It is necessary to select events in the energy range below the ρ-meson peak. ISR shifts the effective energy in c.m.s. and as a result the cross section falls down the more rapidly, the more hard photon is radiated.

It is clear, the second instrument which can help to push down the events with ISR is the acollinearity cut. The photons radiated from initial state mostly fly along the beam direction. It results in that space angle between tracks of pions differs from 180° as bigger as harder photon was emitted. The FSR also breaks the space angle between tracks, but not so strongly. When the threshold on photon energy is about 150 MeV, more the ratio arrives the value about 5. It means, that the relative admixture of events due to ISR makes up $\sim 20\%$ only. It is worth to remind that the spectrum form at high-photon energies is just the subject of interest. So, the fraction of admixture events at the end of spectrum falls down whereas the relative weight of the events with FSR increases. The comparison of the simulated spectrum with experimental one can elucidate this problem.

1.1.5. Initial-State Radiation in Pion-Pair Production. Let us consider now the ISR effects in pion-pair production. Performing the calculations similar to the case of muon-pair production we have:

$$\frac{d\sigma_{\rm ISR}^{e^+e^-\to\pi^+\pi^-\gamma}}{d\nu} = \\ = \frac{\alpha^3}{3s}\frac{1+(1-\nu)^2}{(1-\nu)^2\nu}(l_e-1)(\beta^2-\nu)\sqrt{\frac{\beta^2-\nu}{1-\nu}}|F_\pi(s(1-\nu))|^2, \quad (1.29)$$

where $q^2 = (q_+ + q_-)^2 = s(1-\nu)$. Using integrals presented above we can obtain the following contribution to the total cross section:

$$\sigma_{\rm ISR}^h = \frac{2\alpha^3\beta^3}{3s}(l_e-1)\biggl\{\ln\frac{1}{\Delta} + 2\ln\left(\frac{2\beta}{1+\beta}\right) - \\ - \frac{4}{3} - \frac{1}{2\beta^2} + \frac{1-3\beta^2+4\beta^3}{4\beta^3}L_\beta\biggr\}, \quad (1.30)$$

where $l_e = \ln(s/m^2)$. Here we had assumed the pions to be point-like, i.e., $F_\pi = 1$. Contributions of virtual- and soft-photon emission have the form:

$$\sigma_{\rm ISR}^{v+s} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\left\{(l_e-1)\ln\Delta + \frac{3}{4}l_e - 1 + \xi_2\right\}. \quad (1.31)$$

The total cross section accounted for initial-state radiation can be presented as

$$\sigma_{\rm ISR}^{e^+e^-\to\pi^+\pi^-\gamma} = \frac{2\alpha^3\beta^3}{3s}\Delta_{\rm ISR}^{\pi^+\pi^-}(\beta), \quad (1.32)$$

$$\Delta_{\rm ISR}^{\pi^+\pi^-}(\beta) = (l_e-1)\left[2\ln\frac{2\beta}{1+\beta} - \frac{4}{3} - \frac{1}{2\beta^2} + \frac{1-3\beta^2+4\beta^3}{4\beta^3}L_\beta\right] + \\ + \frac{3}{4}l_e - 1 + \xi_2. \quad (1.33)$$

In ultrarelativistic limit in point-like approximation for pions we have:

$$\sigma_{\rm ISR+FSR}^{e^+e^-\to\pi^+\pi^-\gamma}\Big|_{\beta\to1} = \frac{2\alpha^3}{3s}\left\{\frac{1}{2}l_el_\pi - \frac{1}{2}l_\pi + \frac{3}{2}l_e + \frac{1}{6} + \xi_2\right\}, \quad (1.34)$$

where $l_\pi = \ln(s/M_\pi^2)$.

1.1.6. Estimation of Accuracy. The theoretical precision of the cross sections with $\mathcal{O}(\alpha)$ corrections given above is defined by the unaccounted higher order corrections and is estimated to be at 0.2% level. It is worth to notice that vacuum polarization effects in photon propagator are not considered in this paper and can be found, for instance, in [78]. Let us list the corrections which were omitted in the current formulae.

Weak interactions not considered here arise from replacement of virtual photon Green function by Z-boson one. It results in

$$d\sigma \to d\sigma \left[1 + \mathcal{O}\left(\left(\frac{s}{M_Z^2}\right)^2, \frac{M_\mu^2}{M_Z^2}\right)\right] \tag{1.35}$$

which for $\sqrt{s} \leqslant 10$ GeV is of the order of or smaller than 0.1% in charge-blind experimental setup, when we can omit the $\gamma - Z$ interference contribution.

Here we systematically omit the terms of order $(m/M_\mu)^2$ compared to 1

$$\mathcal{O}\left(\frac{m^2}{M_\mu^2}\right) \leqslant 0.1\%. \tag{1.36}$$

The higher-order contributions (not considered here) can be separated by two classes. One, leading by large logarithm $l_e = \ln(s/m^2)$, is connected with ISR:

$$d\sigma\left[1 + \mathcal{O}\left(\frac{\alpha}{\pi} l_e\right) + \mathcal{O}\left(\frac{\alpha}{\pi}\right)\right], \qquad \mathcal{O}\left(\frac{\alpha}{\pi}\right) \sim 0.5\%. \tag{1.37}$$

This corrections are dominant and can reach 5%. This kind of contributions can, in principle, be taken into account by structure function approach (see [97]).

The higher-order contributions connected with FSR give

$$d\sigma\left[1 + \mathcal{O}\left(\frac{\alpha}{\pi} l_\beta\right)\right], \qquad \mathcal{O}\left(\frac{\alpha}{\pi} l_\beta\right) \sim 2\%, \tag{1.38}$$

in ultrarelativistic limit they $l_\beta \to \ln(s/M_\mu^2)$ as well can be taken into account by structure function method.

We do not consider C-odd interference in real- and virtual-photons emission — it gives zero contribution to the total cross section.

One can see that corrections to Born cross sections $(2\alpha/\pi)\Delta$ can reach several percent near threshold.

In regions where $\beta \sim \alpha$, formulae must be modified [30]. Taking into account that $\Delta^{(i)}(\beta) \sim \pi^2/4\beta$, $\beta \to 0$, we must replace

$$1 + \frac{2\alpha}{\pi}\Delta^{(i)}(\beta) \to \left(1 + \frac{2\alpha}{\pi}\left(\Delta^{(i)}(\beta) - \frac{\pi^2}{4\beta}\right)\right) f(z),$$

where $f(z) = z/(1 - \mathrm{e}^{-z})$ is the Sommerfeld–Sakharov factor, $z = (\pi\alpha/\beta)$.

1.2. Radiative Muon-Pair Production in High-Energy Electron–Positron Annihilation and Crossed Channel Processes. Process of muon-pair production as well as radiative muon-pair production [15] at high energy in electron–positron collisions is commonly used for calibration purposes. This process was investigated in detail in Born approximation in series of papers of Baier and Khoze [40, 41], where the mechanism of returning to resonant region was found.

One of the motivations of our investigation is the high theoretical accuracy required for description of differential cross section. An additional interest appears in the case of small invariant mass of the muon pair. For this case, the radiative muon-pair production is provided by the initial-state hard-photon emission kinematics. It can be used as a calibration process in studying the hadronic systems of small invariant masses created by virtual photon. The lowest order RC in that kinematics to Born cross section [101] as well as the LL and NL contributions in all orders of PT were considered in paper [42].

Besides the practical applications [43, 44], we pursue another aim [15]. The problem is to check the validity RG predictions concerning hard processes of type $2 \to 3$.

Basing on exact (with power accuracy of $\mathcal{O}(M_\mu^2/s)$) calculations, we confirm the Drell–Yan form of the cross section of radiative muon-pair production in LLA. Estimation of nonleading contributions for several kinematical points is given in Table 1.

1.2.1. Born Cross Section and RC. For the process

$$e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-) + \gamma(k_1) \tag{1.39}$$

we use the following kinematics:

$$\begin{gathered}
\chi_\pm = 2k_1p_\pm, \quad \chi'_\pm = 2k_1q_\pm, \quad s = (p_- + p_+)^2, \\
s_1 = (q_- + q_+)^2, \quad t = (p_- - q_-)^2, \quad t_1 = (p_+ - q_+)^2, \\
u = (p_- - q_+)^2, \quad u_1 = (p_+ - q_-)^2, \\
p_\pm^2 = m^2, \quad q_\pm = M^2, \quad k^2 = 0,
\end{gathered} \tag{1.40}$$

where $M(m)$ is the muon (electron) mass. Here all kinematical invariants are much larger than muon (electron) mass, but we take into account terms of order $\ln(M/m)$:

$$\begin{gathered}
s_1 \sim s \sim -t \sim -t_1 \sim -u \sim -u_1 \sim \chi_\pm \gg M^2 \gg m^2, \\
s + s_1 + t + t_1 + u + u_1 = 0.
\end{gathered} \tag{1.41}$$

The differential cross section of the process with the lowest order RC has the form:

$$
\begin{aligned}
\frac{d\sigma_0}{d\Gamma} &= \frac{\alpha^3}{2\pi^2 s} T_0 \left[1 + \frac{\alpha}{\pi}(\Delta_{\rm vac} + \Delta_{ff} + \Delta_{\rm vert} + \Delta_{\rm box} + \Delta_{\rm soft})\right], \\
d\Gamma &= \frac{d^3q_+ d^3q_- d^3k_1}{\varepsilon_+\varepsilon_-\omega_1}\delta^4(p_+ + p_- - q_+ - q_- - k_1).
\end{aligned} \tag{1.42}
$$

It is convenient to separate, starting from Born level definite contributions from hard-photon emission by electron, muon block and their interference:

$$
T_0 = T_0^e + T_0^\mu + T_0^{\rm int}, \tag{1.43}
$$

where

$$
\begin{aligned}
T_0^e &= A\frac{s}{\chi_-\chi_+}, \quad T_0^\mu = A\frac{s_1}{\chi'_-\chi'_+}, \quad A = \frac{t^2 + t_1^2 + u^2 + u_1^2}{ss_1}, \\
T_0^{\rm int} &= A\left[-\frac{t}{\chi_-\chi'_-} - \frac{t_1}{\chi_+\chi'_+} + \frac{u_1}{\chi_+\chi'_-} + \frac{u}{\chi_-\chi'_+}\right].
\end{aligned} \tag{1.44}
$$

The standard evaluation of additional soft-photon emission contribution gives:

$$
\begin{aligned}
\frac{d\sigma_{\rm soft}}{d\sigma_0} &= -\frac{\alpha}{4\pi^2}\int \frac{d^3k_2}{\omega_2}\left(-\frac{p_-}{p_-k_2} + \frac{p_+}{p_+k_2} + \frac{q_-}{q_-k_2} - \frac{q_+}{q_+k_2}\right)^2\bigg|_{\omega_2<\Delta\varepsilon\ll\varepsilon} = \\
&= \frac{\alpha}{\pi}(\Delta_s^e + \Delta_s^\mu + \Delta_s^{\rm int}) = \frac{\alpha}{\pi}\Delta_{\rm soft}.
\end{aligned} \tag{1.45}
$$

Here we denote:

$$
\begin{aligned}
\Delta_s^e &= 2(\rho_s + L - 1)\ln\frac{m\Delta\varepsilon}{\lambda\varepsilon} + \frac{1}{2}(\rho_s + L)^2 - \frac{\pi^2}{3}, \\
\Delta_s^\mu &= 2(\rho_{s_1} - L - 1)\ln\frac{M\Delta\varepsilon}{\lambda\sqrt{\varepsilon_+\varepsilon_-}} + \frac{1}{2}(\rho_{s_1} - L)^2 - \\
&\qquad - \frac{1}{2}\ln^2\frac{\varepsilon_+}{\varepsilon_-} - \frac{\pi^2}{3} + {\rm Li}_2\left(\frac{1+c}{2}\right), \\
\Delta_s^{\rm int} &= \frac{1}{2}(\rho_{t_1} + \rho_u)\ln\frac{t_1}{u} + \frac{1}{2}(\rho_t + \rho_{u_1})\ln\frac{t}{u_1} + \\
&\qquad + 2\ln\frac{t_1}{u}\ln\frac{\sqrt{mM}\Delta\varepsilon}{\lambda\sqrt{\varepsilon\varepsilon_+}} + 2\ln\frac{t}{u_1}\ln\frac{\sqrt{mM}\Delta\varepsilon}{\lambda\sqrt{\varepsilon\varepsilon_-}} + \\
&\qquad + {\rm Li}_2\left(\frac{1+c_-}{2}\right) + {\rm Li}_2\left(\frac{1-c_+}{2}\right) - {\rm Li}_2\left(\frac{1+c_+}{2}\right) - {\rm Li}_2\left(\frac{1-c_-}{2}\right),
\end{aligned}
$$

where

$$
\begin{gathered}
L = \ln\frac{M}{m}, \quad \rho_\lambda = \ln\frac{mM}{\lambda^2}, \quad \rho_s = \ln\frac{s}{mM}, \\
\rho_{s_1} = \ln\frac{s_1}{mM}, \quad \rho_t = \ln\frac{-t}{mM}, \quad \rho_{t_1} = \ln\frac{-t_1}{mM}, \\
\rho_u = \ln\frac{-u}{mM}, \quad \rho_{u_1} = \ln\frac{-u_1}{mM}, \quad c_\pm = \cos(\mathbf{p}_-\mathbf{q}_\pm), \quad c = \cos(\mathbf{q}_+\mathbf{q}_-),
\end{gathered}
\tag{1.46}
$$

and $\varepsilon, \varepsilon_\pm$ are the energies (in c.m.s.) of electron, muon, and λ is the «photon mass».

Let us now consider RC arising from the Dirac form factor of leptons and vacuum polarization (the Pauli form-factor contribution is suppressed by a factor m^2/s in our kinematics). They are:

$$
\begin{aligned}
\Delta_{ff} + \Delta_{\text{vac}} = \frac{2T_0^e + T_0^{\text{int}}}{T_0}\left(\operatorname{Re}\Gamma\left(\frac{s_1}{M^2}\right) + \operatorname{Re}\Pi(s_1)\right) + \\
+ \frac{2T_0^\mu + T_0^{\text{int}}}{T_0}\left(\operatorname{Re}\Gamma\left(\frac{s}{m^2}\right) + \operatorname{Re}\Pi(s)\right)
\end{aligned}
\tag{1.47}
$$

with

$$
\begin{aligned}
\operatorname{Re}\Gamma\left(\frac{s}{m^2}\right) &= \left(\ln\frac{m}{\lambda} - 1\right)(1 - \rho_s - L) - \frac{1}{4}(\rho_s + L)^2 - \frac{1}{4}(\rho_s + L) + \frac{\pi^2}{3}, \\
\operatorname{Re}\Gamma\left(\frac{s_1}{M^2}\right) &= \left(\ln\frac{M}{\lambda} - 1\right)(1 - \rho_{s_1} + L) - \frac{1}{4}(\rho_{s_1} - L)^2 - \frac{1}{4}(\rho_{s_1} - L) + \frac{\pi^2}{3}, \\
\operatorname{Re}\Pi(s_i) &= \operatorname{Re}\Pi^e(s_j) + \operatorname{Re}\Pi^\mu(s_j) + \operatorname{Re}\Pi^\tau(s_j) + \operatorname{Re}\Pi^h(s_j), \\
\operatorname{Re}\Pi^e(s_j) &= \frac{1}{3}(\rho_{s_j} + L) - \frac{5}{9}, \quad \operatorname{Re}\Pi^\mu(s_j) = \frac{1}{3}(\rho_{s_j} - L) - \frac{5}{9}.
\end{aligned}
$$

Here s_j is the kinematical invariant s or s_1. The contributions from the vacuum polarization from the heavy lepton τ and hadrons Π^τ, Π^h are given in [78].

1.2.2. Calculations of Box-Type RC. Consider now amplitudes arising from box-type FD. There are twelve FD of such a kind, or 48 squared matrix elements. In calculation we restrict ourselves by consideration of only three of box-type FD. Really the total contribution of interference of box-type and Born amplitudes can be expressed in the form:

$$
\operatorname{Re}\Sigma M_{\text{box}} M_0^\star = (1 + P_1)[(1 - P_2)B^e(M_0^e)^* + (1 + P_2)B^e(M_0^\mu)^*], \tag{1.48}
$$

with $M_0^e + M_0^\mu = M_0$, $M_0^e(M_0^\mu)$ being electron (muon) block emission part of the Born matrix element; B^e is the electron emission part of contribution to the box-type amplitude with uncrossed photon legs. Note that calculating the B^e we must consider the pentagon-type FD and two remaining ones.

The substitution operators $P_{1,2}$ work as

$$\begin{aligned} &P_1 f(p_+, p_-; q_+, q_-, k_1) = f(q_+, q_-; p_+, p_-; -k_1), \\ &P_2 f(p_+, p_-; q_+, q_-, k_1) = f(p_+, p_-; q_-, q_+, k_1). \end{aligned} \tag{1.49}$$

The operator P_1 «changes» the photon emission from electron line to muon line. The application of operator P_2 permits one to obtain the contribution from FD with crossed virtual photon lines. As a result, we obtain:

$$\Delta_{\text{box}} = -(\rho_s + \rho_\lambda) \ln \frac{tt_1}{uu_1} + \Delta_B^{\text{NL}}. \tag{1.50}$$

The expression for Δ_B^{NL} is rather cumbersome.

1.2.3. Vertex-Type FD. Let as now consider the contribution arising from FD with vertex-type insertions V^e. The other vertex contributions appear from these ones by using substitutions

$$\operatorname{Re} \Sigma M_{\text{vert}} M_0^\star = (1 + P_1)(1 + P_3) V^e (M_0^e)^*, \tag{1.51}$$

with operator P_3 defined as

$$P_3 f(p_+, p_-; q_+, q_-, k_1) = f(p_-, p_+, q_+, q_-; k_1). \tag{1.52}$$

The total answer for vertex-type contribution reads:

$$\begin{aligned} \Delta_{\text{vert}} = -\frac{1}{2} \frac{T_e + (1/2)T_i}{m_0} [(\rho_s + L)^2 + 2(\rho_s + L)(\rho_\lambda + L) - 3(\rho_s + L) + \Delta_v^{\text{NL}}(s)] - \\ - \frac{1}{2} \frac{T_\mu + (1/2)T_i}{T_0} [(\rho_{s_1} - L)^2 + 2(\rho_{s_1} - L)(\rho_\lambda - L) - 3(\rho_{s_1} - L) + \Delta_v^{\text{NL}}(s_1)]. \end{aligned}$$

1.2.4. Master Formula. Extracting the explicate dependence on vacuum polarization in the form $\dfrac{1}{|1 - \Pi|^2}$ and collecting the leading and nonleading terms arising from soft-photon emission, vertex and box-type FD contributions, as well as lepton form factors, we arrive to the formula:

$$\Delta_{\text{soft}} + \Delta_{\text{box}} + \Delta_{\text{vert}} + \Delta_{ff} = \Delta_{\text{lead}} + \Delta_{\text{NL}}. \tag{1.53}$$

This expression is free from the infrared singularities as well as from squares of large logarithms. The form of Δ_{lead} is consistent with renormalization group prescriptions:

$$\begin{aligned} 1 + \frac{\alpha}{\pi} \Delta_{\text{lead}} = \left(1 + \frac{\alpha}{2\pi} \ln \frac{s}{m_e^2} P_\Delta(\varepsilon)\right)^2 \times \\ \times \left(1 + \frac{\alpha}{2\pi} \ln \frac{s_1}{M^2} P_\Delta(\varepsilon_+)\right) \left(1 + \frac{\alpha}{2\pi} \ln \frac{s_1}{M^2} P_\Delta(\varepsilon_-)\right) + \mathcal{O}(\alpha^2), \end{aligned} \tag{1.54}$$

with P_Δ being the δ part of the kernel of evolution equation:

$$\begin{aligned} P_\Delta(\varepsilon) &= 2\ln\frac{\Delta\varepsilon}{\varepsilon} + \frac{3}{2}, \\ P_\Delta(\varepsilon_\pm) &= 2\ln\frac{\Delta\varepsilon}{\varepsilon_\pm} + \frac{3}{2}. \end{aligned} \tag{1.55}$$

An additional hard-photon emission contribution in leading logarithmical order can be taken into account using the quasi-real electron's method [5]. It results in the replacement P_Δ by the whole kernel of evolution equation of twist-2 operators

$$\begin{gathered} P(z) = P^{(1)}(z) = \lim_{\Delta\to 0}[P_\Delta\delta(1-z) + P_\Theta(z)], \\ P_\Delta = 2\ln\Delta + \frac{3}{2}, \quad P_\Theta(z) = \Theta(1-\Delta-z)\frac{1+z^2}{1-z}. \end{gathered} \tag{1.56}$$

As a result, we arrive to compact form of the cross section:

$$\begin{gathered} \frac{d\sigma^{e^+e^-\to\mu^+\mu^-\gamma}(p_-,p_+,q_-,q_+,k_1)}{d\Gamma} = \int\limits_{x_m}^{1} dx_1 \int\limits_{x_m}^{1} dx_2 \times \\ \times \int\limits_{y_-}^{1}\frac{dz_-}{z_-}\int\limits_{y_+}^{1}\frac{dz_+}{z_+} D_e(x_1,\beta)D_e(x_2,s\beta)D_\mu\left(\frac{y_-}{z_-},\beta_1\right)D_\mu\left(\frac{y_+}{z_+},\beta_1\right)\times \\ \times\frac{(1+\frac{\alpha}{\pi}K)}{|1-\Pi(sx_1x_2)|^2}\frac{d\sigma^{e^+e^-\to\mu^+\mu^-\gamma}(x_1p_-,x_2p_+,Q_-,Q_+,k_1)}{d\Gamma_1}, \\ Q_\pm = \frac{z_\pm}{y_\pm}q_\pm,\ y_\pm = \frac{\varepsilon_\pm}{\varepsilon},\ \beta = \frac{\alpha}{2\pi}\left(\ln\frac{s}{m^2}-1\right),\ \beta_1 = \frac{\alpha}{2\pi}\left(\ln\frac{s_1}{M^2}-1\right) \end{gathered} \tag{1.57}$$

and the structure functions $D(x,s)$ having the standard form:

$$\begin{aligned} D_e(x,\beta) &= \delta(1-x) + P^{(1)}(x)\beta + \ldots, \\ D_\mu(y,\beta_1) &= \delta(1-y) + P^{(1)}(y)\beta_1 + \ldots \end{aligned} \tag{1.58}$$

The phase volumes entering the left and right parts of master equation are different:

$$\begin{gathered} d\Gamma = \frac{d^3q_-}{\varepsilon_-}\frac{d^3q_+}{\varepsilon_+}\frac{d^3k_1}{\omega_1}\delta(p_++p_--q_+-q_--k_1), \\ d\Gamma_1 = \frac{d^3Q_-}{E_-}\frac{d^3Q_+}{E_+}\frac{d^3k_1}{\omega_1}\delta(x_2p_++x_1p_--Q_+-Q_--k_1), \\ E_\pm = \frac{z_\pm}{y_\pm}\varepsilon_\pm. \end{gathered} \tag{1.59}$$

The lower limits of the energy fractions integrations x_m, y_m are determined by the experiment setup conditions. The quantity K (the so-called K factor) collects all the nonleading contributions. It has contributions from virtual-, soft-, and hard-photon emission terms. In Table 1 we give its value for typical experimental points of the considered process keeping all contributions except ones arising from additional hard-photon emission.

Table 1. **Numerical estimation of K factor**

N	y_-	y_+	c_-	c_+	$\Delta_{\rm NL}$
1	0.59	0.66	0.29	–0.06	6.77
2	0.67	0.67	0.50	0.30	3.24
3	0.68	0.65	0.69	–0.50	8.68
4	0.59	0.56	–0.30	–0.30	8.35

Without additional calculations we can obtain, by the analogy with the result given above, the cross section of crossing process — radiative electron–muon scattering:

$$e_-(p_1) + \mu_-(q_1) \to e_-(p_2) + \mu_-(q_2) + \gamma(k_1) + (\gamma). \tag{1.60}$$

It can be constructed in complete analogy with the Drell–Yan form of cross section of above considered process $e_+e_- \to \mu_+\mu_-\gamma$, using in the right-hand side as a hard subprocess the Born cross section:

$$\frac{d\sigma_B^{e\mu\gamma}(p_1,q_1;p_2,q_2,k_1)}{d\Gamma_{e\mu\gamma}} = \\ = \frac{\alpha^3}{16\pi^2(p_1q_1)}\frac{(p_1q_2)^2+(p_1q_1)^2+(p_2q_1)^2+(p_2q_2)^2}{(p_1p_2)(q_1q_2)}W, \tag{1.61}$$

with

$$\begin{aligned} d\Gamma_{e\mu\gamma} &= \frac{d^3q_2}{q_{20}}\frac{d^3p_2}{p_{20}}\frac{d^3k_1}{\omega_1}\delta^4(p_1+q_1-p_2-q_2-k_1), \\ W &= -\left(\frac{p_1}{p_1k_1}+\frac{q_1}{q_1k_1}-\frac{p_2}{p_2k_1}-\frac{q_2}{q_2k_1}\right)^2. \end{aligned} \tag{1.62}$$

It is worth to note that the value of K factor for the last process is not known.

All the used 1-loop integrals of scalar, vector, and tensor types are presented in Subsec. 3.1.

1.3. Target Normal Spin Asymmetry and Charge Asymmetry for $e\mu$ Elastic Scattering and the Crossed Processes. We give here an accurate description of the process $e\bar{\mu} \to e\bar{\mu}(\gamma)$; $e\bar{e} \to \mu\bar{\mu}(\gamma)$ in the framework of QED, in order to provide a basis for the comparison with experimental data [14]. High precision experiments on the processes $e\bar{e} \to \tau\bar{\tau}$ and $e\bar{e} \to p\bar{p}$ are planned in future $c - \tau$ facilities. Moreover, the possibility of colliding $e\mu$ beam facilities has been discussed in the framework of programs on verification of SM prediction.

The obtained results can also be applied, as a realistic model, to electron (positron) scattering on a point-like hadron (proton).

It is known that in Born approximation the differential cross section of elastic proton–electron scattering

$$e(p_1) + p(p) \to e(p_1') + p(p') \tag{1.63}$$

can be expressed in terms of two-proton form factors, $F_{1,2}(q^2)$, which are functions of a single argument, the momentum transfer squared, $q^2 = t$.

Taking into account two (and more) photon exchanges leads to a generalization of the Born picture, namely the amplitude of ep scattering depends on two Mandelstam variables, the total energy s and t. The virtual photon Compton scattering amplitude is a rather complex object, which can be expressed in terms of 12 chiral amplitudes. Nevertheless, taking into account parity conservation and omitting the terms of order m_e/m_μ (which are responsible for chirality violation), we can reduce the number of relevant amplitudes to three [33]:

$$\begin{aligned} M^{(2)} &= \frac{i\alpha^2}{t}\bar{u}(p_1')\gamma_\mu u(p_1)\times \\ &\times \bar{u}(p')\left[F_1(s,t)\gamma_\mu - \frac{F_2(s,t)}{2M}\gamma_\mu\hat{q} + \frac{1}{t}F_3(s,t)(\hat{p}_1 + \hat{p}_1')(p+p')_\mu\right]u(p), \end{aligned} \tag{1.64}$$

with $q = p_1 - p_1'$, $s = 2p_1 p$.

The explicit calculation, given in this Subsection, permits one to extract the individual contributions F_1, F_2, and F_3, in the framework of QED. The infrared divergency is cancelled when the relevant soft-photon emission is correctly taken into account.

Charge-odd and backward–forward asymmetries appear naturally from the interference of one- and two-photon exchange amplitudes in the framework of QED and SM due to Z_0-boson exchange in the Born approximation. But at the energy range reachable at $c - \tau$ factories, the relevant contribution of SM type is [32]:

$$\frac{d\sigma_Z^{\mathrm{odd}}}{d\sigma_{\mathrm{QED}}} \approx \frac{s}{M_Z^2}a_v a_a \approx 5\cdot 10^{-5}, \quad 3 < \sqrt{s} < 5 \text{ GeV}, \tag{1.65}$$

which is quite small compared to QED effects.

The accuracy of results given below is determined by

$$\mathcal{O}\left(\frac{m_e^2}{m_\mu^2}, \frac{m_e^2}{m_\tau^2}, \frac{m_e^2}{m_p^2}\right) \sim 0.1\% \tag{1.66}$$

and the contribution of higher orders of QED $\alpha/\pi \approx 0.5\%$. Moreover, we assume that all the velocities of the final heavy particles are finite in the annihilation as well as in the scattering channels. This is the reason why Coulomb factors are neglected.

1.3.1. Process $e^+ + e^- \to \mu^+ + \mu^-(\gamma)$ *TPE Amplitude.* At first, we consider the process of creation of $\mu^+\mu^-$ pairs in electron–positron annihilation:

$$e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-). \tag{1.67}$$

The cross section in the Born approximation can be written as

$$\frac{d\sigma_B}{dO_{\mu_-}} = \frac{\alpha^2}{4s}\beta(2 - \beta^2 + \beta^2 c^2), \tag{1.68}$$

with $s = (p_+ + p_-)^2 = 4E^2$, $\beta^2 = 1 - 4m^2/s$, E is the electron beam energy in the center-of-mass reference frame (implied for this process below); m, m_e are the masses of muon and electron; $c = \cos\theta$, and θ is the angle of μ_--meson emission to the electron beam direction.

The interference of the Born amplitude

$$M_B = \frac{i4\pi\alpha}{s}\bar{v}(p_+)\gamma_\mu u(p_-)\bar{u}(q_-)\gamma_\mu v(q_+),$$

with the TPE amplitude M_{box} results in parity violating contributions to the differential cross section, i.e., the ones, changing the sign at $\theta \to \pi - \theta$. As a consequence of charge-odd correlations we can construct:

$$A(\theta, \Delta E) = \frac{d\sigma(\theta) - d\sigma(\pi - \theta)}{d\sigma_B(\theta)}. \tag{1.69}$$

Here we take into account as well the emission of an additional soft real photon with energy not exceeding some small value ΔE, so that $A(\theta, \Delta E)$ is free from the infrared singularities.

There are two TPE Feynman amplitudes (Fig. 1). We calculate only one of them, the uncrossed diagram with matrix element

$$M_a = i\alpha^2 \int \frac{d^4k}{i\pi^2}\frac{\bar{u}(q_-)Tv(q_+) \times \bar{v}(p_+)Zu(p_-)}{(\Delta)(Q)(P_+)(P_-)},$$
$$(\Delta) = (k - \Delta)^2 - m_e^2, \quad (Q) = (k - Q)^2 - m^2, \quad (P_\pm) = (k \mp P)^2 - \lambda^2, \tag{1.70}$$

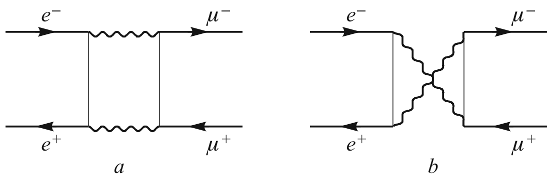

Fig. 1. Feynman diagrams for two-photon exchange in $e\bar{e} \to \mu\bar{\mu}$ process: box diagram (*a*) and crossed box diagram (*b*)

with λ-«photon» mass and

$$\begin{gathered} T = \gamma_\alpha(\hat{k} - \hat{Q} + m)\gamma_\beta, \quad Z = \gamma_\beta(\hat{k} - \hat{\Delta})\gamma_\alpha, \\ \Delta = \frac{1}{2}(p_+ - p_-), \quad Q = \frac{1}{2}(q_+ - q_-), \quad P = \frac{1}{2}(p_+ + p_-). \end{gathered} \tag{1.71}$$

We will assume

$$m^2 = \frac{s}{4}(1 - \beta^2) \sim s \sim -t \sim -u. \tag{1.72}$$

The explicit form of kinematical variables used below is:

$$\begin{gathered} \Delta^2 = -P^2 = -\frac{s}{4}, \quad Q^2 = -\frac{1}{4}s\beta^2, \quad \sigma = \Delta Q = \frac{1}{4}(u - t), \\ u = (p_- - q_+)^2 = -\frac{s}{4}(1 + \beta^2 + 2\beta c), \quad t = (p_- - q_-)^2 = -\frac{s}{4}(1 + \beta^2 - 2\beta c). \end{gathered} \tag{1.73}$$

The contribution to the cross section of the amplitude arising from the crossed Feynman diagram (Fig. 1, b), M_b, can be obtained from M_a by the crossing relation

$$\frac{d\sigma_a(s,t)}{d\Omega_\mu} = -\frac{d\sigma_b(s,u)}{d\Omega_\mu}, \tag{1.74}$$

which has the form

$$\frac{d\sigma_a(s,t)}{d\Omega_\mu} = \frac{\beta\alpha^3}{2\pi s^2}\,\mathrm{Re}\,[R(s,t)], \tag{1.75}$$

with

$$R(s,t) = \int \frac{d^4k}{i\pi^2}\frac{1}{(\Delta)(Q)(P_+)(P_-)}\frac{1}{4}\,\mathrm{Tr}\,[(\hat{q}_- + m)T(\hat{q}_+ - m)\,\gamma_\mu] \times \\ \times \frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_+ Z \hat{p}_- \gamma_\mu)\,. \tag{1.76}$$

The scalar, vector, and tensor loop momentum integrals are defined as

$$J; J_\mu; J_{\mu\nu} = \int \frac{d^4k}{i\pi^2}\frac{1; k_\mu; k_\mu k_\nu}{(\Delta)(Q)(P_+)(P_-)}. \tag{1.77}$$

Using symmetry properties, the vector and tensor integrals can be written as

$$J_\mu = J_\Delta \cdot \Delta_\mu + J_Q \cdot Q_\mu, \tag{1.78}$$

$$J_{\mu\nu} = K_0 g_{\mu\nu} + K_P P_\mu P_\nu + K_Q Q^\mu Q^\nu + K_\Delta \Delta_\mu \Delta_\nu + K_x (Q_\mu \Delta_\nu + Q_\nu \Delta_\mu). \tag{1.79}$$

The quantity $R(s,t)$ can be expressed as a function of polynomials P_i as

$$R = P_1 J + P_2 J_\Delta + P_3 J_Q + P_4 K_0 + P_5 K_\Delta + P_6 K_Q + P_7 K_P + P_8 K_x, \tag{1.80}$$

where the explicit form of polynomials is given in [29]. Using the explicit expression for the coefficients $J_\Delta, \ldots, K_x$ (see Subsec. 3.1) we obtain

$$\begin{aligned} R(s,t) &= 4(\sigma - \Delta^2)(2\sigma - m^2)F + 16(\sigma - \Delta^2)(\sigma^2 + (\Delta^2)^2 - m^2\Delta^2)J + \\ &+ 4[(\Delta^2)^2 - 3\Delta^2\sigma + 2\sigma^2 - m^2\sigma]F_Q + 4[2(\Delta^2)^2 - 2\Delta^2\sigma + 2\sigma^2 - m^2\Delta^2]F_\Delta + \\ &+ 4[(\Delta^2)^2 + \Delta^2\sigma + m^2\Delta^2]G_Q + 4[-(\Delta^2)^2 + \sigma^2 - 2m^2\Delta^2]H_Q, \end{aligned} \tag{1.81}$$

with the quantities $F - H_Q$ given in Subsec. 3.1. Finally the charge-odd part of differential cross section has the form

$$\left(\frac{d\sigma^{e\bar{e}}_{\text{virt}}(s,t)}{d\Omega_\mu}\right)_{\text{odd}} = -\frac{\alpha^3\beta}{2\pi s}\mathcal{D}^{\text{ann}},$$

$$\mathcal{D}^{\text{ann}} = \frac{1}{s}[R(s,t) - R(s,u)] = (2 - \beta^2 + \beta^2c^2)\ln\left(\frac{1+\beta c}{1-\beta c}\right)\ln\frac{s}{\lambda^2} + \mathcal{D}^{\text{ann}}_V \tag{1.82}$$

and

$$\begin{aligned} \mathcal{D}^{\text{ann}}_V &= \left(1 - 2\beta^2 + \beta^2c^2\right)\left[\frac{1}{1+\beta^2+2\beta c}\left(\ln\frac{1+\beta c}{2} + \ln\frac{s}{m^2}\right) - \right. \\ &\left. - \frac{1}{1+\beta^2-2\beta c}\left(\ln\frac{1-\beta c}{2} + \ln\frac{s}{m^2}\right)\right] + \\ &+ \beta c\left[\varphi(\beta)\left(\frac{1}{2\beta^2} - 1 - \frac{\beta^2}{2}\right) - \frac{1}{\beta^2}\ln\frac{s}{m^2} - \frac{\pi^2}{6} + \frac{1}{2}\ln^2\frac{s}{m^2} - \right. \\ &- \frac{1}{2}\ln^2\frac{1-\beta c}{2} - \frac{1}{2}\ln^2\frac{1+\beta c}{2} + \text{Li}_2\left(\frac{1+\beta^2+2\beta c}{2(1+\beta c)}\right) + \\ &\left. + \text{Li}_2\left(\frac{1+\beta^2-2\beta c}{2(1-\beta c)}\right)\right] - \frac{m^2}{s}\left[\ln^2\frac{1-\beta c}{2} - \ln^2\frac{1+\beta c}{2} + \right. \\ &\left. + 2\text{Li}_2\left(\frac{1+\beta^2+2\beta c}{2(1+\beta c)}\right) - 2\text{Li}_2\left(\frac{1+\beta^2-2\beta c}{2(1-\beta c)}\right)\right], \end{aligned} \tag{1.83}$$

where $\varphi(\beta) = sF_Q$, F_Q is given in Subsec. 3.1. The quantity $\mathcal{D}^{\text{ann}} - \mathcal{D}^{\text{ann}}_V$ suffers from infrared divergences, which will be compensated taking into account the soft-photon contribution (see below).

1.3.2. Scattering Channel. Let us consider now the elastic electron muon scattering

$$e(p_1) + \mu(p) \to e(p_1') + \mu(p')$$

which is the crossed process of (1.67). The Born cross section is the same for the scattering of electrons and positrons on the same target. Taking the experimental data from the scattering of electron and positron on the same target (muon or proton), one can measure the difference of the corresponding cross sections which is sensitive to the interference of the one- and two-photon exchange amplitudes. For the case of proton target, in the Laboratory (Lab) frame, the differential cross section as a function of the energy of the initial electron, E, and of the electron scattering angle, θ_e, was derived in [23]:

$$\frac{d\sigma^{ep}}{d\Omega} = \frac{\alpha^2}{4E^2}\frac{\cos^2\theta_e/2}{\sin^4\theta_e/2}\frac{1}{\rho}\left[\frac{F_E^2+\tau F_M^2}{1+\tau} + 2\tau F_M^2 \tan^2\frac{\theta_e}{2}\right],$$
$$\rho = 1 + \frac{2E}{m}\sin^2\frac{\theta_e}{2}, \quad \tau = \frac{-t}{4m^2} = \frac{E^2}{m^2\rho}\sin^2\frac{\theta_e}{2}, \tag{1.84}$$

and it is known as the Rosenbluth formula. The Sachs electric and magnetic proton form factors, F_E and F_M, are related to the Pauli and Dirac form factors by $F_E = F_1 - \tau F_2$, $F_M = F_1 + F_2$. For the scattering on muon, one replaces $F_1 = 1$, $F_2 = 0$ and Eq. (1.84) becomes

$$\frac{d\sigma_B^{e\mu}}{d\Omega} = \frac{\alpha^2(s^2+u^2+2tm^2)}{2m^2\rho^2 t^2},$$
$$s = 2p_1 p = 2mE, \quad t = -2p_1 p_1', \quad u = -2pp_1' = -\frac{s}{\rho}. \tag{1.85}$$

The charge-odd contribution to the cross section of $e\mu$-elastic scattering is:

$$\left(\frac{d\sigma_{\rm virt}^{e\mu}}{d\Omega_e}\right)_{\rm odd} = -\frac{\alpha^3}{2\pi m^2\rho^2}\,\mathrm{Re}\,\mathcal{D}^{\rm sc},$$
$$\mathcal{D}^{\rm sc} = \frac{1}{t}[\mathcal{D}(s,t) - \mathcal{D}(u,t)] = \frac{2}{t^2}[s^2+u^2+2tm^2]\ln\frac{-u}{s}\ln\frac{-t}{\lambda^2} + \mathcal{D}_{\rm virt}^{\rm sc}, \tag{1.86}$$

with

$$\mathcal{D}_{\rm virt}^{\rm sc} = \frac{s-u}{t}\left[\frac{1}{2}\ln^2\left(\frac{-t}{m^2}\right) - \frac{\tau}{1+\tau}\ln\left(\frac{-t}{m^2}\right) + \mathcal{Z}\left(6\tau + 2 - \frac{2\tau^2}{1+\tau}\right)\right] +$$
$$+\frac{s}{t}\left[-\ln^2\frac{s}{-t} + \pi^2 + 2\mathrm{Li}_2\left(1+\frac{m^2}{s}\right)\right] - \frac{u}{t}\left[-\ln^2\frac{u}{t} + 2\mathrm{Li}_2\left(1+\frac{m^2}{u}\right)\right] +$$

$$+\frac{(1-2\tau)}{(-4\tau)}\left[2\ln\left(\frac{s}{-u}\right)\ln\left(\frac{-t}{m^2}\right)+\ln^2\left(\frac{-u}{m^2}\right)-\ln^2\left(\frac{s}{m^2}\right)+\pi^2+\right.$$
$$\left.+2\mathrm{Li}_2\left(1+\frac{m^2}{s}\right)-2\mathrm{Li}_2\left(1+\frac{m^2}{u}\right)\right]+\left(2m^2-\frac{su}{t}\right)\times$$
$$\times\left[\frac{\ln\, s/m^2}{m^2+s}-\frac{\ln\, -u/m^2}{m^2+u}\right] \tag{1.87}$$

and

$$\mathcal{Z}=-\frac{1}{4\sqrt{\tau(1+\tau)}}\left[\pi^2+\ln(4\tau)\ln x+\mathrm{Li}_2{-2}\sqrt{\tau x}-\mathrm{Li}_2\frac{2\sqrt{\tau}}{\sqrt{x}}\right], \tag{1.88}$$
$$x=\frac{\sqrt{1+\tau}+\sqrt{\tau}}{\sqrt{1+\tau}-\sqrt{\tau}}.$$

1.3.3. Soft-Photon Emission in the Scattering Channel. Charge Asymmetry. In this Subsection the emission of soft real photons in the Lab reference frame for $e\mu$ scattering is calculated. Following the recept of 't Hooft and Weltman, [21], see also [34], we obtain for the odd part of the cross section

$$\frac{d\sigma_{e\mu}^{\mathrm{soft}}}{d\Omega}=-\frac{\alpha^3}{\pi}\frac{(s^2+u^2+2tm^2)}{2m^2\rho^2t^2}\left\{2\ln\rho\ln\left[\frac{(2\rho\Delta E)^2}{\lambda^2 x}\right]+\mathcal{D}_{\mathrm{soft}}^{\mathrm{sc}}\right\}, \tag{1.89}$$
$$\mathcal{D}_{\mathrm{soft}}^{\mathrm{sc}}=-2\mathrm{Li}_2\left(1-\frac{1}{\rho x}\right)+2\mathrm{Li}_2\left(1-\frac{\rho}{x}\right),\quad \rho=\frac{s}{-u}.$$

The sum $(d\sigma_{e\mu}^{\mathrm{virt}}+d\sigma_{e\mu}^{\mathrm{soft}})_{\mathrm{odd}}$ has the form

$$\left(\frac{d\sigma_{e\mu}^{\mathrm{virt}}}{d\Omega_e}+\frac{d\sigma_{e\mu}^{\mathrm{soft}}}{d\Omega_e}\right)_{\mathrm{odd}}=\frac{\alpha^3}{2\pi m^2\rho^2}\frac{(s^2+u^2+2tm^2)}{t^2}\left[-2\ln\rho\ln\frac{(2\rho\Delta E)^2}{-tx}+\Xi\right], \tag{1.90}$$
$$\Xi=\mathrm{Re}\left[-\frac{t^2\mathcal{D}_{\mathrm{virt}}^{\mathrm{sc}}}{s^2+u^2+2tm^2}-\mathcal{D}_{\mathrm{soft}}^{\mathrm{sc}}\right]$$

and it is independent of the photon mass λ.

The function Ξ is shown in Fig. 2 as a function of $\cos\theta_e$ for given E/m.

The ratio between the difference and the sum (corresponding to the Born cross section) of the cross sections for $e^{\pm}\mu$ scattering is:

$$\frac{d\sigma^{e^-\mu\to e^-\mu(\gamma)}-d\sigma^{e^+\mu\to e^+\mu(\gamma)}}{d\sigma^{e^+\mu\to e^+\mu(\gamma)}+d\sigma^{e^-\mu\to e^-\mu(\gamma)}}=\frac{\alpha}{\pi}\left[\Xi-2\,\ln\,\rho\ln\frac{(2\rho\Delta E)^2}{-tx}\right]. \tag{1.91}$$

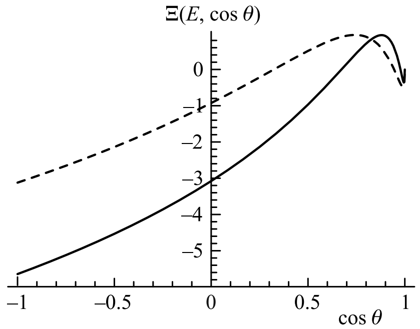

Fig. 2. $\Xi(s, \cos\theta)$ for $E = 5m$ (dashed line) and $E = 10m$ (solid line), m is muon mass

1.3.4. Soft-Photon Emission in Annihilation Channel. Charge Asymmetry. The odd contributions to the differential cross section for the process $e^+ + e^- \to \mu^+ + \mu^-$, due to soft-photon emission, has the form:

$$(d\sigma_{\rm soft}^{e^+e^-\to\mu^+\mu^-(\gamma)})_{\rm odd} =$$
$$= d\sigma_0 \left(-\frac{\alpha}{2\pi^2}\right) \int \frac{d^3k}{\omega} \left(-\frac{p_-}{p_-k} + \frac{p_+}{p_+k}\right) \left(\frac{q_+}{q_+k} - \frac{q_-}{q_-k}\right)_{S_0,\omega<\Delta\varepsilon}. \quad (1.92)$$

Again, the integration must be performed in the special frame S^0, where $\bar{p}_+ + \bar{p}_- - \bar{q}_+ = \bar{q}_- + \bar{k} = 0$. In this frame we have

$$(q_- + k)^2 - m^2 = 2(E_- + \omega)\omega \approx 2m\omega =$$
$$= (p_+ + p_- - q_+)^2 - m^2 = 4E(E - \varepsilon_+), \quad (1.93)$$
$$E - \varepsilon_+ = \frac{m}{2E}\Delta\varepsilon.$$

In the elastic case $E - \varepsilon_+^{\rm el} = 0$, and the photon energy in the Lab system is

$$\Delta E = \varepsilon_+^{\rm el} - \varepsilon_+ = \frac{m}{2E}\Delta\varepsilon. \quad (1.94)$$

The 't Hooft–Veltman procedure for soft-photon emission contribution leads to

$$\frac{d\sigma_{\rm ann}^{\rm soft}}{d\Omega} = \frac{d\sigma_0}{d\Omega}\frac{2\alpha}{\pi}\left[\ln\left(\frac{4E\Delta E}{m\lambda}\right)^2 \ln\frac{1+\beta c}{1-\beta c} + \mathcal{D}_S^{\rm ann}\right] \quad (1.95)$$

with

$$\mathcal{D}_s^{\rm ann} = \frac{1}{2}{\rm Li}_2\left(\frac{-2\beta(1+c)}{(1-\beta)(1-\beta c)}\right) + \frac{1}{2}{\rm Li}_2\left(\frac{2\beta(1-c)}{(1+\beta)(1-\beta c)}\right) - \\ - \frac{1}{2}{\rm Li}_2\left(\frac{-2\beta(1-c)}{(1-\beta)(1+\beta c)}\right) - \frac{1}{2}{\rm Li}_2\left(\frac{2\beta(1+c)}{(1+\beta)(1+\beta c)}\right). \quad (1.96)$$

The total contribution (virtual and soft) is free from infrared singularities and has the form

$$\frac{d\sigma_{\rm ann}}{d\Omega} = \frac{\alpha^3\beta}{2\pi s}(2-\beta^2+\beta^2c^2)\Upsilon, \quad \Upsilon = 2\ln\frac{1+\beta c}{1-\beta c}\ln\left(\frac{2\Delta E}{m}\right) + \Phi(s,\cos\theta), \\ \Phi(s,\cos\theta) = \mathcal{D}_S^{\rm ann} - \frac{\mathcal{D}_V^{\rm ann}}{2-\beta^2+\beta^2c^2}, \quad (1.97)$$

with $\mathcal{D}_V^{\rm ann}$ given in 1.83. The quantity $\Phi(s, \cos\,\theta)$ is presented in Fig. 3.

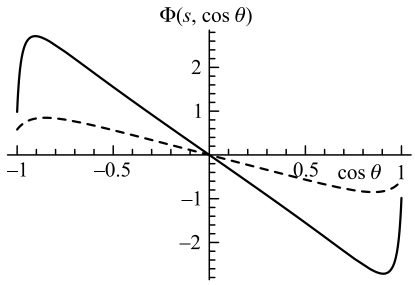

Fig. 3. $\Phi(s, \cos\,\theta)$, for $s = 10m^2$ (dashed line) and $s = 20m^2$ (solid line), m is muon mass

The relevant asymmetry can be constructed from (1.69)

$$A = \frac{4\alpha}{\pi}\Upsilon. \quad (1.98)$$

1.3.5. Crossing Symmetry. In this Subsection we formally consider the relations between the kinematical variables in the scattering and in the annihilation channel, $e^+ + e^- \to p + \bar{p}$. The reduced form of the differential elastic ep-scattering cross section, commonly used, is defined as $\sigma_{\rm red} = \tau F_M^2 + \varepsilon F_E^2$, and it is related to the differential cross section by

$$\frac{d\sigma}{d\Omega} = \sigma_M\sigma_{\rm red}, \quad \sigma_M = \frac{\alpha^2\cos^2\theta_e/2}{4E^2\sin^4\theta_e/2}\frac{1}{\rho\varepsilon(1+\tau)}, \\ \varepsilon = \frac{1}{1+2(1+\tau)\,\tan^2\theta_e/2}, \quad (1.99)$$

where ε is the transverse (linear) polarization of a virtual photon and varies from $\varepsilon = 0$, for $\theta_e = \pi$, to $\varepsilon = 1$, for $\theta_e = 0$.

The crossing relation between the scattering channel $e + p \to e + p$ and the annihilation channel $e^+ + e^- \to p + \bar{p}$ consists in replacing the variables of the scattering channel $s = 2p_1 p = 2Em$ and $Q^2 = -t$ according to

$$m^2 + s \to t = -2E^2(1 - \beta c), \quad Q^2 \to -s = -4E^2, \quad c = \cos\theta, \qquad (1.100)$$

where θ is the angle of the antiproton with respect to the incident electron, in the c.m.s.

The following relation holds for the annihilation channel:

$$\cos^2\theta = \frac{(t-u)^2}{s(s-4M^2)}, \quad s + t + u = 2M^2. \qquad (1.101)$$

On the other hand, in the scattering channel, one has:

$$\frac{1+\varepsilon}{1-\varepsilon} = \frac{\cot^2\theta_e/2}{1+\tau} + 1 = \frac{(s-u)^2}{Q^2(Q^2+4M^2)}, \quad Q^2 = s + u. \qquad (1.102)$$

Therefore one proves the validity of the crossing relation:

$$\cos\theta = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \equiv y, \qquad (1.103)$$

based on the analytical continuation from the annihilation channel to the scattering one. This relation was derived in [36]. Using this relation and the property of the 2γ contribution to the annihilation cross section $\left(\frac{d\sigma}{d\Omega}(\theta)\right)_{2\gamma} = -\left(\frac{d\sigma}{d\Omega}(\pi - \theta)\right)_{2\gamma}$, i.e., $\left(\frac{d\sigma}{d\Omega}(\theta)\right)_{2\gamma} = \cos\theta f(\cos^2\theta, s)$, we have for 2γ contribution to ep-elastic scattering

$$\frac{d\Delta\sigma}{d\Omega_e}(e^- p \to e^- p) = y f(y^2, Q^2),$$
$$f(y^2, Q^2) = c_0(Q^2) + y^2 c_1(Q^2) + y^4 c_2(Q^2) + \ldots \qquad (1.104)$$

This property follows from the change of the sign of the contribution for virtual and real photon emission when the $(s \leftrightarrow u)$ transformation is applied (see Eqs. (1.87), (1.89), (1.90) and relation (1.102)).

This form of the contribution of the interference of Born and TPE amplitudes to the differential cross section derives explicitly from C invariance and crossing symmetry of electromagnetic interactions and excludes any linear function of ε for a possible parameterization of such contribution.

Let us note that not only the elastic channel must be taken into account: the interference of the amplitudes corresponding to the emission of a photon by electron and by proton must be considered, too.

Evidently, the relations derived above are valid for the considered processes with electrons and heavy lepton setting, respectively, $F_E = F_M = 1$.

1.3.6. Derivation of the Additional Structure: Annihilation Channel. Let us start from the following form of the matrix element for the process $e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-)$ in the presence of 2γ exchange (see (1.64) [33]):

$$M_2 = \frac{i\alpha^2}{s}\bar{v}(p_+)\gamma_\mu u(p_-)\times \\ \times \bar{u}(q_-)\left(G_1\gamma_\mu - \frac{G_2}{m}\gamma_\mu\hat{P} + 4\frac{1}{s}G_3\hat{\Delta}Q_\mu\right)v(q_+), \quad (1.105)$$

where the amplitudes G_i are complex functions of the two kinematical variables s and t.

To calculate the structure G_3 from the 2γ amplitude (see Eq. (1.70)), diagrams (Figs. 1, a and b) must be taken into account. Only one of them can be calculated explicitly (the uncrossed one), whereas the other, b, can be obtained from uncrossed one by appropriate replacements.

To extract the structure G_3 we multiply Eq. (1.105) subsequently by

$$\begin{aligned} &\bar{u}(p_-)\gamma_\lambda v(p_+) \times \bar{v}(q_+)\gamma_\lambda u(q_-), \\ &\bar{u}(p_-)\hat{Q}v(p_+) \times \bar{v}(q_+)u(q_-), \\ &\bar{u}(p_-)\hat{Q}v(p_+) \times \bar{v}(q_+)\hat{\Delta}u(q_-), \end{aligned} \quad (1.106)$$

and perform the summation on fermions spin states.

Solving the algebraical set of equations we find

$$\begin{aligned} G_1^a &= \frac{1}{\beta^4\sin^4\theta}\left\{(8B^a + A^a\beta^2\sin^2\theta)(1-\beta^2\cos^2\theta) - \right. \\ &\qquad \left. -4C^a\beta\cos\theta\left[2-\beta^2(1+\cos^2\theta)\right]\right\}, \\ G_2^a &= \frac{1}{\beta^4\sin^4\theta}\left\{\beta(1-\beta^2)(A^a\beta\sin^2\theta - 8C^a\cos\theta) + \right. \\ &\qquad \left. +4B^a\left[2-\beta^2(1+\cos^2\theta)\right]\right\}, \\ G_3^a &= \frac{1}{\beta^3\sin^4\theta}\left[-A^a\beta^2\sin^2\theta\cos\theta - 8B^a\cos\theta + 4\beta C^a(1+\cos^2\theta)\right], \end{aligned} \quad (1.107)$$

with

$$A^a = \int \frac{d^4k}{i\pi^2} \frac{1}{(\Delta)(Q)(P_+)(P_-)} \frac{1}{s} \operatorname{Tr}(\hat{p}_+ Z \hat{p}_- \gamma_\lambda) \frac{1}{4} \operatorname{Tr}[(q_- + m)T(\hat{q}_+ - m)\gamma_\lambda],$$
$$B^a = \int \frac{d^4k}{i\pi^2} \frac{1}{(\Delta)(Q)(P_+)(P_-)} \frac{m}{s^2} \operatorname{Tr}(\hat{p}_+ Z \hat{p}_- \hat{Q}) \frac{1}{4} \operatorname{Tr}[(\hat{q}_- + m)T(\hat{q}_+ - m)], \quad (1.108)$$
$$C^a = \int \frac{d^4k}{i\pi^2} \frac{1}{(\Delta)(Q)(P_+)(P_-)} \frac{1}{s^2} \operatorname{Tr}(\hat{p}_+ Z \hat{p}_- \hat{Q}) \frac{1}{4} \operatorname{Tr}[(\hat{q}_- + m)T(\hat{q}_+ - m)\hat{\Delta}].$$

The explicit value for G_3^a is:

$$\begin{aligned} G_3^a = \frac{2s}{\beta^3(1-c^2)^2} \Bigg\{ & \frac{1}{2} G_Q (1-c^2)\beta^3(1-\beta c) + \\ & + \frac{1}{2} H_Q \beta^2 (1-c^2)[c(-3+5\beta^2) - \beta - \beta c^2] + \\ & + F_\Delta c[1 - 4\beta^2 + 2\beta^4 + c^2\beta^2(3-4\beta^2) - 2\beta c(1-2\beta^2)] + \\ & + F_Q \beta \left[-c^2 + \beta c \left(-\frac{1}{2} - 4\beta^2 c^2 + \frac{5}{2} c^2 \right) + \beta^2 \left(-\frac{1}{2} + 2\beta^2 c^2 + \frac{3}{2} c^2 \right) \right] - \\ & - 2Js\beta^2 c(1-c^2)(1-\beta^2)(1-\beta c) + \\ & + Fc\left[1 + \beta^2 c^2 - 2\beta^4 - 4\beta^4 c^2 + \beta c(-3 + 4\beta^2 + 2\beta^4 + \beta^2 c^2)\right] \Bigg\}. \end{aligned} \quad (1.109)$$

The contributions from the crossed Feynman diagram can be obtained from Eq. (1.109) by

$$(A^b, B^b, C^b)_{\text{cros}} = -[A^a, B^a, C^a(\cos\theta \to -\cos\theta)]_{\text{uncros}}. \quad (1.110)$$

The parameterization (1.105) for the contribution to the matrix element arising from box-type diagrams in terms of three additional functions $G_i(s,t)$, $i = 1, 2, 3$, suffers from infrared divergencies, which can be eliminated by taking into account soft-photon emission expressed in terms of structures G_1, G_2, G_3. This procedure results in replacing $\ln(m/\lambda)$ with $\ln \Delta$.

1.3.7. One-Spin Asymmetry Caused by Proton Polarization. Let us consider now the process of electron interaction with a heavy lepton (point-like proton). For clearness, the expressions are written for the proton case. For the interaction of electrons with heavy leptons, μ or τ, one should take $G_E = G_M = 1$ and use the relevant mass replacement.

The target spin asymmetry for heavy-fermion production process $e^+(p_+) + e^-(p_-) \to p(q_+) + \bar{p}(q_-)$ (in c.m.s. frame) is defined as

$$\frac{d\sigma^\uparrow - d\sigma^\downarrow}{d\sigma^\uparrow + d\sigma^\downarrow} = (\mathbf{an})R_n, \tag{1.111}$$

where $\mathbf{a}$ is the proton polarization vector; $\mathbf{n} = (\mathbf{q}_- \times \mathbf{p}_-)/\left|\mathbf{q}_- \times \mathbf{p}_-\right|$ is the unit vector normal to the scattering plane; $d\sigma^\uparrow$ is the cross section of processes with proton polarization vector $\mathbf{a}$, $d\sigma^\downarrow$ is the cross section of processes with proton polarization vector $-\mathbf{a}$. Thus the denominator in the left-hand side of Eq. (1.111) is the unpolarized cross section of process $e^+e^- \to p + \bar{p}$ which is well known [38] (for the case of proton–antiproton creation):

$$\frac{d\sigma^{e^+e^-\to p\bar{p}}}{d\Omega} = \frac{\alpha^2\beta}{4s}\left[\left(1+\cos^2\theta\right)\left|G_M\right|^2 + \left(1-\beta^2\right)\left|G_E\right|^2\sin^2\theta\right], \tag{1.112}$$

where $\beta = \sqrt{1 - 4M^2/s}$ is the velocity of proton in c.m. frame; s is the total energy square, and θ is the angle between vectors $\mathbf{q}_-$ and $\mathbf{p}_-$.

The difference of cross sections in (1.111) is originated by the s-channel discontinuity of interference of the Born amplitude with TPE amplitude

$$d\sigma^\uparrow - d\sigma^\downarrow \sim \mathrm{Re}\sum(A_{\mathrm{el}}^+ \cdot A_{\mathrm{TPE}} + A_{\mathrm{el}} \cdot A_{\mathrm{TPE}}^+). \tag{1.113}$$

Using the density matrix of final proton $u(p)\bar{u}(p) = (\hat{p} + M)(1 - \gamma_5\hat{a})$ one gets

$$\mathrm{Re}\sum\left(A_{\mathrm{el}}^+ \cdot A_{\mathrm{TPE}} + A_{\mathrm{el}} \cdot A_{\mathrm{TPE}}^+\right) = 32\frac{(4\pi\alpha)^3\,(2\pi i)^2}{s\pi^2}\,\mathrm{Re}\,(Y),$$

$$Y = \int\frac{dk}{i\pi^2}\frac{1}{(\Delta)(Q)(+)(-)} \times \frac{1}{4}\,\mathrm{Tr}\left[\hat{p}_1\gamma^\alpha\hat{p}_1'\gamma^\mu(\hat{k} - \hat{\Delta})\gamma^\nu\right] \times$$
$$\times\frac{1}{4}\,\mathrm{Tr}\left[(\hat{p} - M)(-\gamma_5\hat{a})\gamma_\alpha(\hat{p}' + M)\gamma_\nu(\hat{k} - \hat{Q} + M)\gamma_\mu\right]. \tag{1.114}$$

Performing the loop-momenta integration, the right-hand side of Eq. (1.114) can be expressed in terms of basic integrals (see Subsec. 3.1)

$$\mathrm{Re}\,(Y) = 4M(a, \Delta, Q, P)\,\mathrm{Im}\,(F_Q - G_Q + H_Q), \tag{1.115}$$

where $(a, \Delta, Q, P) \equiv \varepsilon^{\mu\nu\rho\sigma}a_\mu\Delta_\nu Q_\rho P_\sigma = (\sqrt{s}/2)^3(\mathbf{an})\beta\sin\theta$. Using the expressions listed in Subsec. 3.1 we have:

$$\mathrm{Im}\,(F_Q - G_Q + H_Q) = \frac{\pi}{s}\psi(\beta) = \frac{\pi}{s\beta^2}\left(\frac{1-\beta^2}{\beta}\,\ln\frac{1+\beta}{1-\beta} - 2\right). \tag{1.116}$$

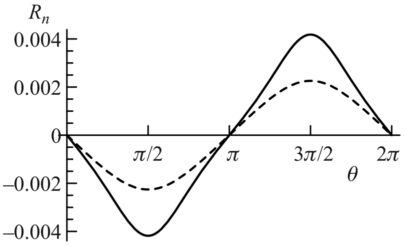

Fig. 4. Asymmetry R_n for the case of structureless proton for energies $s = 5$ GeV^2 (dashed line) and $s = 15$ GeV^2 (solid line)

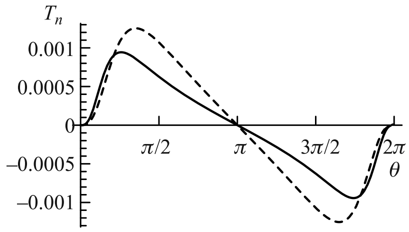

Fig. 5. Asymmetry T_n in the case of structureless proton for $E = 5$ GeV (dashed line) and $E = 10$ GeV (solid line)

Thus, after some algebra, the following expression for spin asymmetry can be obtained for the processes $e^+ + e^- \to \mathbf{p} + \bar{p}$:

$$R_n = 2\alpha \frac{M}{\sqrt{s}} \, \frac{\beta\psi(\beta)\sin\theta}{2 - \beta^2 \sin^2\theta}, \tag{1.117}$$

and it is shown in Fig. 4, as a function of θ at several values of s for the case of structureless proton.

Such considerations apply to the scattering channel when the initial protons are polarized. Similarly to (1.116) one finds

$$\mathrm{Im}_s(\bar{F}_Q - \bar{G}_Q + \bar{H}_Q) = -\frac{\pi}{s + M^2} \tag{1.118}$$

(note that the s-channel imaginary part vanishes for the crossed photon diagram amplitude). The contribution of the polarization vector appears in the same combination

$$(a, \Delta, Q, P) = \frac{1}{2}(a, p, p_1, q) = \frac{ME^2}{2\rho} \sin\theta(\mathbf{an}). \tag{1.119}$$

The single-spin asymmetry for the process $e^- + \mathbf{p} \to e^- + p$ (the initial proton is polarized) has the form:

$$\frac{d\sigma^{\uparrow} - d\sigma^{\downarrow}}{d\sigma^{\uparrow} + d\sigma^{\downarrow}} = (\mathbf{an}) T_n \tag{1.120}$$

with

$$T_n = \frac{\alpha}{2M^2} \frac{s^2}{s + M^2} (1 + \tau) \frac{\epsilon}{\rho\sigma_{\mathrm{red}}} \sin\theta \, \tan^2\frac{\theta}{2}. \tag{1.121}$$

This quantity is shown in Fig. 5 for the case of structureless proton as a function of θ, for two values of s. The asymmetry decreases when the c.m.s. energy growth, so in experiment it is useful to measure the asymmetry near the threshold of proton (or heavy lepton) production.

The results obtained here, for the processes $e^{\pm}p \to e^{\pm}p(\gamma)$ are particularly interesting in view of the experiments planned at Novosibirsk and at JLab as well as $e^+ + e^- \to N\bar{N}(\gamma)$, which can be investigated at Frascati and Bejing (see [37] and references therein).

1.4. Compton Tensor with Heavy Photon in the Case of Longitudinally Polarized Fermion. In the case of unpolarized fermions the Compton tensor with heavy photon was calculated in papers [8] years ago. It accumulates a considerable part of radiative corrections and can be used as a building block in calculations of various processes. This tensor was used for precision calculations of radiative corrections to Bhabha scattering at LEP, cross section of deep inelastic scattering with tagged photons [76], and other.

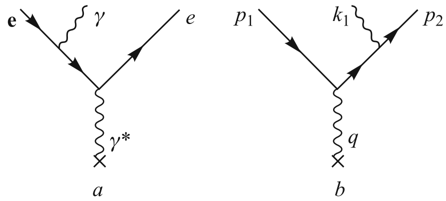

Fig. 6. The Born-level Feynman diagrams (a — for t-type, b — for s-type)

We will restrict ourselves here in considering only that part of the Compton tensor which contains the degree of polarization of the initial electron. The absence in literature of a closed expression for this quantity and the importance of it for many applications is the motivation for this investigation.

Let us consider the process (see Fig. 6)

$$\begin{gathered}\gamma^*(q) + e(p_1) \longrightarrow \gamma(k_1) + \mathbf{e}(p_2), \\ q^2 < 0, \quad k_1^2 = 0, \quad p_1^2 = p_2^2 = m^2, \quad p_1 + q = p_2 + k_1,\end{gathered} \tag{1.122}$$

where m is the electron mass.

The Compton tensor is defined as

$$K_{\rho\sigma} = (8\pi\alpha)^{-2} \Sigma M_\rho^{\mathbf{e}\gamma^* \to e\gamma} (M_\sigma^{\mathbf{e}\gamma^* \to e\gamma})^*, \tag{1.123}$$

where the matrix element M describes the Compton scattering process (1.122). It reads

$$\begin{gathered}M_\rho = M_{0\rho} + M_{1\rho} = \bar{u}(p_2) O_{\rho\mu} u(p_1) e_\lambda^\mu(k_1), \quad O_{\rho\mu} = O_{\rho\mu}^{(0)} + \frac{\alpha}{4\pi} O_{\rho\mu}^{(1)}, \\ O_{\rho\mu}^{(0)} = \gamma_\rho \frac{(\hat{p}_2 - \hat{q} + m)}{t} \gamma_\mu + \gamma_\mu \frac{(\hat{p}_1 + \hat{q} + m)}{s} \gamma_\rho, \\ s = 2p_2k_1, \quad t = -2p_1k_1, \quad u = -2p_1p_2, \quad q^2 = s + t + u.\end{gathered} \tag{1.124}$$

The quantities $O_{\rho\mu}^{(0)}$ and $O_{\rho\mu}^{(1)}$ take into account the lowest and the first orders of perturbation theory, respectively.

Calculating the first order correction, we will assume that all kinematical invariants of the process to be large in comparison with the electron mass squared:

$$s \sim -t \sim -u \sim -q^2 \gg m^2. \tag{1.125}$$

So, we will neglect the electron mass in all places, where possible. Note that for the unpolarized case in [8] the mass was taken into account.

The Compton tensor defined in (1.123) is a Hermitian one by construction:

$$K_{\rho\sigma} = K^*_{\sigma\rho}. \tag{1.126}$$

We will separate the contributions, associated with the electron polarization:

$$\begin{gathered} K_{\rho\sigma} = K^0_{\rho\sigma} + \frac{\alpha}{4\pi}(K^1_{\rho\sigma} + K^{1*}_{\sigma\rho}), \\ K^0_{\rho\sigma} = B_{\rho\sigma} + \xi P^0_{\rho\sigma}, \qquad K^1_{\rho\sigma} = T_{\rho\sigma} + \xi P^1_{\rho\sigma}, \end{gathered} \tag{1.127}$$

where ξ is the degree of the initial electron polarization. Quantities $B_{\rho\sigma}$ and $T_{\rho\sigma}$ correspond to the case of unpolarized electron:

$$\begin{gathered} B_{\rho\sigma} = B_g \tilde{g}_{\rho\sigma} + B_{11}\tilde{p}_{1\rho}\tilde{p}_{1\sigma} + B_{22}\tilde{p}_{2\rho}\tilde{p}_{2\sigma}, \\ B_g = \frac{1}{st}[(s+u)^2 + (t+u)^2] - 2m^2q^2\left(\frac{1}{s^2} + \frac{1}{t^2}\right), \\ B_{11} = \frac{4q^2}{st} - \frac{8m^2}{s^2}, \quad B_{22} = \frac{4q^2}{st} - \frac{8m^2}{t^2}, \end{gathered} \tag{1.128}$$

where the new variables

$$\tilde{g}_{\rho\sigma} = g_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2}, \quad \tilde{p}^\rho_{1,2} = p^\rho_{1,2} - \frac{p_{1,2}q}{q^2}q^\rho \tag{1.129}$$

provide an explicit fulfillment of gauge conditions: $q_\rho K^{\rho\sigma} = 0$, $q_\sigma K^{\rho\sigma} = 0$. Quantity $T_{\rho\sigma}$ has a rather cumbersome form, it is given in [8].

For the case of the most general form for the electron polarization vector

$$\Sigma u(p)\bar{u}(p) = (\hat{p} + m)(1 - \xi\gamma_5\hat{a}), \tag{1.130}$$

one obtains (see also [51, 52])

$$\begin{aligned} P^0_{\rho\sigma} = 4m\biggl\{(p_1 q\rho\sigma)\frac{qa - 2p_2a}{st} + (p_2 q\rho\sigma)\left[\frac{qa}{t^2} + \frac{p_2a}{t}\left(\frac{1}{s} - \frac{1}{t}\right)\right] + \\ + (qa\rho\sigma)\left[\frac{q^2}{st} - \frac{1}{s} - \frac{1}{t} - m^2\left(\frac{1}{s^2} + \frac{1}{t^2}\right)\right]\biggr\}, \end{aligned} \tag{1.131}$$

where we used the notation

$$(abcd) \equiv i\epsilon_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma d^\delta. \tag{1.132}$$

This object obeys the Shouten identity:

$$(abcd)ef = (fbcd)ae + (afcd)be + (abfd)ce + (abcf)de. \tag{1.133}$$

Below we restrict ourselves by considering only the case of longitudinally polarized fermion:

$$\Sigma u(p_1)\bar{u}(p_1) = \hat{p}_1(1 - \xi\gamma_5). \tag{1.134}$$

This is the most interesting case for physical applications. In the Born approximation we obtain

$$\begin{gathered} P_{\rho\sigma} = \xi\left[P^0_{\rho\sigma} + \frac{\alpha}{4\pi}P^1_{\rho\sigma}\right], \\ P^0_{\rho\sigma} = P^{0t}_{\rho\sigma} + P^{0s}_{\rho\sigma} = \frac{2}{st}[(u+t)(p_1q\rho\sigma) + (u+s)(p_2q\rho\sigma)]. \end{gathered} \tag{1.135}$$

The upper indexes t and s mean the contributions of Feynman diagrams (see Fig. 6). It is useful to present the explicit expressions for $P^{0t,s}_{\rho\sigma}$:

$$\begin{aligned} P^{0t}_{\rho\sigma} &= \frac{1}{st}[4(p_1p_2q\sigma)(p_{1\rho} + p_{2\rho}) + 2(t-s)(p_1p_2\rho\sigma) + 2(s+u)(p_2q\rho\sigma)], \\ P^{0s}_{\rho\sigma} &= \frac{1}{st}[-4(p_1p_2q\sigma)(p_{1\rho} + p_{2\rho}) + 2(s-t)(p_1p_2\rho\sigma) + 2(s+t)(p_1q\rho\sigma)]. \end{aligned} \tag{1.136}$$

It is easy to check the following relations:

$$q_\rho P^0_{\rho\sigma} = q_\sigma P^0_{\rho\sigma} = 0, \quad (P^{0s,t}_{\sigma\rho})^* = P^{0s,t}_{\rho\sigma}, \quad P^{0s,t}_{\rho\sigma}q_\rho = 0, \quad P^{0s,t}_{\rho\sigma}q_\sigma \neq 0. \tag{1.137}$$

Note now that we may consider in calculations only half of the full set of eight Feynman diagrams in 1-loop level drawn in Fig. 7, namely, the diagrams *a*, *b*,

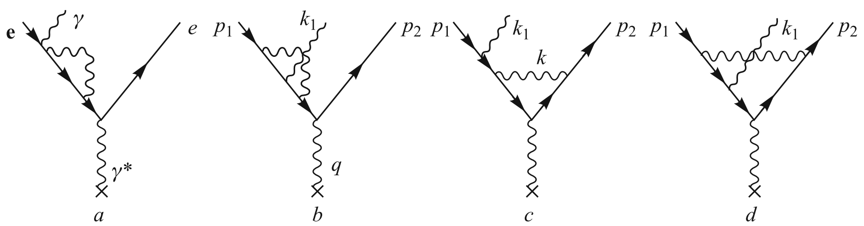

Fig. 7. One-loop virtual t-type Feynman diagrams with photon emission by the *initial* electron

c, *d*. Really the whole contribution may be obtained knowing the values of the contributions arising from Feynman diagrams (Fig. 7) using the rearrangement operator:

$$P^1_{\rho\sigma} = (1+\hat{H})(1-\hat{P})(P^{a,b} + P^{1c} + P^{1d})_{\rho\sigma} + P^{\text{soft}}_{\rho\sigma}, \tag{1.138}$$

where the operator $\hat{P}$ is defined as

$$\hat{P}F(\rho,\sigma,p_1,p_2,q,s,t) = F(\rho,\sigma,p_2,p_1,-q,t,s), \tag{1.139}$$

and the Hermitization operator $\hat{H}$ acts as

$$\hat{H}a_{\rho\sigma} = a^*_{\sigma\rho}. \tag{1.140}$$

Note that $\hat{P}P^{0s,t}_{\rho\sigma} = -P^{0t,s}_{\rho\sigma}$. The last term in Eq. (1.138) describes the contribution due to the emission of additional soft photon [8]:

$$\begin{aligned}
P^{\text{soft}}_{\rho\sigma} &= P^0_{\rho\sigma}\delta^{\text{soft}},\\
\delta^{\text{soft}} &= -\frac{4\pi\alpha}{16\pi^3}\int\frac{d^3k}{\omega}\left(\frac{p_1}{p_1k} - \frac{p_2}{p_2k}\right)^2\Bigg|_{\omega<\Delta\varepsilon} = \frac{\alpha}{\pi}\left[(L_u-1)\ln\frac{m^2(\Delta\varepsilon)^2}{\lambda^2\varepsilon_1\varepsilon_2} + \right.\\
&\left.+\frac{1}{2}L_u^2 - \frac{1}{2}\ln^2\frac{\varepsilon_1}{\varepsilon_2} - \frac{\pi^2}{3} + \mathrm{Li}_2\left(1+\frac{u}{4\varepsilon_1\varepsilon_2}\right)\right], \quad L_u = \ln\frac{-u}{m^2}.
\end{aligned} \tag{1.141}$$

Here $\Delta\varepsilon$ is the maximal energy of additional soft photon escaping the detector; quantities $\varepsilon_{1,2} = p^0_{1,2}$ are the energies of the initial and the final electron in the laboratory reference frame (rest reference frame of the target).

Considering the contribution of Feynman diagrams (Fig. 7, *a*, *b*), we may use the result given in the preprint of paper [8], namely

$$\begin{aligned}
(M^a_\sigma + M^b_\sigma)(-i(4\alpha\pi)^2)^{-1} &= \\
&= \frac{\alpha}{2\pi}\bar{u}(p_2)\gamma_\sigma\left[mA_1\left(\hat{e} - \hat{k}_1\frac{p_1e}{p_1k_1}\right) + A_2\hat{k}_1\hat{e}\right]u(p_1).
\end{aligned} \tag{1.142}$$

Note that this result may be reproduced using the loop integrals list given in the Appendix of [57] and the standard renormalization procedure. We see that only structure in front of coefficient A_2 survives in the limit $m \to 0$. After simple algebra we obtain:

$$\begin{gathered}
P^{a,b}_{\rho\sigma} = 2\frac{2L_t-1}{st}[2(p_1p_2q\sigma)p_{2\rho} + (u+s)((p_2q\rho\sigma) - (p_1p_2\rho\sigma))],\\
L_t = \ln\frac{-t}{m^2}.
\end{gathered} \tag{1.143}$$

The remaining Fig. 7, c, d contributions have the form:

$$P^{1c}_{\rho\sigma} = \frac{1}{t}\int \frac{d^4k}{i\pi^2}\,\frac{1}{a_0 a_2 a_q}\,\frac{1}{4}\,\mathrm{Tr}\,\hat{p}_2\gamma_\lambda(\hat{p}_2-\hat{k})\gamma_\sigma(\hat{p}_2-\hat{q}-\hat{k})\times \\ \times\,\gamma_\lambda(\hat{p}_2-\hat{q})\gamma_\mu\hat{p}_1\gamma_5\tilde{O}^0_{\rho\mu} \quad (1.144)$$

and

$$P^{1d}_{\rho\sigma} = \int \frac{d^4k}{i\pi^2}\,\frac{1}{a_0 a_1 a_2 a_q}\,\frac{1}{4}\,\mathrm{Tr}\,\hat{p}_2\gamma_\lambda(\hat{p}_2-\hat{k})\gamma_\sigma(\hat{p}_2-\hat{q}-\hat{k})\times \\ \times\,\gamma_\mu(\hat{p}_1-\hat{k})\gamma_\lambda\hat{p}_1\gamma_5\tilde{O}^0_{\rho\mu}, \quad (1.145)$$

where

$$a_0 = k^2-\lambda^2, \quad a_1 = k^2-2p_1k, \\ a_2 = k^2-2p_2k, \quad a_q = (p_2-q-k)^2-m^2, \quad (1.146)$$

and the matrix $\tilde{O}^0_{\rho\mu}$ differs from $O^0_{\rho\mu}$ (see Eq. 1.124) by reversal order of gamma matrices. Using the integrals given in the Appendix of [57] one may perform the loop-momenta integration in right-hand parts of expressions for P^{1c}, P^{1d} and obtain the total expression for the Compton tensor. Its explicit form will be given below.

Now we will concentrate our attention on the terms containing the infrared singularities. There are three sources of them. The first one is the renormalization constant

$$Z_1 = 1-\frac{\alpha}{2\pi}\left(\frac{1}{2}L_\Lambda + 2\ln\frac{\lambda}{m}+\frac{9}{4}\right), \quad L_\Lambda = \ln\frac{\Lambda^2}{m^2}, \quad (1.147)$$

which is needed to remove the ultraviolet divergence of the vertex function, entering into P^{1c}. The next source is a part of the box contribution P^{1d}, which comes from the terms from the numerator which does not contain loop momenta. Really for the Feynman diagram Fig. 7, d they are associated with the scalar integral,

$$I = \int \frac{d^4k}{i\pi^2}\,\frac{1}{a_0a_1a_2a_q} = \frac{1}{tu}\left[2L_u\ln\frac{m}{\lambda} - L_q^2 + 2L_tL_u - \frac{\pi^2}{6} - 2\mathrm{Li}_2\left(1-\frac{q^2}{u}\right)\right], \\ L_q = \ln\frac{-q^2}{m^2}. \quad (1.148)$$

The third source is the emission of additional soft photons, which was given above. The infrared singularities are cancelled in the total sum.

Let us consider the contribution from one–loop corrections (see Fig. 7, *a–d*)

$$P^t_{\rho\sigma} = (P^{a,b} + P^{1c} + P^{1d})_{\rho\sigma}. \qquad (1.149)$$

Extracting the leading logarithmic terms and infrared singularities, we may present it as follows:

$$P^t_{\rho\sigma} = P^{0t}_{\rho\sigma}\left[-L_u^2 - 4(L_u-1)\ln\frac{m}{\lambda} + 3L_u\right] + R^t_{\rho\sigma}. \qquad (1.150)$$

After Hermitization and rearrangement operations and adding of the soft-photon contribution we arrive to the result

$$P_{\rho\sigma} = P^0_{\rho\sigma}\left\{1 + \frac{\alpha}{\pi}\left[(L_u-1)\ln\frac{(\Delta\varepsilon)^2}{\varepsilon_1\varepsilon_2} + \frac{3}{2}L_u - \right.\right.$$
$$\left.\left. - \frac{1}{2}\ln^2\frac{\varepsilon_2}{\varepsilon_1} - \frac{\pi^2}{3} + \mathrm{Li}_2\left(\cos^2\frac{\theta}{2}\right)\right]\right\} + \frac{\alpha}{4\pi}R_{\rho\sigma}.$$

Quantities $R^t_{\rho\sigma}$ and $R_{\rho\sigma}$ collect nonleading terms. They are free from infrared singularities.

Tensor $R^t_{\rho\sigma}$ can be presented in the form:

$$R^t_{\rho\sigma} = A(2q\sigma\rho) + B(1q\sigma\rho) + C(12q\sigma)p_{1\rho} +$$
$$+ D(12q\sigma)p_{2\rho} + E(12q\sigma)q_\rho + F(12\sigma\rho).$$

The coefficients $A - F$ have a rather cumbersome form, we are not going to present them here. Note only that they obey the condition

$$Cp_1q + Dp_2q + Eq^2 - F = 0, \qquad (1.151)$$

because of gauge invariance in respect to index ρ.

The rearrangement operation acts as (we use here the designations $(1abc) = (p_1abc)$, etc.):

$$(1-\hat{P})R^t_{\rho\sigma} = (A+\tilde{B})(2q\sigma\rho) + (B+\tilde{A})(1q\sigma\rho) + (C-\tilde{D})(12q\sigma)p_{1\rho} +$$
$$+ (D-\tilde{C})(12q\sigma)p_{2\rho} + (E+\tilde{E})(12q\sigma)q_\rho + (F+\tilde{F})(12\sigma\rho) \equiv$$
$$\equiv A_1(1q\sigma\rho) + A_2(2q\sigma\rho) + B_1(12q\sigma)p_{1\rho} + B_2(12q\sigma)p_{2\rho} +$$
$$+ C_1(12q\sigma)q_\rho + F_1(12\sigma\rho). \qquad (1.152)$$

Tests of gauge invariance gives an important check of our calculations:

$$q^\rho(1-\hat{P})R_{\rho\sigma} = B_1(12q\sigma)p_1q + B_2(12q\sigma)p_2q + C_1(12q\sigma)q^2 +$$
$$+ F_1(12\sigma q) = 0, \quad q^\sigma(1-\hat{P})R_{\rho\sigma} = F_1(12q\rho) = 0. \qquad (1.153)$$

The above conditions yield

$$
\begin{aligned}
&F_1 = 0, \quad C_1 = -B_1 \frac{p_1 q}{q^2} - B_2 \frac{p_2 q}{q^2}, \\
&B_1 p_{1\rho} + B_2 p_{2\rho} + C_1 q_\rho = B_1 \tilde{p}_{1\rho} + B_2 \tilde{p}_{2\rho}.
\end{aligned} \tag{1.154}
$$

By straightforward calculations we checked these relations to be valied.

The Hermitization gives

$$
\begin{aligned}
R_{\rho\sigma} = (1+\hat{H})(1-\hat{P}) R^t_{\rho\sigma} &= (A_1 + A_1^*)(1q\sigma\rho) + (A_2 + A_2^*)(2q\sigma\rho) \\
&+ (12q\sigma)[B_1 \tilde{p}_{1\rho} + B_2 \tilde{p}_{2\rho}] - (12q\rho)[B_1^* \tilde{p}_{1\sigma} + B_2^* \tilde{p}_{2\sigma}],
\end{aligned} \tag{1.155}
$$

where ($a = s+t$, $b = s+u$, $c = t+u$)

$$
\begin{aligned}
A_1 = \frac{2}{st} \bigg[&\frac{2u(2s-u)}{a} L_{qu} + \frac{4us}{a}\left(\frac{u}{a} L_{qu} - 1\right) + \frac{ub}{c} + \frac{2u^2 + us - s^2}{c} L_{sq} + \\
&+ \frac{usb}{c^2} L_{sq} - 2c\zeta(2) - 2cL_{tu} + (2s-c) L_{qu} - \frac{uc}{s} G + \\
&+ \left(\frac{ub}{t} + c\right) \tilde{G} + 5c - 2s \bigg],
\end{aligned}
$$

$$
\begin{aligned}
B_1 = \frac{2}{st} \bigg[&\frac{8u}{a}\left(1 - \left(\frac{u}{a} + 1\right) L_{qu}\right) + \frac{6t}{b} L_{qt} + \frac{2(u^2 - 2s^2 - su)}{cu} L_{sq} + \\
&+ \frac{2b}{c}\left(1 + \frac{s}{c} L_{sq}\right) + \frac{2}{s}(2c-s) L_{tu} + \left(-2 - \frac{4c^2}{st} - \frac{12b}{t} - \frac{4s^2}{ut}\right) L_{qu} + \\
&+ \frac{4b^2}{tu} L_{su} + \left(-2 + \frac{2uc}{s^2} - \frac{2t}{s}\right) G + \left(\frac{2b}{t} + \frac{2b^2}{t^2}\right) \tilde{G} + 6 \bigg],
\end{aligned} \tag{1.156}
$$

$$
G = (L_q - L_u)(L_q + L_u - 2L_t) - \frac{\pi^2}{3} - 2\mathrm{Li}_2\left(1 - \frac{q^2}{u}\right) + 2\mathrm{Li}_2\left(1 - \frac{t}{q^2}\right),
$$

$$
A_2 = (s \leftrightarrow t) A_1, \quad B_2 = -(s \leftrightarrow t) B_1, \quad \tilde{G} = (s \leftrightarrow t) G.
$$

Note that the above expressions are free from kinematical singularities. Really, in the limits $a \to 0$, $b \to 0$, and $c \to 0$, the quantities are finite. The symmetry between A_1, B_1 and A_2, B_2 is because of the initial symmetry between p_1 and p_2 in the traces.

Thus we calculated the part of the leptonic tensor, proportional to the initial longitudinal polarization. This tensor describes Compton scattering with one off-shell photon, which is connected with a certain target.

The calculation allows one to obtain the correction coming from one-loop effects to quantities observable in different polarization experiments. Let us consider for definiteness the task of calculation of α^2 order radiative correction in polarized deep inelastic scattering. The results for the lowest order QED correction for nucleon and nuclear targets can be found in [51, 52]. Both the Born cross section (σ_B) and the cross section at the level of radiative corrections ($\sigma_{\rm RC}$) can be split into unpolarized and polarized parts

$$\sigma_{B,\rm RC} = \sigma^{\rm unp}_{B,\rm RC} + \xi_b \xi_t \sigma^{\rm pol}_{B,\rm RC}, \qquad (1.157)$$

where ξ_b and ξ_t are polarization degrees of beam and target. The correction to asymmetry ($A = \sigma^{\rm pol}/\sigma^{\rm unp}$):

$$\Delta A = \frac{\sigma^{\rm pol}_{\rm RC}\sigma^{\rm unp}_{B} - \sigma^{\rm unp}_{\rm RC}\sigma^{\rm pol}_{B}}{\sigma^{\rm unp}_{B}(\sigma^{\rm unp}_{B} + \sigma^{\rm unp}_{\rm RC})} \qquad (1.158)$$

is usually not large because of mutual cancellation of large factorizing terms in Eq. (1.158). It is clear that in such cases when relatively small correction is obtained as a difference of two large terms, the radiative correction cross section has to be calculated with the most possible accuracy, and special attention has to be paid to nonfactorizing terms like (1.156).

Now the new methods of experimental data processing, where experimental information about spin observable is extracted directly from polarized part of cross section (difference of observed cross sections with opposite spin configurations) [53], are actively developed. It makes new requirements for accuracy of radiative correction calculation. We note that there is no any cancellation of leading contributions in this case, and factorizing terms in (1.150) give the basic contribution.

The kinematical regions with very high y ($y \sim 0.9$) can be reachable in the current polarization experiments on DIS [54, 55]. In this region, radiative correction to cross section is comparable or larger than that of Born cross section. Basically it is originated by contributions of radiative tails from elastic and quasi-elastic peaks. This calculation firstly allows one to obtain the contribution of these tails with taking into account loop effects in the next-to-leading approximation.

There is one particular interesting phenomenon. Note, that $P^{(1)}_{\rho\sigma}$ contains not only the imaginary part, but also a certain real part, which comes from the imaginary parts of A_1 and B_1. The conversion of this real part of $P^{(1)}_{\rho\sigma}$ with the ordinary symmetrical part of the hadronic tensor will give rise to one-spin azimuthal asymmetry for the final electron [56]. The asymmetry is proportional to the degree of polarization of the initial electron. It is small because of the extra power of $\alpha_{\rm QED}$ and the absence of large logarithms.

Compton tensor for longitudinally polarized electron was first obtained in [57]. Some application was considered in [58].

1.5. Double Logarithmical Corrections to Beam Asymmetry in Polarized Electron–Proton Scattering. The beam asymmetry due to transversal polarization of an electron beam scattered on (unpolarized) protons is a pure quantum effect arising from the interference of the Born amplitude (with one photon exchange) and the imaginary part of the two-photon exchange amplitude (Fig. 8).

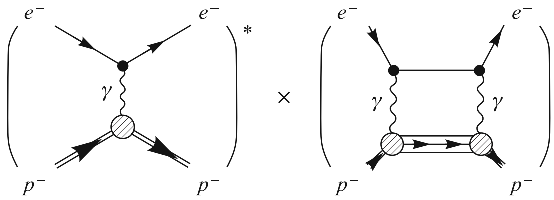

Fig. 8. The beam asymmetry due to transversal polarization of an electron beam in the lowest order

The corresponding contribution to the differential cross section as well as to the beam asymmetry is proportional to the electron mass. Therefore, the presence of this contribution does not contradict the Kinoshita–Lee–Nauenberg theorem [18] about cancellation of mass singularities, since the corresponding cross sections are suppressed by the lepton mass.

We show that the main contribution arises from the kinematical region of loop momenta when the energy of the electron in the intermediate state, being on mass shell, is small in the reference frame of the center of mass of the initial particles.

The kinematics is determined by the conservation laws

$$e^-(a, p_1) + P(p) + (\gamma) \to e^-(p_1'') + X + (\gamma) \to e(p_1') + P(p') + (\gamma), \quad (1.159)$$

where a is an electron spin. In general, the emission of real photons must be considered, in order to avoid infrared divergences, when higher orders of PT are taken into account. For the case of one-proton intermediate state in the TPE, the energies of the initial intermediate state and the final elastically scattered electrons are equal

$$\begin{gathered} p_{10} = p_{10}'' = p_{10}' = \epsilon = \frac{s - M^2}{2\sqrt{s}}, \quad s = (p_1 + p)^2, \\ t = (p_1 - p_1')^2 = -2\epsilon^2(1 - c), \quad p_1^2 = (p_1')^2 = m^2, \quad p^2 = (p')^2 = M^2, \end{gathered} \quad (1.160)$$

where $c = \cos\theta$ and θ is the scattering angle in the center-of-mass frame. The absolute values of the photon momenta squared, in TPE amplitude, can reach zero:

$$|t_1| = |(p_1 - p_1'')^2| = 2\epsilon^2(1 - c_1), \quad |t_2| = |(p_1' - p_1'')^2| = 2\epsilon^2(1 - c_2), \quad (1.161)$$

where $c_{1,2}$ are the cosines of the angles between the initial and intermediate electron momenta and the intermediate and final ones, respectively.

For the case of inelastic hadronic intermediate state (for instance, a nucleon and a pion) the energy of the electron in the intermediate state ϵ'' does not exceed $\epsilon : m < \epsilon'' < \epsilon$. The exchanged photon momenta squared become:

$$t_{1,2} = -2\epsilon\epsilon''(1 - bc_{1,2}), \quad 1 - b^2 = \frac{m^2}{(\epsilon'')^2}\left(1 - \frac{\epsilon''}{\epsilon}\right)^2. \tag{1.162}$$

The main contribution arises from two regions $-t_1 \ll t_2 = t$ and $-t_2 \ll t_1 = t$.

Moreover, we will show that the energy of the intermediate electron is much lower than the electron energy corresponding to the elastic case $\epsilon'' \ll \epsilon$.

Neglecting the dependence on p_1'' from the remaining part of the amplitude, we find the main DL asymptotic behavior of the amplitude:

$$(M^*_{\rm box} M_B)^{(DL)} \approx \int \frac{\epsilon''\, d\epsilon''\, dO''}{2\pi t_1 t_2} \approx \frac{-1}{4t} L^2. \tag{1.163}$$

We do not distinguish here the two kinds of «large logarithms»

$$L_s = \ln\frac{s}{m^2} - i\pi, \quad L_t = \ln\frac{-t}{m^2} = \ln\frac{2\epsilon^2(1-c)}{m^2}. \tag{1.164}$$

The result of exact calculation consists in the replacement

$$L^2 \to L_t L_s. \tag{1.165}$$

1.5.1. Calculation in Double-Logarithm Approximation. Let us investigate the question about the size of radiative corrections to the cross section for the scattering of transversally polarized electron and to the relevant beam asymmetry. The corrections in the lowest order can be of several types (see Figs. 9, 10, 11, 13, 14).

Let us consider first the RC at the lowest order. The contribution from the emission of virtual photons is twofold. Firstly, it is due to the vertex functions in the kinematics where both electrons are on mass shell and the photon mass squared is negative and large (in absolute value) compared to the electron mass squared (see Fig. 9). The main contribution arises from the Dirac form factor. In the scattering channel we have:

$$F_1(q^2) = 1 + \frac{\alpha}{\pi}\left[\ln\frac{m}{\lambda}(1 - L_q) - 1 + \frac{3}{4}L_q - \frac{1}{4}L_q^2 + \frac{\pi^2}{12}\right] = 1 + \frac{\alpha}{\pi}\delta_v(q^2),$$
$$q^2 = t, t_1, t_2, \quad L_q = L_t, L_1, L_2, \quad L_1 = \ln\frac{-t_1}{m^2}, \quad L_2 = \ln\frac{-t_2}{m^2}. \tag{1.166}$$

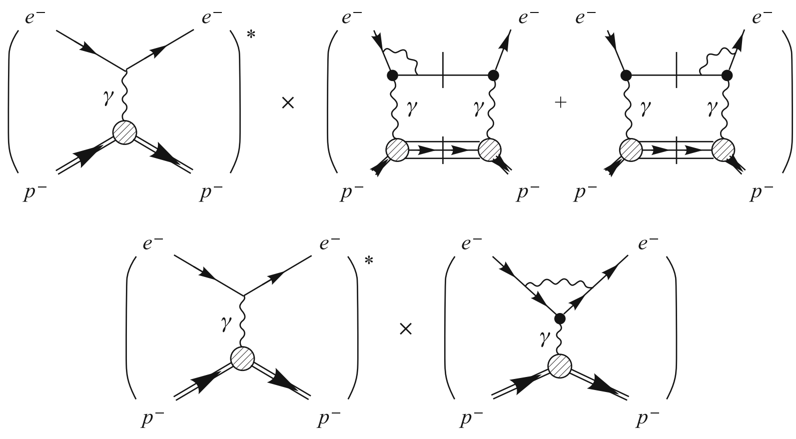

Fig. 9. Vertex corrections δ_v which contribute to the asymmetry

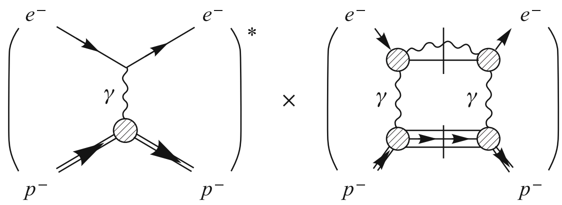

Fig. 10. Inner bremsstrahlung in two-photon diagram

The second class of contributions arises from emission and absorption of real photons as an intermediate state of the leptonic block.

Let firstly restrict our considerations to the emission of soft real intermediate photons (see Fig. 10).

For the corresponding contribution we have

$$\frac{\alpha}{\pi}\delta_s = -\frac{4\pi\alpha}{16\pi^3}\int\frac{d^3k}{\omega}\left(-\frac{p_1}{p_1k}+\frac{p_1''}{p_1''k}\right)\left(\frac{p_1''}{p_1''k}-\frac{p_1'}{p_1'k}\right)\bigg|_{\omega<\Delta_1}, \ \Delta_1 \ll \epsilon. \quad (1.167)$$

Using the expressions

$$-\frac{4\pi\alpha}{16\pi^3}\int\frac{d^3k}{\omega}\frac{m^2}{(p_1''k)^2}\bigg|_{\omega<\Delta_1} = -\frac{\alpha}{\pi}\ln\frac{m\Delta_1}{\lambda\epsilon''},$$

$$\frac{4\pi\alpha}{16\pi^3}\int\frac{d^3k}{\omega}\frac{2p_1p_1''}{(p_1k)(p_1''k)}\bigg|_{\omega<\Delta_1} =$$

$$= \frac{\alpha}{2\pi}\left[L_1\ln\left(\frac{m^2\Delta_1^2}{\lambda^2\epsilon\epsilon''}\right)+\frac{1}{2}L_1^2-\frac{1}{2}\ln^2\left(\frac{\epsilon''}{\epsilon}\right)-\frac{\pi^2}{3}+\mathrm{Li}_2\left(\frac{1+c_1}{2}\right)\right],$$

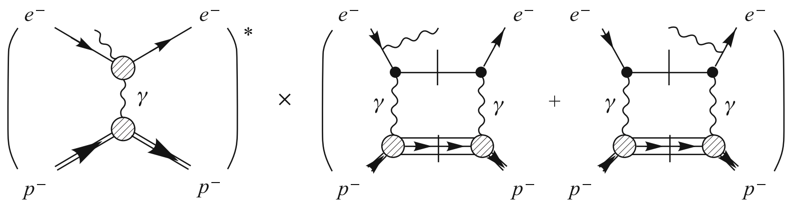

Fig. 11. Inelastic two-photon contribution $\delta_s^{\rm inel}$

where λ is a fictitious «photon mass», the resulting contribution

$$\delta_{vs} = \delta_v(t) + \delta_v(t_1) + \delta_v(t_2) + \delta_s \tag{1.168}$$

suffers from infrared divergences. To remove these divergences, we must take into account the inelastic process of electron–proton scattering with emission of additional soft (or hard) real photons by initial and final electrons (see Fig. 11):

$$\frac{\alpha}{\pi}\delta_s^{\rm inel} = -\frac{4\pi\alpha}{16\pi^3}\int \frac{d^3k}{\omega}\left(-\frac{p_1}{p_1k} + \frac{p_1'}{p_1'k}\right)^2\Bigg|_{\omega<\Delta_2}, \quad \Delta_2 \ll \epsilon. \tag{1.169}$$

The total sum $\delta = \delta_{vs} + \delta_s^{\rm inel}$ is free from infrared divergences:

$$\delta = 2(L_t-1)\ln\frac{\Delta_2}{\epsilon} + \frac{1}{2}(L_1+L_2)\ln\frac{\Delta_1^2}{\epsilon\epsilon''} - \ln\frac{\Delta_1}{\epsilon''} - L_t\ln\frac{\Delta_1}{\epsilon} + \frac{3}{4}(L_t+L_1+L_2) - \\ - 3 - \frac{\pi^2}{4} + \frac{1}{2}\left[\mathrm{Li}_2\left(\frac{1+c_1}{2}\right) + \mathrm{Li}_2\left(\frac{1+c_2}{2}\right) + \mathrm{Li}_2\left(\frac{1+c}{2}\right)\right] - \\ - \frac{1}{2}\ln^2\left(\frac{\epsilon''}{\epsilon}\right). \tag{1.170}$$

Keeping in mind that the dominant (DL) contribution arises from the region of intermediate state with a soft lepton in the presence of photons, we can generalize the above result including the emission of «hard» internal and external photons. This can be realized by the choice

$$\Delta_1 \sim \Delta_2 \sim \epsilon''. \tag{1.171}$$

In this case, the hadronic block remains the same as for the lowest order of PT.

In DL approximation we have

$$\delta^{\rm DL} = \left[L_t + \frac{1}{2}(L_1+L_2)\right]\ln\frac{\epsilon''}{\epsilon} - \frac{1}{2}\ln^2\frac{\epsilon''}{\epsilon}. \tag{1.172}$$

Up to now we calculate the contribution of box diagram to the matrix element. For the calculation of the asymmetry we must take into account imaginary pairs of the contribution. By general arguments of analyticity [59], the matrix diagram with two-photon exchange and insertion of vertex function with correction of α^n order in DL approximation is proportional to $L_s L_t^{2n-1}$.

It is natural to expect (keeping in mind the arguments in favor of exponentiation of soft-photon emission contributions proved by Yenne, Frautchi, and Suura [13]) that this result can be generalized to all orders of PT:

$$R_\infty = \frac{A^{\mathrm{corr}}}{A^B} = \frac{\mathrm{Im}\,(M^*_{\mathrm{box}} M_B)^{(\mathrm{DL})+\mathrm{corr}}}{\mathrm{Im}\,(M^*_{\mathrm{box}} M_B)^{(\mathrm{DL})}} = = \left[\int_0^{L_t} \frac{(1-c)\,dO''\,dl}{2\pi(1-bc_1)(1-bc_2)} \exp\left(\frac{\alpha}{\pi}\delta^{DL}\right)\right] \times \times \left[\int_0^{L_t} \frac{(1-c)\,dO''\,dl}{2\pi(1-bc_1)(1-bc_2)}\right]^{-1}. \quad (1.173)$$

Born approximation reads:

$$A^{\mathrm{Born}} = -\frac{m_e\sqrt{Q^2}\sigma^{\gamma p}_{\mathrm{tot}}}{8\pi^2} \frac{G_e}{\tau G_M^2 + \epsilon G_E^2}\left(\ln\frac{Q^2}{m_e^2} - 2\right). \quad (1.174)$$

We can use here (see Subsec. 3.2)

$$dO'' = \frac{2dc_1\,dc_2}{\sqrt{1-c_1^2-c_2^2-c^2+2cc_1c_2}},$$
$$\delta^{DL} = \frac{3}{4}L_t l + \frac{1}{8}l^2 - \frac{7}{8}L_t^2 + \frac{1}{4}(l-L_t)\left[\ln(1-bc_1) + \ln(1-bc_2)\right], \quad (1.175)$$
$$l = \ln\frac{2(\epsilon'')^2(1-c)}{m^2}.$$

The calculation in the lowest order of PT leads to

$$R_{\mathrm{LO}} \approx 1 - \frac{\alpha}{\pi}\frac{7L_t^2}{24}. \quad (1.176)$$

The numerical results for R in the lowest order, R_{LO}, and in higher orders, R_∞, are presented in Fig. 12. One can see they are sizable and should be taken into account.

The asymmetry at the lowest order of PT has been calculated in previous papers (see, for instance, [60]).

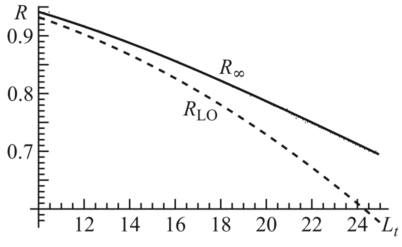

Fig. 12. Numerical results for R, Eq. (1.176). $R_{\rm LO}$ (dashed line) is the result of the calculation with RC in the lowest order of PT, R_∞ (solid line) when all orders of PT are taken into account. We assume $c = 1/2$

Mass suppressed amplitudes connected with Higgs production and decay in DL approximation were calculated in the paper [61].

The contribution to the imaginary part of the amplitude from the square of the box diagram (Fig. 13) is of order $(\alpha/\pi)^2 L$, and can be omitted in DL approximation. This holds also for the interference of the born diagram with the two-loop box diagram (Fig. 14) [2].

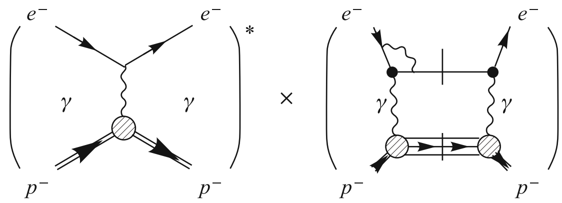

Fig. 13. Square box contribution to the asymmetry

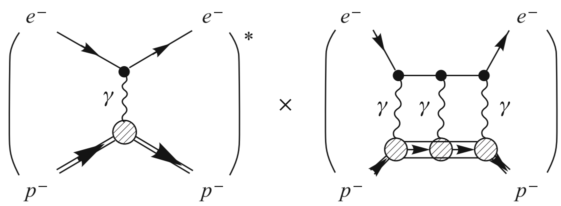

Fig. 14. Contribution to the asymmetry of the interference of the Born diagram with the two-loop box diagram

We note that the limiting case $t \to 0$ cannot be obtained using the approach described above. Contrary to the hadronic block, where the $t \to 0$ limit can be put smoothly, the leptonic block drastically depends on the parameter $(-t/m^2)$. For the $t = 0$ case, the exact cancellation of RC related to the internal emission of virtual and real photons takes place, as it was shown in Appendix D of [1].

The presence of infrared singularities in RC to the impact factor of the electron was previously noted in the paper of one of us [7]. In the present paper, the way to eliminate such singularities is explicitly shown.

1.6. Possible Method to Measure the Ratio of Proton Form Factors in Processes with Proton Spin Transmission. The investigation of the proton electromagnetic form factors, which are very important characteristics of this fundamental object, provides a deeper insight into the structure of the proton and the properties of the interaction between the constituent quarks.

Since the mid-1950s [23,71], to obtain the experimental data on the behavior of the proton electric, $G_E(Q^2)$, and magnetic, $G_M(Q^2)$, form factors (Sachs form factors) and to analyze the electromagnetic structure of the proton, the electron–proton elastic scattering has been used. For the case of the unpolarized electrons and protons, all experimental data on the proton form factors were obtained with the Rosenbluth formula [23] corresponding to the differential cross section of the elastic $ep \to ep$ process,

$$\overline{\sigma} = \frac{d\sigma}{d\Omega} = \frac{\alpha^2 E_e' \cos^2(\theta_e/2)}{4E_e^3 \sin^4(\theta_e/2)} \frac{1}{1+\tau} \left(G_E^2 + \frac{\tau}{\varepsilon} G_M^2 \right). \tag{1.177}$$

Here, $\tau = Q^2/4M^2$, where M is the proton mass and $Q^2 = -q^2 = 4E_e E_e' \times \sin^2(\theta_e/2)$; E_e and E_e' are the energies of the initial and final electrons, respectively; θ_e is the electron scattering angle in the rest frame of the initial proton; and the degree of the transverse (linear) polarization of a virtual photon, ε, is determined by the expression $\varepsilon^{-1} = 1 + 2(1+\tau)\tan^2(\theta_e/2)$.

According to the Rosenbluth formula, the leading contribution to the $ep \to ep$ cross section for high Q^2 values comes from the term proportional to the proton magnetic form factor squared $G_M^2(Q^2)$; therefore, the accuracy of the separation of the $G_E^2(Q^2)$ contribution decreases. For this reason, the use of the Rosenbluth formula for experimental determining the form factors $G_E(Q^2)$ and $G_M(Q^2)$ gives significant uncertainties at $Q^2 \geqslant 1$ GeV^2.

Note that Rosenbluth formula (1.177) valid in the laboratory reference frame, where the initial proton is at rest, is naturally represented as the sum of two terms proportional to the squares of the Sachs form factors, G_E^2 and G_M^2,

$$\overline{\sigma} = \sigma_{\uparrow\uparrow} + \sigma_{\uparrow\downarrow}, \quad \sigma_{\uparrow\uparrow} = \kappa\, G_E^2, \quad \sigma_{\uparrow\downarrow} = \kappa \frac{\tau}{\varepsilon}\, G_M^2. \tag{1.178}$$

Here, κ is the factor in front of the parentheses in Eq. (1.177). However, the physical meaning of these terms is not explained in the literature and is unknown for most researchers. To elucidate the physical meaning of the terms $\sigma_{\uparrow\uparrow}$ and $\sigma_{\uparrow\downarrow}$, the following simple consideration is sufficient. The scattering cross section disregarding the polarizations of the initial and final protons can always be represented as the sum of the cross sections without and with the spin flip of the initial proton, which should be fully polarized along a certain direction determined by the process kinematics. Since the initial proton is at rest, this separated direction can be only the direction of the motion of the scattered proton. Then, according to the commonly known additional reasonings (see, e.g., Eqs. (4.55) from [63] or Eqs. (8.55) and (8.56) from [64]), in the Breit system of the initial and final protons, the matrix element of the proton current for the case of the transition with change in helicity (without spin flip) is expressed only in terms of the electric form factor G_E, whereas the matrix element of the proton current for the case of the transition without change in helicity (with spin flip) is expressed only in terms of the magnetic form factor G_M. Thus, the terms $\sigma_{\uparrow\uparrow}$ and $\sigma_{\uparrow\downarrow}$ in Eq. (1.178) are the cross sections without and with the spin flip for the case where the initial proton is fully polarized in the direction of motion of the final proton. Below, we demonstrate that our simple physical consideration is based on rigorous mathematical results obtained using the approach of the diagonal spin basis [65, 66]. Since the cross sections without and with the proton spin flip in Eq. (1.178) are expressed only in terms of one of the Sachs form factors, they can be attractable for performing direct experiments on their measurement and acquiring new independent data on the behavior of G_E^2 and G_M^2 as functions of Q^2.

Akhiezer and Rekalo [67] proposed a method for measuring the ratio of the Sachs form factors that is based on the polarization transfer from the longitudinally polarized initial electron to the final proton and is independent of the Rosenbluth technique. In [67], it was shown that the ratio of the degrees of the longitudinal, P_l, and transverse, P_t, polarizations of the scattered proton is proportional to the ratio of the proton electric and magnetic form factors:

$$\frac{P_l}{P_t} = -\frac{G_M}{G_E}\frac{E_e + E_e'}{2M}\tan\left(\frac{\theta_e}{2}\right). \qquad (1.179)$$

The experiments based on the method of the polarization transfer from the initial electron to the final proton were recently performed with high accuracy by the Bates [68] and JLAB [69] Collaborations. They gave surprising results according to which $G_E(Q^2)$ decreases with an increase in Q^2 faster than $G_M(Q^2))$ does; this contradicts the data acquired by means of the Rosenbluth technique according to which $G_E(Q^2)$ and $G_M(Q^2))$ up to several GeV^2 approximately follow the dipole form and, hence, $\mu_p G_E(Q^2)/G_M(Q^2) \approx 1$.

Here it is proposed a new independent method for measuring the squared Sachs form factors [62]. In this approach, they can be determined separately and independently by direct measurements of the cross sections without and with spin flip of the initial proton, which should be at rest and fully polarized in the direction of the motion of the scattered proton. For the case of the partially polarized initial proton, we propose measuring the ratio of these cross sections, which makes it possible to determine the ratio of the squared Sachs form factors.

According to Eq. (1.178), when the initial and final protons are fully polarized, the ratio of the cross sections without and with proton spin flip has the extremely simple form

$$\frac{d\sigma_{\uparrow\uparrow}}{d\sigma_{\uparrow\downarrow}} = \frac{\varepsilon}{\tau}\,\frac{G_E^2}{G_M^2}. \tag{1.180}$$

The simplest way for verifying the correctness of Eqs. (1.178) and (1.180) is to use the method for calculating the matrix elements in the diagonal spin basis [66] (see Eq. (1.201)), which allows one to project the spins in the initial and final states of the particles onto one common direction. The generalization of Eq. (1.180) for the most general case of the partially polarized initial proton is given below (see Eq. (1.190)).

Our proposals are based on the results obtained in [65], where it was shown that the matrix elements of the proton current in the diagonal spin basis that correspond to the transitions without and with proton spin flip are expressed only in terms of the electric, G_E, and magnetic, G_M, Sachs form factors, respectively. Note that our terminology for the cross sections (amplitudes) with and without proton spin flip is not conventional, but has an absolute physical meaning, because we choose one common direction of the spin projection for the initial and detected protons; this direction coincides with the direction of the motion of the scattered proton.

The corresponding experiment on the measurement of the squared Sachs form factors in the processes with and without proton spin flip can be performed as follows. The initial proton at rest and detected proton should be partially polarized along the direction of the scattered proton or in the opposite direction. Measuring the corresponding differential cross section, one can determine the ratio of the squared Sachs form factors. The proposed method can be applied to the elastic muon–proton scattering and implemented in the COMPASS experiment. The mechanism under consideration is also present in the radiative ep scattering. In the Bethe–Heitler kinematics, where the leading contribution to the process cross section comes from two diagrams corresponding to the emission of a photon by an electron, the above consideration for the elastic ep scattering remains applicable. The ratio of the squares of the proton electric and magnetic form factors can also be measured for the process of the photoproduction of lepton pairs on a polarized proton in the Bethe–Heitler kinematics. Here we consider only the mechanism of the single-photon exchange between the electron and proton. Our consideration is inapplicable for the two-photon exchange. However, the contribution of the two-photon mechanism (caused by the interference of the amplitudes with the exchange by one and two photons) is about 0.5% of the contribution from the single-photon mechanism.

1.6.1. Matrix Elements of the Proton Current in the Diagonal Spin Basis. In the Born approximation, the matrix element corresponding to the electron–proton elastic scattering,

$$e(p_1) + p(p, a) \to e(p_2) + p(p', a'), \tag{1.181}$$

where a and a' are the polarization 4-vectors of the initial and final protons, has the form

$$M_{ep\to ep} = \overline{u}(p_2)\gamma^{\mu}u(p_1)\cdot\overline{u}(p')\Gamma_{\mu}(q^2)u(p)\frac{1}{q^2}, \tag{1.182}$$

$$\Gamma_{\mu}(q^2) = F_1\gamma_{\mu} + \frac{F_2}{4M}(\hat{q}\gamma_{\mu} - \gamma_{\mu}\hat{q}), \quad q = p' - p, \tag{1.183}$$

with the mass-shell conditions $p_1^2 = p_2^2 = m^2$ for electrons and $p^2 = p'^2 = M^2$ for protons. The matrix elements of the proton current corresponding to the transitions without and with spin flip,

$$(J_p^{\pm\delta,\delta})_{\mu} = \overline{u}^{\pm\delta}(p')\Gamma_{\mu}(q^2)u^{\delta}(p), \tag{1.184}$$

calculated in the diagonal spin basis [65, 66] can be expressed in terms of the Sachs form factors,

$$G_E = F_1 - \tau F_2, \quad G_M = F_1 + F_2, \tag{1.185}$$

where F_1 and F_2 are the Dirac and Pauli proton form factors, respectively. The matrix elements of the proton current given by Eq. (1.184) in the diagonal spin basis have the form [65, 66]

$$(J_p^{\delta,\delta})_{\mu} = 2G_E M(b_0)_{\mu}, \tag{1.186}$$

$$(J_p^{-\delta,\delta})_{\mu} = -2\delta M\sqrt{\tau}G_M(b_{\delta})_{\mu}, \tag{1.187}$$

where

$$b_0 = \frac{(p+p')}{\sqrt{(p+p')^2}}, \quad b_3 = \frac{q}{\sqrt{Q^2}},$$
$$(b_1)_{\mu} = \varepsilon_{\mu\nu\kappa\sigma}b_0^{\nu}b_3^{\kappa}b_2^{\sigma}, \quad (b_2)_{\mu} = \varepsilon_{\mu\nu\kappa\sigma}p^{\nu}p'^{\kappa}\frac{p_1^{\sigma}}{\rho}, \tag{1.188}$$
$$b_{\delta} = b_1 + i\delta b_2, \quad \delta = \pm 1, \quad b_1^2 = b_2^2 = b_3^2 = -b_0^2 = -1.$$

Here $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor ($\varepsilon_{0123} = -1$); ρ is determined from the normalization conditions, and the set of unit 4-vectors b_0, b_1, b_2, b_3 is an orthonormalized basis, $b_{\delta}^* = b_1 - i\delta b_2$.

Therefore, the matrix elements of the proton current in the diagonal spin basis that correspond to the transitions with and without proton spin flip given by Eqs. (1.186) and (1.187) are expressed in terms of the electric, G_E, and magnetic, G_M, form factors, respectively (see [65, 66]). In the Breit system of the initial and final protons, which is a particular case of the diagonal spin basis, Eqs. (1.186) and (1.187) coincide with similar Eq. (4.55) from [63] and Eqs. (8.55) and (8.56) from [64].

In the diagonal spin basis [65,66], the spin 4-vectors a and a' of the protons with the 4-momenta p and p' ($ap = a'p' = 0$, $a^2 = a'^2 = -1$), respectively, lie in the hyperplane formed by the 4-vectors p and p':

$$a = -\frac{(vv')v - v'}{\sqrt{(vv')^2 - 1}}, \quad a' = \frac{(vv')v' - v}{\sqrt{(vv')^2 - 1}}, \quad v = \frac{p}{M}, \quad v' = \frac{p'}{M}. \tag{1.189}$$

Spin 4-vectors (1.189) obviously remain unchanged under the transformations of the small Lorentz group common for the particles with the 4-momenta p and p'. Thus, the diagonal spin basis provides the description of the spin states of the system of two particles by means of the spin projections onto one common direction. Note that this common direction of the projection of the spins of the initial and final protons in the rest frame of the initial proton is the direction of the motion of the final proton (see Subsubsec. 1.6.3).

The fundamental fact of the existence of the small Lorentz group common for the particles with the momenta p and p' in the diagonal spin basis gives rise to a number of remarkable features. First, the particles with the 4-momenta p (before interaction) and p' (after interaction) in the diagonal spin basis have common spin operators. This makes it possible to separate the interactions with and without change in the spin states of the particles involved in the reaction in the covariant form and, thus, to trace the dynamics of the spin interaction. The spin states of massless particles in the diagonal spin basis coincide up to sign with helical states [66]; in this case, the diagonal spin basis formalism is equivalent to the CALKUL group method [4].

1.6.2. Generalization of the Diagonal Spin Basis to the Case of Partially Polarized Protons. The general expression for the ratio of the cross sections for the $ep \to ep$ process with and without spin flip for the partially polarized initial and final protons has the form:

$$\frac{d\sigma_{\uparrow\uparrow}}{d\sigma_{\uparrow\downarrow}} = \frac{G_E^2(1+\eta) + (\tau/\varepsilon)G_M^2(1-\eta)}{G_E^2(1-\eta) + (\tau/\varepsilon)G_M^2(1+\eta}, \quad \eta = \lambda\lambda', \tag{1.190}$$

where λ and λ' are the degrees of the polarization of the initial and final protons in the direction of the motion of the final proton ($\lambda, \lambda' \leqslant 1$).

For the peripheral processes of the radiative electron–proton scattering and production of pairs by a photon on a polarized proton at high energies, we can set $\varepsilon = 1$. In this approximation, relation (1.190) is valid not only for the elastic process $ep \to ep$, but also for the $ep \to ep\gamma$ and $\gamma p \to e\bar{e}p$ processes (see below). To separate elastic events on a proton, it is necessary to measure the spectrum of elastic electrons in the radiative ep scattering or distribution over the fraction of the energy of the component of a pair produced in the photoproduction process.

Let us consider the radiative electron–proton scattering

$$e(p_1) + p(p, a) \to e(p_2) + p(p', a') + \gamma(k, e), \tag{1.191}$$

where a and a' are the polarization 4-vectors of the initial and final protons, respectively, and e is the polarization 4-vector of the photon such that $ap = a'p' = ek = 0$. In the peripheral (Bethe–Heitler) kinematics specified by the relations

$$s_e = 2p_1 p \gg Q^2 = -q^2 = -(p - p')^2 \sim M^2, \tag{1.192}$$

the matrix element of process (1.191) has the factorized form (this is easily shown using the Gribov identity [1]):

$$M_e = 2s_e \frac{(4\pi\alpha)^{3/2}}{q^2} N_e N_p^e, \tag{1.193}$$

where

$$N_p^e = \frac{1}{s_e} \bar{u}(p', a') \hat{p}_1 \left(F_1(q^2) - \frac{1}{2M} F_2(q^2) \hat{q} \right) u(p, a),$$
$$N_e = \frac{1}{s_e} \bar{u}(p_2) \left(\hat{p} \frac{\hat{p}_2 - \hat{q} + m}{d_2} \hat{e} + \hat{e} \frac{\hat{p}_2 - \hat{q} + m}{d_2} \hat{p} \right) u(p_1).$$

Similar expressions can be written for the matrix element for the production of pairs by a photon on a polarized proton:

$$\gamma(k, e) + p(p, a) \to e^+(p_+) + e^-(p_-) + p(p', a')\,. \tag{1.194}$$

In the Bethe–Heitler kinematics,

$$s_\gamma = 2kp \gg Q^2 \sim M^2,$$

the matrix element of process (1.194) has the form

$$M_\gamma = 2s_\gamma \frac{(4\pi\alpha)^{3/2}}{q^2} N_\gamma N_p^\gamma, \tag{1.195}$$

where

$$N_p^\gamma = \frac{1}{s_\gamma} \bar{u}(p', a') \hat{k} \left(F_1(q^2) - \frac{1}{2M} F_2(q^2) \hat{q} \right) u(p, a),$$
$$N_\gamma = \frac{1}{s_\gamma} \bar{u}(p_-) \left(\hat{p} \frac{\hat{k} - \hat{p}_+ + m}{d_+} \hat{e} + \hat{e} \frac{\hat{p}_- - \hat{k} + m}{d_-} \hat{p} \right) v(p_+)\,.$$

Calculating the squared absolute values of the matrix elements of the proton current, $|N_p^e|^2$ and $|N_p^\gamma|^2$, for processes $ep \to ep\gamma$ (1.191) and $\gamma p \to e\bar{e}p$ (1.194) with and without spin flip, we arrive at the same expressions

$$|N_p^e(a, \pm a')|^2 = |N_p^\gamma(a, \pm a')|^2 = 4G_\pm,$$
$$G_\pm = \frac{1}{2(1+\tau)}\left(G_E^2(1 \pm \eta) + \tau G_M^2(1 \mp \eta)\right).$$

Averaging and summing the expression for $|N_e|^2$ over the spin states of the electrons and photon, we obtain

$$\sum |N_e|^2 = 4D_e,$$
$$D_e = x(1-x)^2\left(\frac{Q^2(1+x^2)}{d_1 d_2} - 2m^2 x(\frac{1}{d_1} - \frac{1}{d_2})^2\right),$$

where

$$d_1 = m^2(1-x)^2 + \mathbf{p_2}^2,$$
$$d_2 = m^2(1-x)^2 + (\mathbf{p}_2 + \mathbf{q}(1-x))^2,$$

and x is the energy fraction carried by the scattered electron; $\mathbf{p}_2$ is the momentum component transverse to the electron beam, and $\mathbf{q}$ is the transverse momentum component transferred to the proton.

Finally, averaging and summing the expression for $|N_\gamma|^2$ over the spin states of the components of a pair and the photon for $\gamma p \to e\bar{e}p$ process (1.194), we obtain

$$\sum |N_\gamma|^2 = 4D_\gamma,$$
$$D_\gamma = x_+ x_-\left(\frac{Q^2(x_+^2 + x_-^2)}{d_+ d_-} + 2m^2 x_+ x_-\left(\frac{1}{d_+} - \frac{1}{d_-}\right)^2\right),$$

where x_- and x_+ are the energy fractions carried by the electron and positron ($x_+ + x_- = 1$), respectively; $d_\pm = \mathbf{p}_\pm^2 + m^2$; and $\mathbf{p}_-$ and $\mathbf{p}_+$ are the transverse momenta of the components of the pair ($\mathbf{p}_- + \mathbf{p}_+ = \mathbf{q}$).

Relation (1.190) is valid for processes $ep \to ep\gamma$ (1.191) and $\gamma p \to e\bar{e}p$ (1.194), because the differential cross sections for these processes in the Bethe–Heitler kinematics have the form

$$d\sigma^{ep\to ep\gamma}(a, \pm a') = \frac{2\alpha^3}{\pi^2(Q^2)^2} D_e G_\pm \frac{d^2q\, d^2p_2\, dx}{x(1-x)}, \tag{1.196}$$
$$d\sigma^{\gamma p\to e\bar{e}p}(a, \pm a') = \frac{2\alpha^3}{\pi^2(Q^2)^2} D_\gamma G_\pm \frac{d^2q\, d^2p_-\, dx_-}{x_- x_+}. \tag{1.197}$$

The integration of differential cross section (1.196) with respect to the transverse momentum of the final electron yields

$$\frac{d\sigma^{ep\to ep\gamma}}{dQ^2 dx}(a, \pm a') = \frac{2\alpha^3}{(Q^2)^2} G_\pm \times [[\tau_1(1+x^2) + x]R(\tau_1) - x], \tag{1.198}$$

where $\tau_1 = Q^2/m^2$ and

$$R(z) = \frac{1}{\sqrt{z(1+z)}} \ln\left(\sqrt{1+z} + \sqrt{z}\right). \tag{1.199}$$

A similar expression for the cross section for the photoproduction of pairs on the proton has the form

$$\frac{d\sigma^{\gamma p \to e\bar{e}p}}{dQ^2\, dx_-}(a, \pm a') = \frac{2\alpha^3}{(Q^2)^2} G_\pm \times \\ \times [[\tau_1(x_+^2 + x_-^2) - x_+ x_-] R(\tau_1) + x_+ x_-]. \tag{1.200}$$

The details of the $ep \to ep$ kinematics in the rest frame of the initial proton are discussed in Subsubsec. 1.6.3.

1.6.3. Kinematics of the Elastic Process in the Laboratory Reference Frame. In the laboratory reference frame where the initial proton is at rest, $p = M(1,0,0,0)$, the polarization 4-vectors a and a' of the initial and final protons, respectively, in the diagonal spin basis given by Eqs. (1.189) and the 4-momentum of the final proton p' have the form

$$a = (0, \mathbf{a}) = (0, \mathbf{n}), \quad a' = \frac{1}{M}(p, E'\mathbf{n}),$$
$$p' = (E', p\mathbf{n}), \quad E' = M(1+2\tau), \quad p = 2M\sqrt{\tau(1+\tau)},$$

where $\mathbf{n}$ is the unit vector along the direction of the motion of the final proton. Thus, the spin 4-vectors of the diagonal spin basis for the initial and final protons are defined so that the axes of the spin projections of the protons in the laboratory reference frame under consideration coincide and are directed along the momentum of the final proton: $\mathbf{a} = \mathbf{n}$. In this case, the spin state of the final proton is helical. The 4-momentum of the initial electron has the form $p_1 = E_e(1,0,0,1)$.

Calculating the convolution $I = L^{\mu\nu} P_{\mu\nu}$ of the lepton,

$$L^{\mu\nu} = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu + \frac{q^2 g^{\mu\nu}}{2},$$

and proton,

$$P_{\mu\nu} = \frac{1}{4} \operatorname{Tr}(\hat{p}' + M)(1 - \gamma_5 \hat{a}')\gamma_\mu \left(F_1 - \hat{q}\frac{1}{2M}F_2\right) \times \\ \times (\hat{p} + M)(1 - \gamma_5 \hat{a})\gamma_\nu \left(F_1 + \hat{q}\frac{1}{2M}F_2\right),$$

tensors and using the kinematic relations

$$(aa') = -1 - 2\tau, \quad (ap')(a'p) = -4M^2\tau(1+\tau),$$
$$(ap')(ap_1) = \tau[2M^2(1+2\tau) - s],$$

$$(ap_1)(a'p) = -2M(E+M)\tau,$$
$$(ap_1)(a'p_1) = \frac{\tau}{1+\tau}(E+M)[-E+M(1+2\tau)],$$

we arrive at the result (1.180). To generalize the diagonal spin basis to the case of the partially polarized states of the initial and final protons and to derive the corresponding general expressions for the ratio of the cross sections with and without proton spin flip, it is necessary to change $a \to \lambda a$ and $a' \to \lambda' a'$, $\lambda, \lambda' \leqslant 1$. As a result, we obtain Eq. (1.190).

The energy and 3-momentum of the recoil proton in the laboratory reference frame are expressed in terms of its scattering angle with respect to the direction of the motion of the initial electron θ_p (see [70]) as

$$\frac{E'}{M} = \frac{1+\cos^2\theta_p}{\sin^2\theta_p}, \quad \frac{p}{M} = \frac{2\cos\theta_p}{\sin^2\theta_p}.$$

The use of the matrix elements of the proton current given by Eqs. (1.186) and (1.187) reduces the calculation of the probability of the $ep \to ep$ process to the calculation of the trivial trace

$$\begin{aligned} &\mid T \mid^2 = \\ &= \frac{4M^2}{q^4}\frac{1}{8}\sum_\delta \mathrm{Tr}\left(G_E^2(\hat{p}_2+m)\hat{b}_0(\hat{p}_1+m)\hat{b}_0 + \tau G_M^2(\hat{p}_2+m)\hat{b}_\delta(\hat{p}_1+m)\hat{b}_\delta^*\right). \end{aligned}$$

As a result, we arrive at the following expression coinciding with the result presented in [23]:

$$\begin{gathered} d\sigma = \frac{\alpha^2 do}{4w^2}\frac{1}{1+\tau}(G_E^2 Y_{\mathrm{I}} + \tau G_M^2 Y_{\mathrm{II}})\frac{1}{q^4}, \\ Y_{\mathrm{I}} = (p_+q_+)^2 + q_+^2 q^2, \quad Y_{\mathrm{II}} = (p_+q_+)^2 - q_+^2(q^2+4m^2), \\ p_+ = p_1 + p_2, \quad q_+ = p + p', \end{gathered} \tag{1.201}$$

where do is the solid angle element, and w is the total energy in the center-of-mass system.

The ratio of the cross sections for the processes with and without proton spin flip is given by the expression

$$\frac{d\sigma_{\uparrow\uparrow}}{d\sigma_{\uparrow\downarrow}} = \frac{Y_{\mathrm{I}}}{\tau Y_{\mathrm{II}}}\frac{G_E^2}{G_M^2}. \tag{1.202}$$

In the rest system of the initial proton, neglecting the electron mass, we obtain Eqs. (1.178) and (1.180).

2. RADIATIVE PROCESSES IN e^+e^- AND $\gamma e^{\pm}$ COLLISIONS

2.1. Hadronic Cross Sections in Electron–Positron Annihilation with Tagged Photon. Experiments with tagged photons, radiated from the initial state in electron–proton and electron–positron collisions become particularly attractive. The reason is [72] that these radiative processes will permit one to extract information about the final states at continuously varying values of the collision energy. To investigate deep inelastic scattering, the authors of [73] suggested to use radiative events instead of using the colliders at reduced beam energies. The method takes advantage of a PD placed in the very forward direction from the incoming electron beam. The effective beam energy, for each radiative event, is determined by the energy of the hard photon observed in PD. In fact, radiative events were already used to measure the structure function F_2 down to $Q^2 \geqslant 1.5$ GeV^2 [74]. The specific theoretical work concerns the evaluation of QED radiative corrections [75, 76] to the radiative Born cross section. With an accurate determination of the cross sections and of the possible sources of background, we believe that the use of radiative events may become particularly useful to carry investigations at various present and future machines.

A detailed analysis of bremstrahlung cross section including the case of polarized photon was considered in papers [94, 95].

The important role of the initial-state radiation in the process of electron–positron annihilation was underlined in a series of papers by V. N. Baier and V. A. Khoze [39, 40, 77], where the radiative process was studied in detail in the Born approximation. In these papers the mechanism of returning to a resonant region was discovered. This mechanism consists in the preferable emission of photons from the initial particles, which provides a resonant kinematics of a subprocess. A utilization of radiative events can become a common type of investigations at various machines.

In this Section we derive explicit formulae for the spectrum of tagged photons. The calculations are performed with an accuracy of the per-mill order. Formulae can be used at electron–positron colliders to investigate, for instance, hadronic final states at intermediate energies. A measurement of the total hadronic cross section at low energies is essential as a high precision test of the Standard Model particularly for a precise determination of the fine structure constant $\alpha_{\mathrm{QED}}(M_Z)$ and of the muon anomalous magnetic moment $(g-2)_\mu$. The largest contribution to the errors for these quantities comes from the large indetermination still present on the measurement of the total hadronic cross section in electron–positron annihilation at the center-of-mass energies of a few GeV.

We will consider here the RC-corrected cross section for the radiative electron–positron annihilation process

$$e^-(p_1) + e^+(p_2) \longrightarrow \gamma(k) + X(q), \quad k = (1-z)p_1, \tag{2.1}$$

where X is a generic hadronic state. The hard photon hitting of the PD has a momentum k and an energy fraction $1-z$ with respect to the beam energy. In the following, we assume that the PD is placed along the electron beam direction, and has an opening angle $2\theta_0 \ll 1$, such that $\varepsilon^2\theta_0^2 \gg m^2$, with m — the electron mass, and ε — the beam energy. Below we first consider the cross section of the process (2.1) in the Born approximation. We give formulae suitable to study both the differential distributions in hadronic channels, as well as the total (in terms of quantity R),

$$R = \frac{\sigma_{e\bar{e}\to X}}{\sigma_{e\bar{e}\to\mu\bar{\mu}}}, \tag{2.2}$$

and inclusive (in terms of hadron fragmentation functions) hadronic cross sections. Further we calculate separate contributions into radiative corrected cross section of the process (2.1) within the next-to-leading accuracy due to virtual- and soft-photon emission. The case, when additional hard photon hits a PD is considered as well.

2.1.1. The Born Approximation. In order to obtain the Born approximation for the cross section of the process (2.1), when the PD is placed in front of electron (positron) beam, we can use the quasi-real electron method [5, 6]. It gives

$$d\sigma(k, p_1, p_2) = dW_{p_1}(k)\, \sigma_0(p_1 - k, p_2), \tag{2.3}$$

where $dW_{p_1}(k)$ is the probability to radiate photon with energy fraction $1-z$ inside a narrow cone with the polar angle not exceeding $\theta_0 \ll 1$ around the incoming electron, and $d\sigma_0$ is the differential cross section for the radiationless process of electron–positron annihilation into hadrons at the reduced electron beam energy. The form of both, $dW_{p_1}(k)$ and $\sigma_0(p_1-k, p_2)$ is well known:

$$\begin{gathered} dW_{p_1}(k) = \frac{\alpha}{2\pi} P_1(z, L_0)\, dz, \\ P_1(z, L_0) = \frac{1+z^2}{1-z}(L_0 - 1) + (1-z), \quad L_0 = \ln\frac{\varepsilon^2\theta_0^2}{m^2}. \end{gathered} \tag{2.4}$$

We need further the general form of the lowest order cross section σ_0 for the process $e^+(z_1 p_2) + e^-(z p_1) \to \gamma^*(q) +$ hadrons boosted along the beam axis ($\mathbf{p}_1$):

$$\begin{gathered} \sigma_0(z, z_1) = \frac{8\pi^2\alpha^2}{q^2|1-\Pi(q^2)|^2} \int T(q)\, d\Gamma(q), \quad T(q) = \frac{L_{\rho\sigma} H_{\rho\sigma}}{(q^2)^2}, \\ L_{\rho\sigma} = \frac{q^2}{2}\widetilde{g}_{\rho\sigma} + 2z^2 \widetilde{p}_{1\rho}\widetilde{p}_{1\sigma}, \quad d\Gamma(q) = (2\pi)^4 \delta\left(q - \sum q_j\right) \prod \frac{d^3 q_j}{2\varepsilon_j (2\pi)^3}, \\ q = z p_1 + z_1 p_2, \quad q^2 = s z_1 z, \quad \widetilde{g}_{\rho\sigma} = g_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2}, \quad \widetilde{p}_{1\rho} = p_{1\rho} - \frac{p_1 q}{q^2} q_\rho, \end{gathered} \tag{2.5}$$

where q is the full 4-momentum of final hadrons; q_j is 4-momentum of an individual hadron; $s = 2p_1p_2 = 4\varepsilon^2$ is the full center-of-mass energy squared, and $H_{\rho\sigma}$ is the hadronic tensor. The vacuum polarization operator $\Pi(q^2)$ of the virtual photon with momentum q is a known function [78] and will not be specified here.

Tensors $H_{\rho\sigma}$ and $L_{\rho\sigma}$ obey the current conservation conditions once saturated with the 4-vector q. The differential cross section with respect to the tagged photon energy fraction z can be obtained by performing the integration on the hadrons phase space. It takes the form

$$\frac{d\sigma}{dz} = \frac{\alpha}{2\pi} P_1(z, L_0)\,\sigma_0(z, 1). \tag{2.6}$$

Each hadronic state is described by its own hadronic tensor. The cross section in Eqs. (2.4) and (2.5) is suitable for different uses and, as mentioned above, it can be used to check different theoretical predictions.

The sum of the contributions of all hadronic channels by means of the relation

$$\sum_h \int H_{\rho\sigma}\, d\Gamma = f_h(q^2)\widetilde{g}_{\rho\sigma}, \tag{2.7}$$

can be expressed in terms of the ratio of the total cross section for annihilation into hadrons and muons $R = \sigma_h/\sigma_\mu$. For the $\mu^+\mu^-$ final state we get

$$f_\mu = \frac{q^2}{6\pi} K(q^2), \quad K(q^2) = \left(1 + \frac{2m_\mu^2}{q^2}\right)\sqrt{1 - \frac{4m_\mu^2}{q^2}},$$

and so,

$$f_h(q^2) = \frac{q^2 R(q^2)}{6\pi} K(q^2). \tag{2.8}$$

Substituting this expression into the right-hand side of Eqs. (2.4), (2.5) results in the replacement $\sigma_0(z, z_1) = R(q^2)4\pi\alpha^2 K(q^2)/(3q^2)$.

In experiments of semi–inclusive type one fixes an hadron with 3-momentum $\mathbf{q}_1$ energy ϵ_1 and mass M in every event and sums over all the rest. In this case instead of Eq. (2.7) we will have (similarly to the DIS case [76]):

$$\begin{gathered}
\sum_{h'} \int H_{\rho\sigma}\, d\Gamma = H^{(1)}_{\rho\sigma} \frac{d^3 q_1}{2\varepsilon_1 (2\pi)^3}, \\
H^{(1)}_{\rho\sigma} = F_1(\eta, q^2)\widetilde{g}_{\rho\sigma} - \frac{4}{q^2} F_2(\eta, q^2)\widetilde{q}_{1\rho}\widetilde{q}_{1\rho}, \\
\eta = \frac{q^2}{2qq_1} > 1, \quad \tilde{q}_1 = q_1 - q\frac{qq_1}{q^2},
\end{gathered} \tag{2.9}$$

where we have introduced two dimensionless functions $F_1(\eta, q^2)$ and $F_2(\eta, q^2)$ in a way similar to the DIS case.

By introducing the dimensionless variable $\chi = 2qq_1/(2zp_1q_1)$, we can write the corresponding cross section for radiative events in e^+e^- annihilation in the same form as in the case of deep inelastic scattering with a tagged photon [76]:

$$\begin{gathered}\frac{d\sigma}{dz} = \frac{\alpha^2(q^2)}{2\pi}\,\frac{\alpha}{2\pi}P_1(z, L_0)\Sigma(\eta, \chi, q^2)\frac{1}{(q^2)^2}\,\frac{d^3q_1}{\varepsilon_1},\\ \Sigma(\eta, \chi, q^2) = F_1(\eta, q^2) + \frac{2F_2(\eta, q^2)}{\eta^2\chi^2}\left(\chi - 1 - \frac{M^2}{q^2}\eta^2\chi^2\right).\end{gathered} \tag{2.10}$$

2.1.2. Corrections Due to the Virtual and Real Soft Photons. The interference of Born and one-loop contributions to the amplitude of the initial-state radiation in annihilation of e^+e^- into hadrons can be obtained from the analogous quantity of hard-photon emission in electron–proton scattering [76]. We do that by using the crossing transformation. The contribution coming from the emission of real soft photons is:

$$\begin{gathered}\frac{d\sigma^S}{d\sigma_0} = \frac{\alpha}{\pi}\left[2(L_s - 1)\ln\frac{m\Delta\varepsilon}{\lambda\varepsilon} + \frac{1}{2}L_s^2 - \frac{\pi^2}{3}\right],\\ L_s = \ln\frac{s}{m^2} = L_0 + L_\theta, \quad L_\theta = \ln\frac{4}{\theta^2}.\end{gathered} \tag{2.11}$$

where λ is the «photon mass», $\Delta\varepsilon$ is the energy in c.m.s. carried by the soft photon. The sum of the two contributions is free from infrared singularities. It reads

$$d\sigma^{V+S} = \frac{8\pi^2\alpha^2}{s|1 - \Pi(q^2)|^2}\frac{\alpha}{2\pi}[\rho B_{\rho\sigma}(q) + A_{\rho\sigma}(q)]\frac{H_{\rho\sigma}(q)\,d\Gamma(q)}{(q^2)^2}\frac{\alpha}{4\pi^2}\frac{d^3k}{\omega}, \tag{2.12}$$

where

$$\rho = 4(L_s - 1)\ln\Delta + 3L_q - \frac{\pi^2}{3} - \frac{9}{2}, \quad L_q = L_s + \ln z, \quad \Delta = \frac{\Delta\varepsilon}{\varepsilon} \ll 1, \tag{2.13}$$

where k and ω are the 3-momentum and the energy of the hard photon, respectively. The tensors $A_{\rho\sigma}$ and $B_{\rho\sigma}$ have a rather involved form. The first can be obtained from the corresponding expressions of [8]. The tensor $B_{\rho\sigma}$ coincides with the one of the Born approximation. In the kinematical region where the hard photon is emitted close to the initial electron direction of motion one has

$$\begin{gathered}B_{\rho\sigma} = \frac{2}{z}\left(\frac{1+z^2}{y_1(1-z)} - \frac{2m^2z}{y_1^2}\right)L_{\rho\sigma}(q),\\ A_{\rho\sigma} = \frac{2}{q^2}A_gL_{\rho\sigma}(q), \quad q = zp_1 + p_2,\end{gathered} \tag{2.14}$$

where tensor $L_{\rho\sigma}$ is given in Eq. (2.5), $y_1 = 2kp_1$, and quantity A_g reads

$$A_g = \frac{4zsm^2}{y_1^2} L_s \ln z + \frac{s}{y_1}\left[\frac{1+z^2}{1-z}\left(-2L_s \ln z - \ln^2 z + 2\mathrm{Li}_2(1-z) + \right.\right.$$
$$\left.\left. + 2\ln\frac{y_1}{m^2}\ln z\right) + \frac{1+2z-z^2}{2(1-z)}\right]. \quad (2.15)$$

Further integration over the hard-photon phase space can be performed within the logarithmic accuracy by using the integrals

$$\int \frac{d^3k}{2\pi k_0}\left[\frac{1}{y_1}, \frac{m^2}{y_1^2}, \frac{\ln(y_1/m^2)}{y_1}\right] = \int \left[\frac{1}{2}L_0, \frac{1}{2(1-z)}, \frac{1}{4}L_0^2 + \frac{1}{2}L_0 \ln(1-z)\right] dz.$$

The final expression for the Born cross section corrected for the emission of soft and virtual photons has the form

$$\frac{d\sigma^{B+V+S}}{dz} = \sigma_0(z,1)\left[\frac{\alpha}{2\pi}P_1(z,L_0) + \left(\frac{\alpha}{2\pi}\right)^2 (\rho P_1(z,L_0) + N)\right],$$
$$N = -\frac{1+z^2}{1-z}\left[(L_0 + \ln z)\ln z - \frac{\pi^2}{3} + 2\mathrm{Li}_2(z)\right] L_0 - 2P_1(z,L_0)\ln\frac{\theta_0^2}{4} +$$
$$+ \frac{1+2z-z^2}{2(1-z)}L_0 + \frac{4z}{1-z}L_0 \ln z. \quad (2.16)$$

2.1.3. Two Hard Photons Are Tagged by the Detector. If an additional hard photon emitted by the initial-state electron hits the PD, we cannot use the quasi-real electron method and have to calculate the corresponding contribution starting from the Feynman diagrams.

We can use double hard photon spectra as given in [79] for annihilation diagrams only and write the cross section under consideration as follows:

$$\frac{d\sigma_{c1}^H}{dz} = \sigma_0(z,1)\left(\frac{\alpha}{2\pi}\right)^2 L_0 \times$$
$$\times \int\limits_{\Delta}^{1-z-\Delta} \frac{dx}{\xi}\left[\frac{\gamma\tau}{2}L_0 + (z^2 + (1-x)^4)\ln\frac{(1-x)^2(1-z-x)}{zx} + \right.$$
$$\left. + zx(1-z-x) - x^2(1-x-z)^2 - 2\tau(1-x)\right], \quad (2.17)$$
$$\xi = x(1-x)^2(1-z-x), \quad \gamma = 1 + (1-x)^2, \quad \tau = z^2 + (1-x)^2.$$

Here the variable x under the integral is the energy fraction of one hard photon. The quantity $1-z-x$ is the energy fraction of the second hard photon provided that their total energy fraction equals $1-z$. We write the index $c1$ in the left-hand side of Eq. (2.17) to emphasize that this contribution arises from the collinear kinematics, when the additional hard photon is emitted along the initial electron with 4-momentum p_1.

The integration in the right-hand side of Eq. (2.17) leads to the result

$$\frac{d\sigma^H_{c1}}{dz}=\sigma_0(z,1)\left(\frac{\alpha}{2\pi}\right)^2\frac{L_0}{2}\left\{\left[P^{(2)}_\Theta(z)+2\frac{1+z^2}{1-z}\left(\ln z-\frac{3}{2}-2\ln\Delta\right)\right]L_0+\right.$$
$$\left.+\,6(1-z)+\frac{3+z^2}{1-z}\ln^2 z-\frac{4(1+z)^2}{1-z}\ln\frac{1-z}{\Delta}\right\}, \quad (2.18)$$

where the quantity $P^{(2)}_\Theta(z)$ represents the so-called Θ term of the second-order electron structure function:

$$P^{(2)}_\Theta(z)=2\frac{1+z^2}{1-z}\left(\ln\frac{(1-z)^2}{z}+\frac{3}{2}\right)+(1+z)\ln z-2(1-z). \quad (2.19)$$

2.1.4. Additional Hard Photon Is Emitted outside PD. If an additional hard photon, emitted from the initial state, does not hit the PD situated in the direction of motion of the initial electron, we distinguish the case when it is emitted in the direction close, within a small cone with angle $\theta'\ll 1$, to the direction of the initial positron. In this case we obtain:

$$\frac{d\sigma^H_{c2}}{dz}=\frac{\alpha}{2\pi}P_1(z,L_0)\int\limits_{\Delta}^{1-\delta/z}\frac{\alpha}{2\pi}P_1(1-x,L')\sigma_0(z,1-x)\,dx, \quad (2.20)$$

where $L'=L_s+\ln(\theta'^2/4)$, $\delta=M^2/s$, and M^2 is the minimal hadron mass squared. We suppose that $z\sim 1$.

We have introduced the additional auxiliary parameter $\theta'\ll 1$ which, together with θ_0, separates collinear and semicollinear kinematics of the second hard photon. Contrary to θ_0, which is supposed to determine the PD acceptance, θ' will disappear in the sum of the collinear and semicollinear contributions of the second photon. This last kinematical region gives

$$\frac{d\sigma^H_{\mathrm{sc}}}{dz}=\left(\frac{\alpha}{2\pi}\right)^2P_1(z,L_0)\int\frac{d^3k_1}{2\pi\omega_1^3}\frac{16\pi^2\alpha^2}{(1-c^2)z^2}T(c,z,x),$$
$$T(c,z,x)=\int\frac{H_{\rho\sigma}(q_2)d\Gamma(q_2)}{s(q_2^2)^2|1-\Pi(q_2^2)|^2}\left[\frac{s}{2}((z-x_2)^2+z^2(1-x_1)^2)g_{\rho\sigma}+\right.$$
$$\left.+\,2(z(1-x_1)-x_2)(z^2p_{1\rho}p_{1\sigma}+p_{2\rho}p_{2\sigma})\right], \quad (2.21)$$
$$x_1=\frac{x}{2}(1-c),\quad x_2=\frac{x}{2}(1+c),\quad q_2=zp_1+p_2-k_1,\quad c=\cos\widehat{\mathbf{k}_1\mathbf{p}_1}.$$

The phase volume of the second photon is parameterized as

$$\int \frac{d^3k_1}{2\pi\omega^3} = \int\limits_{\Delta}^{\hat{x}} \frac{dx}{x} \int\limits_{0}^{2\pi} \frac{d\varphi}{2\pi} \int\limits_{-1+\theta'^2/2}^{1-\theta_0^2/2} dc, \quad \hat{x} = \frac{2(z-\delta)}{1+z+c(1-z)}. \tag{2.22}$$

Explicitly extracting the angular singularities we represent this expression as

$$\frac{d\sigma_{\rm sc}^H}{dz} = \left(\frac{\alpha}{2\pi}\right)^2 P_1(z,L_0)\left[\Sigma_{\rm sc}(z) + \ln\frac{4}{\theta_0^2}\int\limits_{\Delta}^{z-\delta}\frac{dx}{x}\frac{z^2+(z-x)^2}{z^2}\sigma_0(z-x,1) + \right.$$
$$\left. + \ln\frac{4}{\theta'^2}\int\limits_{\Delta}^{1-\delta/z}\frac{dx}{x}(1+(1-x)^2)\sigma_0(z,1-x)\right], \tag{2.23}$$

$$\Sigma_{\rm sc} = \frac{8\pi^2\alpha^2}{z^2}\int\limits_{-1}^{1} dc \int\limits_{\Delta}^{\hat{x}} \frac{dx}{x}\left[\frac{T(c,z,x)-T(1,z,x)}{1-c} + \frac{T(c,z,x)-T(-1,z,x)}{1+c}\right].$$

2.1.5. Complete QED Correction and Leading Logarithmic Approximation. The final result in the order $\mathcal{O}(\alpha)$ for radiative corrections to radiative events can be written as follows:

$$\frac{d\sigma}{dz} = \frac{\alpha}{2\pi}P_1(z,L_0)\sigma_0(z,1)(1+r) =$$
$$= \frac{\alpha}{2\pi}P_1(z,L_0)\sigma_0(z,1) + \left(\frac{\alpha}{2\pi}\right)^2\left\{L_0\left(\frac{1}{2}L_0P^{(2)}(z)+G\right)\sigma_0(z,1) + P_1(z,L_0)\times\right.$$
$$\left.\times\left[\int\limits_0^{1-\delta/z} C_1(x)\sigma_0(z,1-x)\,dx + L_\theta\int\limits_0^{z-\delta} C_2(z,x)\sigma_0(z-x,1)\,dx + \Sigma_{\rm sc}\right]\right\}, \tag{2.24}$$

where the last term is defined by Eq. (2.23), $L_\theta = \ln 4/\theta_0^2$ and

$$C_1(x) = P_1(1-x,L_s)\Theta(x-\Delta) + (L_s-1)\left(2\ln\Delta+\frac{3}{2}\right)\delta(x),$$
$$C_2(z,x) = \frac{z^2+(z-x)^2}{z^2x}\Theta(x-\Delta) + \left(2\ln\Delta+\frac{3}{2}-2\ln z\right)\delta(x), \tag{2.25}$$
$$G(z) = \frac{1+z^2}{1-z}(3\ln z - 2{\rm Li}_2(z)) + \frac{1}{2}(1+z)\ln^2 z - \frac{2(1+z)^2}{1-z}\ln(1-z) +$$
$$+ \frac{1-16z-z^2}{2(1-z)} + \frac{4z\ln z}{1-z}.$$

In order to include the higher order leading corrections to the tagged photon differential cross section and show the agreement of our calculation with the well-known Drell–Yan representation for the total hadronic cross section at electron–positron annihilation [11]

$$\sigma(s) = \int\limits_{\delta}^{1} dx_1 \int\limits_{\delta/x_1}^{1} dx_2\, D(x_1, \alpha_{\text{eff}}) D(x_2, \alpha_{\text{eff}}) \sigma_0(x_1 x_2 s), \tag{2.26}$$

where the electron structure functions include both nonsinglet and singlet parts

$$D(x_1, \alpha_{\text{eff}}) = D^{\text{NS}}(x, \alpha_{\text{eff}}) + D^{S}(x_1, \alpha_{\text{eff}}), \tag{2.27}$$

it is convenient to introduce the quantity

$$\Sigma = D(z, \bar{\alpha}_{\text{eff}}) \int\limits_{\delta/z}^{1} dx_1 \int\limits_{\delta/zx_1}^{1} dx_2\, D(x_1, \widetilde{\alpha}_{\text{eff}}) D(x_2, \hat{\alpha}_{\text{eff}}) \sigma_0(z x_1 x_2 s). \tag{2.28}$$

Note that we use here another definition of nonsinglet structure function [80, 81], which enters into the right-hand side of Eq. (2.27)

$$D^{\text{NS}}(x, \alpha_{\text{eff}}) = \delta(1-x) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\alpha_{\text{eff}}}{2\pi} \right)^{n} P_1^{\otimes n}(x), \tag{2.29}$$

$$D^{S}(x, \alpha_{\text{eff}}) = \frac{1}{2!} \left(\frac{\alpha_{\text{eff}}}{2\pi} \right)^{2} R(x) + \frac{1}{3!} \left(\frac{\alpha_{\text{eff}}}{2\pi} \right)^{3} \left[2P_1 \otimes R(x) - \frac{2}{3} R(x) \right], \tag{2.30}$$

with $P_1(x)$ given above (see [97]) and

$$R(x) = 2(1+x)\ln x + \frac{1-x}{3x}(4 + 7x + 4x^2), \tag{2.31}$$

$$P_1(x) \otimes R(x) = \int\limits_{x}^{1} P_1(t) R\left(\frac{x}{t} \right) \frac{dt}{t}, \tag{2.32}$$

and the effective electromagnetic couplings in the right-habd side of Eq. (2.28) are

$$\begin{aligned} \bar{\alpha}_{\text{eff}} &= -3\pi \ln\left(1 - \frac{\alpha}{3\pi} L_0 \right), \\ \widetilde{\alpha}_{\text{eff}} &= -3\pi \ln \left(\frac{1 - \frac{\alpha}{3\pi} L_s}{1 - \frac{\alpha}{3\pi} L_0} \right), \\ \hat{\alpha}_{\text{eff}} &= -3\pi \ln\left(1 - \frac{\alpha}{3\pi} L_s \right). \end{aligned} \tag{2.33}$$

At fixed values of z $(z < 1)$, the quantity Σ defines the leading logarithmic contributions into differential cross section for the events with tagged particles. That corresponds to only Θ terms in the expansion of the structure function $D(z, \bar{\alpha}_{\mathrm{eff}})$ before the integral in Eq. (2.28). If we consider photon corrections (as in the previous Sections) it needs to restrict ourselves with the nonsinglet part of the electron structure functions and with the first order terms in the expansion of all effective couplings, namely:

$$\bar{\alpha}_{\mathrm{eff}} \to \alpha L_0, \quad \widetilde{\alpha}_{\mathrm{eff}} \to \alpha L_\theta, \quad \hat{\alpha}_{\mathrm{eff}} \to \alpha L_s. \tag{2.34}$$

It is easy to see that in this case the leading contribution into differential cross section (2.24) can be obtained as an expansion of the quantity $\Sigma(z < 1)$ by the powers of α, keeping the terms of the order α^2 in the production of D functions.

To include the contribution due to e^+e^--pair (real and virtual) production, it is required [82] to use both nonsinglet and singlet structure functions and effective couplings defined by Eq. (2.33). Note that the insertion into consideration of higher order corrections rises additional questions about experimental conditions concerning registration of events with e^+e^- pairs.

The total hadronic cross section in e^+e^- annihilation can be obtained by integration of quantity Σ over z (see (2.28))

$$\sigma(s) = \int\limits_{\delta}^{1} dz\, D(z, \bar{\alpha}_{\mathrm{eff}}) \int\limits_{\delta/z}^{1} dx_1 \times \\ \times \int\limits_{\delta/zx_1}^{1} dx_2\, D(x_1, \widetilde{\alpha}_{\mathrm{eff}}) D(x_2, \hat{\alpha}_{\mathrm{eff}}) \sigma(zx_1x_2s). \tag{2.35}$$

We can integrate the expression in the right-hand side of Eq. (2.35) over the variable z provided the quantity $zx_1 = y$ fixed

$$\int\limits_{\delta}^{1} dz\, D(z, \bar{\alpha}_{\mathrm{eff}}) \int\limits_{\delta/z}^{1} dx_1\, D(x_1, \widetilde{\alpha}_{\mathrm{eff}}) = \int\limits_{\delta}^{1} dz \int\limits_{y}^{1} dy\, D(z, \bar{\alpha}_{\mathrm{eff}}) D\left(\frac{y}{z}, \widetilde{\alpha}_{\mathrm{eff}}\right) = \\ = \int\limits_{\delta}^{1} dy\, D(y, \bar{\alpha}_{\mathrm{eff}} + \widetilde{\alpha}_{\mathrm{eff}}), \quad \bar{\alpha}_{\mathrm{eff}} + \widetilde{\alpha}_{\mathrm{eff}} = \hat{\alpha}_{\mathrm{eff}}. \tag{2.36}$$

Using this result and definition of $\hat{\alpha}_{\mathrm{eff}}$ we indicate the equivalence of the Drell–Yan form of the total cross section as given by Eq. (2.26) and the representation of the cross section by Eq. (2.35).

Let us show now that D functions in expression for the quantity Σ have effective couplings as given by Eq. (2.33). By definition, the nonsinglet electron structure function satisfies the equation [9, 12]

$$D(x, s, s_0) = \delta(1-x) + \frac{1}{2\pi}\int\limits_{s_0}^{s} \frac{ds_1}{s_1}\alpha(s_1)\int\limits_{x}^{1}\frac{dz}{z}D(z)D\left(\frac{x}{z}, \frac{s_1}{s_0}\right), \qquad (2.37)$$

where $\alpha(s_1)$ is the electromagnetic running coupling

$$\alpha(s_1) = \alpha\left(1 - \frac{\alpha}{3\pi}\ln\frac{s_1}{m^2}\right)^{-1},$$

and $s_0(s)$ is the minimal (maximal) virtuality of the particle, which radiates photons and e^+e^- pairs.

The structure function $D(z, \bar{\alpha}_{\text{eff}})$ describes the photon emission and pair production inside narrow cone along the electron beam direction. In this kinematics $s_0 = m^2$, $s = \varepsilon^2\theta_0^2$. The corresponding iterative solution of Eq. (2.37) has the form (2.29) with $\alpha_{\text{eff}} = \bar{\alpha}_{\text{eff}}$. The structure function $D(x_1, \widetilde{\alpha}_{\text{eff}})$ describes the events, when emitted (by the electron) particles escape this narrow cone. In this case $s_0 = \varepsilon^2\theta_0^2$, $s = 4\varepsilon^2$. The corresponding solution of Eq. (2.37) gives the structure function with $\alpha_{\text{eff}} = \widetilde{\alpha}_{\text{eff}}$. At last, the structure function $D(x_2, \hat{\alpha}_{\text{eff}})$ is responsible for the radiation off the positron into the whole phase space. In this case $s_0 = m^2$, $s = 4\varepsilon^2$. Therefore we obtain D function with $\alpha_{\text{eff}} = \hat{\alpha}_{\text{eff}}$. The analogous consideration can be performed for the singlet part of structure functions.

When writing the representation (2.35) for the total cross section we, in fact, divide the phase space of the particles emitted by the electron on the regions inside and outside the narrow cone along electron beam direction. Therefore, we can use this representation to investigate the events with tagged particles in both these regions. As we saw before, the differential cross section for events with tagged particles inside the narrow cone is defined by the quantity $\Sigma(z < 1)$. In order to obtain the corresponding differential cross section for events with tagged particles outside this narrow cone we have to change the places of $\bar{\alpha}_{\text{eff}}$ and $\widetilde{\alpha}_{\text{eff}}$ in expression for $\Sigma(z, 1)$. This follows from the symmetry of representation (2.35) relative to such a change.

2.1.6. Numerical Estimation. In Fig. 15, we show the cross section $d\sigma/dz$ as a function of z. The beam energy is chosen to be $E_{\text{beam}} = 0.5$ GeV. The region of z values is limited by the pion production threshold at the left, and by the threshold of photon detection (we choose 50 MeV) at the right. The peak in the middle corresponds to the large contribution of the ρ meson. Values of R, used for numerical estimations, were taken from [20]. The values of corrections r (see Eq. (2.36)) in percent are shown in Fig. 16.

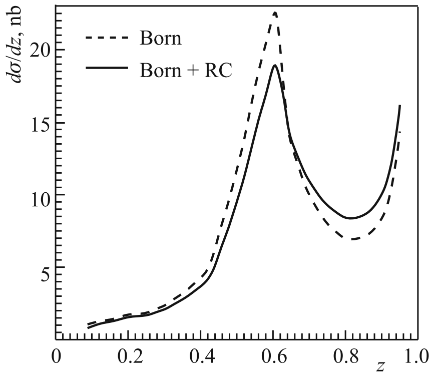

Fig. 15. The cross section of $e^+e^- \to$ hadrons with tagged photon

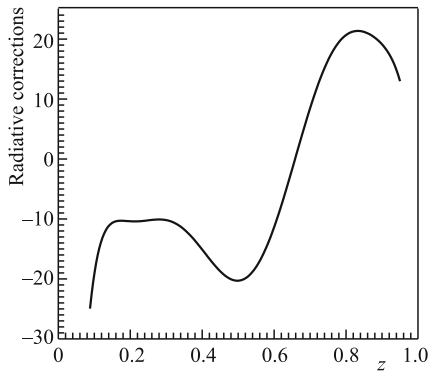

Fig. 16. The radiative corrections to $e^+e^- \to$ hadrons with tagged photon

So, we calculated the cross section of e^+e^- annihilation with detection of a hard photon at small angles in respect to the electron beam. The general structure of a measured cross section, from which one should extract the annihilation cross section σ_0, looks as follows:

$$\sigma = \sigma_0 \left[a_1 \frac{\alpha}{\pi} L + b_1 \frac{\alpha}{\pi} + a_2 \left(\frac{\alpha}{\pi}\right)^2 L^2 + b_2 \left(\frac{\alpha}{\pi}\right)^2 L + c_2 \left(\frac{\alpha}{\pi}\right)^2 + \mathcal{O}(\alpha^3) \right], \quad (2.38)$$

where L denotes some large logarithm. We calculated the terms a_1, b_1, a_2, b_2 and some contributions to c_2. The generalized formula (2.28) allows one to involve the leading terms of the order $\mathcal{O}(\alpha^3 L^3)$. In this way our formulae provide high theoretical precision.

Similar formulae can be obtained for an experimental setup by tagging a definite hadron. Keeping in mind such installations as BEPS, DAΦNE [83], VEPP, CLEO, SLAC-B/factory and others with luminosity of order 10^{33} cm$^{-2} \cdot$ s^{-1}, one may be in principle able to scan, by measuring the initial-state radiation spectrum, the whole energy region of hadron production with an effective luminosity of the order of 10^{31} cm$^{-2} \cdot$ s^{-1}. We hope further study will follow on these issues both from the experiments and theory.

2.2. High-Accuracy Description of Radiative Return Production of Low-Mass Muon and Pion Pairs at e^+e^- Colliders. Radiative return method, when the hard initial-state radiation is used to reduce the invariant mass of a hadronic system produced in the high-energy electron–positron annihilation, provides an important tool to study various hadronic cross sections in a wide range of invariant masses without actually changing the center-of-mass energy of the collider [39,77,85]. The very high luminosity of the modern meson factories makes the method competitive with the more conventional energy scan approach [86,87]. Preliminary experimental studies both at KLOE [88], BABAR [89], and BELLE confirm the excellent potential of the radiative return method. The case of resonance production by this mechanism was first considered in [90]. Recently, considerable efforts were devoted to elucidate the theoretical understanding of the radiative return process, especially for the case of low-energy pion-pair production (see, for example, [44, 91, 92]). In this Subsection [84] we, by means of explicit calculations, confirm the Drell–Yan form of the cross section. This fact allows us to take into account all leading and next-to-leading terms in all orders of PT and as a consequence to improve the accuracy of theoretical predictions.

The case, when the invariant mass of hadron system $\sqrt{s_1}$ is small compared to the center-of-mass total energy $\sqrt{s} = 2\varepsilon$, represents a special interest. Such situation is realized, for example, in the BABAR radiative return studies, where such interesting physical quantities as form factors of the pion and the nucleon can be investigated. The processes of radiative annihilation into muon and pion pairs, considered here, play a crucial role in such studies, both for the normalization purposes and as one of the principal hadron-production process at low energies.

We specify the kinematics of the radiative muon (pion)-pair creation process

$$e_-(p_-) + e_+(p_+) \to \mu_-(q_-) + \mu_+(q_+) + \gamma(k_1) \tag{2.39}$$

as follows:

$$\begin{gathered} p_\pm^2 = m^2, \quad q_\pm^2 = M^2, \quad k_1^2 = 0, \\ \chi_\pm = 2k_1 \cdot p_\pm, \quad \chi'_\pm = 2k_1 \cdot q_\pm, \quad s = (p_- + p_+)^2, \\ s_1 = (q_- + q_+)^2, \quad t = -2p_- \cdot q_-, \quad t_1 = -2p_+ \cdot q_+, \\ u = -2p_- \cdot q_+, \quad u_1 = -2p_+ \cdot q_-, \end{gathered} \tag{2.40}$$

where m and M are the electron and muon (pion) masses, respectively. Throughout the paper we will suppose

$$s \sim -t \sim -t_1 \sim -u \sim -u_1 \gg 4M^2 \gg m^2, \tag{2.41}$$

and $s \gg s_1 > M^2$.

We will systematically omit the terms of the order of M^2/s and m^2/s_1 compared with the leading ones. In $\mathcal{O}(\alpha)$ radiative corrections, we will drop also terms suppressed by the factor s_1/s. Also we imply that the invariant mass $s_1 \ll M_z^2$, where M_Z is the mass of Z boson (due to we do not consider the weak interactions).

In this Subsection we will consider only the charge-even part of the differential cross section, which can be measured in an experimental setup blind to the charges of the created particles. So we omit the contribution from box-type FD, when additional virtual photon connects both muon and electron line. In the final state we consider corrections only in the lowest order of PT including 1-loop virtual corrections, soft real ones and hard real photon contributions. We imply that invariant mass of the final pairs with photon emission is small compared with $\sqrt{s}$.

2.2.1. The Born-Level Cross Section. Within the Born approximation, the corresponding contribution to the cross section is

$$\begin{gathered}\frac{d\sigma_B^j}{d\Gamma} = \frac{\alpha^3}{8\pi^2 s s_1^2} R^j, \quad R^j = B^{\rho\sigma} i_{\rho\sigma}^{(0j)}, \quad j = \mu, \pi, \\ B_{\rho\sigma} = B_g g_{\rho\sigma} + B_{11}(p_-p_-)_{\rho\sigma} + B_{22}(p_+p_+)_{\rho\sigma},\end{gathered} \tag{2.42}$$

$$\begin{gathered}B_g = -\frac{(s_1+\chi_+)^2 + (s_1+\chi_-)^2}{\chi_+\chi_-}, \quad B_{11} = -\frac{4s_1}{\chi_+\chi_-}, \\ B_{22} = -\frac{4s_1}{\chi_+\chi_-},\end{gathered} \tag{2.43}$$

where we have used the shorthand notations $(qq)_{\rho\sigma} = q_\rho q_\sigma$, $(pq)_{\rho\sigma} = p_\rho q_\sigma + q_\rho p_\sigma$. For muon-pair final state

$$i_{\rho\sigma}^{(0\mu)} = 4\left[(q_+q_-)_{\rho\sigma} - g_{\rho\sigma}\frac{s_1}{2}\right]. \tag{2.44}$$

For the case of pions,

$$i_{\rho\sigma}^{(0\pi)} = |F_\pi^{\mathrm{str}}(s_1)|^2 (q_- - q_+)_\rho (q_- - q_+)_\sigma, \tag{2.45}$$

where $F_\pi^{\mathrm{str}}(s_1)$ is the pion strong interaction form factor, in which we include all the effects of strong interactions in two-pion formation.

Note that the Born-level cross section for the $e^+e^- \to \mu^+\mu^-\gamma$ process was calculated in [29, 93].

For the case of small invariant mass of the created pair $s_1 \ll s$, the phase space volume of the final particles can be rewritten as

$$d\Gamma = \pi^2 dx_- \, dc \, ds_1 \tag{2.46}$$

(note that s_1 is small ($s_1 \approx 4\mu^2$) due to $c \to 1$, but the energy of muon pair is large: $sx_\pm^2 \gg 4M^2$) and approximately

$$x_\pm = \frac{\varepsilon_\pm}{\varepsilon}, \quad x_+ + x_- = 1, \quad c = \cos(\widehat{\mathbf{p}_-\mathbf{q}_-}) = \cos\theta. \tag{2.47}$$

We will assume that the emission angle of the hard photon lies outside the narrow cones around the beam axis: $\theta_0 < \theta < \pi - \theta_0$, with $\theta_0 \ll 1$, $\theta_0\varepsilon \gg M$.

When the initial-state radiation dominates, the Born cross section takes a rather simple form:

$$d\sigma_B^{(\mu)}(p_-,p_+;k_1,q_-,q_+) = \frac{\alpha^3(1+c^2)}{ss_1(1-c^2)}[2\sigma+1-2x_-x_+]\,dx_-\,dc\,ds_1,$$

$$d\sigma_B^{(\pi)}(p_-,p_+;k_1,q_-,q_+) = \frac{\alpha^3(1+c^2)}{ss_1(1-c^2)}\,|F_\pi^{\mathrm{str}}(s_1)|^2\,[-\sigma+x_-x_+]\,dx_-dc\,ds_1,$$

$$\frac{1}{2}(1-\beta) < x_- < \frac{1}{2}(1+\beta), \quad \beta = \sqrt{1-\frac{4M^2}{s_1}}, \quad \sigma = \frac{M^2}{s_1}.$$

Here β is the velocity of the pair component in the center-of-mass reference frame of the pair.

2.2.2. Radiative Corrections. Radiative corrections can be separated into three gauge-invariant parts. They can be taken into account by the formal replacement (see (2.42)):

$$\frac{R^j}{s_1^2} \longrightarrow \frac{K^{\rho\sigma}J^j_{\rho\sigma}}{s_1^2|1-\Pi(s_1)|^2}, \tag{2.48}$$

where $\Pi(s_1)$ describes the vacuum polarization of the virtual photon, $K^{\rho\sigma}$ is the initial-state emission Compton tensor with RC taken into account; $J^j_{\rho\sigma}$ is the final-state current tensor with $\mathcal{O}(\alpha)$ RC.

First we consider the explicit formulae for RC due to virtual, soft, and hard collinear final-state emission. As concerns RC to the initial state for the charge-blind experimental setup considered here, we will use the explicit expression for the Compton tensor with heavy photon $K^{\rho\sigma}$ calculated in paper [8] for the scattering channel and apply the crossing transformation. Possible contribution due to emission of an additional real photon from the initial state will be taken into account, too.

2.2.3. Corrections Connected with the Final State. The third part is related to the lowest order RC to the muon (pion) current

$$J_{\rho\sigma} = i^{(v)}_{\rho\sigma} + i^{(s)}_{\rho\sigma} + i^{(h)}_{\rho\sigma}. \tag{2.49}$$

The virtual photon contribution $i^{(v)}_{\rho\sigma}$ takes into account the Dirac and Pauli form factors of the muon current

$$B^{\rho\sigma}i^{(v\mu)}_{\rho\sigma} = B_g\sum_{\mathrm{pol}}|J^{(v\mu)}_\rho|^2 + B_{11}\left[\sum_{\mathrm{pol}}|p_-\cdot J^{(v\mu)}|^2 + \right.$$
$$\left. + \sum_{\mathrm{pol}}|p_+\cdot J^{(v\mu)}|^2\right]. \tag{2.50}$$

Here Σ means a sum over the muon spin states and

$$\sum_{\mathrm{pol}} |J_\rho^{(v\mu)}|^2 = \frac{\alpha}{\pi}\left[-8(s_1+2M^2)f_1^{(\mu)} - 12 s_1 f_2^{(\mu)}\right],$$
$$\sum_{\mathrm{pol}} |J^{(v\mu)}\cdot p_\pm|^2 = \frac{\alpha}{\pi}s^2(1\pm c)^2\left(x_+x_-f_1^{(\mu)} + \frac{1}{4}f_2^{(\mu)}\right). \qquad (2.51)$$

Explicit expressions for the Dirac and Pauli form factors of muon are well known [3]:

$$F_1(s) = 1 + \frac{\alpha}{\pi}F_1^{(2)}(s), \quad F_2(s) = \frac{\alpha}{\pi}F_2^{(2)}(s), \quad f_{1,2} = \mathrm{Re}\, F_{1,2}^{(2)}. \qquad (2.52)$$

For the pion final state we have

$$B^{\rho\sigma}i_{\rho\sigma}^{(v\pi)} = 2\frac{\alpha}{\pi}B^{\rho\sigma}i_{\rho\sigma}^{(0\pi)}f_\pi^{\mathrm{QED}},$$
$$B^{\rho\sigma}i_{\rho\sigma}^{(0\pi)} = 2\frac{\alpha}{\pi}|F_\pi^{\mathrm{str}}(s_1)|^2\times$$
$$\times\left[(4M^2-s_1)B_g + \frac{1}{8}s^2B_{11}(x_+-x_-)^2(1+c^2)\right]f_\pi^{\mathrm{QED}}.$$

Explicit expression for the f_π^{QED} form factor of pion is [3]:

$$\mathrm{Re}\left[F_\pi^{\mathrm{QED}}(s_1)\right] = 1 + \frac{\alpha}{\pi}f_\pi^{\mathrm{QED}}(s_1),$$
$$f_\pi^{\mathrm{QED}}(s_1) = \left(\ln\frac{M}{\lambda}-1\right)\left(1-\frac{1+\beta^2}{2\beta}L_\beta\right) +$$
$$+\frac{1+\beta^2}{2\beta}\left(-\frac{1}{4}L_\beta^2 + L_\beta\ln\frac{1+\beta}{2\beta} + \frac{\pi^2}{3} + \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right), \qquad (2.53)$$
$$L_\beta = \ln\frac{1+\beta}{1-\beta}.$$

Soft-photon emission correction to the final-state currents reads

$$i_{\rho\sigma}^{(s\pi)} = \frac{\alpha}{\pi}\Delta_{1'2'}i_{\rho\sigma}^{(0\pi)}, \quad i_{\rho\sigma}^{(s\mu)} = \frac{\alpha}{\pi}\Delta_{1'2'}i_{\rho\sigma}^{(0\mu)},$$
$$\Delta_{1'2'} = -\frac{1}{4\pi}\int\frac{d^3k}{\omega}\left(\frac{q_+}{q_+k}-\frac{q_-}{q_-k}\right)^2\Bigg|_{\omega\leqslant\Delta\varepsilon} =$$
$$= \left(\frac{1+\beta^2}{2\beta}\ln\frac{1+\beta}{1-\beta}-1\right)\ln\frac{(\Delta\varepsilon)^2M^2}{\varepsilon^2x_+x_-\lambda^2} + \frac{1+\beta^2}{2\beta}\times$$
$$\times\left[-g-\frac{1}{2}\ln^2\frac{1+\beta}{1-\beta} - \ln\frac{1+\beta}{1-\beta}\ln\frac{1-\beta^2}{4} - \frac{\pi^2}{6} - 2\mathrm{Li}_2\left(\frac{\beta-1}{\beta+1}\right)\right],$$

$$g = 2\beta \int_0^1 \frac{dt}{1-\beta^2 t^2} \ln\left(1 + \frac{1-t^2}{4} \frac{(x_+ - x_-)^2}{x_+ x_-}\right) =$$

$$= \ln\left(\frac{1+\beta}{1-\beta}\right) \ln\left(1 + z - \frac{z}{\beta^2}\right) + \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta/r}\right) + \mathrm{Li}_2\left(\frac{1-\beta}{1-\beta/r}\right) -$$

$$- \mathrm{Li}_2\left(\frac{1+\beta}{1-\beta/r}\right) - \mathrm{Li}_2\left(\frac{1+\beta}{1+\beta/r}\right),$$

$$\beta = \sqrt{1 - \frac{4M^2}{s_1}}, \quad z = \frac{1}{4}\left(\sqrt{\frac{x_+}{x_-}} - \sqrt{\frac{x_-}{x_+}}\right)^2, \quad r = |x_+ - x_-|\,.$$

This formulae provide the generalization of known expression (see (25), (26) in [22]) for the case of small invariant mass $4M^2 \sim \sqrt{s_1} \ll \varepsilon_\pm$.

The contribution of an additional hard-photon emission (with momentum k_2) by the muon block, provided $\tilde{s}_1 = (q_+ + q_- + k_2)^2 \sim s_1 \ll s$, can be found by the expression

$$B^{\rho\sigma} i_{\rho\sigma}^{(h\mu)} = \frac{\alpha}{4\pi^2} \int \frac{d^3 k_2}{\omega_2} B^{\rho\sigma} \sum J_\rho^{(\gamma)} (J_\sigma^{(\gamma)})^* \bigg|_{\omega_2 \geqslant \Delta\varepsilon}, \tag{2.54}$$

with

$$\sum |J_\rho^{(\gamma)}|^2 = 4Q^2(s_1 + 2k_2 \cdot q_- + 2k_2 \cdot q_+ + 2M^2) - 8\frac{(k_2 \cdot q_-)^2 + (k_2 \cdot q_+)^2}{(k_2 \cdot q_-)(k_2 \cdot q_+)},$$

$$Q = \frac{q_-}{q_- \cdot k_2} - \frac{q_+}{q_+ \cdot k_2},$$

and

$$\sum |J^{(\gamma)} \cdot p_\pm|^2 = -8Q^2(q_- \cdot p_\pm)(q_+ \cdot p_\pm) +$$

$$+ 8(p_\pm \cdot k_2)\left(Q \cdot q_+ \frac{p_\pm \cdot q_-}{q_+ \cdot k_2} - Q \cdot q_- \frac{p_\pm \cdot q_+}{q_- \cdot k_2}\right) + 8(p_\pm \cdot k_2)\left(\frac{p_\pm \cdot q_-}{q_+ \cdot k_2} + \frac{p_\pm \cdot q_+}{q_- \cdot k_2}\right) +$$

$$+ 8(p_\pm \cdot Q)(p_\pm \cdot q_+ - p_\pm \cdot q_-) - 8\frac{(k_2 \cdot p_\pm)^2 M^2}{(k_2 \cdot q_+)(k_2 \cdot q_-)}.$$

For the case of charged pion-pair production, the radiative current tensor has the form

$$i_{\rho\sigma}^{(h\pi)} = -\frac{\alpha}{4\pi^2} \int |F_\pi^{\mathrm{str}}(\tilde{s}_1)|^2 \frac{d^3 k_2}{\omega_2} \left[\frac{M^2}{\chi_{2-}^2}(Q_1 Q_1)_{\rho\sigma} + \frac{M^2}{\chi_{2+}^2}(Q_2 Q_2)_{\rho\sigma} -\right.$$

$$\left.\left. - \frac{q_+ q_-}{\chi_{2+}\chi_{2-}}(Q_1 Q_2)_{\rho\sigma} + g_{\rho\sigma} - \frac{1}{\chi_{2-}}(Q_1 q_-)_{\rho\sigma} + \frac{1}{\chi_{2+}}(Q_2 q_+)_{\rho\sigma}\right]\right|_{\omega_2 > \Delta\varepsilon}, \tag{2.55}$$

$$Q_1 = q_- - q_+ + k_2, \quad Q_2 = q_- - q_+ - k_2, \quad \chi_{2\pm} = 2k_2 \cdot q_\pm.$$

One can check that the Bose symmetry and the gauge invariance condition are valid for the pion current tensor. Namely, it is invariant with regard to the permutation of the pion momenta and turns to zero after conversion with 4-vector q.

The sum of soft- and hard-photon corrections to the final current does not depend on $\Delta\varepsilon/\varepsilon$.

2.2.4. Corrections Connected with the Initial State. Let us now consider the Compton tensor with RC, which describes virtual corrections to the initial state. In our kinematical region it will be convenient to rewrite the tensor explicitly extracting large logarithms. We will distinguish two kinds of large logarithms:

$$L_s = \ln\frac{s}{m^2}, \quad l_1 = \ln\frac{s}{s_1}. \tag{2.56}$$

We rewrite the Compton tensor [8] in the form:

$$\begin{gathered}
K_{\rho\sigma} = \left(1+\frac{\alpha}{2\pi}\rho\right)B_{\rho\sigma} + \frac{\alpha}{2\pi}\Big[\tau_g g_{\rho\sigma} + \tau_{11}(p_-p_-)_{\rho\sigma} + \\
+ \tau_{22}(p_+p_+)_{\rho\sigma} - \frac{1}{2}\tau_{12}(p_-p_+)_{\rho\sigma}\Big], \\
\rho = -4\ln\frac{m}{\lambda}(L_s-1) - L_s^2 + 3L_s - 3l_1 + \frac{4}{3}\pi^2 - \frac{9}{2},
\end{gathered} \tag{2.57}$$

with τ_i given in [8].

The infrared singularity (the presence of the photon mass λ in ρ) is compensated by taking into account soft-photon emission from the initial particles:

$$\begin{gathered}
d\sigma^{\text{soft}} = d\sigma_0\frac{\alpha}{\pi}\Delta_{12}, \\
\Delta_{12} = -\frac{1}{4\pi}\int\frac{d^3k}{\omega}\left(\frac{p_+}{p_+k} - \frac{p_-}{p_-k}\right)^2\Bigg|_{\omega\leqslant\Delta\varepsilon} = 2(L_s-1)\ln\frac{m\Delta\varepsilon}{\lambda\varepsilon} + \frac{1}{2}L_s^2 - \frac{\pi^2}{3}.
\end{gathered} \tag{2.58}$$

As a result, the quantity ρ in formula (2.57) will change to

$$\rho \to \rho_\Delta = \left(4\ln\frac{\Delta\varepsilon}{\varepsilon} + 3\right)(L_s-1) - 3l_1 + \frac{2\pi^2}{3} - \frac{3}{2}.$$

Cross section of two hard photons emission for the case when one of them is emitted collinearly to the incoming electron or positron can be obtained by means of the quasi-real electron method [5]:

$$\begin{aligned}
\frac{d\sigma^j_{\gamma\gamma,\,\text{coll}}}{dx_-\,dc\,ds_1} = dW_{p_-}(k_3)&\frac{d\tilde\sigma^j_B(p_-(1-x_3),p_+;k_1,q_+,q_-)}{dx_-\,dc\,ds_1} + \\
&+ dW_{p_+}(k_3)\frac{d\tilde\sigma^j_B(p_-,p_+(1-x_3);k_1,q_+,q_-)}{dx_-\,dc\,ds_1},
\end{aligned}$$

with

$$dW_p(k_3) = \frac{\alpha}{\pi}\left[\left(1 - x_3 + \frac{x_3^2}{2}\right)\ln\frac{(\varepsilon\theta_0)^2}{m^2} - (1 - x_3)\right]\frac{dx_3}{x_3},$$
$$x_3 = \frac{\omega_3}{\varepsilon}, \quad x_3 > \frac{\Delta\varepsilon}{\varepsilon}.$$

Here we suppose that the polar angle θ_3 between the directions of the additional collinear photon and the beam axis does not exceed some small value $\theta_0 \ll 1$, $\varepsilon\theta_0 \gg m$.

The *boosted* differential cross section $d\tilde{\sigma}_B^j(p_-x, p_+y; k_1, q_+, q_-)$ with reduced momenta of the incoming particles reads (compare with Eq. (2.21))

$$\frac{d\tilde{\sigma}_B^{\mu}(p_+x_2, p_-x_1; k_1, q_+, q_-)}{dx_-\, dc\, ds_1} =$$
$$= \frac{\alpha^3(1 + 2\sigma - 2\nu_-(1 - \nu_-))(x_1^2(1-c)^2 + x_2^2(1+c)^2)}{s_1 s x_1^2 x_2^2(1 - c^2)(x_1 + x_2 + c(x_2 - x_1))}, \tag{2.59}$$
$$\frac{d\tilde{\sigma}_B^{\pi}(p_+x_2, p_-x_1; k_1, q_+, q_-)}{dx_-\, dc\, ds_1} = \frac{\alpha^3(\nu_-(1 - \nu_-) - \sigma)(x_1^2(1-c)^2 + x_2^2(1+c)^2)}{s_1 s x_1^2 x_2^2(1 - c^2)(x_1 + x_2 + c(x_2 - x_1))},$$
$$\nu_- = \frac{x_-}{y_2}, \quad y_2 = \frac{2x_1x_2}{x_1 + x_2 + c(x_2 - x_1)}.$$

In a certain experimental situation, an estimate of the contribution of the additional hard-photon emission outside the narrow cones around the beam axes is needed. It can be estimated by

$$\frac{d\sigma^j_{\gamma\gamma,\,\mathrm{noncoll}}}{dx_-\, dc\, ds_1} = \frac{\alpha}{4\pi^2}\int\frac{d^3k_3}{\omega_3}\left[\frac{\varepsilon^2 + (\varepsilon - \omega_3)^2}{\varepsilon\omega_3}\right]\times$$
$$\times\left\{\frac{1}{k_3\cdot p_-}\frac{d\sigma_B^j(p_-(1 - x_3), p_+; k_1, q_+, q_-)}{dx_-\, dc\, ds_1} + \right.$$
$$\left. + \frac{1}{k_3\cdot p_+}\frac{d\sigma_B^j(p_-, p_+(1 - x_3); k_1, q_+, q_-)}{dx_-\, dc\, ds_1}\right\}, \quad x_3 = \frac{\omega_3}{\varepsilon},$$

where the integration was done under condition that $\theta_3 \geqslant \theta_0$, $\Delta\varepsilon < \omega_3 < \omega_1$.

It is a simplified expression for the two-photon initial state emission cross section. Deviation, for the case of a large angle emission, of our estimate from the exact result is small. It does not depend on s and slightly depends on θ_0. For $\theta_0 \sim 10^{-2}$ we have

$$\frac{\pi}{\alpha}\left|\frac{\int(d\sigma^j_{\gamma\gamma,\,\mathrm{noncoll}} - d\sigma^j_{\gamma\gamma,\,\mathrm{noncoll\ exact}})}{\int d\sigma_B^j}\right| \lesssim 10^{-1}. \tag{2.60}$$

2.2.5. Master Formula. By summing up all contributions for the charge-even part, we can put the cross section of the radiative production in the form:

$$\frac{d\sigma^j(p_+,p_-;k_1,q_+,q_-)}{dx_-\,dc\,ds_1} = \int\limits^1\int\limits^1 \frac{dx_1\,dx_2}{|1-\Pi(sx_1x_2)|^2}\times$$
$$\times\frac{d\tilde{\sigma}^j_B(p_+x_2,p_-x_1;k_1,q_+,q_-)}{dx_-\,dc\,ds_1}D(x_1,\beta)D(x_2,\beta)\left(1+\frac{\alpha}{\pi}K^j\right)+$$
$$+\frac{\alpha}{2\pi}\int\limits_\Delta^1 dx\left[\frac{1+(1-x)^2}{x}\ln\frac{\theta_0^2}{4}+x\right]\left[\frac{d\tilde{\sigma}^j_B(p_-(1-x),p_+;k_1,q_+,q_-)}{dx_-\,dc\,ds_1}\right.$$
$$\left.+\frac{d\tilde{\sigma}^j_B(p_-,p_+(1-x);k_1,q_+,q_-)}{dx_-\,dc\,ds_1}\right]+\frac{d\sigma^j_{\gamma\gamma,\,\mathrm{noncoll}}}{dx_-\,dc\,ds_1}, \quad (2.61)$$

$$D(x,\beta)=\delta(1-x)+\beta P^{(1)}(x)+\ldots,$$
$$\Delta=\frac{\Delta\varepsilon}{\varepsilon},\quad P^{(1)}(x)=\left(\frac{1+x^2}{1-x}\right)_+,\quad j=\mu,\pi.$$

The boosted cross sections $d\tilde{\sigma}$ are defined above in Eq. (2.59). The lower limits of the integrals over $x_{1,2}$ depend on the experimental conditions.

The structure function D includes all dependence on the large logarithm L_s. The so-called K factor reads

$$K^j=\frac{1}{R^j}B^{\lambda\sigma}\left(i^{(vj)}_{\lambda\sigma}+i^{(sj)}_{\lambda\sigma}+i^{(hj)}_{\lambda\sigma}\right)+R^{(j)}_{\mathrm{compt}}. \quad (2.62)$$

Quantities $R^{(j)}_{\mathrm{compt}}$ include the «nonleading» contributions from the initial-state radiation. Generally, they are rather cumbersome expressions for the case $s_1\sim s$. For the case $s_1\sim M^2\ll s$ we obtain

$$R^{(\mu)}_{\mathrm{compt}}=R^{(\pi)}_{\mathrm{compt}}+\frac{c^2}{(1-2x_-x_++2\sigma)(1+c^2)},$$

$$R^{(\pi)}_{\mathrm{compt}}=\frac{1-c^2}{4(1+c^2)}\left\{\frac{5+2c+c^2}{1-c^2}\ln^2\left(\frac{2}{1+c}\right)-\right.$$
$$-\frac{5-c}{1+c}\ln\left(\frac{2}{1+c}\right)+\frac{5-2c+c^2}{1-c^2}\ln^2\left(\frac{2}{1-c}\right)-$$
$$\left.-\frac{5+c}{1-c}\ln\left(\frac{2}{1-c}\right)-4\frac{c^2}{1-c^2}\right\}+\frac{\pi^2}{3}. \quad (2.63)$$

Here we see the remarkable phenomena: the cancellation of terms containing $\ln(s/s_1)$. In such a way only one kind of large logarithm L_s enters into the final result. This fact is the consequence of the renormalization group invariance.

This result shows that the cross section in our quasi $2 \to 2$ kinematics can be written in the form of the cross section of the Drell–Yan process. Thus the results of Structure Function Approach (see [97]) for the RC to the one-photon e^+e^- annihilation into hadrons are generalized to the situation when a hard photon at a large angle is present in the final state.

This generalization is not a trivial fact because the two types of *large logarithms* are present in the problem.

Possible background from the peripheral process $e\bar{e} \to e\bar{e}\mu\bar{\mu}$ is negligible in our kinematics: it is suppressed by the factor $\alpha/\pi s_1/s$ and, besides, can be eliminated if the registration of the primary hard photon (see Eq. (2.39)) is required by the experimental cuts for event selection.

2.3. Compton and Double Compton Scattering Processes at Colliding Electron–Photon Beams. The Compton scattering process [45]

$$\begin{gathered}\gamma(k_1) + e^-(p_1) \to \gamma(k_2) + e^-(p_2),\\ k_1^2 = k_2 = 0, \quad p_1^2 = p_2^2 = m^2,\\ \kappa_1 = 2p_1k_1 = 4\epsilon_1\omega_1, \quad \kappa_1' = 2p_2k_1 = 2\epsilon_2\omega_1(1-c),\\ s_1 = 2p_1p_2 = 2\epsilon_1\epsilon_2(1+c),\\ \kappa_1 \sim \kappa_1' \sim s_1 \gg m^2, \quad \epsilon_2 = \frac{2\epsilon_1\omega_1}{\omega_1(1-c)+\epsilon_1(1+c)}\end{gathered} \tag{2.64}$$

(where $\epsilon_{1,2}, \omega_1$ are the energies of initial and scattered electrons and the initial photon; $c = \cos\theta$, θ is the angle between $\mathbf{p}_2, \mathbf{k}_1$) plays an important role as a perspective calibration process at high-energy photon–electron colliders [46]. Modern methods based on the renormalization group approach in combination with the lowest order radiative corrections (RC) permit us to obtain a differential cross section with the leading $(((\alpha/\pi)L)^n \sim 1$, with «large logarithm» $L = \ln(s_1/m^2))$ and the next-to-leading approximation (i.e., keeping the terms of the order $(\alpha/\pi)^n L^{n-1}$). So the accuracy of the formulae given below is determined by the terms of the order (which are systematically omitted)

$$\frac{m^2}{\kappa_1}, \quad \frac{\alpha^2}{\pi^2}L, \quad \alpha\frac{\kappa_1}{M_Z^2}, \tag{2.65}$$

compared with the terms of the order of unity and is at the level of per mill for typical experimental conditions [46] $\theta \sim 1, \kappa_1 \sim 10$ GeV^2. We consider the energies of initial particles to be much less than the Z-boson mass M_Z and, therefore, the weak corrections to the Compton effect in our consideration are beyond our accuracy.

First papers devoted to calculation of radiative corrections to Compton scattering were published in 1952 by Brown and Feynman [16] (virtual and soft real photon emission contribution), and Mandl and Skyrme [47] (emission of an additional hard photon).

In the work of H. Veltman [48], the lowest order radiative corrections to the polarized Compton scattering were calculated in nonrelativistic kinematics. This case of kinematics was also considered in the paper of M. Swartz [49].

In the papers of A. Denner and S. Dittmaier [50,80], the lowest order radiative corrections in the framework of Standard Model were calculated for the case of polarized electron and photon.

In this Subsection we consider the case of high-energy electron and photon Compton scattering (CMS energy supposed to be much higher than the electron mass but much less than the Z-boson mass). We found that the cross section with radiative corrections of all orders of PT taken into account could be written down in the form of the Drell–Yan process. Both leading and next-to-leading contributions are derived explicitly.

We imply the kinematics when the initial photon and electron move along the z axis in the opposite directions The energy of the scattered electron will be the function of its scattering angle:

$$z_0 = \frac{\varepsilon_2}{\omega_1} = \frac{2\rho}{a}, \quad a = a(c,\rho) = 1 - c + \rho(1+c), \quad \rho = \frac{\varepsilon_1}{\omega_1}. \tag{2.66}$$

Hereafter we imply the kinematic case $\rho < 1$. The case $\rho > 1$ will be considered in Subsubsec. 2.3.6.

The differential cross section in the Born approximation will be

$$\frac{d\sigma_B}{dc}(p_1,\theta) = \frac{\pi\alpha^2 U_0}{\omega_1^2 a^2}, \quad U_0 = \frac{a}{1-c} + \frac{1-c}{a}. \tag{2.67}$$

When taking into account RC of higher orders (arising from both emission of virtual and real photons), a simple relation between the scattered electron energy and the scattering angle changes, so the differential cross section is in general dependent on the energy fraction z of the scattered electron. Accepting the Drell–Yan form of cross section, we can put it in the form

$$\frac{d\sigma}{dc\,dz}(p_1,\theta,z) = \\ = \int\limits_0^1 dx\, D(x,\beta) \int\limits_z^{z_0} \frac{dt}{t} D\left(\frac{z}{t},\beta\right) \frac{d\sigma_h}{dc\,dt}(xp_1,\theta,t)\left(1 + \frac{\alpha}{\pi}K\right), \tag{2.68}$$

where the structure function $D(x,\beta)$ (see [97]) describes the probability of finding the electron (considered as a parton) inside the electron, K is the so-called

K factor which can be calculated from the lowest RC orders, and the «hard» cross section is

$$\frac{d\sigma_h}{dc\,dt}(xp_1,\theta,t) = \frac{d\sigma_B}{dc}(xp_1,\theta)\delta(t-t(x)),$$

$$\frac{d\sigma_B(xp_1,\theta)}{dc} = \frac{\pi\alpha^2}{\omega_1^2}\frac{1}{(1-c+\rho x(1+c))^2}\times \\ \times\left(\frac{1-c}{1-c+\rho x(1+c)}+\frac{1-c+\rho x(1+c)}{1-c}\right), \quad (2.69)$$

$$t(x) = \frac{2x\rho}{1-c+\rho x(1+c)},$$

with K specified below (see (2.71), (2.81), (2.87)).

The cross section being written in the Drell–Yan form explicitly satisfies the Kinoshita–Lee–Nauenberg theorem about mass singularities cancellation [18]. Really, being integrated on the scattered electron energy fraction z, the structure function corresponding to the scattered electron turns to unity due to its property

$$\int\limits_0^1 dz\int\limits_z^1 \frac{dt}{t}D\left(\frac{z}{t},\beta\right)f(t) = \int\limits_0^1 dt\, f(t). \quad (2.70)$$

Mass singularities associated with the initial lepton structure function remain.

So our master formula for the cross section with RC taken into account is

$$\frac{d\sigma}{dz\,dc}(p_1,p_2) = \int\limits_{x_0}^1 \frac{dx}{t(x)}D(x,\beta)\frac{d\sigma_B}{dc}(xp_1,\theta)D\left(\frac{z}{t(x)},\beta\right) + \\ + \frac{\alpha}{\pi}\frac{d\sigma_B(p_1,\theta)}{dc}[K_{\mathrm{SV}}\delta(z-z_0)+K_h], \quad (2.71)$$

$$z = \frac{\epsilon_2'}{\omega_1} < z_0, \quad x_0 = \frac{z(1-c)}{\rho(2-z(1+c))}, \quad L = \ln\frac{2\omega_1^2 z_0\rho(1+c)}{m^2}.$$

The second term in the r.h.s. of (2.71) collects all the nonleading contributions from virtual, soft- and hard-photons emission, with K_{SV} given in Subsubsec. 2.3.1 where the virtual, soft real and additional hard-photon emission is considered. We introduce an auxiliary parameter θ_0 to distinguish the collinear and noncollinear kinematics of photon emission. Also, we put the expression for the hard-photon contribution K_h.

2.3.1. Contribution of Virtual and Soft Real Photons. To obtain the explicit form of the K factor, we reproduce the lowest order RC calculation. It consists of virtual photon emission contribution and the contribution from the real (soft and hard) photon emission taken into account. The virtual- and soft-photon emission contribution was first calculated in the famous paper of the 1952 year by Laura Brown and Richard Feynman [16]. The result has the form

$$\frac{d\sigma_{\mathrm{virt}}}{d\sigma_B} = -\frac{\alpha}{\pi}\frac{U_1}{U_0}, \tag{2.72}$$

with (see [16], kinematic case II):

$$\frac{U_1}{U_0} = (1-L)\left(\frac{3}{2}+2\ln\frac{\lambda}{m}\right)+\frac{1}{2}L^2-\frac{\pi^2}{6}-K_V, \quad U_0=\frac{\kappa_2}{\kappa_1}+\frac{\kappa_1}{\kappa_2}, \tag{2.73}$$

with K_V (virtual photon contribution to the K factor):

$$K_V = -\frac{1}{U_0}\left[\left(1-\frac{\kappa_2}{2\kappa_1}-\frac{\kappa_1}{\kappa_2}\right)\left(\ln^2\frac{s_1}{\kappa_1}-\ln\frac{s_1}{\kappa_1}+2\ln\frac{\kappa_2}{\kappa_1}\right)+\right. \tag{2.74}$$

$$\left.+\left(1-\frac{\kappa_1}{2\kappa_2}-\frac{\kappa_2}{\kappa_1}\right)\left(\ln^2\frac{s_1}{\kappa_2}-\ln\frac{s_1}{\kappa_1}-\ln\frac{\kappa_1}{\kappa_2}+\pi^2\right)\right], \tag{2.75}$$

and

$$\frac{\kappa_2}{\kappa_1}=\frac{z_0(1-c)}{2\rho}, \quad \frac{s_1}{\kappa_1}=\frac{z_0(1+c)}{2}, \quad \frac{s_1}{\kappa_2}=\frac{\rho(1+c)}{1-c}. \tag{2.76}$$

The soft-photon emission for our kinematics has the form

$$\frac{d\sigma_{\mathrm{soft}}}{d\sigma_B} = -\frac{4\pi\alpha}{16\pi^3}\int\frac{d^3k}{\omega}\left(\frac{p_1}{p_1k}-\frac{p_2}{p_2k}\right)^2_{\omega=\sqrt{\mathbf{k}^2+\lambda^2}<\Delta\epsilon\ll\epsilon_1\sim\epsilon_2}. \tag{2.77}$$

Standard calculations lead to the result

$$\frac{d\sigma_{\mathrm{soft}}}{d\sigma_B} = $$
$$= \frac{\alpha}{\pi}\left[(L-1)\ln\left(\frac{m^2\Delta\varepsilon^2}{\lambda^2\varepsilon_1\varepsilon_2}\right)+\frac{1}{2}L^2-\frac{1}{2}\ln^2\frac{\varepsilon_1}{\varepsilon_2}-\frac{\pi^2}{3}+\mathrm{Li}_2\left(\frac{1-c}{2}\right)\right]. \tag{2.78}$$

The resulting contribution to the cross section from virtual and soft real photons does not depend on the fictitious «photon mass» λ as well as the terms of L^2 type. It can be written in the form

$$\left(\frac{d\sigma}{dz\,dc}\right)_{sv} = \frac{d\sigma_{\mathrm{virt}}+d\sigma_{\mathrm{soft}}}{dc}\delta(z-z_0) =$$
$$= \frac{\alpha}{2\pi}\frac{d\sigma_B(p_1,\theta)}{dc}\big[(L-1)(P_{1\Delta}+P_{2\Delta})+2K_{\mathrm{SV}}\big]\delta(z-z_0), \tag{2.79}$$

where we have introduced the notation

$$P_{1\Delta} = \frac{3}{2} + 2\ln\frac{\Delta\varepsilon}{\varepsilon_1}, \quad P_{2\Delta} = \frac{3}{2} + 2\ln\frac{\Delta\varepsilon}{\varepsilon_2}. \tag{2.80}$$

We can see that the terms proportional to the «large» logarithm L have the form conforming with the RG prescription of the structure function. The contribution of nonleading terms K_{SV} is

$$K_{\mathrm{SV}} = -\frac{\pi^2}{6} + \mathrm{Li}_2\left(\frac{1-c}{2}\right) - \frac{1}{2}\ln^2\frac{z_0}{\rho} + K_V. \tag{2.81}$$

2.3.2. Contribution of the Hard Collinear Real Photon Emission. The dependence on the auxiliary parameter $\Delta\epsilon$ will be eliminated when taking into account the emission of real additional hard photon with 4-momentum k and the energy ω exceeding $\Delta\epsilon$.

It is convenient to consider the kinematics when this additional photon moves within the narrow cone of the angular size $m/\epsilon_1 \ll \theta_0 \ll 1$ along the directions of the initial or scattered electrons. The contribution of these kinematic regions can be obtained by using the «Quasi-Real Electron Method» [5] instead of using a general (rather cumbersome) expression for the cross section of the DC scattering process [47].

In the case when the collinear photon is emitted along the initial electron the result has the form:

$$\left(\frac{d\sigma}{dz\,dc}\right)_{\mathbf{k}||\mathbf{p}_1} = \frac{\alpha}{2\pi}\int\limits_0^{1-\frac{\Delta\epsilon}{\epsilon_1}} dx\frac{d\sigma_B}{dc}(xp_1,\theta)\left[\frac{1+x^2}{1-x}(L_1-1)+1-x\right]\delta(z-t(x)),$$
$$L_1 = \ln\frac{\theta_0^2\epsilon_1^2}{m^2} = L + \ln\frac{\theta_0^2\rho}{2z_0(1+c)}. \tag{2.82}$$

When the photon is emitted along the scattered electron, we have:

$$\left(\frac{d\sigma}{dz\,dc}\right)_{\mathbf{k}||\mathbf{p}_2} = \frac{\alpha}{2\pi}\frac{d\sigma_B}{dc}(p_1,\theta)\times$$
$$\times \int\limits_{z\left(1+\frac{\Delta\epsilon}{\epsilon_2}\right)}^{z_0} \frac{dt}{t}\delta(t-z_0)\left[\frac{1+z^2/t^2}{1-z/t}(L_2-1)+1-\frac{z}{t}\right], \tag{2.83}$$
$$L_2 = \ln\frac{\epsilon_2'^2\theta_0^2}{m^2} = L + \ln\frac{\theta_0^2 z^2}{2\rho(1+c)z_0},$$

where $z = \varepsilon_2'/\omega_1 < z_0$ is the energy fraction of the scattered electron (after emission of the collinear photon).

It is convenient to write down the contribution of the collinear kinematics in the form

$$\left(\frac{d\sigma_h}{dz\,dc}\right)_{\text{coll}} = \frac{\alpha}{2\pi}(L-1)\left[\int\limits_0^1 dx\frac{1+x^2}{1-x}\theta(1-x-\Delta_1)\frac{d\sigma_B(xp_1,\theta)}{dc}\delta(z-t(x))+\right.$$
$$\left.+\int\limits_z^{z_0}\frac{dt}{t}\frac{1+(z/t)^2}{1-z/t}\theta\left(1-\frac{z}{t}-\Delta_2\right)\frac{d\sigma_B(p_1,\theta)}{dc}\delta(t-z_0)\right]+\frac{df^{(1)}}{dz\,dc}+\frac{df^{(2)}}{dz\,dc},$$

with

$$\frac{df^{(1)}}{dz\,dc} = \frac{\alpha^3}{4\rho(1-c)\omega_1^2}\left(\frac{2-z(1+c)}{2}+\frac{2}{2-z(1+c)}\right)\times$$
$$\times\left[\frac{1+x^2}{1-x}\ln\frac{\rho\theta_0^2}{2z_0(1+c)}+1-x\right]_{x=x_0}\theta(1-x-\Delta_1),$$
$$\frac{df^{(2)}}{dz\,dc} = \frac{\alpha^3}{4\rho a\omega_1^2}\left(\frac{1-c}{a}+\frac{a}{1-c}\right)\times$$
$$\times\left[\frac{1+z^2/t^2}{1-z/t}\ln\frac{z^2\theta_0^2}{2\rho(1+c)z_0}+1-\frac{z}{t}\right]_{t=z_0}\theta\left(1-\frac{z}{t}-\Delta_2\right),\quad \Delta_{1,2}=\frac{\Delta\varepsilon}{\varepsilon_{1,2}}. \tag{2.84}$$

We use here the relation $\delta(z-t(x)) = (2x_0^2\rho/(z^2(1-c))\delta(x-x_0)$.

Again we can see that the terms containing large logarithm L have the form conforming with the structure function. So our suggestion (2.68) is confirmed.

The auxiliary parameter θ_0 dependence vanishes when taking into account the contribution of noncollinear kinematics of the additional hard-photon emission (see Subsubsec. 2.3.4).

2.3.3. Noncollinear Kinematics Contribution. Double Compton Scattering Process. The general expression for the cross section of the DC scattering process

$$\begin{gathered}\gamma(k_1)+e^-(p_1)\to\gamma(k_2)+\gamma(k)+e^-(p_2),\\ \kappa=2kp_1,\quad \kappa'=2kp_2,\quad \kappa_2=2k_2p_1,\quad \kappa_2'=2k_2p_2\end{gathered} \tag{2.85}$$

was obtained years ago by Mandl and Skyrme [47]. The expression for the cross section presented in this paper is exact but, unfortunately, too complicated. Instead, we use the expression for the differential cross section calculated (by the methods of chiral amplitudes (see [98])) with the assumption that all kinematic invariants are large compared with the electron mass squared $\kappa \sim \kappa' \sim \kappa_i \sim \kappa_i' \gg m^2$

$$\frac{\varepsilon_2 d\sigma_0^{\mathrm{DC}}}{d^3 p_2} = \frac{1}{2!}\frac{\alpha^3}{2\pi^2\kappa_1} R\, d\Phi, \quad d\Phi = \frac{d^3k_2}{\omega_2}\frac{d^3 k}{\omega}\delta^4(p_1 + k_1 - p_2 - k_2 - k),$$
$$R = s_1 \frac{\kappa\kappa'(\kappa^2 + \kappa'^2) + \kappa_1\kappa_1'(\kappa_1^2 + \kappa_1'^2) + \kappa_2\kappa_2'(\kappa_2^2 + \kappa_2'^2)}{\kappa\kappa'\kappa_1\kappa_1'\kappa_2\kappa_2'}. \tag{2.86}$$

The explicit expression for the contribution to the K factor from hard-photon emission K_h is

$$\frac{\alpha}{\pi}\frac{d\sigma_B}{dc} K_h = \frac{d\sigma_{\theta_0}^{\mathrm{DC}}}{dz\, dc} + \frac{df^{(1)}}{dz\, dc} + \frac{df^{(2)}}{dz\, dc}, \tag{2.87}$$

with

$$\frac{d\sigma_{\theta_0}^{\mathrm{DC}}}{dz\, dc} = \frac{\alpha^3 z}{2! 4\pi\rho}\int R\, d\Phi, \tag{2.88}$$

and the phase volume $d\Phi$ is restricted by the conditions $\omega, \omega_2 > \Delta\epsilon$ and the requirement that the angles between 3-vectors $\mathbf{k}_2, \mathbf{k}$ and 3-vectors $\mathbf{p}_1$, $\mathbf{p}_2$ exceed θ_0.

The values of K_h calculated numerically are given in Tables 2, 8. We show numerically and analytically (see Subsubsec. 2.3.4) the independence of K_h on the auxiliary parameters $\theta_0, \Delta\varepsilon$.

Table 2. **The value of K_h as a function of z, $\cos\theta$ (calculated for $\rho = 0.4$)**

z	$\cos\theta$								
	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8
0.1	–2.82	–2.61	–2.39	–2.19	–2.09	–1.89	–1.87	–2.06	–2.75
0.2	–2.77	–2.47	–2.17	–1.90	–1.65	–1.46	–1.39	–1.56	–2.30
0.3	–3.43	–2.98	–2.55	–2.14	–1.77	–1.47	–1.30	–1.38	–2.13
0.4	–4.96	–3.87	–3.23	–2.65	–2.13	–1.67	–1.34	–1.30	–2.02

The cross section of the DC scattering process in an inclusive experimental setup with the leading logarithmic approximation in terms of structure functions has the form

$$d\sigma^{\mathrm{DC}}(p_1, k_1; p_2, k, k_2) = \int_0^1 dx\, D(x, \beta)\times$$
$$\times \int_z^1 D\left(\frac{z}{t}, \beta\right)\frac{dt}{t} d\sigma_0^{\mathrm{DC}}\left(xp_1, k_1; \frac{tp_2}{z}, k, k_2\right), \tag{2.89}$$

where the structure functions could be found in [97] and

$$d\sigma_0^{\rm DC}(p_1,k_1;p_2,k,k_2) = \\ = \frac{\alpha^3}{4\pi^2\kappa_1} R \frac{d^3k_2\, d^3k\, d^3p_2}{\omega_2\omega\epsilon_2}\delta^4(p_1+k_1-p_2-k_2-k). \quad (2.90)$$

2.3.4. Cancellation of θ Dependence. Performing the integration over k_2 of the phase volume

$$d\Phi = \frac{d^3k}{\omega}\frac{d^3k_2}{\omega_2}\delta^4(Q-k-k_2), \quad Q = p_1+k_1-p_2, \quad (2.91)$$

we can put it in the form

$$d\Phi = \frac{\omega d\omega}{\omega_1^2}\frac{2dc_1dc_2}{\sqrt{D}}\times \\ \times\delta\left[2\rho-\rho z(1+c)-z(1-c)-\frac{\omega}{\omega_1}(\rho(1-c_1)-z(1-c_2)+1+c_1)\right],$$

where $D = 1-c_1^2-c_2^2-c^2-2cc_1c_2$; c_1, c_2 are the cosines of the angles between $\mathbf{k}$ and $\mathbf{p}_1$, $\mathbf{p}_2$, respectively.

For collinear kinematics the following relation can be useful:

1. $k \approx (1-x)p_1$

$$R_1 = R|_{\mathbf{k}||\mathbf{p}_1} = \left(\frac{2x\rho}{z(1-c)}+\frac{z(1-c)}{2x\rho}\right)\frac{1+x^2}{(1-x)^2}\frac{1}{2\rho^2(1-c_1)x\omega_1^2},$$

$$d\Phi_1 = d\Phi|_{\mathbf{k}||\mathbf{p}_1} = 2\frac{d^3k}{\omega}\delta((xp_1+k_1-p_2)^2) = 2\pi\frac{\rho(1-x)dx\,dc_1}{2-z(1+c)}\delta(x-x_0), \quad (2.92)$$

$$\frac{d\sigma_h^1}{dz\,dc} = \frac{\alpha^3 z}{2!4\pi\rho}\int R_1\,d\Phi_1 = \frac{\alpha^3}{4\rho\omega_1^2(1-c)}\frac{1+x_0^2}{1-x_0}\times \\ \times\left(\frac{2x_0\rho}{z(1-c)}+\frac{z(1-c)}{2x_0\rho}\right)\ln\left(\frac{4}{\theta_0^2}\right).$$

In the last equation we take into account the same contribution from the region $k_2 \approx (1-x)p_1$.

2. For the case $k \approx (t/z-1)p_2$, we obtain

$$R_2 = R|_{\mathbf{k}||\mathbf{p}_2} = \left(\frac{2x\rho}{z(1-c)}+\frac{z(1-c)}{2x\rho}\right)\frac{1+x^2}{(1-x)^2}\frac{1}{2\rho^2(1-c_1)x\omega_1^2}, \quad (2.93)$$

$$d\Phi_2 = d\Phi|_{\mathbf{k}||\mathbf{p}_2} = 2\frac{d^3k}{\omega}\delta((xp_1+k_1-p_2)^2) = 2\pi\frac{\rho(1-x)\,dx\,dc_1}{2-z(1+c)}\delta(x-x_0).$$

So the contribution of the case $\mathbf{k}||\mathbf{p}_2$ $(\mathbf{k}_2||\mathbf{p}_2)$ has the form

$$\frac{d\sigma_h^2}{dz\,dc} = \frac{\alpha^3 z}{2!4\rho\omega_1^2}\int R_2\,d\Phi_2 = \frac{\alpha^3}{4\rho a\omega_1^2}\times \\ \times\left(\frac{1-c}{a}+\frac{a}{1-c}\right)\frac{1+z^2/t^2}{1-z/t}\ln\left(\frac{4}{\theta_0^2}\right). \quad (2.94)$$

Comparing formulae (2.92), (2.94) with (2.84) we can see explicit cancellation of the θ_0 dependence.

2.3.5. Numerical Estimation. The characteristic form «reverse radiative tail» (see Tables 3, 5) of the differential cross section on the energy fraction z can be reproduced if one uses the «smoothed» expression for nonsinglet structure functions which includes the virtual electron pair production (see [97]).

Table 3. **The value of $\omega_1^2/\alpha^2\,d\sigma/(dc\,dz)$ (leading contribution, first term in the right-hand side of the master formula (2.71)) as a function of z, $\cos\theta$ (calculated for $\rho = 0.4$, $\omega_1 = 5$ GeV)**

z	$\cos\theta$								
	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8
0.1	0.211	0.237	0.265	0.299	0.345	0.413	0.526	0.754	1.450
0.2	0.337	0.357	0.378	0.405	0.445	0.508	0.618	0.850	1.576
0.3	0.703	0.669	0.643	0.634	0.644	0.685	0.782	1.013	1.784
0.4	3.883	2.153	1.554	1.264	1.113	1.054	1.090	1.296	2.122

In Fig. 17 we put the magnitude of RC in the leading approximation

$$R(\theta) = \left(\frac{d\sigma_B}{dc}\right)^{-1}\times \\ \times\left(\int dz\frac{d\sigma}{dz\,dc}-\frac{d\sigma_B}{dc}\right). \quad (2.95)$$

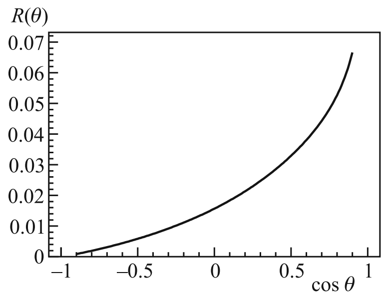

Fig. 17. The leading order radiative corrections as $\cos\theta$ distribution (see formulae (2.95))

The results cited above imply the experimental setup without additional e^+e^-, $\mu^+\mu^-$, $\pi^+\pi^-$ real pairs in the final state.

The accuracy of the formulae given above is determined by the order of magnitude of the terms omitted (see (2.65)) compared to the terms of order of unity, i.e., is of the order of 0.1% for typical experimental conditions. In particular, it is the reason why we omit the evolution effect of the K-factor terms.

Table 4. **Born cross section (2.67) (without factor α^2/ω_1^2) for $\rho = 0.4$**

	$\cos\theta$								
	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8
$\frac{\omega_1^2}{\alpha^2}\frac{d\sigma_B}{dc}$	1.779	2.038	2.365	2.796	3.389	4.266	5.721	8.669	17.881

Table 5. **The value of $\varepsilon_1^2/\alpha^2 d\tilde{\sigma}/(dc\,dy)$ (leading contribution, first term in the right-hand side of the master formula (2.98)) as a function of z, $\cos\theta$ (calculated for $\omega_1 = 400$ MeV, $\varepsilon_1 = 6$ GeV)**

y	$\cos\theta$							
	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6
0.05	9.658	11.110	13.626	17.513	23.678	34.116	53.669	98.208
0.10	11.350	15.024	22.633	39.297	86.017			
0.15	13.839	23.190	56.097					
0.20	17.735	45.672						
0.25	24.303							

Table 6. **Born cross section (2.99) (without factor α^2/ω_1^2) for $\omega_1 = 400$ MeV, $\varepsilon_1 = 6$ GeV**

	$\cos\theta$								
	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.88
$\frac{\varepsilon_1^2}{\alpha^2}\frac{d\tilde{\sigma}_B}{dc}$	93.317	60.706	49.42 8	44.994	44.351	47.084	54.584	72.444	129.944

The numerical value of K_h, leading contributions, and the Born cross section for different kinematic regions are presented as a functions of z, c in Tables 2–6, 8.

2.3.6. Kinematical Region $\rho > 1$. Here we put the different case of kinematic region for ρ, z.

All the above formulae were considered for the case $\rho < 1$, and the possible region for the variable z was determined by the equation $x_0 < 1$

$$z \leqslant \frac{2\rho}{1 - c + \rho(1 + c)}, \tag{2.96}$$

which means that the low boundary of integration in formula (2.71) is less than 1. In the case of $\rho > 1$ it is convenient to put the new variable

$$\eta = \frac{\omega_1}{\varepsilon_1}, \quad y = \frac{\varepsilon_2'}{\varepsilon_1}, \quad y_0 = \frac{\varepsilon_2}{\varepsilon_1} = \frac{2\eta}{1 + c + \eta(1 - c)}, \quad \eta < 1. \tag{2.97}$$

The master equation (2.71) for the case $\rho > 1$ (or $\eta < 1$) reads

$$\frac{d\tilde{\sigma}}{dy\,dc}(p_1,p_2) = \int\limits_{\tilde{x}_0}^{1} \frac{dx}{\tilde{t}(x)} D(x,\tilde{\beta}) \frac{d\tilde{\sigma}_B(xp_1,\theta)}{dc} D\left(\frac{y}{\tilde{t}(x)},\tilde{\beta}\right) + \\ + \frac{\alpha}{\pi}\frac{d\tilde{\sigma}_B(p_1,\theta)}{dc}\left[\tilde{K}_{\mathrm{SV}}\delta(y-y_0)+\tilde{K}_h\right], \tag{2.98}$$

$$\tilde{x}_0 = \frac{y\eta(1-c)}{2\eta - y(1+c)}, \quad \tilde{L} = \ln\frac{2\varepsilon_1^2 y_0(1+c)}{m^2}, \quad \tilde{t}(x) = \frac{2\eta x}{x(1+c)+\eta(1-c)},$$
$$\tilde{\beta} = \frac{\alpha}{2\pi}(\tilde{L}-1),$$

with the possible values for energy fraction of the scattered electron $y(\tilde{x}_0 < 1)$: $y \leqslant y_0$. The Born cross section (2.67), (2.69) and formulae for hard-photon emission, $\tilde{K}_{\mathrm{SV}}$, $\tilde{K}_h$ for the case $\rho > 1$ appear just by appropriate exchange $\rho \to \eta^{-1}$:

$$\frac{d\tilde{\sigma}_B(xp_1,\theta)}{dc} = \frac{\pi\alpha^2}{\varepsilon_1^2}\frac{1}{(\eta(1-c)+x(1+c))^2}\times \\ \times\left(\frac{\eta(1-c)}{\eta(1-c)+x(1+c)} + \frac{\eta(1-c)+x(1+c)}{\eta(1-c)}\right). \tag{2.99}$$

Large amounts of the leading contribution near the kinematic bound can be understood as manifestation of the $\delta(y-y_0)$ character of the differential cross section. The y_0, z_0 dependence is given in Table 7.

Table 7. **The value of y_0, z_0 as a function of c for $\eta = 0.064$ and $\rho = 0.4$**

	$\cos\theta$								
	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8
y_0	0.417	0.263	0.192	0.152	0.125	0.106	0.093	0.082	0.074
z_0	0.423	0.455	0.489	0.526	0.571	0.625	0.690	0.769	0.870

Table 8. **The value of $\tilde{K}_h$ as a function of y, $\cos\theta$ (calculated for $\eta = 0.064$)**

y	$\cos\theta$								
	–0.8	–0.6	–0.4	–0.2	0.0	0.2	0.4	0.6	0.8
0.05	0.70	–1.97	–7.41	–15.54	–26.90	–42.70	–65.40	–100.64	–166.21
0.10	0.36	–3.20	–9.85	–18.38	–18.35				
0.15	0.03	–3.38	–1.34						
0.20	–0.20	0.29							
0.25	–0.25								

3. TABLE OF INTEGRALS. ONE-LOOP FEYNMAN INTEGRALS

3.1. Integrals for Process $e^+e^- \to \mu^+\mu^-$. Considering the lowest order radiative corrections to the amplitude of process annihilation of electron–positron pair to muon–antimuon pair

$$e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-), \quad p_\pm^2 = m_e^2, \quad q_\pm^2 = m^2, \tag{3.1}$$

it is convenient to use such linear combinations of 4-vectros:

$$\Delta = \frac{1}{2}(p_+ - p_-), \quad Q = \frac{1}{2}(q_+ - q_-), \quad P = \frac{1}{2}(p_+ + p_-) \tag{3.2}$$

and kinematical invariants:

$$\begin{gathered} s = (p_+ + p_-)^2 = 4E^2, \quad t = (p_- - q_-)^2 = -\frac{s}{4}[1 + \beta^2 - 2\beta c], \\ u = (p_- - q_+)^2 = -\frac{s}{4}[1 + \beta^2 + 2\beta c], \end{gathered} \tag{3.3}$$

where E is the energy of electron in the center-of-mass initial particles frame: $c = \cos\theta$, θ is the angle between 3-momentum of electron and negatively charged muon, $\beta = \sqrt{1 - (4m^2)/s}$ is the muon velocity. We will suppose below the value β not to be specially close to unity

$$\beta \sim 1, \quad \frac{m_e^2}{s} \ll 1.$$

Another bilinear combinations are

$$\Delta^2 = -P^2 = -\frac{s}{4}, \quad Q^2 = -\frac{s}{4}\beta^2, \quad \sigma = \Delta Q = \frac{1}{4}(u - t). \tag{3.4}$$

We use the following set of scalar integrals with three and four denominators defined as [29]:

$$(\Delta) = (k - \Delta)^2 - m_e^2, \;\; (Q) = (k - Q)^2 - m^2, \;\; (P_\pm) = (k \pm P)^2 - \lambda^2. \tag{3.5}$$

They have the form [10, 17, 29, 93, 96]:

$$F_\Delta = \frac{-i}{\pi^2}\int \frac{d^4k}{(\Delta)(P_+)(P_-)} = \frac{1}{s}\left[\frac{\pi^2}{6} + \frac{1}{2}\ln^2\frac{s}{m_e^2}\right],$$

$$\begin{aligned} F_Q = \frac{-i}{\pi^2}\int \frac{d^4k}{(Q)(P_+)(P_-)} = \\ = \frac{1}{s\beta}\left[\frac{1}{2}\ln^2\frac{1-\beta}{2} - \frac{1}{2}\ln^2\frac{1+\beta}{2} + \mathrm{Li}_2\left(\frac{1+\beta}{2}\right) - \mathrm{Li}_2\left(\frac{1-\beta}{2}\right)\right], \end{aligned}$$

$$H = \frac{-i}{\pi^2}\int \frac{d^4k}{(\Delta)(Q)(P_+)} = G = \frac{-i}{\pi^2}\int \frac{d^4k}{(\Delta)(Q)(P_-)} = -\frac{1}{2(m^2-t)}\times$$
$$\times\left[\ln^2\frac{m^2-t}{m^2} + \left(2\ln\frac{m^2-t}{m^2} + \ln\frac{m^2}{m_e^2}\right)\times\right.$$
$$\left.\times\ln\frac{m^2}{\lambda^2} - \frac{1}{2}\ln^2\frac{m^2}{m_e^2} - 2\mathrm{Li}_2\left(-\frac{t}{m^2-t}\right)\right], \quad (3.6)$$
$$F = \frac{1}{2}sJ - G = -\frac{1}{2(m^2-t)}\left[\left(2\ln\frac{m^2-t}{m^2} + \ln\frac{m^2}{m_e^2}\right)\ln\frac{s}{m^2} - \right.$$
$$\left. - \ln^2\frac{m^2-t}{m^2} + \frac{1}{2}\ln^2\frac{m^2}{m_e^2} + 2\mathrm{Li}_2\left(-\frac{t}{m^2-t}\right)\right],$$
$$J = \frac{-i}{\pi^2}\int\frac{d^4k}{(\Delta)(Q)(P_+)(P_-)} = -\frac{1}{s(m^2-t)}\left(2\ln\frac{m^2-t}{m^2} + \ln\frac{m^2}{m_e^2}\right)\ln\frac{s}{\lambda^2}.$$

The terms proportional to m_e^2/s, m_e^2/m_μ^2 were neglected.

The vector integrals with three denominators are:

$$\frac{1}{i\pi}\int\frac{k^\mu d^4k}{(\Delta)(Q)(P_+)} = H_P P^\mu + H_\Delta \Delta^\mu + H_Q Q^\mu, \quad H_Q = \frac{1}{t}\ln\frac{m^2-t}{m^2},$$
$$H_\Delta = \frac{1}{m^2-t}\left(-\ln\frac{m^2}{m_e^2} - \frac{m^2+t}{t}\ln\frac{m^2-t}{m^2}\right),$$
$$H_P = H + \frac{1}{m^2-t}\left(\ln\frac{m^2}{m_e^2} + 2\ln\frac{m^2-t}{m^2}\right), \quad (3.7)$$
$$\frac{1}{i\pi}\int\frac{k^\mu\, d^4k}{(\Delta)(P_+)(P_-)} = G_\Delta \Delta^\mu, \quad G_\Delta = \frac{1}{s}\left(-2\ln\frac{s}{m_e^2} + \frac{1}{2}\ln^2\frac{s}{m_e^2} + \frac{\pi^2}{6}\right),$$
$$\frac{1}{i\pi}\int\frac{k^\mu d^4k}{(Q)(P_+)(P_-)} = G_Q Q^\mu, \quad G_Q = \frac{1}{s-4m^2}\left(-2\ln\frac{s}{m^2} + sF_Q\right).$$

Four denominator vector and tensor integrals have the form:

$$J_\mu; J_{\mu\nu} = \int\frac{d^4k}{i\pi^2}\frac{k_\mu; k_\mu k_\nu}{((\Delta)(Q)(P_+)(P_-)}, \quad J_\mu = J_\Delta\Delta_\mu + J_Q Q_\mu,$$
$$J_{\mu\nu} = K_0 g_{\mu\nu} + K_P P_\mu P_\nu + K_Q Q_\mu Q_\nu + K_\Delta \Delta_\mu\Delta_\nu + K_x(Q_\mu\Delta_\nu + Q_\nu\Delta_\mu). \quad (3.8)$$

The relevant coefficients are:

$$J_\Delta = \frac{1}{2d}\left[(F+F_\Delta)\sigma - Q^2(F+F_Q)\right],$$
$$J_Q = \frac{1}{2d}\left[(F+F_Q)\sigma - \Delta^2(F+F_\Delta)\right], \quad d = \Delta^2 Q^2 - \sigma^2,$$

$$K_0 = -\frac{1}{2\sigma}\left[\sigma(F - G + H_P + H_\Delta + H_Q) + H_\Delta(\sigma - \Delta^2) - H_Q(\sigma - Q^2) + \right.$$
$$\left. + 2P^2(\Delta^2 - 2\sigma)J_\Delta + \Delta^2 G_\Delta - Q^2 G_Q - 2P^2 Q^2 J_Q\right],$$

$$K_\Delta = -\frac{1}{2\sigma d}\left[Q^2\sigma(G - F - H_P - 3H_\Delta + 6P^2 J_\Delta) + \right.$$
$$\left. + (\Delta^2 Q^2 + \sigma^2)(H_\Delta - 2P^2 J_\Delta - G_\Delta) - (Q^2)^2(H_Q - 2P^2 J_Q - G_Q)\right], \quad (3.9)$$

$$K_P = \frac{1}{2P^2\sigma}\left[2\sigma(H_\Delta - 2P^2 J_\Delta + H_p + \frac{1}{2}F - \frac{1}{2}G) + Q^2(H_Q - 2P^2 J_Q - G_Q) - \right.$$
$$\left. - \Delta^2(H_\Delta - 2P^2 J_\Delta - G_\Delta)\right],$$

$$K_Q = -\frac{1}{2\sigma d}\left[-\Delta^2\sigma A_P + 2(\Delta^2)^2 A_\Delta + (\sigma^2 - 2\Delta^2 Q^2)A_Q\right],$$
$$K_x = -\frac{1}{2d}\left(\sigma A_P + Q^2 A_Q - 2\Delta^2 A_\Delta\right),$$

where we used

$$A_\Delta = H_\Delta + 2\Delta^2 J_\Delta - G_\Delta, \quad A_Q = H_Q + 2\Delta^2 J_Q - G_Q,$$
$$A_P = F - G + H_P + 3H_\Delta + 6\Delta^2 J_\Delta.$$

3.2. Some Useful One-Fold Integrals. Trace Conversion. Loop momentum integrals by means of Feynman joining denominators trick

$$\frac{1}{y_1^a y_2^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int\limits_0^1 \frac{dx x^{a-1}(1-x)^{b-1}}{(y_1 x + y_2(1-x))^{a+b}} \quad (3.10)$$

can integrate over Feynman parameters. We put some of them below:

$$\int\limits_0^1 \frac{dz}{R(z)} = \frac{1}{\sqrt{d}}L, \quad \int\limits_0^1 \frac{z\,dz}{R(z)} = \frac{1}{2a}\ln\frac{R(1)}{c} - \frac{b}{2a\sqrt{d}}L,$$
$$\int\limits_0^1 \frac{dz}{R^2(z)} = \frac{b^2 + ab - 2ac}{R(1)\,c\,d} - \frac{2a}{\sqrt{d^3}}L, \quad \int\limits_0^1 \frac{z\,dz}{R^2(z)} = -\frac{2a+b}{R(1)\,d} + \frac{b}{\sqrt{d^3}}L, \quad (3.11)$$
$$\int\limits_0^1 \frac{z^2 dz}{R^2(z)} = \frac{2c+b}{dR(1)} - \frac{2c}{\sqrt{d^3}}L,$$

with

$$R(z) = az^2 + bz + c, \quad d = b^2 - 4ac > 0, \quad L = \ln\frac{(b + 2c + \sqrt{d})^2}{4cR(1)}. \quad (3.12)$$

In problems with two fixed axes, the angular phase volume for the case of one axis $\int dO_n = \int d\varphi \sin\theta \, d\theta$ must be replaced by

$$dO_n = 2\int \frac{dx_1\, dx_2}{\sqrt{D}}, \quad D = 1 - a^2 - x_1^2 - x_2^2 + 2ax_1x_2 > 0, \tag{3.13}$$

with $a = \cos\theta_0, x_i = \cos\theta_i, i = 1,2$ and θ_0 being the angle between the axes directions, θ_i — the angles between the current three-vector $\mathbf{n}$ and one of axes directions. We present below some onefold integrals. Writing $D = (x_1 - x_-)(x_+ - x_1), x_\pm = x_2 a \pm \sqrt{(1-a^2)(1-x_2^2)}$ and using the Euler substitution $t^2 = (x_+ - x_1)/(x_1 - x_-)$ we obtain:

$$\int\limits_{x_-}^{x_+} \frac{dx_1}{\sqrt{D}}[1; x_1; x_1^2] = \pi\left[1; ax_2; a^2x_2^2 + \frac{1}{2}(1-x_2^2)(1-a^2)\right],$$

$$\int\limits_{x_-}^{x_+} \frac{dx_1}{\sqrt{D}}\left[\frac{1}{1-\beta x_1}; \frac{1}{(1-\beta x_2)^2}\right] = \pi\left[\frac{1}{r}, \frac{1-\beta a x_2}{r^3}\right], \quad |\beta| < 1, \tag{3.14}$$

$$r = \sqrt{(1-\beta x_-)(1-\beta x_+)} = \sqrt{(a-\beta x_2)^2 + (1-\beta^2)(1-x_2^2)}.$$

Also the next twofold integrals colud be useful:

$$\int\limits_{-1}^{1} dx_2 \int\limits_{x_-}^{x_+} \frac{dx_1}{(1-\beta_1 x_1)(1-\beta_2 x_2)\sqrt{D}} =$$

$$= \frac{\pi}{R}\left[\ln\frac{4}{1-\beta_1^2} + \ln\frac{4}{\beta_2^2} + 2\ln\frac{1-\beta_1\beta_2 a + R}{4}\right], \tag{3.15}$$

$$\int\limits_{-1}^{1} dx_2 \int\limits_{x_-}^{x_+} \frac{dx_1}{(1-\beta x_1)\sqrt{D}} = \frac{\pi}{\beta}\ln\frac{1+\beta}{1-\beta},$$

with

$$R = \sqrt{(1-\beta_1\beta_2 a)^2 - (1-\beta_1^2)(1-\beta_2^2)} =$$

$$= \sqrt{(\beta_1-\beta_2)^2 + \beta_1\beta_2(1-a)[2-\beta_1\beta_2(1+a)]}, \tag{3.16}$$

here we assume that $0 < \beta,\ \beta_{1,2} < 1$.

The following relation of converted tensor can be useful in hand calculation of traces:

$$\frac{1}{4}\,\mathrm{Tr}\,[\hat{a}\gamma_\sigma\hat{b}\gamma_\nu\hat{c}\gamma_\lambda]\frac{1}{4}\,\mathrm{Tr}\,[\hat{d}\gamma^\sigma\hat{e}\gamma^\nu\hat{f}\gamma^\lambda] = 8ad\cdot be\cdot cf + 2\cdot\frac{1}{4}\,\mathrm{Tr}\,[\hat{c}\hat{d}\hat{b}\hat{f}\hat{a}\hat{e}], \tag{3.17}$$

or in more general form

$$\frac{1}{4}\,\mathrm{Tr}\,[\gamma_\lambda \hat{a}_2 \gamma_\sigma \hat{b}_2 \gamma_\nu \hat{c}_2(\alpha_2+\beta_2\gamma_5)]\frac{1}{4}\,\mathrm{Tr}\,[\gamma_\lambda \hat{a}_1 \gamma_\sigma \hat{b}_1 \gamma_\nu \hat{c}_1(\alpha_1+\beta_1\gamma_5)] =$$
$$= 8(\alpha_1\alpha_2+\beta_1\beta_2)a_1a_2\cdot b_1b_2\cdot c_1c_2 +$$
$$+ 2\cdot\frac{1}{4}\,\mathrm{Tr}\,[\hat{c}_1\hat{a}_2\hat{b}_1\hat{c}_2\hat{a}_1\hat{b}_2(\alpha_1\alpha_2-\beta_1\beta_2+\gamma_5(\alpha_1\beta_2-\alpha_2\beta_1))].$$

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