ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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We consider several applications of the simplest nonlinear QED phenomena described by the light-by-light (LBL) scattering tensor. Among the relevant processes we present the splitting of high energy photon in a Coulomb field, calculate the asymptotics of differential photon–photon elastic scattering. We show that the LBL mechanism of the four-photon mode of neutral-pion decay have a dominant role compared, for instance, with the quark-loop Feynman amplitude contribution. The mechanisms of creation of two and three gluon jets at colliding electron–positron beams is analyzed. We calculate also the contribution of the LBL mechanism to the orthopositronium decay width. One of the important applications is the analytic calculation of the QED contribution to the anomalous magnetic moment of the muon arising from the LBL mechanism realized through electron–positron loops, which is enhanced by the logarithm of the ratio of muon to electron masses. The modification of the QED kernel, which takes into account the QED polarization operator, is used to extract the pure strong-interaction contribution. We consider the problem of the Coulomb law modification as well. In the second part of the review we consider the Möller scattering process and radiation corrections (RC) to it. We show that RC are in agreement with the renormalization-group approach and could be taken into account in the form of Drell–Yan process cross section.

Рассмотрен простейший нелинейный процесс в КЭД, описываемый тензором рассеяния света на свете. Изучено расщепление фотона в поле ядра на два фотона, а также асимптотика фотон-фотонного дифференциального сечения. Показано, что механизм рассеяния света на свете в процессе распада нейтрального пиона на четыре фотона доминирует над остальными вкладами, в частности, над вкладом от кварковой петли. Рассмотрен вклад от тензора рассеяния света на свете в образование двух и трех струйных глюоных джетов при электрон-позитронном столкновении, а также поправки к ширине распада ортопозитрония. Рассчитан в аналитическом виде вклад механизма рассеяния света на свете, реализованный через электрон-позитронную петлю, в аномальный магнитный момент мюона. Показано, что данный вклад усилен большим логарифмом отношения мюонной и электронной масс. Эффекты поляризации вакуума лептонами приняты во внимание при выделении вклада сильных взаимодействий в аномальный магнитный момент мюона. Рассмотрена задача модификации кулоновских сил при взаимодействии заряженных частиц за счет механизма рассеяния света на свете. Во второй части рассматриваются радиационные поправки к рассеянию Мёллера. Произведен явный расчет дополнительного вклада в сечение процесса от излучения дополнительного жесткого фотона. Показано, что сумма вкладов в сечение от виртуальных поправок, излучения дополнительного мягкого и жёсткого фотонов может быть представлена в виде сечения процесса Дрелла–Яна. В явном виде представлен вклад от нелидирующих слагаемых (в том числе и от неколлинеарного излучения дополнительного фотона).

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To the blessed memory of Teachers Alexandr Ilich Akhiezer, Sergej Semenovich Sannikov, and Vladimir Naumovich Gribov

INTRODUCTION

Some applications of the LBL scattering tensor which contributes from electron–positron closed loop are considered. Among them are the total cross section of LBL scattering at high energies [4] (and references therein), contribution to muon anomalous magnetic moment [5–7], four-photon channel of neutral-pion decay [8], splitting of photon to two photons in Coulomb field [9], and the crossing process of photons fusion [10]. In modern experiments the contribution of LBL mechanisms to orthopositronium total width [11] becomes important. Creation of two and three gluon jets in electron–positron collisions [12] can be investigated. We also mention the Delbrück process — scattering of photon on Coulomb field started from [13] with further development in [7, 32]. In spite of rather cumbersome form of LBL tensor, the processes considered in this section are described in a compact form.

At small photon energies, where high-intensity sources can be applied, the cross section of photon on photon elastic scattering is very small $\sigma_{\gamma\gamma} \sim (0.1\alpha^2 r_0^2/\pi)(\omega/m)^6$, with m, r_0, ω being respectively mass, classical radii of electron and center-of-mass photon energy. So for $\omega = 1$ MeV the cross section has an order $\sigma_{\gamma\gamma} \sim 10^{-65}$ cm^2. The cross section is maximal $\sigma_{\gamma\gamma} \sim 1.2\pi\alpha^2 r_0^2 \sim 1.6 \cdot 10^{-30}$ cm^2 for $\omega \sim m$. For large values of photon energies $\omega \gg m$, the cross section (in lowest order of perturbation theory) decreases as $\sigma_{\gamma\gamma} \sim (20\alpha^2 r_0^2/\pi)(m/\omega)^2$.

Additional process of annihilation of e^+e^- pair through one virtual photon to three real photons cannot be directly measured due to large background of direct annihilation to three photons. Nevertheless, its contribution to the width of orthopositronium can in principle be measured. A similar process in the framework of QCD annihilation to three gluons in the region of energies without narrow resonances can also be used to investigate nonlinear effects. The problem of calculation of anomalous magnetic moment of muon also requires the knowledge of LBL scattering tensor in 6 and 8 orders of PT due to the fact that the corresponding contribution is now within experimental accuracy.

A lot of attention was paid to calculations of LBL tensor and investigation of manifestations of nonlinear phenomena. We send the reader to the paper by Costantini, De Tollis and Pistoni [4] with almost complete list of relevant literature. We do not pretend to the complete description of this problem. Some applications to the questions mentioned above are given below.

In the second part of the rewiew we consider the Möller scattering process and radiative corrections (RC) to it. In Sec. 2 we calculate the contribution of additional hard-photon emission and, by using the well-known result for RC from soft-photon emission and virtual RC, we show that all corrections are in agreement with the renormalization-group approach and could be taken into account in the form of Drell–Yan process. Also, we put the explicit form of nonleading terms (including the compensation term from additional hard-photon emission) in the form of the so-called K factor.

Throughout our paper we use the following designations: FD — Feynman diagram; LBL — light by light; QCD — quantum chromodynamics; QED — quantum electrodynamics; RC — radiative corrections; SM — Standard Model.

1. LIGHT-BY-LIGHT SCATTERING TENSOR AND VACUUM POLARIZATION

1.1. Photon Splitting in a Coulomb Field. Photon–Photon Elastic Scattering. Consider first the splitting of photon on an atomic electron [9]:

$$\gamma(k_1,\lambda_1)+Y(p)\to Y(p')+\gamma(k_3,\lambda_3)+\gamma(k_4,\lambda_4). \tag{1.1}$$

The cross section in Weizsäcker–Williams approximation will be

$$d\sigma_{\gamma\to\gamma\gamma}=\frac{\alpha}{\pi}\frac{dr}{r}L\,d\sigma_{\gamma\gamma\to\gamma\gamma}, \tag{1.2}$$

with differential photon–photon elastic scattering cross section

$$d\sigma_{\gamma\gamma\to\gamma\gamma}=\frac{\alpha^4}{2\pi r}|M|^2\frac{d^3k_3}{\omega_3}\frac{d^3k_4}{\omega_4}\delta^4(k_1+k_2-k_3-k_4), \tag{1.3}$$

and

$$L=\int\limits_{q^2_{\rm min}}^{q^2_{\rm max}}\frac{dz}{z}(1-F(z))^2,\quad k_2=p-p', \tag{1.4}$$

where $q^2_{\rm min,max}$ are minimal and maximal transverse momenta squared to the nuclei, which are determined by experiment; $F(z)$ is the atomic form factor. The kinematical invariants are defined as

$$\begin{gathered}r=k_3k_4=\frac{\mathbf{k}_3^2}{2y(1-y)}=\omega_1^2\theta_3^2\frac{y}{2(1-y)},\quad s=-k_1k_3=-\frac{1}{2}\omega_1^2\theta_3^2y,\\ t=-k_1k_4=-\omega_1^2\theta_3^2\frac{y^2}{2(1-y)},\end{gathered} \tag{1.5}$$

where $\omega_3 = y\omega_1$, $\omega_4 = (1-y)\omega_1$, $\theta_3 = \widehat{\mathbf{k}_1, \mathbf{k}_3}$, and ω_i are the energies of corresponding photons. We use here the normalization accepted in [4].

The cross section for the case of unpolarized photons can be written in the form

$$\frac{d\sigma_{\gamma\to\gamma\gamma}}{d\omega_3\, d\Omega_3} = \frac{4Z^2\alpha^5(1-y)}{y\pi^3}\frac{L}{\omega_3^3\theta_3^4}|\bar{M}^2|. \quad (1.6)$$

The total photon splitting cross section in the case of full screening is (we use the numerical estimation of the integral $\int\limits_0^\infty (dr/r)\sigma^{\rm tot}_{\gamma\gamma\to\gamma\gamma}(r) = 5\cdot 10^{-30}$ cm^2)

$$\sigma_{\gamma\to\gamma\gamma} = \frac{2Z^2\alpha}{\pi}\ln(183Z^{1/3})\int\limits_0^\infty \frac{dr}{r}\sigma_{\gamma\gamma\to\gamma\gamma} = \\ = \frac{Z^2\alpha}{\pi}\ln(183Z^{-1/3})\cdot 10^{-29}\ \text{cm}^2. \quad (1.7)$$

For the large-invariant case $r \sim -t \sim -s \gg m^2$ and unpolarized photons, we have

$$|M|^2 \to |\bar{M}|^2 = \frac{1}{2}\big[|M_{++++}|^2 + |M_{++--}|^2 + |M_{+-+-}|^2 + \\ + |M_{+--+}|^2 + 4|M_{+++-}|^2\big], \quad (1.8)$$

with

$$\begin{gathered} M_{++++}(x) = 1 + (2x-1)L_2 + \frac{1}{2}[x^2 + (1-x)^2](L_2^2 + \pi^2), \\ M_{+-+-}(x) = 1 + \left(1 - \frac{2}{x}\right)(L_1 - i\pi) + \frac{1}{2x^2}[1 + (1-x)^2](L_1^2 - 2i\pi L_1), \\ M_{++--} = M_{++-+} = M_{+++-} = M_{+-++} = M_{-+++} = -1, \quad (1.9) \\ M_{+--+}(x) = M_{+-+-}(1-x), \\ L_1 = \ln\frac{1}{1-x}, \quad L_2 = \ln\frac{1-x}{x}, \quad x = -\frac{t}{r}. \end{gathered}$$

Using these asymptotic expressions for chiral amplitudes, one has a possibility to calculate the high center-of-mass energy limit of the cross section of photon on photon (ω is the photon energy in center-of-mass reference frame):

$$\lim_{\omega\to\infty}\omega^2\sigma^{\rm tot}_{\gamma\gamma\to\gamma\gamma}(\omega) = \frac{\alpha^4}{2\pi}\int\limits_0^1 dx|\bar{M}|^2(x) = \\ = \frac{\alpha^4}{2\pi}\left[\frac{108}{5} + \frac{13}{2}\pi^2 - 8\pi^2\xi_3 + \frac{148}{225}\pi^4 - 24\xi_5\right] \approx 1.4\cdot 10^{-8}. \quad (1.10)$$

Taking into account the contribution from muon in fermion loop, this result must be multiplied by a factor of 4.

We do not consider here the contribution of hadrons in the intermediate state (the last one was considered in papers [14, 15].

Using the asymptotic for chiral amplitudes of LBL tensor, given above, we can calculate the ratio of cross sections with equal and different chiral state of initial photons in the limit of large center-of-mass photon energies compared with loop fermion mass:

$$R = \frac{\sigma_{LL} + \sigma_{RR}}{\sigma_{LR} + \sigma_{RL}} = \frac{\int\limits_0^1 dx[|M_{++++}|^2 + 3]}{\int\limits_0^1 dx[2|M_{+-+-}|^2 + 2]}. \tag{1.11}$$

Using

$$\begin{aligned} \int\limits_0^1 dx|M_{++++}|^2 &= \frac{1}{5} + 2\xi_2 + \frac{224}{25}\xi_2^2, \\ \int\limits_0^1 dx|M_{+-+-}|^2 &= -24\xi_5 - 48\xi_3\xi_2 + \frac{96}{5}\xi_2^2 + 38\xi_2 + 19, \end{aligned} \tag{1.12}$$

one obtains

$$R = 1.0480707.$$

1.2. Four-Photon Decay Mode of the Neutral Pion. Next application of LBL tensor is connected with experimental looking for the four-photon decay of the neutral pion [8]. It was recently shown [16] that hadronic mechanisms of this channel contribute too little and lead to branching at the level of 10^{-16}. It turns out that the main contribution arises from the QED mechanism with decay to one real and another virtual photons and the latter one decays through the LBL mechanism to three real photons:

$$\mathrm{Br}\,(\pi_0 \to 4\gamma)_{\mathrm{LBL}} \approx (2.6 \pm 0.1) \cdot 10^{-11}. \tag{1.13}$$

The main contribution arises from the electron (positron) as a fermion in the fermion loop amplitude.

The purpose of experiments is measuring the C-violating decay $\pi_0 \to 3\gamma$, which may be a signal of «new physics». The allowed decay channel $\pi_0 \to 4\gamma$ is a potential background for $\pi_0 \to 3\gamma$. Keeping in mind the Bose statistics of photons, the matrix element can be described by four types of diagrams:

$$\pi_0(q) \to [\gamma(k_1) + (\gamma^* \to \gamma(k_2) + \gamma(k_3) + \gamma(k_4))] + \\ + (k_1 \leftrightarrow k_2) + (k_1 \leftrightarrow k_3) + (k_1 \leftrightarrow k_4).$$

Using the dimensionless version of Kumar's variables [17]:

$$s_1 = \frac{1}{m^2}(q-k_1)^2, \quad s_2 = \frac{1}{m^2}(q-k_1-k_2)^2,$$
$$u_1 = \frac{1}{m^2}(q-k_2)^2, \quad u_2 = \frac{1}{m^2}(q-k_3)^2,$$
$$\xi = \frac{1}{2(1-u_1)\rho}[\rho-(1-s_1)^2+(1-u_1)^2], \;\; \rho = \sqrt{\lambda(1,s_2,1+s_2-u_1-s_1)},$$
$$q^2 = m^2, \;\; k_i^2 = 0, \;\; i=1,2,3,4, \;\; \lambda(a,b,c) = a^2+b^2+c^2-2(ab+ac+bc), \tag{1.14}$$

where m is the pion mass, the branching ratio can be written in the form

$$\mathrm{Br}\,(\pi_0 \to 4\gamma) \approx \frac{\Gamma(\pi_0 \to 4\gamma)}{\Gamma(\pi_0 \to 2\gamma)} = \frac{1}{6\pi}\left(\frac{\alpha}{8\pi}\right)^4 R, \tag{1.15}$$

with

$$R = \int\limits_0^1 ds_1 \int\limits_0^{s_1} ds_2 \int\limits_{s_2/s_1}^{1-s_1+s_2} \frac{du_1}{\rho} \int\limits_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi}} F(s_1,s_2,u_1,u_2,\xi). \tag{1.16}$$

The function $F(s_1,s_2,u_1,u_2,\xi)$ stands for half sum of the squared helicity amplitudes:

$$F(s_1,s_2,u_1,u_2,\xi) = \sum_\lambda |M_{+\lambda_2\lambda_3\lambda_4}|^2.$$

For evaluating these helicity amplitudes, we use the asymptotic form of LBL tensor found in [4] with electron as a fermion in a loop. The main contribution arises from the region where the intermediate (virtual) photon 4-momentum squared $k^2 \sim m^2 \gg m_e^2$ is large compared with electron mass squared. These contributions do not have any problems with infrared divergences.

The contribution of muons in fermion loop is suppressed compared with electron contribution by a factor $(m_e/(4M_\mu))^4$ and can be neglected.

The numerical estimation gives the result (1.13). This value is about three orders of magnitude below the present experimental limits. Note, however, that this value is dominant compared with contributions arising from hadronic mechanisms ones and exceed them by at least 3 orders of magnitude [16].

1.3. Explanation of 1973 Year Experiment on Photon Splitting in Coulomb Field. In the experiment on the elastic and inelastic photon scatterings in a Coulomb field of nuclei (performed in experiment [18]), the excess in number of photon with energy $0.87\omega_0$ was observed for different values of initial photon energy $\omega_0 = 1,7; 3,4; 6.1$ GeV/c.

The authors of the experiment believe that they measure the photon splitting process. In paper [19] it was shown that really the process

$$\gamma(\omega_0) + Y(Z) \to e^+ + e^- + \gamma(\theta, \omega) + Y(Z)$$

was measured. Also, in the experiment the magnetic field was used which excludes the possibility to tag the pair component. As was shown in Subsec. 1.1, the real photon splitting process has the cross section of two orders of magnitude lower than the radiative pair production process. Inclusive on photon, the cross section of radiative pair production process has the form

$$\frac{\omega d\sigma}{d^3k} = \frac{Z^2\alpha^4 x}{\pi^2(k_\perp)^4} L \left[A(x) \ln \frac{k_\perp^2}{m_e^2} + B(x) \right], \tag{1.17}$$

with $k_\perp = x\theta\omega_0, x = \omega/\omega_0$, L is given in (1.4) and

$$A(x) = \frac{1}{3}[20 + 56x - 56x^2 - 20x^3] + 16x(1+x)\ln x,$$

$$\begin{aligned} B(x) = \frac{1}{3}[32 - 96x + 96x^2 - 32x^3] + \frac{1}{3}[-48x + 16x^2 + 40x^3]\ln x - \\ - 16x(1-x)\ln(1-x) + \left(-16x^2 + 16x + \frac{32x}{1-x} \right)(\ln x)(\ln(1-x)) + \\ + \left[-16x^2 - 24x + \frac{16x}{1-x^2} \right] \ln^2 x - \left[16x^2 - 16x + \frac{32x}{1+x} \right] \mathrm{Li}_2(1-x). \end{aligned} \tag{1.18}$$

It was shown in [19] that formulae (1.17) and (1.18) describe satisfactorily the result of experiment [18].

1.4. Creation of Gluon and Quark Jets at Electron–Photon High-Energy Collisions. QED results for LBL can also be used for describing QCD processes with creation of two and three gluons [31]. The cross section of jets caused by quark–antiquark pair creation by photon on electron differs only by factor $N = (\sum Q_q^2)^2$:

$$\begin{aligned} d\sigma^{\gamma e \to \mathrm{jet}, e} &= N d\sigma^{\gamma e \to e_+ e_- e}, \\ \frac{d\sigma^{\gamma e \to e_+ e_- e}}{d\Delta} &= \frac{2\alpha^3}{m^2} \left[1 - \frac{4}{3}\Delta(1-\Delta) \right] \left[2 \ln \left(\frac{s\Delta(1-\Delta)}{m^2} \right) - 1 \right], \end{aligned} \tag{1.19}$$

where $\Delta = \epsilon_+/\omega$ is positron energy fraction; $s = 4E\omega = 4E^2$ is the square of total energy in the center-of-mass frame.

It is interesting to consider the limit of photon-splitting differential cross section for the case that the energy fraction of one of the photons is close to unity. In this case the kinematics is similar to elastic scattering of photon on a Coulomb field of nuclei (Delbrück scattering limit).

In the main logarithmic approximation (Weizsäcker–Williams approximation), the cross sections of processes $\gamma e \to Fe$, $F = e^+e^-\gamma; \bar{q}qg$ (quark–antiquark gluon state) can be expressed in terms of convergent integral:

$$\int\limits_{s_{\rm th}}^{s} \frac{ds_1}{s_1} \sigma^{\gamma\gamma\to F}(s_1).$$

The lower value of two-photon mass squared s_1 in $x \to 1$ limit is $s_{\rm th} = (\mathbf{p}^2 + m^2)/(1-x)$, where x and $\mathbf{p}$ are the energy fraction and transverse component of 4-momentum of final state F. It results in nonvanishing limit of the photon-splitting cross section due to the massless nature of a gluon ($m = 0$).

As a result, the Delbrück limit of photon conversion to gluon jet is not zero:

$$\left.\frac{d\sigma^{\gamma\to gg}}{dx\, d\mathbf{p}^2}\right|_{x\to 1} = \frac{2\alpha^3\alpha_s^2}{\pi^2(\mathbf{p}^2)^2} n_f^2 (C_F C_V)^2 I. \tag{1.20}$$

For the $q\bar{q}g$ jet we obtain

$$\frac{d\sigma^{\gamma\to\bar{q}qg}}{dx\, d\mathbf{p}^2} = \\ = \frac{2\alpha^3\alpha_s}{\pi^2(\mathbf{p}^2)^2}(1-x)\left[\frac{20}{3}\ln\frac{\mathbf{p}^2}{m_q^2} - 32(\ln(1-x)+1)\right]\ln\left(\frac{s(1-x)}{\mathbf{p}^2}\right) n_f C_F C_V,$$

with $C_V = N$, $C_F = (N^2-1)/2N$ being structure constants of color group; n_f is the number of light quarks, and

$$\begin{aligned} I &= \int\limits_0^\infty \frac{dz}{z^2}[I_1 + I_2 + I_3 + \pi^2(I_4 + I_5)], \\ I_1 &= \frac{z}{1-z} + \frac{z^2}{(1-z)^2}\ln z - \mathrm{Li}_2(1) + \mathrm{Li}_2(1-z), \\ I_2 &= \frac{2z^2}{1-z^2} + \frac{2z^2(3-z^2)}{(1-z^2)^2}\ln z + z^2\left[\mathrm{Li}_2(1) - \mathrm{Li}_2\left(1 - \frac{1}{z^2}\right)\right], \\ I_3 &= \frac{z}{1+z} - \frac{z^2}{(1+z)^2}\ln z + \mathrm{Li}_2(1) - \mathrm{Li}_2(1+z), \\ I_4 &= \frac{z^2(3+2z)}{(1+z)^2} + 2z^2\ln\frac{z}{1+z}, \quad I_5 = \frac{z^2}{(1+z)^2} - \ln(1+z). \end{aligned} \tag{1.21}$$

This expression can be inferred from the results of paper [4]. Calculation leads to

$$I = \frac{14}{9}\pi^4 - 12\pi^2 \approx 31. \tag{1.22}$$

We note that the creation of gluon jets in this kinematics provides the independent test of QCD predictions.

1.5. Three-Gluon Jet Production at High-Energy Electron–Positron Annihilation. In the lowest order of PT the creation of three-gluon mechanism consists in conversion of lepton pair to the virtual photon and subsequent conversion to gluons through the LBL mechanism. The difference compared with QED consists in the replacement of fermions by quarks, in the taking into account the gluon color effects and replacement of the coupling constant factor as $\alpha^5 \to \alpha^2\alpha_s^3$. First we note that only heavy quarks are relevant due to the fact that the contribution of light quarks is cancelled as a consequence of zero sum of their charges:

$$\sum_{u,d,s} Q_q = 0. \tag{1.23}$$

The color factor $\mathrm{Tr}\,(t^a t^b t^c) = (1/4)[d^{abc} + i f^{abc}]$, with t^a color group generators, results in factor $(d^{abc})^2/16 = (N^2-1)(N^2-4)/(16N) = 5/6$ for color $SU(3)$ group. Really the structures connected with f^{abc} are cancelled if one takes into account both directions of lines in fermion loop. So we further will consider pure QED process of annihilation, by means of the LBL mechanism, of e^+e^- pair to three photons. The differential cross section has the form ($s = 4E^2$, E is the energy of initial particles in c.m.f.)

$$d\sigma = \frac{\alpha^5}{2^{13}\pi^4 E^6}\left(\sum |M|^2\right) d\Gamma, \tag{1.24}$$

with phase volume

$$d\Gamma = \frac{d^3k_2}{\omega_2}\frac{d^3k_3}{\omega_3}\frac{d^3k_4}{\omega_4}\delta^4(p_+ + p_- - k_2 - k_3 - k_4),$$

and k_i, ω_i, $i = 2,3,4$ being real photons momenta and energies, and

$$\begin{gathered}\sum |M|^2 = 2\sum_{\eta\sigma}(|M^{\eta\sigma}_{+++}|^2 + |M^{\eta\sigma}_{++-}|^2 + |M^{\eta\sigma}_{+-+}|^2 + |M^{\eta\sigma}_{-++}|^2),\\ M^{\eta\sigma}_{\lambda_2\lambda_3\lambda_4} = \bar{v}^\eta(p_+)\gamma^\mu u(p_-)^\sigma G_\mu^{\lambda_2\lambda_3\lambda_4}, \quad \lambda_i = \pm;\end{gathered} \tag{1.25}$$

here $\lambda_i, i = 2,3,4$ are real photon chiralities. The current of heavy-photon decay to three real photons G_μ (see [4] for designations of chiral amplitudes

$E^{(i)}_{\lambda_2\lambda_3\lambda_4}(i_1 i_2 i_3 i_4))$ could be expressed in terms of chiral amplitudes:

$$G_\mu^{\lambda_2\lambda_3\lambda_4} = \frac{i}{\sqrt{32\Delta}}\left[E^{(1)}_{\lambda_2\lambda_3\lambda_4}(1234)\left(k_3 - \frac{\nu_3}{\nu_2}k_2\right)_\mu - \right.$$
$$\left. - E^{(1)}_{\lambda_2\lambda_3\lambda_4}(1243)\left(k_4 - \frac{\nu_4}{\nu_2}k_2\right)_\mu + iE^{(2)}_{\lambda_2\lambda_3\lambda_4}(1234)\epsilon_{\mu\alpha\beta\gamma}q^\alpha k_2^\beta k_3^\gamma\right], \quad (1.26)$$

and $q = p_+ + p_-$, $q^2 = 4E^2$, $\nu_i = \omega_i/E$, $\Delta = E^6(1-\nu_2)(1-\nu_3)(1-\nu_4)$.

To obtain the distribution on the photon energy fractions, we average over the angular variables and sum over photon polarization states the product of two such currents. Due to gauge invariance it has the form

$$\overline{G_\mu G_\nu^*} = \frac{1}{3}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)G_\eta G_\eta^*. \quad (1.27)$$

The evident advantage of this averaging is the possibility to express the right-hand side in terms of photon energy fractions. Expressing the phase volume as

$$\int d\Gamma = 8\pi^2 E^2 \int d\nu_2\, d\nu_3\, d\nu_4 \delta(2 - \nu_2 - \nu_3 - \nu_4), \quad (1.28)$$

the energy fractions distribution could be written in the form

$$\frac{d^2\sigma^{e^+e^-\to 3\gamma}}{d\nu_3 d\nu_4} = \frac{\alpha^5}{3!}\frac{1}{12\pi^2E^2}[R(234) + R(324) + R(423)], \quad (1.29)$$

where dimensionless quantities $R(234) = R(243)$ could be found in Appendix of [12]. The quantity $R(z)$ (r.h.s. of Eq. (1.29)), which is defined as

$$R\left(\frac{m}{E}\right) =$$
$$= \int d\nu_2\, d\nu_3\, d\nu_4[R(234) + R(324) + R(423)]\delta(2 - \nu_2 - \nu_3 - \nu_4), \quad (1.30)$$

can be approximated as a step function

$$R(z) \approx 16, \quad z < 1, \quad R(z) \approx 0, \quad z > 1. \quad (1.31)$$

Now by using the changes which we discuss at the beginning of this subsection, we can consider the case of annihilation to the three gluons. The inclusive distribution over the energy fraction of one of the gluons (we take into account only one sort of quarks with charge $e_q = eQ_q$) has the form

$$\frac{d\sigma}{dx} = \frac{5\alpha^2\alpha_s^3}{432\pi^2E^2}F(x), \quad x = \nu_2, \quad (1.32)$$

the function $F(x)$ has a rather cumbersome analytic form. Its asymptotic expressions are

$$\begin{aligned} &F(x) \approx 4x(\ln^2 x - 2\ln x + 3), \quad x \ll 1, \\ &F(x) \approx 1.93\ln^2(1-x) + 0.4\ln(1-x) + 16, \quad 1 - x \ll 1. \end{aligned} \tag{1.33}$$

Compare now the cross section of gluon-jet production with the cross section of production of quark–antiquark and gluon state in annihilation channel:

$$\frac{d\sigma^{e^+e^-\to\bar{q}qg}}{d\nu_+\,d\nu_-} = \frac{\alpha^2\alpha_s Q_q^2}{3E^2}\left[\frac{2(\nu_+^2+\nu_-^2)}{(1-\nu_+)(1-\nu_-)} + \right.$$
$$\left. + \frac{m_q^2}{E^2}\left[\frac{2(1-\nu)}{(1-\nu_+)(1-\nu_-)} - \frac{\nu^2}{(1-\nu_+)^2(1-\nu_-)^2}\right] - \frac{m_q^4}{2E^4}\frac{\nu^2}{(1-\nu_+)^2(1-\nu_-)^2}\right],$$

with conservation law restrictions

$$\begin{aligned} &\nu_\pm = \frac{E_\pm}{E}, \quad \nu = 2 - \nu_+ - \nu_-, \\ &(1-\nu)(1-\nu_+)(1-\nu_-) > \frac{m_q^2}{4E^2}(1-\nu). \end{aligned} \tag{1.34}$$

It can be concluded that measuring the three-gluon jet production in annihilation channel, at least three orders of magnitude are suppressed compared with quark–antiquark gluon production.

1.6. Gluon-Jet Creation in the Scattering Channel at e^+e^- High-Energy Collisions. Within equivalent photons approximation applied to both electron and positron, the cross section of two-gluon jet creation in e^+e^- scattering has the form [12]

$$d\sigma = \frac{\alpha^4\alpha_s^2}{8\pi^3 p_m^2}\left(\ln\frac{E^2}{m_e^2}\right)^2 \sum_\lambda\sum_q Q_q^2|M^q_{\lambda_1\lambda_2\lambda_3\lambda_4}|^2 \frac{d\cos\theta_1\,d\cos\theta_2}{\sin^4\left(\dfrac{\theta_1+\theta_2}{2}\right)}, \tag{1.35}$$

where p_m is the value of the component, transverse to beams axes, of gluon momentum; $\theta_{1,2}$ are the polar angles of the gluons to the beam axes; $M^q_{\lambda_1\lambda_2\lambda_3\lambda_4}$ are chiral amplitudes of LBL tensor with fermion mass m_q. As well as these amplitudes fall rapidly with increasing fermion mass, one can consider only light-quark contribution,

$$\sum_\lambda\sum_q Q_q^2|M^q_{\lambda_1\lambda_2\lambda_3\lambda_4}|^2 \to \frac{16}{9}|\bar{M}|^2, \tag{1.36}$$

with $|\bar{M}|^2$ given in Eq. (1.8), where we must substitute $x = \sin(\theta_1/2)\cos(\theta_2/2)/\sin((\theta_1+\theta_2)/2)$. The ratio of this cross section to the cross section of quark–antiquark production at the same conditions:

$$\frac{d\sigma^{e^+e^-\to e^+e^-2g}}{d\sigma^{e^+e^-\to e^+e^-2\bar{q}q}} = \frac{\alpha_s^2}{\pi^2}\frac{2|\bar{M}|^2 x(1-x)}{x^2+(1-x)^2}\frac{2\left(\sum Q_i^2\right)^2}{3(\sum Q_i^4)} \tag{1.37}$$

shows that it is contrary to the case of three-gluon jet and it is the quantity of order 0.1 and in principle could be measured.

1.7. LBL Contribution to Orthopositronium Width. In the last decade before the year 2000 a lot of attention was paid to the problem of orthopositronium width. It was realized that the result of QED calculations was in contradiction with experimental data on the level of ten standard deviations. Namely, the most accurate experimental rates

$$\begin{aligned} \Gamma^{\text{exp}} &= (7.0514 \pm 0.0014)\ \mu\text{s}^{-1}, \\ \Gamma^{\text{exp}} &= (7.0482 \pm 0.0016)\ \mu\text{s}^{-1} \end{aligned} \tag{1.38}$$

deviate by 9.4σ, 6.2σ from the most accurate theoretical estimations (up to the order of α/π, see [20] and references therein). Great efforts to compute the next correction $(\alpha/\pi)^2$ of the perturbation theory

$$\begin{gathered} \Gamma^{\text{QED}}(PS \to 3\gamma) = \Gamma_0\left[1 + (-10.2866 \pm 0.0006)\frac{\alpha}{\pi} + C\left(\frac{\alpha}{\pi}\right)^2 + \ldots\right], \\ \Gamma_0 = \alpha^6 m\frac{2(\pi^2-9)}{9\pi} \end{gathered} \tag{1.39}$$

were done. To eliminate this contradiction, the value of C must be too large (about $C \approx 400$), which is unusual for perturbative QED.

The contribution to the width arising from the square of the matrix element describing the conversion of orthopositronium state to a virtual photon with its subsequent conversion to three real photons by the LBL mechanism could be considered [11].

The corresponding matrix element has the form

$$-\frac{\sqrt{4\pi\alpha}}{4m^2}\int\frac{d^4p}{(2\pi)^4}\,\text{Tr}\,[\Psi^{(m)}(p)\gamma_\rho]\,G_\rho^{(\lambda_2,\lambda_3,\lambda_4)}; \tag{1.40}$$

here $G_\rho^{(\lambda_2,\lambda_3,\lambda_4)}$ are the same as in (1.26), and

$$\Psi^{(m)}(p) = (2\pi)\delta(p_0)\sqrt{2m}G_m\Phi(\mathbf{p}), \tag{1.41}$$

where

$$G_m = (0, \boldsymbol{\sigma}\boldsymbol{\epsilon}_m, 0, 0), \tag{1.42}$$

$\boldsymbol{\epsilon}_m$ is the orthopositronium polarization vector and

$$\Phi(\mathbf{p}) = \sqrt{\frac{\gamma^3}{m}} \frac{8\pi\gamma}{\mathbf{p}^2 + \gamma^2}, \quad \gamma = \frac{m\alpha}{2}. \tag{1.43}$$

By integrating an averaging as in Subsec. 1.5 we obain the relevant contribution to the width:

$$\Delta\Gamma = -m\frac{\alpha^8}{2^4 9\pi^3} \int d\nu_2\, d\nu_3\, d\nu_4 [R(234) + R(324) + R(423)]\delta(2 - \nu_2 - \nu_3 - \nu_4) = \\ = 0.17021(10) \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0. \tag{1.44}$$

This kind of contributions of course do not solve the contradiction problem, but, nevertheless, must be taken into account.

1.8. LBL Tensor and Anomalous Magnetic Moment of Muon. We consider below the contribution to the anomalous magnetic moment of muon (amm) $(\Delta a)_\mu$ from the gauge-invariant set of Feynman amplitudes, containing LBL block [5]. It is one of contributions to amm in $(\alpha/\pi)^3$ order of PT which turns out to be the numerically largest. Really, it equals

$$\left(\frac{\alpha}{\pi}\right)^3 \left[\left(\frac{2\pi^2}{3}\right) \ln\left(\frac{M}{m}\right) + \text{const}\right] \approx 21.4(\alpha/\pi)^3, \tag{1.45}$$

(here M, m are the muon and electron masses, respectively), whereas the total contribution in this order is $24.07(\alpha/\pi)^3$. Our result is in agreement with ones obtained numerically [21] and analytically [22].

It can be shown [21] (and references therein) that relevant contribution to amm has the form

$$(\Delta a)_\mu = \frac{1}{48M} \operatorname{Tr} (\hat{p} + M)[\gamma^\rho \gamma^\sigma](\hat{p} + M) M_{\rho\sigma}, \tag{1.46}$$

with

$$M_{\rho\sigma} = -\frac{4\pi\alpha}{(2\pi)^8} \int \frac{d^4k_1\, d^4k_2\, d^4k_3}{k_1^2 k_2^2 k_3^2} \delta^4(k_1 + k_2 + k_3) \times \\ \times J^{\mu\nu\lambda}_{\rho\sigma}(k_1, k_2, k_3) \gamma_\lambda (\hat{p} + \hat{k}_3 + M)^{-1} \gamma_\nu (\hat{p} - \hat{k}_1 + M)^{-1} \gamma_\mu. \tag{1.47}$$

The current $J^{\mu\nu\lambda}_{\rho\sigma}$ is connected with the antisymmetric part of the derivative of the LBL tensor $G^{\mu\nu\lambda\sigma}(\Delta, k_1, k_2, k_3)$:

$$J^{\mu\nu\lambda}_{\rho\sigma} = \lim_{\Delta \to 0} \left(\frac{\partial}{\partial \Delta_{[\rho}}\right) G^{\mu\nu\lambda}_{\sigma]}(\Delta, k_1, k_2, k_3). \tag{1.48}$$

Building the current $J^{\mu\nu\lambda}_{\rho\sigma}(k_1,k_2,k_3) = J^{\mu\nu\lambda}_{\rho\sigma}(123)$, we follow the logical scheme of the famous paper by Karplus and Neuman [23], imposing the Bose-symmetry and gauge-invariance requirements

$$J^{\mu\nu\lambda}_{\rho\sigma}(123) = J^{\nu\lambda\mu}_{\rho\sigma}(231) = J^{\lambda\mu\nu}_{\rho\sigma}(312) = \ldots, \quad k_{1\mu}J^{\mu\nu\lambda}_{\rho\sigma}(123) = 0. \tag{1.49}$$

As a result, we obtain such a general expression in terms of Maxwell tensor

$$\begin{gathered} F^{\mu}_{\alpha\beta}(k_1) = k_{1\alpha}\delta^{\mu}_{\beta} - k_{1\beta}\delta^{\mu}_{\alpha}, \quad F^{\mu}(k_1)k_{1\mu} = 0, \\ F_{\alpha\beta}(k,\epsilon) = k_\alpha\epsilon_\beta - k_\beta\epsilon_\alpha, \end{gathered} \tag{1.50}$$

and six scalar functions $d_1, d_2, d_3, d_4, a, A_1$ («heads» in terms of [23]):

$$\begin{aligned} J^{\mu\nu\lambda}_{\rho\sigma}(123) &= a(123)[12]_{\rho\sigma}(F^{\mu}(k_1)F^{\nu}(k_2)F^{\lambda}(k_3)) + \\ &+ \sum_{\text{perm}}\Bigg[(F^{\nu}(k_2)F^{\lambda}(k_3))\Bigg[\frac{1}{4}F^{\mu}_{\rho\sigma}(k_1k_3A_1(321) - d_3(132)) - \\ &\qquad - \frac{1}{2}A_1(231)((\bar{k}_2)_{\rho\sigma}F^{\mu}(k_1)k_2)\Bigg] - \\ &- \frac{1}{2k_1k_3}d_2(123)(k_1F^{\lambda}(k_3)k_2)(F^{\mu}(k_1)F^{\nu}(k_2))_{\gamma\eta}[\gamma\eta]_{\rho\sigma} + \\ &+ \frac{1}{2k_1k_2}d_4(123)[(F^{\mu}(k_1)F^{\lambda}(k_3)F^{\nu}(k_2))_{\eta\gamma}[\eta\gamma]_{\rho\sigma}k_1k_2 + \\ &+ ((\bar{k}_1)_{\rho\sigma}F^{\nu}(k_2)F^{\mu}(k_1)F^{\lambda}(k_3)k_3) - ((\bar{k}_2)_{\rho\sigma}F^{\mu}(k_1)F^{\nu}(k_2)F^{\lambda}(k_3)k_1)\Bigg], \end{aligned}$$

with

$$(\bar{k}^{\mu}_i)_{\rho\sigma} = k_{i\rho}\delta^{\mu}_{\sigma} - k_{i\sigma}\delta^{\mu}_{\rho}, \quad [\nu\lambda]_{\rho\sigma} = \delta^{\nu}_{\rho}\delta^{\lambda}_{\sigma} - \delta^{\nu}_{\sigma}\delta^{\lambda}_{\rho},$$

and conversion on lower indices implied

$$(F^{\nu}(k_2)F^{\mu}(k_1)) = F^{\nu}_{\alpha\beta}(k_2)F^{\mu}_{\beta\alpha}(k_1), \quad (aF^{\mu}b) = a^{\alpha}F^{\mu}_{\alpha\beta}b^{\beta}, \ldots \tag{1.51}$$

This form is general and does not depend on theoretical model. We now can identify the six functions by comparing the characteristic terms in Feynman amplitudes and fermion loop diagrams describing the LBL scattering.

Let us now discuss the details of integration over virtual photons momenta. The main contribution leading to «large logarithm» $\int\limits_m^M (dk/k)$ arises from such a region of 3-momenta of virtual photons variation (muon rest frame implied):

$$m \ll |\mathbf{k}_i| \ll M, \quad |k_{0i}| \sim \frac{\mathbf{k}_i^2}{M} \ll |\mathbf{k}_i|, \quad i = 1,2,3. \tag{1.52}$$

It corresponds to almost real muon intermediate states. Besides, the main contribution arises from «nearest singularities» — small values of $y_i = k_{0i}/m$, $y_1 + y_2 + y_3 = 0$. To provide the convergence of y_i integrations, we must use the symmetrization procedure. It consists in using the relation

$$\frac{1}{3!}\int J(y_1, y_2, y_3) d^3 y \delta\left(\sum y_i\right)\left[\frac{1}{-y_1 + i0}\frac{1}{y_3 + i0} + \right.$$
$$+ \frac{1}{-y_1 + i0}\frac{1}{y_2 + i0}\frac{1}{-y_2 + i0}\frac{1}{y_1 + i0} + \frac{1}{-y_2 + i0}\frac{1}{y_3 + i0} +$$
$$\left. + \frac{1}{-y_3 + i0}\frac{1}{y_1 + i0}\frac{1}{-y_3 + i0}\frac{1}{y_2 + i0}\right] = -4\pi^2 J(0,0,0). \quad (1.53)$$

Further manipulations are straightforward but tedious. After that we arrive at an expression valid in logarithmic approximation

$$(\Delta a)_\mu = -\frac{2\pi}{3}\left(\frac{\alpha}{\pi}\right)^3 \ln\frac{M^2}{m^2} I,$$

$$I = \int_0^1 \frac{dx}{x^2}\int_0^\infty \frac{dt}{t^3}[\arctan(xt) - xt]\times$$
$$\times\left[5x - 2x^2 + \frac{2x(1 - 4x(1-x))}{xt^2 + 1} - \frac{(1-x)(1-2x)}{(xt^2+1)^2}\right].$$

Analytic calculation of I gives the value $-\pi/2$, reproducing Eq. (1.45).

As a check of our calculations we consider the case of small photons virtualities $|k_i^2| \ll m^2$. In this case we have $A_1 = a = 0$, $d_1 = d_2 = d_3 = 8\alpha^2/9$, $d_4 = 28\alpha^2/(45m^4)$ and explicit expression for current J is in agreement with Heisenberg–Euler result for the LBL tensor [24].

1.9. New Formulation of $(g-2)_\mu$ Hadronic Contribution. *1.9.1. Motivation.* Anomalous magnetic moment of muon a_μ is very sensitive laboratory to search for new physics beyond the Standard Model (SM) (see [25] and references therein). However, before deriving any premature conclusions about new physics, careful calculations of hadronic uncertainties in amm should be done [26]. The estimation of theoretical and experimental ones becomes very important.

Here we suggest a new, more natural form of inclusion of hadronic vacuum polarization effects. The theoretical as well as the systematic experimental uncertainties are expected to be considerably reduced.

The SM contributions are usually split into three parts: $a_\mu = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm hadr}$. The part of $a_\mu^{\rm hadr}$ takes into account only vacuum polarization effects (we do not consider hadronic contributions of light-by-light type) and usually is presented in the form (see, for instance, [27] and references therein)

$$a_\mu^{\rm hadr} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_{4m_\pi^2}^{\infty} \frac{ds}{s} R(s)\left[K^{(1)}(s) + \frac{\alpha}{\pi}K^{(2)}(s)\right], \tag{1.54}$$

with

$$K^{(1)}(s) = \int\limits_0^1 dx \frac{x^2(1-x)}{x^2+\rho(1-x)}, \quad \rho = \frac{s}{M^2}, \tag{1.55}$$

where M is the muon mass and

$$R(s) = \frac{\sigma_0^{e^+e^-\to{\rm hadr}}(s)}{\sigma^{e^+e^-\to\mu^+\mu^-}(s)} = 12\pi\,{\rm Im}\,\Pi_h(s), \quad \sigma^{e^+e^-\to\mu^+\mu^-}(s) = \frac{4\pi\alpha^2}{3s}. \tag{1.56}$$

The quantity $\sigma_0^{e^+e^-\to{\rm hadr}}(s)$, which enters the quantity $R(s)$, is rather unphysical one and it does not take into account the effects of vacuum polarization of virtual photon ($e^+ + e^- \to \gamma^* \to$ hadr.). The physical one can be obtained by the replacement

$${\rm Im}\,\Pi_h(s) \to {\rm Im}\left(\frac{\Pi(s)}{1-\Pi(s)}\right) = \frac{{\rm Im}\,\Pi_h(s)}{|1-\Pi(s)|^2}, \quad \Pi(s) = \Pi_l(s) + \Pi_h(s), \tag{1.57}$$

where $\Pi_l(s)$, $\Pi_h(s)$ are leptonic and hadronic contributions to the vacuum polarization operator. Namely, the quantity $\sigma_{\rm exp}^{e^+e^-\to{\rm hadr}}(s)$, defined as

$$\sigma_{\rm exp}^{e^+e^-\to{\rm hadr}}(s) = \frac{\sigma_0^{e^+e^-\to{\rm hadr}}(s)}{|1-\Pi(s)|^2}, \tag{1.58}$$

is more relevant to experiment, contrary to Born one $\sigma_0^{e^+e^-\to{\rm hadr}}(s)$. In the region of narrow resonances the application of this formula must be performed with some care [28].

1.9.2. Second-Order Kernel Modification. Keeping this definition in mind, one must revise the formulae for $a_\mu^{\rm hadr}$, cited above. Really, one must replace in integrands of $a_\mu^{\rm hadr}$: $\sigma_0^{e^+e^-\to{\rm hadr}}(s) \to \sigma_{\rm exp}^{e^+e^-\to{\rm hadr}}(s)$. The kernel $K^{(1)}(s)$ remains the same, but the kernel $K^{(2)}(s)$ must be modified to avoid the double counting. The modification consists in eliminating of contributions of all Feynman diagrams containing two kinds of polarization of vacuum insertions (hadronic, leptonic sort and the mixed ones). It results in omitting the contributions of $K^{(2b,2c)}(s)$ in terminology of [27]. As for $K^{(2a)}(s)$, it must be modified in

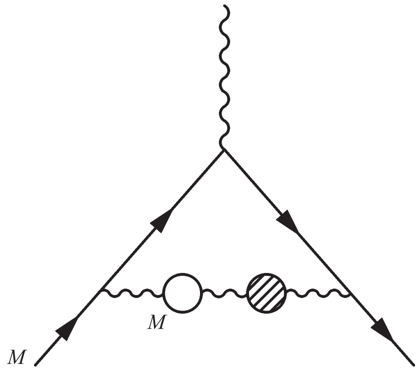

Fig. 1. Subtracting Feynman diagram

such a way as to extract the contribution of the Feynman diagram (see Fig. 1) that contains polarization operator for the muon case with hadronic one. So, our result consists in replacement of $K^{(2a)}(s)$ [29, 30]:

$$\begin{aligned}
K^{(2a)}(s) = 2\Bigg\{ & -\frac{139}{144} + \frac{115}{72}\rho + \left(\frac{19}{12} - \frac{7}{36}\rho + \frac{23}{144}\rho^2 + \frac{1}{\rho-4}\right) L + \\
& + \frac{1}{\Delta}\left(-\frac{4}{3} + \frac{127}{36}\rho - \frac{115}{72}\rho^2 + \frac{23}{144}\rho^3\right)\ln y + \left(\frac{9}{4} + \frac{5}{24}\rho - \frac{1}{2}\rho^2 - \frac{2}{\rho}\right)\xi_2 + \\
& + \frac{5}{96}\rho^2 L^2 + \frac{1}{\Delta}\left(-\frac{1}{2}\rho + \frac{17}{24}\rho^2 - \frac{7}{48}\rho^3\right) L \ln y + \\
& + \left(\frac{19}{24} + \frac{53}{48}\rho - \frac{29}{96}\rho^2 - \frac{1}{3\rho} + \frac{2}{\rho-4}\right)\ln^2 y + \\
& + \frac{1}{\Delta}\left(-2\rho + \frac{17}{6}\rho^2 - \frac{7}{12}\rho^3\right) D_p(\rho) + \\
& + \frac{1}{\Delta}\left(\frac{13}{3} - \frac{7}{6}\rho + \frac{1}{4}\rho^2 - \frac{1}{6}\rho^3 - \frac{4}{\rho-4}\right) D_m(\rho) + \\
& + \left(\frac{1}{2} - \frac{7}{6}\rho + \frac{1}{2}\rho^2\right) T(\rho)\Bigg\}, \qquad (1.59)
\end{aligned}$$

with $L = \ln(s/M^2)$, $\Delta = \sqrt{\rho(\rho-4)}$, $\xi_2 = \pi^2/6$ and

$$y = \frac{\sqrt{\rho} - \sqrt{\rho-4}}{\sqrt{\rho} + \sqrt{\rho-4}},$$

$$D_p(\rho) = \mathrm{Li}_2(y) + \ln y \, \ln(1-y) - \frac{1}{4}\ln^2 y - \xi_2,$$

$$D_m(\rho) = \mathrm{Li}_2(-y) + \frac{1}{4}\ln^2 y + \frac{1}{2}\xi_2,$$

$$T(\rho) = -6\mathrm{Li}_3(y) - 3\mathrm{Li}_3(-y) + \ln^2 y\, \ln(1-y) + \\ + \frac{1}{2}(\ln^2 y + 6\xi_2)\ln(1+y) + 2\ln\, y(\mathrm{Li}_2(-y) + 2\mathrm{Li}_2(y)),$$

$$\mathrm{Li}_3(y) = \int\limits_0^y \frac{dx}{x}\mathrm{Li}_2(x), \tag{1.60}$$

by the new one:

$$\bar{K}^{(2)}(s) = K^{(2a)}(s) - K^{(2b)}(s)\Big|_{m_f=M}, \tag{1.61}$$

with

$$K^{(2b)}(s)_{m_f=M} = 2\int\limits_0^1 dx \frac{x^2(1-x)}{x^2+\rho(1-x)}\Pi(1,x), \\ \Pi(1,x) = -\frac{8}{9} + \frac{b^2}{3} - b\left(\frac{1}{2} - \frac{b^2}{6}\right)\ln\frac{b-1}{b+1}, \quad b = \frac{2-x}{x}. \tag{1.62}$$

The quantity $K^{(2b)}(s)_{m_f=M}$ can be calculated analytically:

$$K^{(2b)}(s)_{m_f=M} = \frac{2}{\rho}\left[\frac{8}{9}\rho^2 + \frac{35}{36}\rho - \frac{4}{3}\xi_2 - \frac{1}{\Delta}[L_-P_1(x_-) - L_+P_1(x_+)] - \right. \\ \left. - \frac{1}{\Delta}[\mathrm{Li}_-P_2(x_-) - \mathrm{Li}_+P_2(x_+)]\right], \tag{1.63}$$

with $x_\pm = (\rho \pm \Delta)/2$ and

$$L_\pm = \ln\frac{x_\pm}{x_\pm - 1}, \quad \mathrm{Li}_\pm = \mathrm{Li}_2(1 - x_\mp), \\ P_1(z) = -\frac{5}{9}z^4 - \frac{4}{3}z^3 + \frac{4}{3}z^2, \quad P_2(z) = \frac{1}{3}z^4 - 2z^2 + \frac{4}{3}z. \tag{1.64}$$

For expansion into series by powers of ρ^{-1}, we have

$$\bar{K}^{(2)}(s) = 2\frac{1}{\rho}\left[\bar{a}_1 + \bar{b}_1 L + \frac{1}{\rho}(\bar{a}_2 + \bar{b}_2 L + \bar{c}_2 L^2) + \frac{1}{\rho^2}(\bar{a}_3 + \bar{b}_3 L + \bar{c}_3 L^2) + \right. \\ \left. + \frac{1}{\rho^3}(\bar{a}_4 + \bar{b}_4 L + \bar{c}_4 L^2) + \frac{1}{\rho^4}(\bar{a}_5 + \bar{b}_5 L + \bar{c}_5 L^2)\right] + O(\rho^{-6}), \tag{1.65}$$

with

$$\begin{aligned}
\bar{a}_1 &= \frac{50}{27} - \frac{2}{3}\xi_2, & \bar{b}_1 &= -\frac{23}{36}, & & \\
\bar{a}_2 &= \frac{9241}{1152} - \frac{103}{24}\xi_2, & \bar{b}_2 &= -\frac{487}{216}, & \bar{c}_2 &= \frac{43}{144}, \\
\bar{a}_3 &= \frac{15256601}{432000} - \frac{803}{40}\xi_2, & \bar{b}_3 &= -\frac{29279}{3600}, & \bar{c}_3 &= \frac{221}{80}, \\
\bar{a}_4 &= \frac{66452261}{432000} - \frac{10829}{120}\xi_2, & \bar{b}_4 &= -\frac{57917}{1800}, & \bar{c}_4 &= \frac{3763}{240}, \\
\bar{a}_5 &= \frac{18433084459}{27783000} - \frac{13877}{35}\xi_2, & \bar{b}_5 &= -\frac{34443349}{264600}, & \bar{c}_5 &= \frac{47651}{630}.
\end{aligned} \tag{1.66}$$

So, our final result for the hadronic contribution to the anomalous magnetic moment of muon is

$$a_\mu^{\text{hadr}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_{4m_\pi^2}^{\infty} \frac{ds}{s} R_{\text{exp}}^h(s)\left[K^{(1)}(s) + \frac{\alpha}{\pi}\bar{K}^{(2)}(s)\right], \tag{1.67}$$

where $R_{\text{exp}}^h(s) = \sigma_{\text{exp}}^{e^+e^-\to\text{hadr}}(s)/\sigma^{e^+e^-\to\mu^+\mu^-}(s)$ and $\bar{K}^{(2)}(s)$ is given above (see (1.61), (1.65)).

The set of Feynman diagrams contribution with lepton and hadron vacuum polarization associated with different virtual photon lines cannot be considered with the method discussed above. Their contribution (see Fig. 2) enhanced by logarithmic factor can be estimated as $\delta a_\mu^{\text{hadr}} \sim (\alpha/\pi)^2(1/3)\ln M^2/m_e^2 \approx 2\cdot 10^{-5}a_\mu^{\text{hadr}}$. Fortunately, this is beyond the modern experimental possibilities.

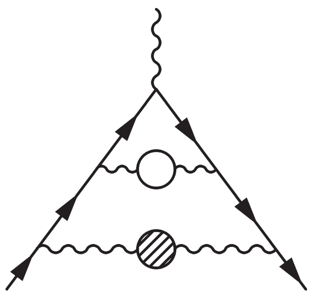

Fig. 2. Typical contribution with lepton and hadron vacuum polarization associated with different virtual photon lines enhanced by logarithmic factor

1.10. LBL Modification of Coulomb Force. The LBL mechanism provides a correction to the Coulomb force of the charged target [33] (see also [1, Ch. 5]). Really, for large distances $mr \sim 1$, using the effective Heisenberg–Euler Lagrangian, one obtains

$$\varphi(r) = \frac{Ze}{4\pi r}\left[1 - \frac{2Z^2\alpha^3}{225\pi}\left(\frac{1}{mr}\right)^4\right], \tag{1.68}$$

with m-electron mass. For example, for $Z = 20$, $r = 10$ fm the expression in the square brackets is negative. It is some kind of antiscreening effect.

It is interesting to estimate the contribution of the light quarks as a fermion in the loop of the LBL-type Feynman amplitude. Keeping in mind the interaction of three gluons with the quarks in the nuclei of a charge Z with the atomic number $N > Z$, we obtain for the ratio of the QCD and QED corrections:

$$\frac{\Delta_{\rm QCD}}{\Delta_{\rm QED}} = \left(\frac{\alpha_s(0)}{\alpha}\right)^3 \left(\frac{m}{M_u}\right)^4 \left(\frac{3N}{Z}\right)^2 C_q, \qquad (1.69)$$

where $M_u \approx M_d \approx 200$ MeV is the constituent light-quark mass; $\alpha_s(0) \approx 2$ [34] is the gluon–quark coupling constant; the color factor C_q is

$$C_q = (Q_u + Q_d)\left(\frac{1}{4}d^{abc}\right)^2 = \frac{5}{18}, \qquad (1.70)$$

with $Q_u = 2/3, Q_d = -1/3$ being the quark charges in units e. We see that the QCD contribution is at least two orders of magnitude suppressed compared with light-fermion ones. The situation can be changed if the critical charge $\alpha_s(0) \approx 20$ [35]. In this case the antiscreening effect can take place already at $r = 15-20$ fm and, in principle, can be measured experimentally — by carefully searching for the energy levels of heavy ions.

2. MÖLLER SCATTERING

We put below the results of calculations of the high-energy electron–electron quasi-elastic scattering in ultrarelativistic approximation (Möller scattering):

$$\begin{gathered} e^-(p_1) + e^-(p_2) \to e^-(p_1') + e^-(p_2'), \\ p_{1,2}^2 = p_{1,2}^{'2} = m^2, \quad s = (p_1+p_2)^2 = 4E^2, \\ u = (p_1 - p_2')^2 \approx -s(1-\chi), \quad t = (p_1 - p_1')^2 \approx -s\chi, \end{gathered} \qquad (2.1)$$

where $\chi = \sin^2(\theta/2)$, θ is the angle between the three momenta of the initial electron $\mathbf{p}_1$ (the direction of z axis) and the scattered electron $\mathbf{p}_1'$. Here and below we imply the center-of-mass reference frame of initial particles.

Taking into account the lowest-order RC, due to emission of virtual and soft real photons, one obtains [3]

$$\begin{aligned} \frac{d\sigma^{e^-e^-\to e^-e^-}}{dO_1} &= \frac{d\bar\sigma_B^{e^-e^-\to e^-e^-}}{dO_1}\times \\ &\times \left[1 + \frac{\alpha}{\pi}\left\{\left(4\ln\frac{\Delta E}{E} + 3\right)L - 4 + 4(\ln(\chi(1-\chi)) - 1)\ln\frac{\Delta E}{E}\right\} + \right. \\ &\left. + \frac{2\alpha}{\pi F_B}F(\chi)\right], \quad L = \ln\frac{s}{m^2}, \end{aligned} \qquad (2.2)$$

with the Born cross section, corrected by vacuum polarization of virtual photon:

$$\frac{d\bar{\sigma}_B^{e^-e^-\to e^-e^-}}{dO_1} = \frac{\alpha^2}{2s\chi^2(1-\chi)^2}F_B,$$

$$F_B = (1-\chi)^2(1+(1-\chi)^2)\frac{1}{(1-\Pi_t)^2} + \\ + \chi^2(1+\chi^2)\frac{1}{(1-\Pi_u)^2} + 2\chi(1-\chi)\frac{1}{(1-\Pi_t)(1-\Pi_u)}. \quad (2.3)$$

Here $\Delta E \ll E$ is the energy of soft photons, escaping the detectors.
The expression for $F(\chi)$, which contains the nonleading terms, is

$$F(\chi) = (1+P(\chi,1-\chi))\Big[-\frac{3\pi^2}{8}\chi(1-\chi) + \frac{1}{6}\chi(11-3\chi+14\chi^2)\ln(1-\chi) + \\ + \frac{1}{4}\chi(-3+8\chi-4\chi^2+3\chi^3)\ln^2\chi + \\ + \frac{1}{4}(-8+19\chi(1-\chi)-8\chi^2(1-\chi)^2)\ln\chi\ln(1-\chi)\Big], \quad (2.4)$$

with $P(\chi,1-\chi)$ being the exchange operator $P(\chi,1-\chi)f(\chi) = f(1-\chi)$.
Emission of hard photon with c.m. energy $\omega > \Delta E$

$$e^-(p_1) + e^-(p_2) \to e^-(p_1') + e^-(p_2') + \gamma(k) \quad (2.5)$$

is described in terms of kinematic invariants:

$$s = (p_1+p_2)^2, \quad t = (p_1-p_1')^2, \quad u = (p_1-p_2')^2, \quad \chi_i = kp_i, \ i = 1,2; \\ s_1 = (p_1'+p_2')^2, \ t_1 = (p_2-p_2')^2, \ u_1 = (p_2-p_1')^2, \ \chi_i' = kp_i', \ i = 1,2; \quad (2.6)$$

$$s + s_1 + t + t_1 + u + u_1 = 0.$$

Here and further we imply

$$s \sim s_1 \sim -t \sim -t_1 \sim -u \sim -u_1 \gg m^2. \quad (2.7)$$

The cross section of the process (2.5) has the form [2]

$$d\sigma^{e^-e^-\to e^-e^-\gamma} = \frac{1}{2}\frac{\alpha^3}{\pi^2 s}R_{ee}\,d\Gamma, \quad (2.8)$$

with phase volume of final state

$$d\Gamma = \frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'}\frac{d^3k}{\omega}\delta^4(p_1+p_2-p_1'-p_2'-k), \quad (2.9)$$

and

$$R_{ee} = \frac{W}{16}\frac{1}{tt_1uu_1}\left[ss_1(s^2+s_1^2)+tt_1(t^2+t_1^2)+uu_1(u^2+u_1^2)\right] - \\ - \frac{m^2}{\chi_1'^2}\left(\frac{t_1}{u}+\frac{u}{t_1}+1\right)^2 - \frac{m^2}{\chi_2'^2}\left(\frac{t}{u_1}+\frac{u_1}{t}+1\right)^2 - \\ - \frac{m^2}{\chi_2^2}\left(\frac{t}{u}+\frac{u}{t}+1\right)^2 - \frac{m^2}{\chi_1^2}\left(\frac{t_1}{u_1}+\frac{u_1}{t_1}+1\right)^2, \quad (2.10)$$

$$W = \frac{s(tu+t_1u_1)+s_1(t_1u+u_1t)+2uu_1(t+t_1)+2tt_1(u+u_1)}{\chi_1\chi_2\chi_1'\chi_2'}.$$

The vacuum polarization factor $1/(1-\Pi)$ has contributions from leptons, heavy vector meson ($W_\pm$) and hadrons. The contribution of light charged leptons (electron–positron) is

$$\Pi_t = \frac{\alpha}{3\pi}\left(L_t - \frac{5}{3}\right), \quad \Pi_u = \frac{\alpha}{3\pi}\left(L_u - \frac{5}{3}\right), \quad L_t = \ln\frac{-t}{m^2}, \quad L_u = \ln\frac{-u}{m^2}. \quad (2.11)$$

The differential cross section with radiative corrections coming from emission of virtual and real soft photons in leading logarithm approximation (the ones containing $\alpha L/\pi$) can be written as

$$\frac{d\sigma^{e^-e^-\to e^-e^-}}{dO_1} = \frac{d\bar\sigma_B^{e^-e^-\to e^-e^-}}{dO_1}\left[1+\frac{\alpha}{2\pi}(L-1)\left(2\ln\frac{\Delta E}{E}+\frac{3}{2}\right)\right]^4. \quad (2.12)$$

Let us note that the last term in the square brackets is the so-called Δ-part of the kernel of the evolution equation of twist-two operators P_Δ (see below).

This observation permits us (using the general factorization theorem) to generalize the result obtained in lowest order of perturbation theory to one valid in the leading and next-to-leading approximation in all orders of perturbation theory:

$$\frac{d\sigma^{e^-e^-\to e^-e^-(\gamma)}}{dO_1\,dy_1\,dy_2} = \int\limits_0^1 D(x_1,\beta)\,dx_1 \int\limits_0^1 D(x_2,\beta)\,dx_2 \times \\ \times \frac{d\bar\sigma_B}{dO_1}(x_1p_1,x_2p_2)\frac{1}{x_1'}D\left(\frac{y_1}{x_1'},\beta\right)\frac{1}{x_2'}D\left(\frac{y_2}{x_2'},\beta\right)\left(1+\frac{\alpha}{\pi}K\right), \quad (2.13)$$

where

$$\beta = \frac{\alpha}{\pi}(L-1), \quad x_1' = \frac{2x_1x_2}{a}, \quad x_2' = \frac{x_1^2+x_2^2+c^2(x_2^2-x_1^2)}{a}, \\ a = x_1(1-c)+x_2(1+c), \quad c=\cos\theta, \quad (2.14)$$

and the structure function $D(x, \beta) = D^{NS}(x, \beta)$ is the nonsinglet lepton structure function considered in [36]:

$$D(x, \beta) = \delta(1 - x) + \beta P^{(1)}(x) + \frac{1}{2!}\beta^2 P^{(2)}(x) + \ldots,$$

$$P^{(n)}(x) = P^{(1)} \otimes P^{(n-1)}(x) = \int\limits_x^1 \frac{dy}{y} P^{(1)}(x) P^{(n-1)}\left(\frac{x}{y}\right), \quad n = 2, 3, \ldots, \tag{2.15}$$

$$P^{(1)}(x) = P(x) = \lim_{\epsilon \to 0} \left[P_\Delta \delta(1 - x) + \Theta(1 - x - \epsilon) P_\Theta(x)\right],$$

$$P_\Delta = 2 \ln \epsilon + \frac{3}{2}, \quad P_\Theta(x) = \frac{1 + x^2}{1 - x}.$$

The values y_1, y_2 are the energy fractions of the detected electrons in the final state, $y_1 < x_1', y_2 < x_2'$. The value of k-factor K is the sum of nonleading terms (of order α/π not enhanced by the large logarithmic factor L) arising from taking into account emission of both virtual and real photons.

Distinguishing the emission of soft and hard real photons, we put it in the form

$$K = K_{\text{SV}} + K_H, \tag{2.16}$$

with

$$K_{\text{SV}} = -1 + 4 \ln\left(\chi(1 - \chi)\right) \ln \frac{\Delta E}{E} + \frac{2F(\chi)}{F_B}. \tag{2.17}$$

The hard-photon emission contribution can be written as

$$K_H = \frac{\pi}{\alpha}\left[\frac{\alpha^3}{2\pi^2 s} \int \tilde{R}_{ee} d\tilde{\Gamma} + K_{\text{comp}}\right]\left(\frac{d\bar{\sigma}_B}{dO_1}(p_1, p_2)\right)^{-1}, \tag{2.18}$$

with

$$\tilde{R}_{ee} = R_{ee}\big|_{m=0},$$

$$d\tilde{\Gamma} = d\Gamma \theta(\theta_1 - \theta_0)\, \theta(\theta_2 - \theta_0)\, \theta(\theta_1' - \theta_0)\, \theta(\theta_2' - \theta_0)\, \theta(\omega - \Delta E), \quad \theta_0 \ll 1; \tag{2.19}$$

here $\theta_1, \theta_2, \theta_1', \theta_2'$ are angles between the photon momentum $\mathbf{k}$ and the momenta of leptons $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_1'$, $\mathbf{p}_2'$, correspondingly.

The compensation term K_{comp} is

$$K_{\text{comp}} = \int\limits_{\Delta E/E}^1 dx \Bigg[\frac{dW_0}{dx}\left[\frac{d\bar{\sigma}_B((1-x)p_1, p_2)}{dO_1} + \frac{d\bar{\sigma}_B(p_1, (1-x)p_2)}{dO_1}\right] + \\ + 2\frac{d\bar{W}_0}{dx}\frac{d\bar{\sigma}_B(p_1, p_2)}{dO_1}\Bigg],$$

with

$$\frac{dW_0}{dx} = \frac{\alpha}{2\pi x}\left[2(1+(1-x)^2)\ln\frac{\theta_0}{2} + x^2\right],$$
$$\frac{d\bar{W}_0}{dx} = \frac{\alpha}{2\pi x}\left[2(1+(1-x)^2)\left(\ln\frac{\theta_0}{2} + \ln(1-x)\right) + x^2\right], \tag{2.20}$$

and

$$\frac{d\bar{\sigma}_B(x_1p_1, x_2p_2)}{dO_1} = \frac{4\alpha^2}{sa^2}\left[\frac{a^2+Y^2}{2X^2}\frac{1}{(1-\Pi(t_1))^2} + \frac{a^2+X^2}{2Y^2}\frac{1}{(1-\Pi(u_1))^2} + \right.$$
$$\left. + \frac{a^2}{YX}\frac{1}{(1-\Pi(t_1))(1-\Pi(u_1))}\right],$$

$$\frac{-t_1}{s} = \frac{x_1^2x_2(1-c)}{a}, \quad \frac{-u_1}{s} = \frac{x_1x_2^2(1+c)}{a}, \quad X = x_1(1-c), \quad Y = x_2(1+c).$$

Analytical or numerical analysis shows that the K_H value does not depend on the auxiliary parameter θ_0 at sufficient small values of θ_0. Moreover, the whole K value does not depend on the other auxiliary parameter $\Delta E/E$.

The value of k-factor depends in general on the experimental setup — details of detection of the scattered electrons. The relevant cuts can be included as additional restrictions on the phase volume of final particles.

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