ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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THEORY OF NEUTRINO MASSES AND MIXING

W. Grimus

University of Vienna, Faculty of Physics, Vienna, Austria

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We motivate the usage of finite groups as symmetries of the Lagrangian. After a presentation of basic group-theoretical concepts, we introduce the notion of characters and character tables in the context of irreducible representations and discuss their applications. We exemplify these theoretical concepts with the groups S_4 and A_4. Finally, we discuss the relation between tensor products of irreducible representations and Yukawa couplings and describe a model for tri-bimaximal lepton mixing based on A_4.

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INTRODUCTION

Motivation for Horizontal Symmetries. The mass spectrum of quarks and leptons is one of the least understood facts of particle physics. However, it was noticed quite early [1] that the Cabbibo angle might be a function of the ratio of down and strange quark mass because numerically one has

$$\sin\theta_c \simeq \sqrt{\frac{m_d}{m_s}}. \tag{1}$$

A very popular possibility to generate fermion masses and mixing is the Higgs mechanism. This has brought about the idea that in such a framework the CKM matrix could be explained by symmetries acting on the three quark families which restrict the Yukawa couplings such that a relation like equation (1) becomes possible. Since the CKM matrix is not far from the unit matrix and the up and down quark mass spectra are strongly hierarchical, it seems at least plausible that the mixing angles are functions of quark mass ratios.

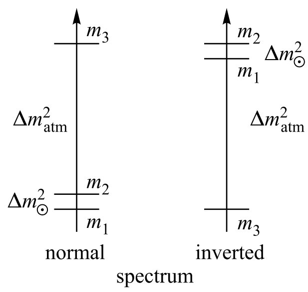
The observation by Harrison, Perkins and Scott [2] that lepton mixing is in good approximation *tri-bimaximal*, i.e., compatible with the mixing matrix

$$U \simeq \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \equiv U_{\mathrm{HPS}}, \tag{2}$$

has given a boost to the idea of family symmetries. In the lepton sector it seems that mixing angles could be related to «pure numbers». At any rate, U is very different from the unit matrix and thus lepton mixing is very different from quark mixing [3].

Neutrino Mass Spectrum. The idea that the elements of U are, in good approximation, pure numbers (and not functions of lepton mass ratios) is in accord with the observation of the neutrino mass spectrum: it is either completely different from the charged-fermion mass spectra or its hierarchy is not so pronounced [3].

We know from neutrino oscillations that the neutrino mass spectrum is nondegenerate. The neutrino mass spectrum is called hierarchical if $m_1 \ll \Delta m^2_\odot$, where m_1 is the smallest neutrino mass and $\Delta m^2_\odot$ is the solar mass-squared difference. Since $\Delta m^2_{\rm atm}/\Delta m^2_\odot \sim 30$, we conclude that $m_3/m_2 \simeq \sqrt{\Delta m^2_{\rm atm}/\Delta m^2_\odot} \sim 5$–$6$ in the hierarchical case which illustrates that a neutrino mass hierarchy can only be rather weak. The quantity $\Delta m^2_{\rm atm}$ is the atmospheric mass-squared difference. An inverted hierarchy is also possible if (by the usual convention) m_3 is the smallest mass with $m_3 \ll \Delta m^2_{\rm atm}$. Experimentally, the question of the neutrino mass spectrum is completely undecided. If the smallest neutrino mass is denoted by m_s, we have a *normal* ordering for $m_s = m_1$ and an *inverted* ordering for $m_s = m_3$. The spectrum is called quasi-degenerate if $m_1 \simeq m_2 \simeq m_3$. Of course, also a spectrum between hierarchical and quasi-degenerate is allowed for both orderings.



Types of neutrino mass spectra

Neutrino Mass Terms and Parameter Counting. In the following we assume that

- neutrinos have Majorana nature and
- the charged-lepton mass matrix is diagonal.

Majorana neutrinos are theoretically more appealing than Dirac neutrinos because many mechanisms for neutrino mass generation, e.g., the seesaw mechanism [4], naturally lead to Majorana nature. The second assumption is used only for the time being for the purpose of parameter counting.

A Majorana neutrino mass term is given by

$$\mathcal{L}_{\rm Maj} = \frac{1}{2}\nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \text{h.c.} \tag{3}$$

with the charge-conjugation matrix C. From the anticommutation property of the neutrino fields we conclude that $\mathcal{M}_\nu = \mathcal{M}_\nu^T$, i.e., $\mathcal{M}_\nu$ is a symmetric but in general complex matrix. For the transformation to the mass eigenfields, the following theorem, specialized to 3×3 matrices, is applied.

Theorem 1 (Schur). *For every complex, symmetric matrix $\mathcal{M}_\nu$ there exists a unitary matrix U with $U^T \mathcal{M}_\nu U = \mathrm{diag}\,(m_1, m_2, m_3)$ and $m_j \geqslant 0$.*

The matrix U diagonalizing the neutrino mass matrix $\mathcal{M}_\nu$ is called the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) or lepton mixing matrix U, provided we are in a basis where the charged-lepton mass matrix is diagonal. The matrix U is usually parameterized as

$$U = \mathrm{e}^{i\hat{\alpha}}\, U_{23} U_{13} U_{12}\, \mathrm{diag}\left(1, \mathrm{e}^{i\beta_2}, \mathrm{e}^{i\beta_3}\right). \tag{4}$$

The diagonal phase matrix $\mathrm{e}^{i\hat{\alpha}} = \mathrm{diag}\,(\mathrm{e}^{i\alpha_1}, \mathrm{e}^{i\alpha_2}, \mathrm{e}^{i\alpha_3})$ is unphysical in the charged-current interaction because it can be absorbed into the charged lepton fields. The matrices U_{23}, U_{13}, and U_{12} are rotations in the subsectors indicated by their subscripts:

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \tag{5}$$

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}\,\mathrm{e}^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}\,\mathrm{e}^{i\delta} & 0 & c_{13} \end{pmatrix}, \tag{6}$$

$$U_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

In the mixing matrix the conventions $0 \leqslant \theta_{ij} \leqslant 90^\circ$ are imposed. As a consequence, one must allow the full range $0 \leqslant \delta < 360^\circ$ of the CP-violating CKM-type phase δ. As for the neutrino masses, one imposes $m_1 < m_2$ with $\Delta m^2_\odot = m_2^2 - m_1^2$. With this convention the sign of $m_3^2 - m_1^2$ is a physical quantity and must eventually be determined by experiment.

In summary there are nine physical parameters in neutrino masses and mixing: three masses, three angles, and the three phases δ, β_2, and β_3. The latter two phases are the so-called Majorana phases; if neutrinos have Dirac nature, they can be removed from the charged-current interactions by absorbing them into the neutrino fields.

Let us compare the number of nine parameters with the number of parameters in $\mathcal{M}_\nu$. There are $6 \times 2 = 12$ real parameters in $\mathcal{M}_\nu$. However, e.g., the first line and first column can be made real by a phase transformation $\mathrm{e}^{i\hat{\alpha}}$ which has no effect in the charged-current interactions — see above. Thus we have nine real physical parameters in $\mathcal{M}_\nu$ corresponding to the nine physical quantities above. As mentioned before, there is also one discrete physical parameter, namely $\mathrm{sign}\,(m_3^2 - m_1^2)$, which is $+1$ for the normal ordering and -1 for the inverted ordering of the neutrino mass spectrum.

Finally, we want to make some remarks concerning the diagonalization of $\mathcal{M}_\nu$ with theorem 1. If we write $U = (u_1, u_2, u_3)$ with an orthonormal (ON) basis u_j of $\mathbb{C}^3$, theorem 1 tells us that

$$\mathcal{M}_\nu u_j = m_j u_j^*. \tag{8}$$

Note the following points:

- In general, u_j is not an eigenvector of $\mathcal{M}_\nu$, this is the case only for real u_j.
- If λ is an eigenvalue of $\mathcal{M}_\nu$, then $|\lambda|$ is in general *not* a neutrino mass.
- However, the neutrino masses can be obtained by $\mathcal{M}_\nu^\dagger \mathcal{M}_\nu u_j = m_j^2 u_j$.

In these lecture notes we will discuss some features of model building for lepton masses and mixing — see, for instance, [5, 6] for reviews. However, we will first delve into useful theoretical aspects of finite groups and review two groups popular in model building. For the general theory of groups we refer the reader, e.g., to [7, 8]. Recent reviews on finite subgroups of $SU(2)$ and $SU(3)$ are presented in [9, 10], for more specialized recent reviews see [11–13].

1. THEORY OF FINITE GROUPS

1.1. Basics. We assume familiarity of the reader with the very basic notions like the definition of a group, representation, irreducible representation (irrep), subgroup, coset and normal subgroup, which can be found in any text-book on group theory, e.g., in [7, 8].

Now we will explain some basic useful concepts. On a group G one always has an equivalence relation via the following definition: g_1 is *conjugate* to g_2 if there exists a $g \in G$ such that $g g_1 g^{-1} = g_2$. The sets of equivalent elements are called *conjugacy classes*. Obviously, $\{e\}$ is a class consisting only of the unit element, and a normal subgroup consists of complete conjugacy classes.

Irreps and Proper Normal Subgroups. Using a symmetry group in physics mostly boils down to applying its irreps to physical objects (multiplets). Therefore, we need to know the irreps or methods how to track them down. A good part of this section is devoted to this subject.

Knowing the proper normal subgroups of G helps in this respect. The notion «proper» means that the subgroup is larger than $\{e\}$ and smaller than G. Let H be a proper normal subgroup of G, then

- the mapping $f : g \in G \to Hg \in G/H$ is a homomorphism, i.e., the relation $f(g)f(g') = f(gg')$ holds $\forall\, g, g' \in G$,
- and any representation D of G/H induces naturally a representation $\bar{D}$ of G via $\bar{D}(g) \equiv D(Hg)$.

Direct Product. With two groups G and G' one can form the direct product group $G \times G'$ with the multiplication law $(g_1, g'_1)(g_2, g'_2) = (g_1 g_2, g'_1 g'_2)$. This is often used in model building. For example, one has a symmetry group like the permutation group S_3 and enlarges it by a sign transformation leading to $S_3 \times \mathbb{Z}_2$, a direct product of S_3 with the cyclic group $\mathbb{Z}_2$.

Semidirect Product. This generalization of the direct product is written as $H \rtimes_\phi G$, which symbolizes that G acts on H via the homomorphism $\phi : G \to \mathrm{Aut}(H)$, where $\mathrm{Aut}\,(H)$ is the group of automorphisms on H. (An automorphism ϕ on H is simply a group isomorphism $\phi : H \to H$.) The multiplication law is given by

$$(h_1,\, g_1)(h_2,\, g_2) = (h_1\, \phi(g_1) h_2,\, g_1 g_2). \tag{9}$$

This is a rather abstract definition and it takes a bit of effort to prove that the multiplication law is associative. We will shortly see that in practice it has a very simple interpretation.

Obviously, for $\phi = \mathrm{id}$ the semidirect product is identical with the direct product. A useful question for model building is whether a group can be decomposed into a semidirect product. Actually, a closer examination of finite groups shows that semidirect products are ubiquitous! The reason is the following theorem.

Theorem 2. *Let us assume that H is a proper normal subgroup of S; and G, a subgroup of S with following properties:*
1) $H \cap G = \{e\}$,
2) Every element $s \in S$ can be written as $s = hg$ with $h \in H$, $g \in G$.
Then the following holds:
- *$S \cong H \rtimes_\phi G$ with $\phi(g)h = ghg^{-1}$,*
- *The decomposition $s = hg$ is unique,*
- *$S/H \cong G$.*

The proof is straightforward. That the homomorphism ϕ has the form given in the theorem simply follows from the multiplication of two elements of S:

$$s_1 s_2 = (h_1 g_1)(h_2 g_2) = (h_1 g_1 h_2 g_1^{-1})(g_1 g_2). \tag{10}$$

1.2. Symmetries in the Lagrangian versus Symmetry Groups. Suppose we have a multiplet of fermion fields $\psi_1, \ldots, \psi_r$ in the Lagrangian $\mathcal{L}$. Then $\mathcal{L}$ has the form

$$\mathcal{L} = i \sum_{j=1}^{r} \bar{\psi}_j \gamma^\mu \partial_\mu \psi_j + \ldots, \tag{11}$$

where the dots indicate the terms beyond the kinetic terms. The symmetries of $\mathcal{L}$ are given by transformations $\psi_j \to A^{(p)}_{jk} \psi_k$ $(p = 1, \ldots, N_{\mathrm{gen}})$. Since the kinetic term has to be invariant, it follows that the matrices $A^{(p)}$ $(p = 1, \ldots, N_{\mathrm{gen}})$ are unitary. There are two approaches to symmetries and Lagrangians:

• We start with $\mathcal{L}$ and impose symmetries $A^{(p)}$ on $\mathcal{L}$. Then the $N_{\rm gen}$ matrices $A^{(p)}$ generate a representation of a symmetry group G from which we can infer the group G.

• We can also take the opposite point of view. We begin with a group G and introduce multiplets of fields which transform according to representations of G. In this way we determine $\mathcal{L}$ from the symmetry group and the multiplets we introduce.

1.3. Useful Theorems for Finite Groups. Finite groups, i.e., groups whose number of elements is finite, are very popular in model building. As expected, infinite groups are more complicated than finite ones: They possess infinitely many inequivalent irreps, and noncompact simple Lie groups G possess no finite-dimensional unitary irreps apart from the trivial ones where every element is mapped onto unity.

Let us, for example, consider $U(1)$ as the simplest infinite group. We readily find its irreps: $\mathrm{e}^{i\alpha} \to \mathrm{e}^{in\alpha}$ with $n \in \mathbb{Z}$. Thus there are infinitely many. The same applies to the simplest non-Abelian group $O(2)$. Its irreps can be found, for instance, in Appendix of [14].

For finite groups, the number of its elements is called order of G and abbreviated by $\operatorname{ord} G$. Finite groups have the following properties:

• They possess a finite number of inequivalent irreps.

• All irreps are equivalent to unitary irreps.

• All numbers concerning properties of the group and its irreps are finite as well; this allows one to derive extremely useful relations which are totally lacking in infinite groups.

Now we list some of the most important theorems for finite groups:

Theorem 3 (Lagrange). *If H is a subgroup of G, then* $\operatorname{ord} H$ *is a divisor of* $\operatorname{ord} G$.

This theorem has a straightforward corollary. Defining the order of an element g of G as the smallest number r such that $g^r = e$, we observe that every element $g \in G$ generates a cyclic subgroup $\mathbb{Z}_r \subseteq G$. Therefore, the order of every element is a divisor of $\operatorname{ord} G$.

Theorem 4. *If we denote the irreps of G by $D^{(\alpha)}$, with dim $D^{(\alpha)} = d_\alpha$ being the dimension of the vector space on which the irrep acts, and if the index α numbers all inequivalent irreps, then it follows that*

$$\sum_\alpha d_\alpha^2 = \operatorname{ord} G. \tag{12}$$

Theorem 5. *The number of inequivalent irreps $D^{(\alpha)}$ equals the number of conjugacy classes of G.*

1.4. Characters and Character Tables. *Orthogonality Relations for Irreps.* One can define the space of functions on G and endow it with the scalar product

$$(f_1|f_2) = \frac{1}{\operatorname{ord} G} \sum_{g \in G} f_1^*(g) f_2(g) \tag{13}$$

in order to make it a unitary space.

Suppose we have an irrep $D^{(\alpha)}$ with dimension d_α. Then with respect to a basis, the irrep consists of matrices and we can conceive the matrix elements $D^{(\alpha)}_{ij}(g)$ as functions on G. With Schur's lemma (not to be confused with theorem 1 (Schur)) it is rather easy to prove the following theorem [7–9]:

Theorem 6. *For irreps $D^{(\alpha)}$ and $D^{(\beta)}$ with dimensions d_α and d_β, respectively, the orthogonality relations*

$$\sum_{g \in G} D^{(\alpha)}_{ij}(g^{-1}) D^{(\beta)}_{kl}(g) = \frac{\operatorname{ord} G}{d_\alpha}\, \delta_{\alpha\beta}\delta_{jk}\delta_{il} \tag{14}$$

hold.

For finite groups we can always assume that the representation matrices are unitary. In this case $D^{(\alpha)}_{ij}(g^{-1}) = ({D^{(\alpha)}}^\dagger)_{ij}(g) = (D^{(\alpha)}_{ji}(g))^*$ is valid and equation (14) can be rewritten as

$$(D^{(\alpha)}_{ji}|D^{(\beta)}_{kl}) = \frac{1}{d_\alpha}\, \delta_{\alpha\beta}\delta_{jk}\delta_{il}. \tag{15}$$

The Character of a Representation. For any representation D its character is defined by the function

$$\chi:\ g \in G \to \chi(g) = \operatorname{Tr} D(g) \in \mathbb{C}, \tag{16}$$

where Tr denotes the trace. The character has the property that it is constant on every class C_k.

Let us move to the characters of irreps. We denote by $\chi^{(\alpha)}$ the character of the irrep $D^{(\alpha)}$. These characters have the following properties:

$$\chi^{(\alpha)}(e) = d_\alpha, \quad \sum_{g \in G} (\chi^{(\alpha)}(g))^* \chi^{(\beta)}(g) = \delta_{\alpha\beta} \operatorname{ord} G. \tag{17}$$

The first relation is trivial, the second one follows from Eq. (14). If we denote by c_k the number of elements in class C_k and by $\chi^{(\alpha)}_k$ the value of $\chi^{(\alpha)}$ on C_k, then the orthogonality relation for the characters of irreps reads

$$\sum_{k=1}^{n} c_k (\chi^{(\alpha)}_k)^* \chi^{(\beta)}_k = \delta_{\alpha\beta} \operatorname{ord} G, \tag{18}$$

where n is the number of classes.

Character Tables. Since according to theorem 5 for every group G the number of classes, n, equals the number of inequivalent irreps, one can depict a quadratic scheme of numbers $\chi_k^{(\alpha)}$, with columns and lines marked by k and α, respectively. Such a scheme is called character table of the group G (see Table 1). Note that this scheme is usually supplemented by two further lines as shown in Table 1, for providing further information on the group.

Table 1. **Schematic description of a character table. In the first line, after the name of the group G, the classes are listed, below each class C_k is its number of elements c_k, and in the second line below the class, the order ν_k of its elements is stated**

G (# C_k) ord (C_k)	C_1 (c_1) ν_1	C_2 (c_2) ν_2	$\cdots$ $\cdots$ $\cdots$	C_n (c_n) ν_n
$D^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$	$\cdots$	$\chi_n^{(1)}$
$D^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_n^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$D^{(n)}$	$\chi_1^{(n)}$	$\chi_2^{(n)}$	$\cdots$	$\chi_n^{(n)}$

It is customary to set $C_1 = \{e\}$, thus in the first column the dimensions $d_\alpha = \chi_1^{(\alpha)}$ of the irreps can be read off. Furthermore, the usual convention is that $D^{(1)}$ is the trivial irrep, therefore, $\chi_k^{(1)} = 1 \; \forall k$. Moreover, the irreps are ordered according to increasing dimensions.

From equation (18) we know that the line vectors

$$\left(\sqrt{\frac{c_1}{\operatorname{ord} G}}\,\chi_1^{(\alpha)}, \ldots, \sqrt{\frac{c_n}{\operatorname{ord} G}}\,\chi_n^{(\alpha)}\right) \tag{19}$$

form an ON basis of $\mathbb{C}^n$. Consequently, also the column vectors

$$\sqrt{\frac{c_k}{\operatorname{ord} G}} \begin{pmatrix} \chi_k^{(1)} \\ \vdots \\ \chi_k^{(n)} \end{pmatrix} \quad (k = 1, \ldots, n) \tag{20}$$

define an ON basis whose orthonormality conditions can be reformulated as

$$\sum_{\alpha=1}^{n} (\chi_k^{(\alpha)})^* \chi_\ell^{(\alpha)} = \frac{\operatorname{ord} G}{c_k}\,\delta_{k\ell}. \tag{21}$$

Equations (18) and (21) are useful for the construction of a character table.

Reducible Representations and Character Tables. Suppose a representation D of a group G is given. Then with its character table it is straightforward to find its decomposition into irreps because the character of a reducible representation is a sum

$$\chi_D = \sum_{\alpha=1}^{n} n_\alpha \chi^{(\alpha)}, \tag{22}$$

where the n_α denote the multiplicities with which the irreps $D^{(\alpha)}$ occur in D. Consequently,

$$n_\alpha = (\chi^{(\alpha)} | \chi_D). \tag{23}$$

This relation is particularly useful for tensor products because the character of the tensor product $D^{(\alpha)} \otimes D^{(\beta)}$ is given by the product of the characters of $D^{(\alpha)}$ and $D^{(\beta)}$:

$$\chi^{(\alpha\otimes\beta)}(g) = \chi^{(\alpha)}(g) \times \chi^{(\beta)}(g). \tag{24}$$

1.5. The Group S_4. Let us examine the symmetric group S_4, i.e., the group of permutations of four objects, in the light of our group-theoretical discussion. We have chosen S_4 for two reasons. First, it is a group which is popular for model building — see, e.g., [15] for a very early paper with S_4 used in the quark sector and two recent papers [16, 17] where this group is a symmetry in the lepton sector. Second, for the symmetric groups S_n there is a general and simple rule how to find their classes*.

The order of S_n is $n!$. Every element $p \in S_n$ can be written as

$$p = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}. \tag{25}$$

This scheme means that i is mapped to p_i $(i = 1, \dots, n)$. One can also present permutations as cycles. A cycle of length r is a mapping

$$(n_1 \to n_2 \to n_3 \to \dots \to n_r \to n_1) \equiv (n_1 n_2 n_3 \cdots n_r) \tag{26}$$

such that all numbers $n_1, \dots, n_r$ are different. Evidently, every permutation is a unique product of cycles which have no common elements. For instance,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix} = (145)(3)(26). \tag{27}$$

Cycles which have no common element commute and a cycle which consists of only one element is identical with the unit element of S_n. The classes of S_n are characterized by the cycle structure [7].

*In general, the problem of finding the classes of a group can be quite tricky, if its order is large.

Theorem 7. *The classes of S_n consist of the permutations with the same cycle structure.*

Let us apply this to S_4. The theorem says that it has five classes corresponding to the cycle structures e, (n_1n_2), $(n_1n_2)(n_3n_4)$, $(n_1n_2n_3)$ and $(n_1n_2n_3n_4)$. Its corresponding classes will be denoted by $C_1, \ldots, C_5$, respectively, in the following. Thus, S_4 has five inequivalent irreps.

There is another useful theorem concerning S_n.

Theorem 8. *S_n has exactly two 1-dimensional irreps: $p \to 1$ and $p \to \mathrm{sgn}\,(p)$.*

The sign of a permutation is $+1$ (-1), if it can be decomposed into an even (odd) number of transpositions, i.e., cycles of length $r = 2$. A cycle of length r is even (odd) if r is odd (even).

Now we can easily find the dimensions of all irreps of S_4. We know already that there are five irreps, with two of them having dimension one. Thus, according to theorem 4, we have the equation $1^2 + 1^2 + d_3^2 + d_4^2 + d_5^2 = 24$. One can easily check that the solution is unique (up to reordering): $d_3 = 2$, $d_4 = d_5 = 3$.

In order to find the remaining three irreps we take advantage of the fact that Klein's four-group

$$K = \{e,\, (12)(34),\, (13)(24),\, (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{28}$$

is a normal, Abelian subgroup of S_4. That it is an Abelian subgroup is easily checked, that K is also normal follows from theorem 7. We observe that S_3 can be conceived as subgroup of S_4 if we consider the permutations of only $2, 3, 4$. One can check that K and the S_3 defined in this way have exactly the properties of H and G of theorem 2. Therefore,

$$S_4 \cong K \rtimes S_3 \tag{29}$$

and every element of S_4 can uniquely be decomposed into $s = kp$ with $k \in K$ and $p \in S_3$.

Taking advantage of equation (29), we find the 2-dimensional irrep as

$$kp \to D_2(p) \quad \text{with} \quad D_2((234)) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad D_2((34)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

Note that D_2 is an irrep of S_3. Clearly, K, which is represented trivially as we have discussed in Subsec. 1.1, and the two cycles in equation (30) generate the full S_4, thus we really have found the complete 2-dimensional irrep.

It remains to construct the two 3-dimensional irreps. We only sketch the procedure. A 3-dimensional representation of $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by

$$\begin{aligned} (12)\,(34) &\to \mathrm{diag}\,(\;\;\,1, -1, -1)\,, \\ (13)\,(24) &\to \mathrm{diag}\,(-1, \;\;\,1, -1)\,, \\ (14)\,(23) &\to \mathrm{diag}\,(-1, -1, \;\;\,1)\,. \end{aligned} \tag{31}$$

We denote the representation of K by $A(k)$. Obviously, the mapping

$$(34) \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (24) \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (23) \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{32}$$

generates a representation of the S_3 which permutes the numbers $2, 3, 4$. We denote this representation by $M_3(p)$. It is not difficult to ckeck that $kp \to A(k)M_3(p)$ is indeed a representation of S_4. Obviously, it is irreducible. The second 3-dimensional irrep is obtained by multiplication of the previous one with $\mathrm{sgn}\,(p)$.

Thus we have the following summary of the S_4 irreps:

$$s = kp \in S_4 \Rightarrow \begin{array}{ll} \mathbf{1}: & kp \to 1, \\ \mathbf{1}': & kp \to \mathrm{sgn}\,(p), \\ \mathbf{2}: & kp \to D_2(p), \\ \mathbf{3}: & kp \to A(k)M_3(p), \\ \mathbf{3}': & kp \to \mathrm{sgn}\,(p)\, A(k)M_3(p). \end{array} \tag{33}$$

Note that $\mathrm{sgn}\,(kp) = \mathrm{sgn}\,(p) = \det\, M_3(p)$.

Table 2. **Character table of** S_4

S_4 (# C_k) ord (C_k)	C_1 (1) 1	C_2 (6) 2	C_3 (3) 2	C_4 (8) 3	C_5 (6) 4
$\mathbf{1}$	1	1	1	1	1
$\mathbf{1}'$	1	−1	1	1	−1
$\mathbf{2}$	2	0	2	−1	0
$\mathbf{3}$	3	1	−1	0	−1
$\mathbf{3}'$	3	−1	−1	0	1

Having all classes and irreps at our disposal, we can write down character Table 2. As an application we compute the decomposition of $\mathbf{3} \otimes \mathbf{3}$ into irreps. The character of $\mathbf{3} \otimes \mathbf{3}$ is given by the square of the line labeled by $\mathbf{3}$ in Table 2:

$$\chi^{\mathbf{3}\otimes\mathbf{3}} = [\,9\;1\;1\;0\;1\,]\,. \tag{34}$$

With Eq. (23) the multiplicities of the irreps in $\mathbf{3} \otimes \mathbf{3}$ are computed. The whole information for this computation is contained in the character table:

$$
\begin{aligned}
n_{\mathbf{1}} &= \frac{1}{24}\,(1\times 1\times 9 + 6\times 1\times 1 + 3\times 1\times 1 + 8\times 1\times 0 + 6\times 1\times 1) = 1,\\
n_{\mathbf{1}'} &= \frac{1}{24}\,(1\times 1\times 9 - 6\times 1\times 1 + 3\times 1\times 1 + 8\times 1\times 0 - 6\times 1\times 1) = 0,\\
n_{\mathbf{2}} &= \frac{1}{24}\,(1\times 2\times 9 + 6\times 0\times 1 + 3\times 2\times 1 - 8\times 1\times 0 + 6\times 0\times 1) = 1,\\
n_{\mathbf{3}} &= \frac{1}{24}\,(1\times 3\times 9 + 6\times 1\times 1 - 3\times 1\times 1 + 8\times 0\times 0 - 6\times 1\times 1) = 1,\\
n_{\mathbf{3}'} &= \frac{1}{24}\,(1\times 3\times 9 - 6\times 1\times 1 - 3\times 1\times 1 + 8\times 0\times 0 + 6\times 1\times 1) = 1.
\end{aligned}
$$

All products of three numbers in this computation are given by

$$c_k \times \chi_k^{(\alpha)} \times \chi_k^{\mathbf{3}\otimes\mathbf{3}}.$$

Thus the result of the decomposition is

$$\mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{2}\oplus\mathbf{3}\oplus\mathbf{3}'. \tag{35}$$

With some experience it is not difficult to guess the Clebsch–Gordan coefficients (for their definition see, e.g., [7,8]). Denoting the Cartesian basis vectors in $\mathbf{3}$ by e_j $(j = 1, 2, 3)$ and defining $\omega = \mathrm{e}^{2\pi i/3}$ we find

$$
\begin{aligned}
&\mathbf{1}: \quad \frac{1}{\sqrt{3}}\,(e_1\otimes e_1 + e_2\otimes e_2 + e_3\otimes e_3)\,,\\
&\mathbf{2}: \left\{\begin{array}{l}
\frac{1}{\sqrt{3}}\left(e_1\otimes e_1 + \omega^2 e_2\otimes e_2 + \omega e_3\otimes e_3\right),\\
\frac{1}{\sqrt{3}}\left(e_1\otimes e_1 + \omega e_2\otimes e_2 + \omega^2 e_3\otimes e_3\right),
\end{array}\right.\\
&\mathbf{3}: \left\{\begin{array}{l}
\frac{1}{\sqrt{2}}\,(e_2\otimes e_3 + e_3\otimes e_2)\,,\\
\frac{1}{\sqrt{2}}\,(e_3\otimes e_1 + e_1\otimes e_3)\,,\\
\frac{1}{\sqrt{2}}\,(e_1\otimes e_2 + e_2\otimes e_1)\,,
\end{array}\right.\\
&\mathbf{3}': \left\{\begin{array}{l}
\frac{1}{\sqrt{2}}\,(e_2\otimes e_3 - e_3\otimes e_2)\,,\\
\frac{1}{\sqrt{2}}\,(e_3\otimes e_1 - e_1\otimes e_3)\,,\\
\frac{1}{\sqrt{2}}\,(e_1\otimes e_2 - e_2\otimes e_1)\,.
\end{array}\right.
\end{aligned}
\tag{36}
$$

1.6. The Group A_4. After the seminal paper by Ma and Rajasekaran [18], this group has become the most popular one in the context of neutrino masses and lepton mixing. We can only list a few early papers here in [19, 20], refer the reader to the review [6] and to citations in recent A_4 papers to get an impression of the bustling activities with respect to model building with A_4. It is worth noting that this group has already been used much earlier in the quark sector [21].

The group A_4 consists of all even permutations of S_4. Therefore, its structure is

$$A_4 \cong K \rtimes \mathbb{Z}_3. \tag{37}$$

Theorem 7 cannot be applied to find the classes, it is however clear that the classes of A_4 must be subsets of the classes of S_4 which consist of even permutations. In this way we obtain

$$\begin{aligned} C_1 &= \{e\}, \\ C_2 &= \{(12)(34),\, (13)(24),\, (14)(23)\}, \\ C_3 &= \{(132),\, (124),\, (234),\, (143)\}, \\ C_4 &= \{(123),\, (142),\, (243),\, (134)\}. \end{aligned} \tag{38}$$

Thus we know that A_4 has four inequivalent irreps. Equation (37) tells us that there are three 1-dimensional irreps stemming from the $\mathbb{Z}_3$, which map K onto 1:

$$\mathbf{1} : (243) \to 1, \quad \mathbf{1}' : (243) \to \omega^2, \quad \mathbf{1}'' : (243) \to \omega. \tag{39}$$

Since A_4 has 12 elements, the remaining irrep must have dimension three. Equations (31) and (32) for S_4 allow one to determine this irrep:

$$(12)(34) \to A \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (243) \to E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{40}$$

For the second relation we have exploited the relation $(243) = (23)(24)$. An alternative definition of A_4 is given by this irrep because the $\mathbf{3}$ is faithful. In this way, A_4 can be considered as a finite subgroup of $SU(3)$ with generators A and E.

Having constructed all irreps we can write down the character table of A_4 (see Table 3). As an example for its usage one can, for instance, compute

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3} \oplus \mathbf{3}. \tag{41}$$

Table 3. **Character table of A_4**

A_4	C_1	C_2	C_3	C_4
(# C_k)	(1)	(3)	(4)	(4)
ord (C_k)	1	2	3	3
$\mathbf{1}$	1	1	1	1
$\mathbf{1}'$	1	1	ω	ω^2
$\mathbf{1}''$	1	1	ω^2	ω
$\mathbf{3}$	3	-1	0	0

The Clebsch–Gordan decomposition of this tensor product is given by

$$\begin{aligned}
\mathbf{1}: &\quad \frac{1}{\sqrt{3}}\left(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3\right), \\
\mathbf{1}': &\quad \frac{1}{\sqrt{3}}\left(e_1 \otimes e_1 + \omega^2 e_2 \otimes e_2 + \omega e_3 \otimes e_3\right), \\
\mathbf{1}'': &\quad \frac{1}{\sqrt{3}}\left(e_1 \otimes e_1 + \omega e_2 \otimes e_2 + \omega^2 e_3 \otimes e_3\right), \\
\mathbf{3}: &\quad e_2 \otimes e_3,\ e_3 \otimes e_1,\ e_1 \otimes e_2, \\
\mathbf{3}: &\quad e_3 \otimes e_2,\ e_1 \otimes e_3,\ e_2 \otimes e_1.
\end{aligned} \tag{42}$$

For the two 3-dimensional irreps one could equivalently use the symmetric and antisymmetric combinations of $e_j \otimes e_k$, or any other weighted, orthogonal combination. Note that for S_4 one does not have this freedom, one must use the symmetric combination for the $\mathbf{3}$ and the antisymmetric combination for the $\mathbf{3}'$ (see Eq. (36)). In the case of A_4 this freedom comes about because the $\mathbf{3}'$ becomes identical with the $\mathbf{3}$ due to the absence of transpositions.

2. MODELS OF NEUTRINO MASSES AND LEPTON MIXING

2.1. Lagrangians and Horizontal Symmetries. We begin with some remarks. The notion «horizontal symmetry» is used synonymously with «family symmetry»:

• We assume that any model we have in mind is an extension of the Standard Model. Therefore, the full symmetry group of the Lagrangian $\mathcal{L}$ is $G_{\rm gauge} \times G_{\rm family}$. ($G_{\rm family}$ could also be gauged, but we do not consider this possibility here.)

• Kinetic and gauge terms in the Lagrangian are automatically invariant under G_{family}.

• Therefore, the effect of G_{family} is felt in the Yukawa Lagrangian and the scalar potential.

• The Yukawa couplings are connected with the Clebsch–Gordan coefficients of the tensor products of the fermion representations, such that for every irrep of scalar fields there is a free Yukawa coupling constant.

• The mass matrices contain, in addition, the vacuum expectation values (VEVs) which are determined by the minimum of the scalar potential.

• With several VEVs one has the problem of *vacuum alignment*. The meaning of this notion is that only specific VEV relations lead to mass matrices which give the desired mixing angles and, sometimes in addition, predictions for the neutrino mass spectrum.

• With family symmetries one has almost necessarily a proliferation of the scalar sector and, in most cases, also additional fermion fields. Thus there is a tension between the introduction of new fields and, as a consequence, unknown constants, which are necessary to realize the symmetry, and the attempted predictions for masses and mixings.

Let us discuss the relation between Clebsch–Gordan coefficients and Yukawa couplings in more detail. Suppose we have a tensor product $D \otimes D' = D_S \oplus \ldots$ with irreps D, D', and D_S. We choose the bases $D : \{e_\alpha\}$ and $D' : \{f_\alpha\}$. Then the basis for irrep D_S has the form $\{b_i = \Gamma_{i\alpha\beta} e_\alpha \otimes f_\beta\}$. With the transformations

$$e_\alpha \to D_{\gamma\alpha} e_\gamma, \quad f_\beta \to D_{\delta\beta} f_\delta, \quad b_i \to (D_S)_{ji} b_j \tag{43}$$

the conditions on the Clebsch–Gordan coefficient matrices Γ_i are obtained as

$$\Gamma_i = \left(D^\dagger \Gamma_j D'^* \right) (D_S)_{ji}. \tag{44}$$

Now we consider generic Yukawa couplings in the Majorana form

$$\mathcal{L}_Y = y\, \psi_\alpha^T C^{-1} \gamma_{i\alpha\beta} S_i \, \psi'_\beta + \text{h.c.}, \tag{45}$$

where ψ and ψ' transform according to D and D', respectively. Comparing with equation (44) we find that

$$\psi \to D\psi, \quad D' \to D'\psi' \Rightarrow S \to D_S^* S, \quad \gamma_i = \Gamma_i^*. \tag{46}$$

That is, the scalar fields transform with the irrep complex conjugate to D_S and the Yukawa couplings are partially determined by the complex conjugate Clebsch–Gordan coefficient matrix, as announced above.

If we have three fermion families, then the fermion multiplets constitute 3-dimensional representations of the horizontal group G. We can distinguish three cases:

i. Abelian case: Only 1-dimensional irreps are present.
ii. Non-Abelian case: 2-dimensional irreps occur, but no 3-dimensional one.
iii. Non-Abelian case: 3-dimensional irreps occur.

An Abelian group G is synonymous with «texture zeros», i.e., a Yukawa coupling is either present and undetermined or it is zero, but there are no relations between different Yukawa couplings. Relations among observables in the mass spectrum and mixing have their origin solely in these zeros. It has been shown [22] that by Abelian symmetries the only extremal mixing angle which can be enforced is $\theta_{13} = 0°$. It is possible to enforce texture zeros in arbitrary entries of the fermion mass matrices by means of Abelian symmetries and an extended scalar sector [23].

In the second case it is possible to enforce $\theta_{13} = 0°$ and $\theta_{23} = 45°$. For tri-bimaximal mixing one needs 3-dimensional irreps. In the next subsection we will discuss one such model based on A_4.

2.2. A Type I Seesaw Model Based on A_4. As a prototype for a renormalizable A_4 model we discuss the model of [20]. It is based on the following A_4 multiplets:

$$\begin{aligned} \text{fermion fields:} \quad & \ell_R \in \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'', \quad D_L \in \mathbf{3}, \quad \nu_R \in \mathbf{3}, \\ \text{scalar fields:} \quad & \phi \in \mathbf{3}, \quad \phi_0 \in \mathbf{1}, \quad \chi \in \mathbf{3}. \end{aligned} \tag{47}$$

In this list the D_L are the usual leptonic left-handed gauge doublets, the ℓ_R are the right-handed charged gauge singlets and the ν_R are the right-handed neutrino singlets. There are four Higgs doublets ϕ and ϕ_0 with hypercharge $+1$ and three real gauge singlets χ.

With the discussion in the previous section it is straightforward to derive the Lagrangian

$$\begin{aligned} \mathcal{L} = \ldots - \big[h_1 \left(\bar{D}_{1L}\phi_1 + \bar{D}_{2L}\phi_2 + \bar{D}_{3L}\phi_3 \right) \ell_{1R} + \\ + h_2 \left(\bar{D}_{1L}\phi_1 + \omega^2 \bar{D}_{2L}\phi_2 + \omega \bar{D}_{3L}\phi_3 \right) \ell_{2R} + \\ + h_3 \left(\bar{D}_{1L}\phi_1 + \omega \bar{D}_{2L}\phi_2 + \omega^2 \bar{D}_{3L}\phi_3 \right) \ell_{3R} + \end{aligned} \tag{48}$$

$$+ h_0 \left(\bar{D}_{1L}\nu_{1R} + \bar{D}_{2L}\nu_{2R} + \bar{D}_{3L}\nu_{3R} \right) \tilde{\phi}_0 + \text{h.c.} \big] + \tag{49}$$

$$+ \frac{1}{2} \left[M \left(\nu_{1R}^T C^{-1} \nu_{1R} + \nu_{2R}^T C^{-1} \nu_{2R} + \nu_{3R}^T C^{-1} \nu_{3R} \right) + \text{h.c.} \right] + \tag{50}$$

$$\begin{aligned} + \frac{1}{2} \big[h_\chi \big(\chi_1 \left(\nu_{2R}^T C^{-1} \nu_{3R} + \nu_{3R}^T C^{-1} \nu_{2R} \right) + \\ + \chi_2 \left(\nu_{3R}^T C^{-1} \nu_{1R} + \nu_{1R}^T C^{-1} \nu_{3R} \right) + \\ + \chi_3 \left(\nu_{1R}^T C^{-1} \nu_{2R} + \nu_{2R}^T C^{-1} \nu_{1R} \right) \big) + \text{h.c.} \big] , \end{aligned} \tag{51}$$

where we have confined ourselves to the Yukawa interactions and mass terms. The dots indicate the kinetic terms, the gauge interactions and the scalar potential.

Through spontaneous symmetry breaking with VEVs v_j, w_j $(j = 1, 2, 3)$ and v_0 of the Higgs doublets and scalar singlets, respectively, the Lagrangian leads to the mass terms

$$-\bar{\ell}_L M_\ell \ell_R - \bar{\nu}_L M_D \nu_R + \frac{1}{2}\nu_R^T C^{-1} M_R \nu_R + \text{h.c.} \tag{52}$$

While $M_D = h_0 v_0^* \mathbb{1}$ is simply proportional to the unit matrix, the other two mass matrices are given by

$$M_\ell = \begin{pmatrix} h_1 v_1 & h_2 v_1 & h_3 v_1 \\ h_1 v_2 & h_2 v_2 \omega^2 & h_3 v_2 \omega \\ h_1 v_3 & h_2 v_3 \omega & h_3 v_3 \omega^2 \end{pmatrix} \text{ and } M_R = \begin{pmatrix} M & h_\chi w_3 & h_\chi w_2 \\ h_\chi w_3 & M & h_\chi w_1 \\ h_\chi w_2 & h_\chi w_1 & M \end{pmatrix}. \tag{53}$$

With general VEVs one cannot obtain tri-bimaximal mixing. It is well known that the vacuum alignment

$$v_1 = v_2 = v_3 \equiv v, \quad w_1 = w_3 = 0, \quad h_\chi w_2 \equiv M' \tag{54}$$

is needed, which gives the mass matrices

$$M_\ell = \sqrt{3} v \, U_\omega^\dagger \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}, \quad M_R = \begin{pmatrix} M & 0 & M' \\ 0 & M & 0 \\ M' & 0 & M \end{pmatrix}. \tag{55}$$

The matrix U_ω can be read off from M_ℓ in Eq. (53). We denote by U_ν the matrix which diagonalizes M_R of Eq. (55). These two unitary matrices are then obtained as

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_\nu = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}. \tag{56}$$

Therefore, up to diagonal phase matrices on the left- and right-hand side, we arrive at the lepton mixing matrix

$$U = U_\omega U_\nu = \text{diag}\,(1, \omega, \omega^2)\, U_{\text{HPS}}\, \text{diag}\,(1, 1, -i). \tag{57}$$

Thus the A_4 symmetry of the Lagrangian together with suitable vacuum alignment leads to tri-bimaximal mixing. The charged-lepton masses are reproduced by choosing the Yukawa couplings appropriately: $m_\alpha = \sqrt{3}|v h_\alpha|$ $(\alpha = e, \mu, \tau)$.

We conclude the discussion of the model of [20] with a few comments. As shown above, vacuum alignment is a very important ingredient for achieving tri-bimaximal mixing. Of course, the symmetry group also restricts the scalar potential and is essential for allowing the required vacuum structure to be a minimum of the scalar potential. Nevertheless, vacuum alignment is usually a tricky problem. In the model we have discussed, it was necessary to break A_4 down to $\mathbb{Z}_3$ generated by the matrix E (see Eq. (40)) in the charged lepton sector, while in the neutrino sector the VEVs of the scalars χ_k break A_4 to a $\mathbb{Z}_2$ generated by $\mathrm{diag}\,(-1, 1, -1)$. This is quite generic for A_4 models with tri-bimaximal mixing and leads to $M_D \propto \mathbb{1}$ and the structure of M_ℓ and M_R of Eq. (55). It was shown in [20] that the vacuum alignment (54) is possible if the scalar potential is CP-conserving. Since in this model ℓ_R is not in the same A_4 multiplet as D_L and ν_R, it cannot be embedded in a Grand Unified Theory. Note, however, that it is possible to put both D_L and ℓ_R into a **3** and to use the type II seesaw mechanism with scalar gauge triplets (see, for instance, [24]) – a scenario which can at least in principle be extended to a Grand Unified Theory.

CONCLUSIONS

A large part of this lecture dealt with the theory of finite groups, the other part with the application of group theory to Lagrangians for the purpose of «explaining» mass and mixing patterns found experimentally or to make predictions in this context. Let us finish with remarks on the second part.

We have tried to demonstrate that symmetries based on finite groups *could* be a way to tackle the mass and mixing problem. However, all models for lepton mixing (and neutrino masses) require complicated and contrived extensions of the Standard Model. Such models are in most cases incompatible with Grand Unification, need vacuum alignment, employ SUSY and nonrenormalizable terms, etc. Here we have confined ourselves to the relatively simple renormalizable model of [20] as a showcase, which manages without SUSY. As for tri-bimaximal mixing (2), for the time being it is compatible with all experimental results. However, it could turn out that $s_{13}^2 \sim 0.01$ [3]. In that case, ideas alternative to tri-bimaximal mixing would be in demand. Or one assumes that tri-bimaximal mixing holds at a high (seesaw) scale and, by the renormalization group evolution of the mixing angles from the high scale down to the electroweak scale, s_{13}^2 evolves sufficiently away from zero; this is possible with a degenerate neutrino mass spectrum.

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REFERENCES

1. *Gatto R., Sartori G., Tonin M.* Weak Self-Masses, Cabibbo Angle, and Broken $SU_2 \times SU_3$ // Phys. Lett. B. 1968. V. 28. P. 128;
Cabibbo N., Maiani L. Dynamical Interrelations of Weak, Electromagnetic and Strong Interactions and the Value of θ // Ibid. P. 131.

2. *Harrison P. F., Perkins D. H., Scott W. G.* Tri-Bimaximal Mixing and the Neutrino Oscillation Data // Phys. Lett. B. 2002. V. 530. P. 167; hep-ph/0202074.

3. *Maltoni M. et al.* Status of Global Fits to Neutrino Oscillations // New J. Phys. 2004. V. 6. P. 122; hep-ph/0405172;
Fogli G. L. et al. Global Analysis of Three-Flavor Neutrino Masses and Mixings // Prog. Part. Nucl. Phys. 2006. V. 57. P. 742; hep-ph/0506083;
Fogli G. L. et al. Hints of $\theta_{13} > 0$ from Global Neutrino Data Analysis // Phys. Rev. Lett. 2008. V. 101. P. 141801; arXiv:0806.1649;
Schwetz T., Tórtola M., Valle J. W. F. Three-Flavour Neutrino Oscillation Update // New J. Phys. 2008. V. 10. P. 113011; arXiv:0808.2016;
Gonzalez-Garcia M. C., Maltoni M., Salvado J. Updated Global Fit to Three Neutrino Mixing: Status of the Hints of $\theta_{13} > 0$ // J. High Energy Phys. 2010. V. 04. P. 056; arXiv:1001.4524.

4. *Minkowski P.* $\mu \to e\gamma$ at a Rate of One out of 10^9 Muon Decays? // Phys. Lett. B. 1977. V. 67. P. 421;
Yanagida T. // Proc. of the Workshop on Unified Theory and Baryon Number in the Universe. Tsukuba, Japan, 1979. KEK Report. V. 79-18;
Glashow S. L. // Quarks and Leptons: Proc. of the Advanced Study Institute, Cargèse, Corsica, 1979. N. Y., 1981;
Gell-Mann M., Ramond P., Slansky R. Complex Spinors and Unified Theories. Supergravity. Amsterdam: North Holland, 1979;
Mohapatra R. N., Senjanovic G. Neutrino Mass and Spontaneous Parity Violation // Phys. Rev. Lett. 1980. V. 44. P. 912.

5. *Grimus W.* Neutrino Physics — Theory // Lect. Notes Phys. 2004. V. 629. P. 169; hep-ph/0307149.

6. *Altarelli G., Feruglio F.* Discrete Flavor Symmetries and Models of Neutrino Mixing. arXiv:1002.0211.

7. *Hamermesh M.* Group Theory and Its Application to Physical Problems. Addison–Wesley, Reading, Massachusetts, 1962.

8. *Ramond P.* Group Theory: A Physicists's Survey. Cambridge, UK: Cambridge Univ. Press, 2010.

9. *Ludl P. O.* Systematic Analysis of Finite Family Symmetry Groups and Their Applications to the Lepton Sector. Diploma Thesis. arXiv:0907.5587.

10. *Ishimori H. et al.* Non-Abelian Discrete Symmetries in Particle Physics // Prog. Theor. Phys. Suppl. 2010. V. 183. P. 1; arXiv:1003.3552.

11. *Grimus W., Ludl P. O.* Principal Series of Finite Subgroups of $SU(3)$ // J. Phys. A. 2010. V. 43. P. 445209; arXiv:1006.0098.

12. *Ludl P. O.* On the Finite Subgroups of $U(3)$ of Order Smaller than 512 // J. Phys. A. 2010. V. 43. P. 395204; arXiv:1006.1479.

13. *Parattu K. M., Wingerter A.* Tri-Bimaximal Mixing from Small Groups. arXiv:1012.2842.

14. *Grimus W., Lavoura L., Neubauer D.* A Light Pseudoscalar in a Model with Lepton Family Symmetry $O(2)$ // JHEP. 2008. V. 07. P. 051; arXiv:0805.1175.

15. *Yamanaka Y., Sugawara H., Pakvasa S.* Permutation Symmetries and the Fermion Mass Matrix // Phys. Rev. D. 1982. V. 25. P. 1895; Erratum // Phys. Rev. D. 1984. V. 29. P. 2135.

16. *Lam C. S.* Determining Horizontal Symmetry from Neutrino Mixing // Phys. Rev. Lett. 2008. V. 101. P. 121602; arXiv:0804.2622.

17. *Grimus W., Lavoura L., Ludl P. O.* Is S_4 the Horizontal Symmetry of Tri-Bimaximal Lepton Mixing? // J. Phys. G. 2009. V. 36. P. 115007; arXiv:0906.2689.

18. *Ma E., Rajasekaran G.* Softly Broken A_4 Symmetry for Nearly Degenerate Neutrino Masses // Phys. Rev. D. 2001. V. 64. P. 113012; hep-ph/0106291.

19. *Babu K. S., Ma E., Valle J. W. F.* Underlying A_4 Symmetry for the Neutrino Mass Matrix and the Quark Mixing Matrix // Phys. Lett. B. 2003. V. 552. P. 207; hep-ph/0206292;
Altarelli G., Feruglio F. Tri-Bimaximal Neutrino Mixing, A_4 and the Modular Symmetry // Nucl. Phys. B. 2006. V. 741. P. 215; hep-ph/0512103;
de Medeiros Varzielas I., King S. F., Ross G. G. Tri-Bimaximal Neutrino Mixing from Discrete Subgroups of $SU(3)$ and $SO(3)$ Family Symmetry // Phys. Lett. B. 2007. V. 644. P. 153; hep-ph/0512313.

20. *He X.-G., Keum Y., Volkas R. R.* A_4 Flavour Symmetry Breaking Scheme for Understanding Quark and Neutrino Mixing Angles // JHEP. 2006. V. 04. P. 039; hep-ph/0601001.

21. *Wyler D.* Discrete Symmetries in the Six Quark $SU(2) \times U(1)$ Model // Phys. Rev. D. 1979. V. 19. P. 3369;
Branco G. C., Nilles H. P., Rittenberg V. Fermion Masses and Hierarchy in Symmetry Breaking // Phys. Rev. D. 1980. V. 21. P. 3417.

22. *Low C. I.* Abelian Family Symmetries and the Simplest Models that Give $\theta_{13} = 0$ in the Neutrino Mixing Matrix // Phys. Rev. D. 2005. V. 71. P. 073007; hep-ph/0501251.

23. *Grimus W. et al.* Symmetry Realization of Texture Zeros // Eur. Phys. J. C. 2004. V. 36. P. 227; hep-ph/0405016.

24. *Ma E.* Suitability of A_4 as a Family Symmetry in Grand Unification // Mod. Phys. Lett. A. 2006. V. 21. P. 2931; hep-ph/0607190.