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CLASSIFICATION OF ARNOLD–BELTRAMI FLOWS AND THEIR HIDDEN SYMMETRIES

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In the context of mathematical hydrodynamics, we consider the group theory structure which underlies the so-called ABC-flows introduced by Beltrami, Arnold and Childress. Main reference points are Arnold's theorem stating that, for flows taking place on compact three manifolds $\mathcal{M}_3$, the only velocity fields able to produce chaotic streamlines are those satisfying the Beltrami equation and the modern topological conception of contact structures, each of which admits a representative contact one-form also satisfying the Beltrami equation. We advocate that the Beltrami equation is nothing else but the eigenstate equation for the first order Laplace–Beltrami operator $\star_g \mathrm{d}$, which can be solved by using time-honored harmonic analysis. Taking for $\mathcal{M}_3$ a torus T^3 constructed as $\mathbb{R}^3/\Lambda$, where Λ is a crystallographic lattice, we present a general algorithm to construct solutions of the Beltrami equation which utilizes as main ingredient the orbits under the action of the point group $\mathfrak{P}_\Lambda$ of three-vectors in the momentum lattice $^\star\Lambda$. Inspired by the crystallographic construction of space groups, we introduce the new notion of a *Universal Classifying Group* $\mathfrak{GU}_\Lambda$ which contains all space groups as proper subgroups. We show that the $\star_g \mathrm{d}$ eigenfunctions are naturally arranged into irreducible representations of $\mathfrak{GU}_\Lambda$, and by means of a systematic use of the branching rules with respect to various possible subgroups $\mathrm{H}_i \subset \mathfrak{GU}_\Lambda$, we search and find the Beltrami fields with nontrivial hidden symmetries. In the case of the cubic lattice, the point group is the proper octahedral group O_{24}, and the Universal Classifying Group $\mathfrak{GU}_{\mathrm{cubic}}$ is a finite group G_{1536} of order $|\mathrm{G}_{1536}| = 1536$ which we study in full detail deriving all of its 37 irreducible representations and the associated character table. We show that the O_{24} orbits in the cubic lattice are arranged into 48 equivalence classes, the parameters of the corresponding Beltrami vector fields filling all the 37 irreducible representations of G_{1536}. In this way we obtain an exhaustive classification of all *generalized* ABC-*flows* and of their hidden symmetries. We make several conceptual comments about the need of a field theory yielding the Beltrami equation as a field equation and/or an instanton equation and on the possible relation of Arnold–Beltrami flows with (supersymmetric) Chern–Simons gauge theories. We also suggest

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linear generalizations of the Beltrami equation to higher odd dimensions that are different from the nonlinear one proposed by Arnold and possibly make contact with M-theory and the geometry of flux compactifications.

В контексте математической гидродинамики мы рассматриваем теоретико-групповую структуру, лежащую в основе так называемых ABC-потоков, введенных Бельтрами, Арнольдом и Чайлдрессом. Главные ориентиры — теорема Арнольда, утверждающая, что для потоков на компактных трехмерных многообразиях $\mathcal{M}_3$ только поля скоростей, которые удовлетворяют уравнению Бельтрами, способны произвести хаотические траектории, и современная топологическая концепция контактных структур, каждая из которых характеризуется контактной одноформой, также удовлетворяющей уравнению Бельтрами. Мы аргументируем, что уравнение Бельтрами является не чем иным, как уравнением на собственные функции оператора первого порядка Лапласа–Бельтрами $\star_g \mathrm{d}$, и может быть решено при помощи проверенного временем гармонического анализа. Рассматривая в качестве многообразия $\mathcal{M}_3$ тор T^3, построенный как $\mathbb{R}^3/\Lambda$, где Λ — кристаллографическая решетка, мы представляем общий алгоритм построения решений уравнения Бельтрами, который базируется на орбитах точечной группы $\mathfrak{P}_\Lambda$ симметрии решетки при действии на три-векторы импульсов дуальной решетки $^\star\Lambda$. Вдохновленные существующим построением кристаллографических пространственных групп, мы вводим новое понятие *универсальной классифицирующей группы* $\mathfrak{G}\mathfrak{U}_\Lambda$, которая содержит все пространственные группы как подгруппы. Мы показываем, что собственные функции оператора $\star_g \mathrm{d}$ естественным образом группируются в неприводимые представления группы $\mathfrak{G}\mathfrak{U}_\Lambda$, и, посредством систематического использования правил их разложения относительно различных возможных подгрупп $\mathrm{H}_i \subset \mathfrak{G}\mathfrak{U}_\Lambda$, мы ищем и находим поля Бельтрами с нетривиальными скрытыми симметриями. В случае кубической решетки точечная группа симметрии — правильная октаэдральная группа O_{24}, а универсальная классифицирующая группа $\mathfrak{G}\mathfrak{U}_{\mathrm{cubic}}$ — конечная группа G_{1536} порядка $|\mathrm{G}_{1536}| = 1536$, которую мы изучаем во всех деталях, строя все ее 37 неприводимых представлений и их характеры. Мы показываем, что O_{24}-орбиты в кубической решетке группируются в 48 классов эквивалентности, а параметры соответствующих векторных полей Бельтрами заполняют все 37 неприводимых представлений группы G_{1536}. Таким образом, мы получаем исчерпывающую классификацию всех *обобщенных* ABC-*потоков* и их скрытых симметрий. Мы делаем несколько концептуальных замечаний относительно необходимости создания полевой теории, содержащей уравнение Бельтрами как уравнение поля и/или инстантонное уравнение, и о возможной связи потоков Арнольда–Бельтрами с (суперсимметричными) калибровочными теориями Черна–Саймонса. Мы также предлагаем линейное обобщение уравнения Бельтрами на случай нечетномерных пространств более высоких размерностей, которое отличается от нелинейного, предложенного Арнольдом, и, возможно, связано с М-теорией и геометрией компактификаций с потоками.

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1. INTRODUCTION

Classical hydrodynamics of ideal, incompressible, inviscid fluids, subject to no external forces, is described by the Euler equation in the three-dimensional

Euclidean space $\mathbb{R}^3$, namely, by

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p; \quad \nabla\cdot\mathbf{u} = 0, \tag{1.1}$$

where $\mathbf{u} = \mathbf{u}\,(x,\,t)$ denotes the local velocity field, and $p(\mathbf{x})$ denotes the local pressure*. In vector notation, Eq. (1.1) reads as follows:

$$\frac{\partial}{\partial t}u^i + u^j\,\partial_j\,u^i = -\partial^i\,p; \quad \partial^\ell\,u_\ell = 0 \tag{1.2}$$

and admits some straightforward rewriting that, notwithstanding the kinder garten arithmetic involved in its derivation, it is at the basis of several profound and momentous theoretical developments which have kept the community of dynamical system theorists busy for already fifty years [1–13].

With the present paper, we aim at introducing into the classical field of mathematical fluid-mechanics a new group-theoretical approach that allows for a more systematic classification and algorithmic construction of the so-called Beltrami flows, hopefully providing new insight into their properties.

1.1. Beltrami Flows and Arnold Theorem. Let us then begin with the rewriting of Eq. (1.2) which is the starting point of the entire adventure. The first step to be taken in our raising conceptual ladder is that of promoting the fluid trajectories, defined as the solutions of the following first-order differential system**:

$$\frac{d}{dt}x^i(t) = u^i(x(t),t), \tag{1.4}$$

to smooth maps

$$\mathcal{S} : \mathbb{R}_t \to \mathcal{M}_g \tag{1.5}$$

from the time real line $\mathbb{R}_t$ to a smooth Riemannian manifold $\mathcal{M}_g$ endowed with a metric g. The classical case corresponds to $\mathcal{M} = \mathbb{R}^3$, $g_{ij}(x) = \delta_{ij}$, but any other Riemannian three-manifold might be used and there exists generalization also to higher dimensions. Adopting this point of view, the velocity field $\mathbf{u}\,(x,\,t)$

*Note that we have put the density $\rho = 1$.

**In mathematical hydrodynamics people distinguish two notions, that of trajectories, which are the solutions of the differential equations (1.4), and that of streamlines. Streamlines are the instantaneous curves that at any time $t = t_0$ admit the velocity field $u^i(x,t_0)$ as tangent vector. Introducing a new parameter τ, streamlines at time t_0 are the solutions of the differential system:

$$\frac{d}{d\tau}x^i(\tau) = u^i(x(\tau),t_0). \tag{1.3}$$

In the case of steady flows, where the velocity field is independent of time, trajectories and streamlines coincide. Since we are mainly concerned with steady flows, in the present paper we often use the word streamlines and trajectories indifferently.

is turned into a time evolving vector field on $\mathcal{M}$, namely, into a smooth family of sections of the tangent bundle $T\mathcal{M}$:

$$\forall\, t \in \mathbb{R} : u^i(x,t)\,\partial_i \equiv \mathrm{U}(t) \in \Gamma\,(T\mathcal{M},\,\mathcal{M})\,. \tag{1.6}$$

Next, using the Riemannian metric, which allows one to raise and lower tensor indices to any $\mathrm{U}(t)$, we can associate a family of sections of the cotangent bundle $CT\mathcal{M}$ defined by the following time evolving one-form:

$$\forall\, t \in \mathbb{R} : \Omega^{[\mathrm{U}]}(t) \equiv g_{ij}\,u^i(x,t)\,dx^j \in \Gamma\,(CT\mathcal{M},\,\mathcal{M})\,. \tag{1.7}$$

Utilizing the exterior differential and the contraction operator acting on differential forms, we can evaluate the Lie-derivative of the one-form $\Omega^{[\mathrm{U}]}(t)$ along the vector field U. Applying definitions (see, for instance, [14], chapter five, page 120 of volume two), we obtain

$$\begin{aligned}\mathcal{L}_{\mathrm{U}}\Omega^{[\mathrm{U}]}(t) &\equiv \mathrm{i}_{\mathrm{U}}\cdot \mathrm{d}\Omega^{[\mathrm{U}]} + \mathrm{d}\left(\mathrm{i}_{\mathrm{U}}\cdot\Omega^{[\mathrm{U}]}\right) = \\ &= \left(u^\ell\partial_\ell\,u^i + g^{ik}\partial_k \underbrace{\parallel \mathrm{U}\parallel^2}_{g_{mn}\,u^m\,u^n}\right) g_{ij}\,dx^j,\end{aligned} \tag{1.8}$$

and the Euler equation can be rewritten in either one of the following two-equivalent index-free reformulations:

$$-\mathrm{d}\left(p - \frac{1}{2}\parallel \mathrm{U}\parallel^2\right) = \partial_t\Omega^{[\mathrm{U}]} + \mathcal{L}_{\mathrm{U}}\Omega^{[\mathrm{U}]}, \tag{1.9}$$

or

$$-\mathrm{d}\left(p + \frac{1}{2}\parallel \mathrm{U}\parallel^2\right) = \partial_t\Omega^{[\mathrm{U}]} + \mathrm{i}_{\mathrm{U}}\cdot \mathrm{d}\Omega^{[\mathrm{U}]}. \tag{1.10}$$

Equation (1.10) is one of the possible formulations of the classical Bernoulli theorem. Indeed from Eq. (1.10) we immediately conclude that

$$H = p + \frac{1}{2}\parallel \mathrm{U}\parallel^2 \tag{1.11}$$

is constant along the trajectories defined by Eq. (1.4). Turning matters around, we can say that in steady flows, where $\partial_t\mathrm{U} = 0$, the fluid trajectories necessarily lay on the level surfaces $H(\mathrm{x}) = h \in \mathbb{R}$ of the function

$$H : \mathcal{M} \to \mathbb{R} \tag{1.12}$$

defined by (1.11). Then, if $H(\mathbf{x})$ has a nontrivial x-dependence, it defines a natural foliation of the n-dimensional manifold $\mathcal{M}$ into a smooth family of $(n-1)$-manifolds (all diffeomorphic among themselves) corresponding to the level surfaces. Furthermore, as already advocated, the trajectories, *i.e.*, the solutions of Eq. (1.4), lay on these surfaces. In other words, the dynamical system encoded in Eq. (1.4) is effectively $(n-1)$-dimensional admitting H as an additional conserved Hamiltonian. In the classical case $n = 3$, this means that the differential system (1.4) is actually two-dimensional, namely, nonchaotic and in some instances even integrable*. Consequently, we reach the conclusion that no chaotic trajectories (or streamlines) can exist if $H(x)$ has a nontrivial x-dependence: the only window open for Lagrangian chaos occurs when H is a constant function. Looking at Eq. (1.10), we realize that in steady flows, where $\partial_t \Omega^{[\mathrm{U}]} = 0$, the only open window for chaotic trajectories is provided by velocity fields that satisfy the condition

$$\mathrm{i}_{\mathrm{U}} \cdot \mathrm{d}\Omega^{[\mathrm{U}]} = 0. \tag{1.13}$$

This weak condition (1.13) is certainly satisfied if the velocity field U satisfies the strong condition

$$\mathrm{d}\Omega^{[\mathrm{U}]} = \lambda \star_g \Omega^{[\mathrm{U}]} \Leftrightarrow \star_g \mathrm{d}\Omega^{[\mathrm{U}]} = \lambda\, \Omega^{[\mathrm{U}]}, \tag{1.14}$$

where $\star_g$ denotes the Hodge duality operator in the metric g:

$$\star_g \Omega^{[\mathrm{U}]} = \epsilon_{\ell mn}\, g^{\ell k}\, \Omega^{[\mathrm{U}]}_k\, dx^m \wedge dx^n = u^\ell\, dx^m \wedge dx^n\, \epsilon_{\ell mn}, \tag{1.15}$$

$$\star_g \mathrm{d}\, \Omega^{[\mathrm{U}]} = \epsilon_{\ell mn}\, g^{mp}\, g^{nq} \partial_p \left(g_{qr} u^r\right) dx^\ell. \tag{1.16}$$

The heuristic argument, which leads to consider velocity fields that satisfy the *Beltrami condition* (1.14) as the unique steady candidates compatible with chaotic trajectories, was transformed by Arnold into a rigorous theorem [1, 5] which, under the strong hypothesis that $(\mathcal{M}, g)$ is a closed, compact Riemannian three-manifold, states the following.

Theorem 1.1 (Arnold). *There are only two possibilities*:

a) Either the form $\Omega^{[\mathrm{U}]}$ *is an eigenstate of the Beltrami operator* $\star_g \mathrm{d}$ *with a nonvanishing eigenvalue* $\lambda \neq 0$*,*

b) or the manifold $\mathcal{M}$ *is subdivided into a finite collection of cells, each of which admits a foliation diffeomorphic to* $T^2 \times \mathbb{R}$ *and every two-torus* T^2 *is an invariant set with respect to the action of the velocity field* U: *in other words, all trajectories lay on some* T^2 *immersed in the manifold* $\mathcal{M}$*.*

Henceforth, the desire to investigate the on-set of chaotic trajectories in steady flows of incompressible fluids motivated the interest of the dynamical system community in the Beltrami vector fields defined by the condition (1.14). Furthermore, in view of the above powerful theorem proved by Arnold, the

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*Here we rely on a general result established by the theorem of Poincaré–Bendixson [15] on the limiting orbits of planar differential systems whose corollary is generally accepted to establish that two-dimensional continuous systems cannot be chaotic.

focus of attention concentrated on the rather unphysical, yet mathematically very interesting case of compact three-manifolds. Within this class, the most easily treatable case is that of flat compact manifolds without boundary, so that the most popular playground turned out to be the three-torus T^3. Reporting literally the words of Robert Ghrist in his very nice review [13]: *on those occasions when compactness is desired and the complexities of boundary conditions are not, the fluid domain is usually taken to be an Euclidean T^3 torus given by quotienting out Euclidean space $\mathbb{R}^3$ by the action of three mutually orthogonal translations*. These slightly ironical words are meant to emphasize the main point which is outspokenly put forward by the same author few lines below: *Since so little is known about the rigorous behavior of fluid flows, any methods which can be brought to bear to prove theorems about their behavior are of interest and potential use*. Certainly, most physical contexts for fluid dynamics do not correspond to the idealized situation of a motion in a compact manifold or, said differently, periodic boundary conditions are not the most appropriate to be applied either in a river, or in the atmosphere, or in the charged plasmas environing a compact star, yet the message conveyed by the Arnold theorem that the Beltrami vector fields play a distinguished role in chaotic behavior is to be taken seriously into account and gives an important hint. Moreover, although boundary terms usually encode relevant physical phenomena, yet the history of periodic boundary conditions is a very rich and noble one in Quantum Mechanics, Classical Field Theory and also in Quantum Field Theory. It suffices to recall that periodic boundary conditions of quantized fields provide a formulation of finite temperature quantum field theory.

In our opinion, such arguments are a sufficient justification for the fifty year long efforts devoted by dozens of authors to the study of steady flows generated by the Beltrami vector fields. On the other hand, what is somewhat surprising is that an overwhelming part of such efforts is focused on a single example constructed on the T^3 torus. The following vector field

$$\mathbf{u}(x,y,z) = \mathbf{V}^{(ABC)}(x,y,z) \equiv \begin{pmatrix} C\cos(2\pi y) + A\sin(2\pi z) \\ A\cos(2\pi z) + B\sin(2\pi x) \\ B\cos(2\pi x) + C\sin(2\pi y) \end{pmatrix}, \qquad (1.17)$$

which satisfies the Beltrami condition with eigenvalue $\lambda = 1$ and which contains three real parameters A, B, C, defines what is known in the literature by the name of an ABC-flow (Arnold–Beltrami–Childress) [1, 2, 16], and during the last half century it was the target of fantastically numerous investigations.

The main aim of our work was to understand the principles underlying the construction of the ABC-flows, use systematically such principles to construct and classify all other Arnold-like Beltrami flows, deriving also, as a bonus, their hidden discrete symmetries.

The issue of symmetries happens to be quite relevant at least in two respects. On the one hand, in the case of the ABC model, it occurs that the choice of parameters ($A : B : C = 1$), which leads to the Beltrami vector field with the largest group of automorphisms, leads also to the most extended distribution of chaotic trajectories. On the other hand, symmetries of the Beltrami flows have proved to be crucial in connection with their use in modeling *magneto-hydrodynamic fast dynamos* [10, 11, 17]. By these words it is understood the mechanism that in a steady flow of charged particles generates a large scale magnetic field whose magnitude might be exponentially increasing with time. No analytic results do exist on fast dynamos, and all studies have been so far numerical, yet, while dealing with these latter, crucial simplifications occur and optimization algorithms become available if the steady flow possesses a large enough group $\mathcal{G}$ of symmetries. In this case the magnetic field can be developed into irreducible representations of $\mathcal{G}$ and this facilitates the numerical determination of growing rates of different modes. It is important to stress that the linearized dynamo equations for the magnetic field $\mathbf{B}$ coincide with the linearized equations for perturbations around a steady flow. Therefore the same development of perturbations into irreps of $\mathcal{G}$ is of great relevance also for the study of fluid instabilities. In plasma physics, the Beltrami flows are known under the name of Force–Free Magnetic Fields [9].

1.2. The Conception of Contact Structures. Last but not least, let us mention that the Beltrami vector fields are intimately related with the mathematical conception of *contact topology*. This latter, vigorously developed in the last two decades starting from classical results of analysis that date back to Darboux, Goursat and other XIX century maitres, is a mathematical theory aiming at providing an intrinsic geometrical-topological characterization of *nonintegrability*, namely of the issues discussed above. As we have seen from our sketch of the Arnold theorem, the main obstacle to the onset of chaotic trajectories has a distinctive geometrical flavor: trajectories are necessarily ordered and nonchaotic if the manifold, where they take place, has a foliated structure $\Sigma_h \times \mathbb{R}_h$, the two-dimensional level sets Σ_h being invariant under the action of the velocity vector field U. In this case each streamline lays on some surface Σ_h. Equally adverse to chaotic trajectories is the case of *gradient flows*, where there is a foliation provided by the level sets of some function $H(x)$ and the velocity field $\mathrm{U} = \nabla H$ is just the gradient of H. In this case all trajectories are orthogonal to the leaves Σ_h of the foliation and their well-aligned tangent vectors are parallel to its normal vector.

In conclusion, in presence of a foliation we have the following decomposition of the tangent space to the manifold $\mathcal{M}$ at any point $p \in \mathcal{M}$:

$$T_p\mathcal{M} = T_p^{\perp}\Sigma_h \oplus T_p^{\|}\Sigma_h, \tag{1.18}$$

and no chaotic trajectories are possible in the region $\mathfrak{S} \subset \mathcal{M}$, where $\mathrm{U}(p) \in T_p^{\perp}\Sigma_h$ or $\mathrm{U}(p) \in T_p^{\perp}\Sigma_h$ for $\forall p \in \mathfrak{S}$ (see Fig. 1).

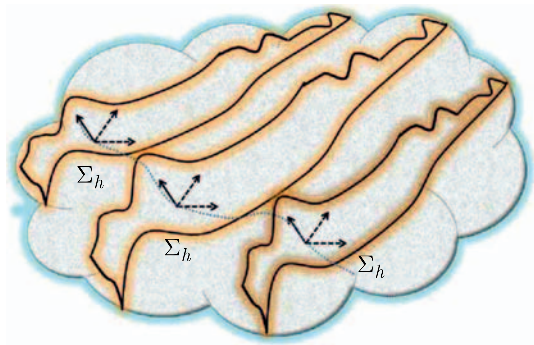

Fig. 1. Schematic view of the foliation of a three-dimensional manifold $\mathcal{M}$. The family of two-dimensional surfaces Σ_h are typically the level sets $H(\mathbf{x}) = h$ of some function $H : \mathcal{M} \to \mathbb{R}$. At each point of $p \in \Sigma_h \subset \mathcal{M}$, the dashed vectors span the tangent space $T_p^{\parallel}\Sigma_h$, while the solid vectors span the normal space to the surface $T_p^{\perp}\Sigma_h$. Equally adverse to chaotic trajectories is the case where the velocity field U lies in $T_p^{\perp}\Sigma_h$ (gradient flow) or in $T_p^{\parallel}\Sigma_h$

This matter of fact motivates an attempt to capture the geometry of the bundle of subspaces orthogonal to the lines of flow by introducing an intrinsic topological indicator that distinguishes necessarily nonchaotic flows from possibly chaotic ones. Let us first consider the extreme case of a gradient flow, where $\Omega^{[\mathrm{U}]} = \mathrm{d}H$ is an exact form. For such flows we have

$$\Omega^{[\mathrm{U}]} \wedge \mathrm{d}\Omega^{[\mathrm{U}]} = \Omega^{[\mathrm{U}]} \wedge \underbrace{\mathrm{d}\,\mathrm{d}H}_{=0} = 0. \tag{1.19}$$

Secondly, let us consider the opposite case, where the velocity field U is orthogonal to a gradient vector field ∇H so that the integral curves of U lay on the level surfaces Σ_h. Furthermore, let us assume that U is selfsimilar on neighboring level surfaces. We can characterize this situation in a Riemannian manifold $(\mathcal{M}, g)$ by the following two conditions:

$$\mathrm{i}_{\nabla H}\Omega^{[\mathrm{U}]} \;\Leftrightarrow\; g\,(\mathrm{U},\, \nabla H) = 0; \quad [U,\, \nabla H] = 0. \tag{1.20}$$

The first of Eqs. (1.20) is obvious. To grasp the second, it is sufficient to introduce, in the neighborhood of any point $p \in \mathcal{M}$, a local coordinate system composed by $(h, x^{\parallel})$, where h is the value of the function H and $x^{\parallel}$ denotes some local coordinate system on the level set Σ_h. The situation we have described corresponds to assuming that

$$\mathrm{U} \;\simeq\; U^{\parallel}(x^{\parallel})\,\partial_{\parallel}; \quad \partial_h\, U^{\parallel}(x^{\parallel}) = 0. \tag{1.21}$$

Under the conditions spelled out in Eq. (1.20) we can easily prove that

$$\mathrm{i}_{\nabla H}\, \mathrm{d}\Omega^{[\mathrm{U}]} = 0. \tag{1.22}$$

Indeed, from the definition of the Lie derivative, we obtain

$$\mathrm{i}_{\nabla H}\, \mathrm{d}\Omega^{[\mathrm{U}]} = \underbrace{\mathcal{L}_{\nabla H}\, \Omega^{[\mathrm{U}]}}_{=\,\Omega^{[[U,\,\nabla H]]}=0} - \mathrm{d}\left(\underbrace{\mathrm{i}_{\nabla H}\Omega^{[\mathrm{U}]}}_{=0}\right). \tag{1.23}$$

Since we have both $\mathrm{i}_{\nabla H}\Omega^{[\mathrm{U}]} = 0$ and $[U,\, \nabla H] = 0$, it follows that also in this case

$$\Omega^{[\mathrm{U}]} \wedge \mathrm{d}\Omega^{[\mathrm{U}]} = 0. \tag{1.24}$$

Indeed, the three-form $\Omega^{[\mathrm{U}]} \wedge \mathrm{d}\Omega^{[\mathrm{U}]}$ has no projection in the direction ∇H and therefore it lives on the two-dimensional surfaces Σ_h: but in two dimensions, any three-form necessarily vanishes. This heuristic arguments motivate the notion of *contact form* and *contact structure* that capture the nonintegrability of a vector field in a *topological, metric independent way*.

Definition 1.1. *Let $\mathcal{M}$ be a smooth three-manifold. A contact form $\alpha \in \Gamma\left(CT\mathcal{M}, \mathcal{M}\right)$ is a one-form such that*

$$\alpha \wedge \mathrm{d}\alpha \neq 0. \tag{1.25}$$

Definition 1.2. *Let $\mathcal{M}$ be a smooth three-manifold and α be a contact form on it. The rank-two vector-bundle of all vector fields X, which satisfy the condition*

$$\mathrm{i}_X\, \alpha = 0, \tag{1.26}$$

is named the ***contact structure*** *$\mathcal{CS}_\alpha$ defined by α.*

In view of what we discussed above, it is clear that the definition of contact structures captures the notion of maximal *nonintegrability*. At each point $p \in \mathcal{M}$, the contact structure is a two-dimensional plane singled out in the tangent space $T_p\mathcal{M}$ by the condition (1.26). This smooth family of planes, however, cannot be considered as the tangent plane of the level surfaces of any foliation. This is ruled out by the condition (1.25).

Let us next introduce the notion of *Reeb-like field*

Definition 1.3 ***(Reeb-like field).*** *Let $\mathcal{M}$ be a smooth three-manifold and α be a contact one-form defining a contact-structure. A Reeb-like field for α is a vector field* U *satisfying the following two conditions:*

$$\mathrm{i}_{\mathrm{U}}\, \alpha\, >\, 0; \quad \mathrm{i}_{\mathrm{U}}\, \mathrm{d}\alpha = 0. \tag{1.27}$$

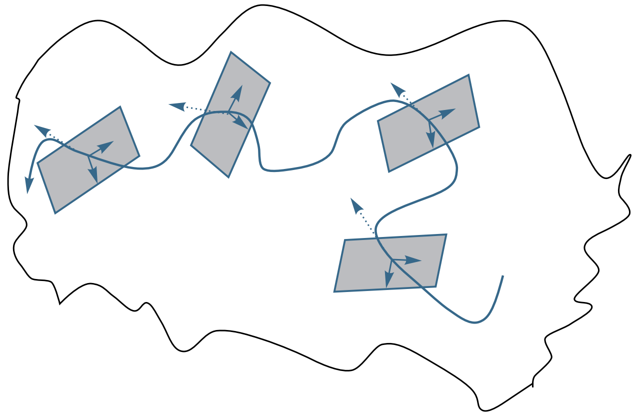

Fig. 2. A schematic picture of a contact structure. Given the Reeb field U, associated with a contact one-form α, we can consider the family of hyperplanes orthogonal, at each point of the manifold, to the vector field U. These hyperplanes are defined as the kernel of the contact form α and constitute the contact structure

Let us stress that this definition makes sense only in view of Eq. (1.25); this latter imposes that $\mathrm{d}\alpha$ has support on the orthogonal complement $W^{\perp} \subset T\mathcal{M}$ of the one-dimensional sub-bundle $W^{\parallel} \subset T\mathcal{M}$ forming the support of α. The second of Eqs. (1.27) imposes that the Reeb-like vector field should have no component along $W^{\perp}$: (see Fig. 2).

As we see, the main reason to introduce the contact form conception is that so doing one liberates the notion of a vector field capable to generate chaotic trajectories from the use of any metric structure. A vector field U is potentially interesting for chaotic regimes if it is a Reeb-like field for at least one contact form α. In this way the mathematical theorems about the classification of contact structures modulo diffeomorphisms (theorems that are metric-free and of topological nature) provide new global methods to capture the topology of hydro-flows.

Instead, if we work in a Riemannian manifold endowed with a metric $(\mathcal{M}, g)$, we can always invert the procedure and define the contact form α that can admit U as a Reeb-like field by identifying

$$\alpha = \Omega^{[\mathrm{U}]}. \tag{1.28}$$

In this way, the first of the two conditions (1.27) is automatically satisfied: $\mathrm{i}_{\mathrm{U}}\Omega^{[\mathrm{U}]} = \parallel \mathrm{U} \parallel^2 > 0$. It remains to be seen whether $\Omega^{[\mathrm{U}]}$ is indeed a contact form, namely, whether $\Omega^{[\mathrm{U}]} \wedge \mathrm{d}\Omega^{[\mathrm{U}]} \neq 0$, and whether the second condition $\mathrm{i}_{\mathrm{U}}\, \mathrm{d}\Omega^{[\mathrm{U}]} = 0$ is also satisfied. Both conditions are automatically fulfilled if U is the Beltrami field, namely, if it is an eigenstate of the operator $\star_g \mathrm{d}$ as advocated in Eq. (1.14). Indeed, the implication $\mathrm{i}_{\mathrm{U}}\, \mathrm{d}\Omega^{[\mathrm{U}]} = 0$ of the Beltrami equation was shown in Eq. (1.13), while from the Beltrami condition it also follows:

$$\begin{aligned}\Omega^{[\mathrm{U}]} \wedge \mathrm{d}\Omega^{[\mathrm{U}]} = \Omega^{[\mathrm{U}]} \wedge \star_g \Omega^{[\mathrm{U}]} = \parallel \mathrm{U} \parallel^2 \ \mathrm{Vol} \neq 0, \\ \mathrm{Vol} \equiv \frac{1}{3!} \times \epsilon_{ijk}\, dx^i \wedge dx^j \wedge dx^k.\end{aligned} \tag{1.29}$$

In this way the conceptual circle closes and we see that all the Beltrami vector fields can be regarded as the Reeb-like fields for a bona-fide contact form. Since the same contact structure (in the topological sense) can be described by different contact forms, once the Beltrami fields have been classified it remains the task to discover how many inequivalent contact structures they actually describe. Yet it is reasonable to assume that every contact structure has a contact form representative that is derived from the Beltrami Reeb-like field. Indeed, a precise correspondence is established by a theorem proved in [12, 13]:

Theorem 1.2. *Any rotational Beltrami vector field on a Riemannian* 3*-manifold is a Reeb-like field for some contact form. Conversely, any Reeb-like field associated to a contact form on a* 3*-manifold is a rotational Beltrami field for some Riemannian metric. A rotational Beltrami field means an eigenfunction of the* $\star_g \mathrm{d}$ *operator corresponding to a nonvanishing eigenvalue* λ*.*

1.3. Beltrami Equation at Large and Harmonic Analysis. All the arguments presented in the previous sections have been instrumental to enlighten the role of the Beltrami vector fields from various viewpoints related with hydrodynamics and other mathematical-physical issues. Let us now consider from a more general point of view the Beltrami equation (1.14) which constitutes the main topic of the present paper. The one here at stake is the case $p = 1$ of an eigenvalue equation that can be written in any $(2\,p + 1)$-dimensional Riemannian manifold $(\mathcal{M}_p,\, g)$, namely*:

$$\star_g \mathrm{d}\omega^{(p)} = \lambda\, \omega^{(p)}. \tag{1.30}$$

The eigenfunctions of the $\star_g \mathrm{d}$ operator are 1-forms for $p = 1$, namely in three dimensions, but they are higher differential forms in higher odd dimensions. A particularly interesting case is that of 7-manifolds, where the eigenfunctions of $\star_g \mathrm{d}$ are three-forms and can be related with a G_2-structure of the manifold. An important general observation is that the relation encoded in theorem 1.2 between Eq. (1.30) and contact structures, as they are defined in current mathematical literature, is true only for $p = 1$ and it is lost for higher p. Indeed, contact

*Note that, since $\star_g \mathrm{d} \star_g \mathrm{d} = (-)^p\, \Delta$ where Δ is the Laplacian, negative definite on compact manifolds, then the eigenvalue λ is real only for $p = 2\nu + 1$ odd. For $p = 2\nu$ even, the Beltrami fields are instead complex. This is the analogue of the well-known properties of instantons in even dimensions and supports the view that the Beltrami fields are the odd-dimensional counterpart of instantons.

structures are always defined in terms of a contact one-form and Eq. (1.25) is replaced by

$$\alpha \wedge \underbrace{d\alpha \wedge d\alpha \ldots d\alpha}_{p-\text{times}} \neq 0. \tag{1.31}$$

Hence, the problem of determining the spectrum and the eigenfunctions of the operator $\star_g d\omega^{(p)}$ is a general one and can be addressed in the same way to all odd-dimensions, yet its relation with flows and contact-structures is peculiar to $d = 3$ and has not a general significance. Whether the contact-structure viewpoint or the pure geometrical view point encoded in Eq. (1.30) is more fundamental is certainly a matter of debate and bears also on personal scientific tastes, yet it is absolutely clear that once the correspondence of theorem 1.2 has been established, the classification of the Beltrami fields is reduced to a classical problem of differential geometry whose solution can be derived within a time honored framework which makes no reference to trajectories, dynamical systems, contact structures and all the rest of the conceptions debated in the previous subsections of this introduction.

The framework we refer to is that of *harmonic analysis* on compact Riemannian manifolds (M, g) and its application to the spectral analysis of the Laplace–Beltrami operators (for reviews, see the book [18] and the articles [19]). As thoroughly discussed in the quoted references there are, on a Riemann manifold (M, g), several invariant differential operators, generically named Laplace–Beltrami some of which are of the second order, some other of the first order. They act on the sections of vector bundles $E \to \mathcal{M}$ of different rank, for instance, the tangent bundle, the bundle of p-forms, the bundle of symmetric two tensors, the spinor bundle, etc. Among the first-order operators, the most important ones are the Dirac operator acting on sections of the spinor bundle and the $\star_g$d-operator acting on p-forms in a $(2p + 1)$-dimensional manifold. The spectrum of all Laplace–Beltrami operators is sensitive both to the topology and to the metric of the underlying manifold. Each eigenspace is organized into irreducible representations of the isometry group G of the metric g, and the eigenfunctions assigned to a particular representation are generically named *harmonics*.

Here comes an important distinction in relation with the nature of the group G. If G is a Lie group and if the manifold $\mathcal{M}$ is homogeneous under its action, then $\mathcal{M} \sim \mathrm{G}/\mathrm{H}$, where $\mathrm{H} \subset \mathrm{G}$ is the stability subgroup of some reference point $p_0 \in \mathcal{M}$. In this case harmonic analysis reduces completely to group-theory and the spectrum of any Laplace–Beltrami operator can be derived in pure algebraic terms without ever using any differential operations. In this case G is not a Lie group and/or $\mathcal{M}$ is not homogeneous under its action, then matters become more complicated and ad hoc techniques have to be utilized case by case to analyze the spectrum of invariant operators.

1.4. Harmonic Analysis on the T^3 Torus and the Universal Classifying Group. The reasons to compactify Arnold–Beltrami flows on the T^3 have already been discussed, and we do not resume the issue. We just observe that $\mathbb{R}^3$ is a noncompact coset manifold so that harmonic analysis over $\mathbb{R}^3$ is a complicated matter of functional analysis. After compactification, namely, after imposing periodic boundary conditions, things drastically simplify. Firstly, as we explain in a detailed way in Sec. 2, the compactification is obtained by quotienting $\mathbb{R}^3$ with respect to a discrete subgroup of the translation group which constitutes a lattice:

$$T^3 = \frac{\mathbb{R}^3}{\Lambda}. \tag{1.32}$$

Secondly, we implement the programme of harmonic analysis by presenting a general algorithm to construct solutions of the the Beltrami equation which utilizes as main ingredient the orbits under the action of the point group $\mathfrak{P}_\Lambda$ of three vectors in the momentum lattice $^\star\Lambda$ which is just the dual of the lattice Λ. In the language of crystallography the point group is just the discrete subgroup $\mathfrak{P}_\Lambda \subset \mathrm{SO}(3)$ of the rotation group which maps the lattice Λ and its dual $^\star\Lambda$ into themselves:

$$\mathfrak{P}_\Lambda\,\Lambda = \Lambda; \quad \mathfrak{P}_\Lambda\,{}^\star\Lambda =^\star \Lambda. \tag{1.33}$$

In the case of the cubic lattice, that is the main example studied in this paper, we have $\mathrm{G}_{\mathrm{cubic}} = \mathrm{O}_{24}$, where $\mathrm{O}_{24} \sim \mathrm{S}_4$ is the proper octahedral group of order $|\mathrm{O}_{24}| = 24$. In the case of the hexagonal lattice, which we also briefly analyze, the point group is the dihedral group $\mathcal{D}_6$ of order $|\mathrm{D}_6| = 12$.

Thirdly, as we explain in detail in Sec. 5, which constitutes the hard-core of the present paper, a general argument, inspired by the logic that crystallographers use to derive and classify space groups, leads us to introduce a large finite group $\mathfrak{GU}_\Lambda$, named by us the *Universal Classifying Group for the Lattice* Λ, made out of discretized rotations and translations that are defined by the structure of Λ. All eigenfunctions of the $\star_g$d-operator can be organized into a finite number of classes, and each class decomposes in a specific unique way into the irreducible representations of $\mathfrak{GU}_\Lambda$. Hence, all Arnold–Beltrami vector fields are in correspondence with the irreps of $\mathfrak{GU}_\Lambda$. Knowing the branching rules of such irreps with respect to its various subgroups $\mathrm{H_i} \subset \mathfrak{GU}_\Lambda$ and selecting the identity representation, one obtains the Arnold–Beltrami vector fields invariant with respect to those $\mathrm{H_i}$ for which we were able to find an identity irrep D_1 in the branching rules. In this way, we can classify all Arnold–Beltrami flows and also uncover their *hidden symmetries*.

In this paper, we consider in an extensive way the case of the cubic lattice and construct the corresponding Universal Classifying Group $\mathfrak{GU}_{\mathrm{cubic}} = \mathrm{G}_{1536}$. This latter is a finite group of order $|\mathrm{G}_{1536}| = 1536$, which we study in full detail deriving all of its 37 irreducible representations and the associated character table. We also analyze a large class of its subgroups $\mathrm{H_i} \subset \mathrm{G}_{1536}$ systematically constructing their irreps and character tables. This allows the derivation of all the branching rules of the 37 G_{1536} irreps with respect to the considered subgroups which are displayed in dedicated tables in Appendices. We show that the O_{24} orbits in the cubic lattice arrange into 48 equivalence classes, the parameters of the corresponding Beltrami vector fields filling all the 37 irreducible representations of G_{1536}. In this way, we obtain an exhaustive classification of all *generalized* ABC-*flows* and of their hidden symmetries. In this way, we fulfill the task of classifying and constructing all possible generalizations of the ABC-flows.

From our analysis emerges the following pattern. The Universal Classifying Group contains at least two* isomorphic but not conjugate subgroups of order 192, namely G_{192} and GF_{192} in our nomenclature. The classical ABC-flows are obtained from the lowest-lying momentum orbit of length 6 which produces an irreducible 6-dimensional representation of the Universal Classifying Group: $D_{23}\left[\mathrm{G}_{1536}, 6\right]$. The three-parameter ABC-flow is just the irreducible 3-dimensional representation $D_{12}\left[\mathrm{GF}_{192}, 3\right]$ in the split $D_{23}\left[\mathrm{G}_{1536}, 6\right] = D_{12}\left[\mathrm{GF}_{192}, 3\right] \oplus D_{15}\left[\mathrm{GF}_{192}, 3\right]$. With respect to the isomorphic but not conjugate subgroup G_{192}, the representation $D_{23}\left[\mathrm{G}_{1536}, 6\right]$ remains instead irreducible: $D_{23}\left[\mathrm{G}_{1536}, 6\right] = D_{20}\left[\mathrm{G}_{192}, 6\right]$, so that there is no proper way of reducing the six parameters to three. The most symmetric case $A : A : A = 1$ simply corresponds to the identity representation of the subgroup $\mathrm{GS}_{24} \subset \mathrm{GF}_{192}$ which occurs in the splitting of the 3-dimensional representation $D_{12}\left[\mathrm{GF}_{192}, 3\right] = D_1\left[\mathrm{GS}_{24}, 1\right] \oplus D_3\left[\mathrm{GS}_{24}, 2\right]$.

All other Beltrami flows arising from different instances of the 48 classes of momentum vectors have similar structures. The result of the construction algorithm produces a representation of the Universal Classifying Group that can be either reducible or irreducible. This latter can be split into irreps of either G_{192} or GF_{192} and apparently all cases of the invariant Beltrami vector fields have invariance groups that are subgroups of one of the two groups G_{192} or GF_{192}. It would be interesting to transform this observation into a theorem. At the moment we have not found an obvious proof.

A much shorter sketch of the Hexagonal Lattice is also discussed, to emphasize the generality of our methods, but we do not address the construction of the

*It is known [20] that there are 4 different Space-Groups Γ^I_{24} $(I = 1, \ldots, 4)$ of order 24, isomorphic to the point group O_{24} but not conjugate one to the other under the action of the continuous translation group. One of them is the point group itself $\Gamma^1_{24} = \mathrm{O}_{24}$ which is a subgroup of the first of the two groups of order 192 identified by us: $\mathrm{O}_{24} \subset \mathrm{G}_{192}$. Another of the four mentioned groups is $\Gamma^2_{24} = \mathrm{GS}_{24}$ which is a subgroup of the second group of order 192 identified by us: $\mathrm{GS}_{24} \subset \mathrm{GF}_{192}$. It remains to see whether Γ^3_{24} and Γ^4_{24} are contained in the two already identified subgroups G_{192} and GF_{192} or if there exist other two such nonconjugate subgroups of order 192 that respectively contain Γ^3_{24} and Γ^3_{24}. We do not know the answer to such a question. Extensive but lengthy calculation can resolve the issue.

Universal Classifying Group for this case, which might be performed along the same lines.

1.5. Organization of the Paper. This very long paper is organized into three parts: Introduction, Appendices, and References. Appendices that fill almost 100 pages are tables whose content is boring, yet it constitutes an essential and indispensable part of the presented results:

1. Appendix A contains the definition of all the relevant groups and subgroups by explicit enumeration of their conjugacy classes of elements.

2. Appendix B contains all the character tables of the relevant groups and subgroups that we have explicitly constructed, since most of them are not available in the literature.

3. Appendix C contains the classification of momentum vectors in the cubic lattice and reports the irreducible representations of the classifying group G_{1536} to which each momentum class leads when solving the Beltrami equation.

4. Appendix D contains all the branching rules of all the irreps of G_{1536} with respect to all considered subgroups. This information is essential to spot all Beltrami vector fields that are invariant with respect to some subgroup $\mathcal{H}_i \subset \mathrm{G}_{1536}$.

5. Appendix E contains the description and the list of conjugacy classes of some additional subgroups that play a role in understanding all cases and subcase of the classical ABC-flows.

6. Appendix F contains some formulae too large for the main text that had to be displayed in landscape format.

As for the Introduction it is divided into the following ten sections:

1. Section 1 is the present conceptual introduction.

2. Section 2 presents in a brief way all the elements of lattice theory and point group theory that are needed in our constructions.

3. Section 3 presents the algorithm for the construction of solutions of the Beltrami equation that has been systematically implemented on a computer by means of a purposely written MATHEMATICA code.

4. Section 4 provides the definition of the cubic lattice and of its octahedral point group that constitute the main example dealt with in the present paper.

5. Section 5 contains the definition and the construction, in the case of the cubic lattice, of the Universal Classifying Group. The same section contains also a detailed description of the induction algorithm utilized to construct all the irreps and the character tables of the relevant groups and subgroups.

6. Section 6 contains the classification of the 48 momentum classes in the cubic lattice and the description of their organization into point group orbits.

7. Section 7 contains a detailed discussion of several examples of the Beltrami fields on the cubic lattice with an in depth analysis of their hidden symmetries.

8. Section 8 contains a brief description of the hexagonal lattice and of its point group $\mathcal{D}_6$. In this case we do not construct the Universal Classifying Group

and we just use the example to illustrate the new features that appear when the lattice is not self-dual as in the case of the cubic one.

9. Section 9 briefly presents some examples of the Beltrami vector field on the hexagonal lattice for illustrative purposes.

10. Section 10, named Conclusions, contains a wide conceptual discussion of the obtained results and of the entire field of ABC-flows from the perspective of authors and readers that do not belong to the community of experts in this field of mathematical hydrodynamics.

To the reader who has no time to follow the technical developments and is rather interested in obtaining a conceptual assessment of the matters dealt within the article we suggest the reading of Introduction and, immediately after, of Conclusions. He can come back to the other sections at another time.

2. BASIC ELEMENTS OF LATTICE AND FINITE GROUP THEORY NEEDED IN OUR CONSTRUCTION

In this section, we summarize the main definitions and we fix our conventions for all those items in Lattice Theory and in Finite Group Theory that we are going to utilize in the sequel and which are essential in our construction.

2.1. Lattices. We begin by fixing our notations for space and momentum lattices that define a three-torus T^3 endowed with a flat metric structure.

Let us consider the standard $\mathbb{R}^3$ manifold and introduce a basis of three linearly independent 3-vectors that are not necessarily orthogonal to each other and of equal length:

$$\vec{\mathbf{w}}_\mu \in \mathbb{R}^3 \quad \mu = 1, \ldots, 3. \tag{2.1}$$

Any vector in $\mathbb{R}$ can be decomposed along such a basis and we have

$$\vec{\mathbf{r}} = r^\mu \vec{\mathbf{w}}_\mu. \tag{2.2}$$

The flat (constant) metric on $\mathbb{R}^3$ is defined by

$$g_{\mu\nu} = \langle \vec{\mathbf{w}}_\mu, \vec{\mathbf{w}}_\nu \rangle, \tag{2.3}$$

where $\langle , \rangle$ denotes the standard Euclidean scalar product. The space lattice Λ consistent with the metric (2.3) is the free Abelian group (with respect to sum) generated by the three basis vectors (2.1), namely:

$$\mathbb{R}^3 \ni \vec{\mathbf{q}} \in \Lambda \Leftrightarrow \vec{\mathbf{q}} = q^\mu \, \vec{\mathbf{w}}_\mu, \quad \text{where} \quad q^\mu \in \mathbb{Z}. \tag{2.4}$$

The momentum lattice is the dual lattice $\Lambda^\star$ defined by the property

$$\mathbb{R}^3 \ni \vec{\mathbf{p}} \in \Lambda^\star \Leftrightarrow \langle \vec{\mathbf{p}}, \vec{\mathbf{q}} \rangle \in \mathbb{Z} \quad \forall \vec{\mathbf{q}} \in \Lambda. \tag{2.5}$$

A basis for the dual lattice is provided by a set of three *dual vectors* $\vec{\mathbf{e}}^{\mu}$ defined by the relations*:

$$\langle \vec{\mathbf{w}}_{\mu},\, \vec{\mathbf{e}}^{\nu} \rangle = \delta^{\nu}_{\mu}, \tag{2.6}$$

so that

$$\forall\, \vec{\mathbf{p}} \in \Lambda^{\star} \quad \vec{\mathbf{p}} = p_{\mu}\, \vec{\mathbf{e}}^{\mu}, \quad \text{where} \quad p_{\mu} \in \mathbb{Z}. \tag{2.7}$$

2.2. The Three-Torus T^3. The three-torus is topologically defined as the product of three circles, namely:

$$\mathrm{T}^3 \equiv \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \equiv \frac{\mathbb{R}}{\mathbb{Z}} \times \frac{\mathbb{R}}{\mathbb{Z}} \times \frac{\mathbb{R}}{\mathbb{Z}}. \tag{2.8}$$

Alternatively we can define the three-torus by modding $\mathbb{R}^3$ with respect to a three-dimensional lattice. In this case, the three-torus comes automatically equipped with a flat constant metric:

$$\mathrm{T}^3_g = \frac{\mathbb{R}^3}{\Lambda}. \tag{2.9}$$

According to (2.9) the flat Riemannian space T^3_g is defined as the set of equivalence classes with respect to the following equivalence relation:

$$\vec{\mathbf{r}}' \sim \vec{\mathbf{r}} \quad \text{iff} \quad \vec{\mathbf{r}}' - \vec{\mathbf{r}} \in \Lambda. \tag{2.10}$$

The metric (2.3) defined on $\mathbb{R}^3$ is inherited by the quotient space and therefore it endows the topological torus (2.8) with a flat Riemannian structure. Seen from another point of view, the space of flat metrics on T^3 is just the coset manifold $\mathrm{SL}(3,\mathbb{R})/\mathrm{O}(3)$ encoding all possible symmetric matrices, alternatively all possible space lattices, each lattice being spanned by an arbitrary triplet of basis vectors (2.1).

2.3. Bravais Lattices. Every lattice Λ yields a metric g and every metric g singles out an isomorphic copy $\mathrm{SO_g}(3)$ of the continuous rotation group $\mathrm{SO}(3)$, which leaves it invariant:

$$M \in \mathrm{SO_g}(3) \quad \Leftrightarrow \quad M^T\, g\, M = g. \tag{2.11}$$

By definition, $\mathrm{SO_g}(3)$ is the conjugate of the standard $\mathrm{SO}(3)$ in $\mathrm{GL}(3,\mathbb{R})$:

$$\mathrm{SO_g}(3) = \mathcal{S}\, \mathrm{SO}(3)\, \mathcal{S}^{-1} \tag{2.12}$$

with respect to the matrix $\mathcal{S} \in \mathrm{GL}(3,\mathbb{R})$ which reduces the metric g to the Kronecker delta:

$$\mathcal{S}^T\, g\, \mathcal{S} = \mathbf{1}. \tag{2.13}$$

*In the sequel for the scalar product of two vectors we utilize also the equivalent shorter notation $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \langle \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \rangle$.

Notwithstanding this, a generic lattice Λ is not invariant with respect to any proper subgroup of the rotation group $\mathrm{G} \subset \mathrm{SO_g}(3) \equiv \mathrm{SO}(3)$. Indeed, by invariance of the lattice one understands the following condition:

$$\forall\, \gamma \in \mathrm{G} \quad \text{and} \quad \forall\, \vec{\mathbf{q}} \in \Lambda : \quad \gamma \cdot \vec{\mathbf{q}} \in \Lambda. \tag{2.14}$$

Lattices that have a nontrivial symmetry group $\mathrm{G} \subset \mathrm{SO}(3)$ are those relevant to Solid State Physics and Crystallography. There are 14 of them grouped in 7 classes that were already classified in the XIX century by Bravais [20]. The symmetry group G of each of these Bravais lattices Λ_B is necessarily one of the well-known finite subgroups of the three-dimensional rotation group $\mathrm{O}(3)$. In the language universally adopted by Chemistry and Crystallography for each Bravais lattice Λ_B, the corresponding invariance group $\mathrm{G_B}$ is named the *Point Group*. For purposes different from our present one, the point group can be taken as the lattice invariance subgroup within $\mathrm{O}(3)$ that, besides rotations, contains also improper rotations and reflections. Since we are interested in the Beltrami equation, which is covariant only under proper rotations, of interest to us are only those point groups that are subgroups of $\mathrm{SO}(3)$.

According to a standard nomenclature, the 7 classes of Bravais lattices are respectively named *Triclinic, Monoclinic, Orthorombic, Tetragonal, Rhombohedral, Hexagonal, and Cubic*. Such classes are specified by giving the lengths of the basis vectors $\vec{\mathbf{w}}_\mu$ and the three angles between them, in other words, by specifying the 6 components of the metric (2.3).

2.4. The Proper Point Groups. Restricting one's attention to proper rotations, the proper point groups that appear in the 7 lattice classes are either the cyclic groups $\mathbb{Z}_h$ with $h = 2, 3, 4$, or the dihedral groups $\mathcal{D}_h$ with $h = 3, 4, 6$, or the tetrahedral group T, or the octahedral group O_{24}. In this paper we restrict our attention to the two lattices with the largest possible point groups, namely, the Hexagonal lattice with d_6 symmetry and the cubic lattice with O_{24} symmetry. We think that these two examples suffice to clarify the principles of the construction we want to present and furthermore provide the potentially more interesting Beltrami flows to be analyzed in connection with the problem of the origin of chaotic trajectories.

2.5. Point Group Characters. Another fundamental ingredient in our construction are the characters of the point group and of other classifying groups that will emerge in our construction.

Given a finite group G, according to standard theory and notations [21], one defines its order and the order of its conjugacy classes as follows:

$$\begin{aligned} g &= |\mathrm{G}| = \#\text{ of group elements}, \\ g_i &= |\mathcal{C}_i| = \#\text{ of group elements in the conjugacy class } \mathcal{C}_i \quad i = i, \ldots, r. \end{aligned} \tag{2.15}$$

If there are r conjugacy classes, one knows from first principles that there is exactly r inequivalent irreducible representation D^μ of dimensions $n_\mu = \dim D^\mu$, such that

$$\sum_{\mu=1}^{r} n_\mu^2 = g. \tag{2.16}$$

For any reducible or irreducible representation of dimension d:

$$\forall\, \gamma \in \mathrm{G} : \gamma \rightarrow \quad \mathfrak{R}\left[\gamma\right] \in \mathrm{Hom}\left[\mathbb{R}^d, \mathbb{R}^d\right], \tag{2.17}$$

the character vector is defined as:

$$\chi^{\mathfrak{R}} = \left\{\mathrm{Tr}\,\left(\mathfrak{R}\left[\gamma_1\right]\right), \mathrm{Tr}\,\left(\mathfrak{R}\left[\gamma_2\right]\right), \dots, \mathrm{Tr}\,\left(\mathfrak{R}\left[\gamma_r\right]\right)\right\}, \quad \gamma_i \in \mathcal{C}_i. \tag{2.18}$$

The choice of a representative γ_i within each conjugacy class $\mathcal{C}_i$ is irrelevant since all representatives have the same trace. In particular, one can calculate the characters of the irreducible representations

$$\chi^\mu = \chi\left[D^\mu\right] = \left\{\mathrm{Tr}\,\left(D^\mu\left[\gamma_1\right]\right), \mathrm{Tr}\,\left(D^\mu\left[\gamma_2\right]\right), \dots, \mathrm{Tr}\,\left(D^\mu\left[\gamma_r\right]\right)\right\}, \quad \gamma_i \in \mathcal{C}_i \tag{2.19}$$

that are named *fundamental characters* and constitute the *character table*. We stick to the widely adopted convention that the first conjugacy class is that of the identity element $\mathcal{C}_1 = \{\mathbf{e}\}$, containing only one member. In this way, the first entry of the character vector is always the dimension d of the considered representation. In the same way we order the irreducible representation starting always with the identity one-dimensional representation which associates to each group element simply the number 1.

It is well known that for any finite group G, the character vectors satisfy the following two fundamental relations:

$$\sum_{\mu=1}^{r} \chi_i^\mu\, \chi_j^\mu = \frac{g}{g_i}\, \delta_{ij} \tag{2.20}$$

and

$$\sum_{i=1}^{r} g_i\, \chi_i^\mu\, \chi_i^\nu = g\, \delta^{\mu\nu}. \tag{2.21}$$

Utilizing these identities one can immediately retrieve the decomposition of any given reducible representation $\mathcal{R}$ into its irreducible components. Suppose that the considered representation is the following direct sum of irreducible ones:

$$\mathfrak{R} = \oplus_{\mu=1}^{r}\, a_\mu\, D^\mu, \tag{2.22}$$

where a_μ denotes the number of times, the irrep D^μ is contained in the direct sum and it is named the *multiplicity*. Given the character vector of any considered representation $\mathfrak{R}$, the vector of its multiplicities is immediately obtained by use of (2.21):

$$a_\mu = \frac{1}{g} \sum_i^r g_i \chi_i^{\mathfrak{R}} \, \chi_i^\mu. \tag{2.23}$$

Furthermore, one can construct the projectors onto the invariant subspaces $a_\mu \, D^\mu$ by means of another classical formula that we will extensively use in the sequel*

$$\Pi_{\mathfrak{R}}^\mu = \frac{\mathrm{dim} D_\mu}{g} \sum_{k=1}^r \chi_k^\mu \sum_{\ell=1}^{g_k} \underbrace{\mathfrak{R}\left[\gamma_\ell\right]}_{\gamma_\ell \in \mathcal{C}_k}. \tag{2.24}$$

3. THE SPECTRUM OF THE $\star$d OPERATOR ON T^3 AND BELTRAMI EQUATION

The main issue of the present paper is the construction of vector fields defined over the three-torus T^3 that are eigenstates of the $\star_g \mathrm{d}$ operator, namely of solutions of the following equations:

$$\begin{aligned} \star_g \mathrm{d}\Omega^{(n;I)} &= m_{(n)} \, \Omega^{(n;I)}, \\ \Omega^{(n;I)} \left[V_{(m;J)}\right] &= \delta^n_m \, \delta^I_J, \end{aligned} \tag{3.1}$$

where d is the exterior differential, and $\star_g$ is the Hodge-duality operator which, differently from the exterior differential, can be defined only with reference to a given metric g. By $\Omega^{(n;i)}$ we denote a one-form:

$$\Omega^{(n;I)} = \Omega^{(n;I)}_\mu \, dx^\mu \tag{3.2}$$

which is declared to be dual to the vector field we are interested in:

$$\begin{aligned} V_{(m;J)} &= V^\mu_{(m;J)} \, \partial_\mu, \\ \Omega^{(n;I)} \left[V_{(m;J)}\right] &\equiv \Omega^{(n;I)}_\mu \, V^\mu_{(m;J)} = \delta^n_m \, \delta^I_J, \end{aligned} \tag{3.3}$$

and by means of the composite index $(n;I)$ we make reference to the quantized eigenvalues $m_{(n)}$ of the $\star_g \mathrm{d}$ operator (ordered in increasing magnitude $|m_{(n)}|$) and to a basis of the corresponding eigenspaces

$$\star_g \mathrm{d}\Omega^{(n)} = m_{(n)} \, \Omega^{(n)} \quad \Rightarrow \quad \Omega^{(n)} = \sum_{I=1}^{d_n} c_I \, \Omega^{(n;I)}, \tag{3.4}$$

---

*We recall that according to standard conventions, the first conjugacy class is always the class of the identity, so that the first component χ^μ of any character is just the dimension of that irrep D_μ. Hence in formula (2.24) $\mathrm{dim} D_\mu = D_\mu$.

the symbol d_n denoting the degeneracy of $|m_{(n)}|$ and c_I being constant coefficients.

Indeed, since T^3 is a compact manifold, the eigenvalues $m_{(n)}$ form a discrete set. Their values and their degeneracies are a property of the metric g introduced on it. Here we outline the general procedure to construct the eigenfunctions of $\star_g \mathrm{d}$, to calculate the eigenvalues and to determine their degeneracies. What follows is an elementary and straightforward exercise in harmonic analysis.

In tensor notation, equation (3.1) has the following appearance:

$$\frac{1}{2}\sqrt{|\det g|}\, g_{\mu\nu}\,\epsilon^{\nu\rho\sigma}\partial_\rho\Omega_\sigma = m\,\Omega_\mu. \tag{3.5}$$

The equation written above is named the Beltrami equation since it was already considered by the great Italian mathematician Eugenio Beltrami in 1881 [16], who presented one of its periodic solutions previously constructed by Gromeka in 1881. Such a solution was inherited by Arnold and it is essentially the basis of his Hydrodynamical Model. Here we will see that the Arnold Model just corresponds to the lowest eigenfunction of the $\star_g\,\mathrm{d}$ operator in the case of the cubic lattice. Many more similar models can be constructed choosing higher eigenvalues, choosing irreducible representation of the point group in their eigenspaces or changing the lattice.

Introducing the basis vectors of the dual lattice $\Lambda^\star$ we can write

$$\Omega = \Omega_\mu\, dr^\mu = \Omega_\mu\, e^\mu_i\, dx^i = \Omega_i\, dx^i, \tag{3.6}$$

where e^μ_i are the components of the vectors $\vec{\mathbf{e}}^\mu$ in a standard orthogonal basis of $\mathbb{R}^3$ and

$$x^i = w^i_\mu\, r^\mu \tag{3.7}$$

are a new set of Euclidean coordinates obtained from the original ones r^μ by means of the components w^i_μ of the basis vectors $\vec{\mathbf{w}}_\mu$ of the space lattice Λ. Recalling that

$$\partial_\mu = \frac{\partial}{\partial r^\mu} = w^i_\mu\,\partial_i; \quad \partial_i = \frac{\partial}{\partial x^i}, \tag{3.8}$$

with a little bit of straightforward algebra, we can rewrite Eq. (3.1) in the equivalent universal way

$$\frac{1}{2}\,\epsilon_{ijk}\partial_j\Omega_k = \mu\,\Omega_i; \quad \mu = m. \tag{3.9}$$

The next task is that of constructing an ansatz for the vector harmonics $Y_i(\mathbf{x})$ that are eigenfunctions of $\star_g \mathrm{d}$. Since such eigenfunctions have to be well defined on T^3, their general form is necessarily the following one:

$$Y_i\,(\mathbf{k}\,|\,\mathbf{x}) = v_i\,(\mathbf{k})\,\cos\left(2\,\pi\,\mathbf{k}\cdot\mathbf{x}\right) + \omega_i\,(\mathbf{k})\,\sin\left(2\,\pi\,\mathbf{k}\cdot\mathbf{x}\right); \quad \mathbf{k}\in\Lambda^\star. \tag{3.10}$$

The condition that the momentum $\mathbf{k}$ lies in the dual lattice guarantees that $Y_i(\mathbf{x})$ is periodic with respect to the space lattice Λ. Indeed, by means of the very definition of dual lattice (2.5) it follows that

$$\forall\, \vec{\mathbf{q}} \in \Lambda : \quad Y_i\,(\mathbf{k}\,|\,\mathbf{x}+\vec{\mathbf{q}}) = Y_i\,(\mathbf{k}\,|\,\mathbf{x})\,. \tag{3.11}$$

Considering next Eq. (3.9) we immediately see that it implies the further condition $\partial^i\, Y_i = 0$. Imposing such a condition on the general ansatz (3.10), we obtain

$$\mathbf{k}\cdot\vec{\mathbf{v}}\,(\mathbf{k}) = 0; \quad \mathbf{k}\cdot\vec{\omega}\,(\mathbf{k}) = 0 \tag{3.12}$$

which reduces the 6 parameters contained in the general ansatz (3.10) to 4. Imposing next the very equation (3.9), we get the following two conditions:

$$\mu\, v_i\,(\mathbf{k}) = \pi\,\epsilon_{ij\ell}\,k_j\,\omega_\ell\,(\mathbf{k})\,, \tag{3.13}$$

$$\mu\,\omega_i\,(\mathbf{k}) = -\pi\,\epsilon_{ij\ell}\,k_j\,v_\ell\,(\mathbf{k})\,. \tag{3.14}$$

The two equations are self-consistent if and only if the following condition is verified:

$$\mu^2 = \pi^2\,\langle\mathbf{k},\,\mathbf{k}\rangle. \tag{3.15}$$

This trivial elementary calculation completely determines the spectrum of the operator $\star_g\,\mathrm{d}$ on T^3_g endowed with the metric fixed by the choice of a lattice Λ. The possible eigenvalues are provided by

$$m_{\mathbf{k}} = \pm\,\pi\,\sqrt{\langle\mathbf{k},\,\mathbf{k}\rangle}, \quad \mathbf{k}\in\Lambda^\star. \tag{3.16}$$

The degeneracy of each eigenvalue is geometrically provided by counting the number of intersection points of the dual lattice $\Lambda^\star$ with a sphere whose center is in the origin and whose radius is

$$r = \sqrt{\langle\mathbf{k},\,\mathbf{k}\rangle}. \tag{3.17}$$

For a generic lattice, the number of solutions of equation (3.17), namely, the number of intersection points of the lattice with the sphere is just two: $\pm\mathbf{k}$, so that the typical degeneracy of each eigenvalue is just 2. If the lattice Λ is one of the Bravais lattices admitting a nontrivial point group G, then the number of solutions of Eq. (3.17) increases since all lattice vectors $\mathbf{k}$ that sit in one orbit of G have the same norm and therefore are located on the same spherical surface. The degeneracy of the $\star_g\,\mathrm{d}$ eigenvalue is precisely the order of the corresponding G-orbit in the dual lattice $\Lambda^\star$. It is appropriate to note that the spectrum of the $\star_g\,\mathrm{d}$ operator on vectors is just the square root of the spectrum of the Laplacian operator on the same space. Indeed, if we have a scalar function $\Phi(\mathbf{x})$ on T^3_g, it admits the expansion in harmonics of the form:

$$Y\,(\mathbf{k}\,|\,\mathbf{x}) = a\,(\mathbf{k})\,\cos\left(2\,\pi\,\mathbf{k}\cdot\mathbf{x}\right) + b\,(\mathbf{k})\,\sin\left(2\,\pi\,\mathbf{k}\cdot\mathbf{x}\right), \quad \mathbf{k}\in\Lambda^\star, \tag{3.18}$$

and calculating the Laplacian, we obtain

$$\Delta_g\, Y\,(\mathbf{k}\,|\,\mathbf{x}) \;\equiv\; \frac{1}{4}\, g^{\mu\nu}\, \partial_\mu\, \partial_\nu\, Y\,(\mathbf{k}\,|\,\mathbf{x}) = \pi^2\, \langle \mathbf{k},\, \mathbf{k}\rangle\, Y\,(\mathbf{k}\,|\,\mathbf{x})\,. \tag{3.19}$$

In particular, all the three components of the vector harmonic (3.10) satisfy the above equation with the above eigenvalue.

3.1. The Algorithm to Construct Arnold–Beltrami Flows. What we described in the previous subsection provides a well-defined algorithm to construct a series of Arnold–Beltrami flows that can be summarized in a few clear-cut steps, and it is quite suitable for a systematic computer aided implementation.

The steps are the following ones:

a) Choose a Bravais Lattice Λ with a nontrivial proper point group $\mathfrak{P}_\Lambda$.

b) Construct the character table and the irreducible representations of $\mathfrak{P}_\Lambda$.

c) Analyze the structure of orbits of $\mathfrak{P}_\Lambda$ on the lattice Λ and determine the number of lattice points contained in each spherical layer $\mathfrak{S}_n$ of the dual lattice $\Lambda^\star$ of quantized radius r_n:

$$\begin{aligned} \mathbf{k}_{(n)} \in \mathfrak{S}_n \quad &\Leftrightarrow \quad \langle \mathbf{k}_{(n)},\, \mathbf{k}_{(n)}\rangle = r_n^2, \\ 2\, P_n &\equiv |\mathfrak{S}_n|\,. \end{aligned} \tag{3.20}$$

The number of lattice points in each spherical layer is always even since if $\mathbf{k} \in \Lambda^\star$, also $-\mathbf{k} \in \Lambda^\star$ and obviously any vector and its negative have the same norm. The spherical layer $\mathfrak{S}_n$ can be composed of one or more $\mathfrak{P}_\Lambda$-orbits. In any case it corresponds to a fixed eigenvalue $m_n = \pi\, r_n$ of the $\star\mathrm{d}$ operator.

d) Construct the most general solution of the Beltrami equation with eigenvalue m_n by using the individual harmonics constructed in Eq. (3.10):

$$V_i\,(\mathbf{x}) = \sum_{\mathbf{x}\in\mathfrak{S}_n} Y_i\,(\mathbf{k}\,|\,\mathbf{x})\,. \tag{3.21}$$

Hidden in each harmonic $Y_i\,(\mathbf{k}\,|\,\mathbf{x})$, there are two parameters that are the remainder of the six parameters $v_i\,(\mathbf{k})$ and $\omega_i\,(\mathbf{k})$ after conditions (3.12), (3.13), (3.14) have been imposed. This would amount to a total of $4\,P_n$ parameters, yet, since the trigonometric functions $\cos\,(\theta)$ and $\sin\,(\theta)$ are mapped into plus or minus themselves under change of sign of their argument and since each spherical layer $\mathfrak{S}_n$ contains lattice vectors in pairs $\pm\mathbf{k}$, it follows that the number of independent parameters is always reduced to $2P_n$. Hence, at the end of the construction encoded in Eq. (3.21), we have the Beltrami vector depending on a set of $2P_n$ parameters that we can call F_I and consider as the components of $2\,P_n$-component vector $\mathbf{F}$. Ultimately we have an object of the following form:

$$\mathbf{V}\,(\mathbf{x}\,|\,\mathbf{F})\,, \tag{3.22}$$

which under the point group $\mathfrak{P}_\Lambda$ necessarily transforms in the following way:

$$\forall\,\gamma \in \mathfrak{P}_\Lambda\,: \quad \gamma^{-1}\cdot \mathbf{V}\left(\gamma\cdot\mathbf{x}\,|\,\mathbf{F}\right) = \mathbf{V}\left(\mathbf{x}\,|\,\mathfrak{R}[\gamma]\cdot\mathbf{F}\right), \tag{3.23}$$

where $\mathfrak{R}[\gamma]$ are $2\,P_n \times 2\,P_n$ matrices that form a representation of $\mathfrak{P}_\Lambda$. Equation (3.23) is necessarily true because any rotation $\gamma \in \mathfrak{P}_\Lambda$ permutes the elements of $\mathfrak{S}_n$ among themselves.

e) Decompose the representation $\mathfrak{R}[\gamma]$ into irreducible representations of $\mathfrak{P}_\Lambda$. Each irreducible subspace $\mathbf{f}_p$ of the $2P_n$ parameter space $\mathbf{F}$ defines a streamline of the Arnold–Beltrami flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{x}(t) = \mathbf{V}\left(\mathbf{x}(t)\,|\,\mathbf{f}_p\right), \tag{3.24}$$

which is worth to analyze.

An obvious question which arises in connection with such a constructive algorithm is the following: How many Arnold–Beltrami flows are there? At first sight it seems that there is an infinite number of such systems since we can arbitrarily increase the radius of the spherical layer and on each new layer it seems that we have new models. Let us however observe that if on two different spherical layers $\mathfrak{S}_{n_1}$ and $\mathfrak{S}_{n_2}$ there are two orbits of lattice vectors $\mathcal{O}_1$ and $\mathcal{O}_2$ that have the same order

$$\ell = |\mathcal{O}_1| = |\mathcal{O}_2|\,, \tag{3.25}$$

and furthermore all vectors $\mathbf{k}_{(n_2)} \in \mathcal{O}_2$ are simply proportional to their analogues in orbit $\mathcal{O}_1$:

$$\mathbf{k}_{(n_2)} = \lambda\,\mathbf{k}_{(n_1)}; \quad \lambda \in \mathbb{Z}, \tag{3.26}$$

then we can conclude that

$$\mathbf{V}_{(n_2)}\left(\mathbf{x}\,|\,\mathbf{f}_p\right) = \mathbf{V}_{(n_1)}\left(\,\lambda\,\mathbf{x}\,|\,\mathbf{f}_p\right). \tag{3.27}$$

By redefining the coordinate fields $\lambda\,\mathbf{x} = \mathbf{x}'$ and rescaling time t, the two differential systems (3.24) respectively constructed from layer n_1 and layer n_2 can be identified.

As we shall demonstrate analyzing the case of the cubic lattice and the orbits of the octahedral group, there is always a finite number of $\mathfrak{P}_\Lambda$-orbit type on each lattice Λ. There is a maximal orbit $\mathcal{O}_{\max}$ that has order equal to the order of the point group:

$$|\mathcal{O}_{\max}| = |\mathfrak{P}_\Lambda|\,, \tag{3.28}$$

and there are a few shortened orbits $\mathcal{O}_i$ ($i = 1, \ldots, s$) that have a smaller order:

$$\ell_i = |\mathcal{O}_i| \,<\, |\mathfrak{P}_\Lambda|\,. \tag{3.29}$$

The fascinating property is that for the shortened orbits which seem to play a role in this context analogue to that of BPS states in another context, property (3.26) is always true. The vectors pertaining to the same orbit $\mathcal{O}_i$ in different spherical layers are always the same up to a multiplicative factor. Hence, from the shortened orbits we obtain always a finite number of the Arnold–Beltrami flows. It remains the case of the maximal orbit for which property (3.26) is not necessarily imposed. How many independent flows do we obtain considering all the layers? The answer to the posed question is hidden in number theory. Indeed, we have to analyze how many different types of triplets of integer numbers satisfy Diophantine equations of the Fermat type. In Sec. 6, we provide a systematic classification of such triplets for the cubic lattice showing that there is a finite number of the Arnold–Beltrami flows.

4. THE CUBIC LATTICE AND THE OCTAHEDRAL POINT GROUP

Let us now consider, within the general frame presented above, the cubic lattice.

The self-dual cubic lattice (momentum and space lattice at the same time) is displayed in Fig. 3.

The basis vectors of the cubic lattice Λ_{cubic} are

$$\vec{\mathbf{w}}_1 = \{1, 0, 0\}; \quad \vec{\mathbf{w}}_2 = \{0, 1, 0\}; \quad \vec{\mathbf{w}}_3 = \{0, 0, 1\}, \tag{4.1}$$

which implies that the metric is just the Kronecker delta

$$g_{\mu\nu} = \delta_{\mu\nu}, \tag{4.2}$$

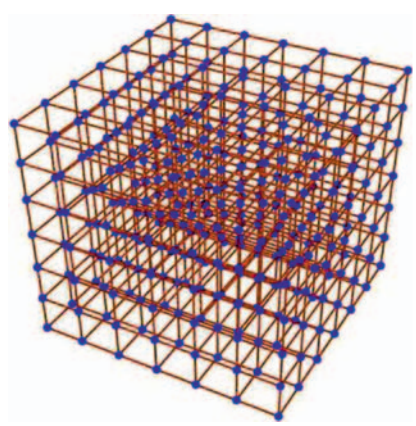

Fig. 3. A view of the self-dual cubic lattice

and the basis vectors $\vec{\mathbf{e}}^{\mu}$ of the dual lattice $\Lambda^{\star}_{\mathrm{cubic}}$ coincide with those of the lattice Λ. Hence the cubic lattice is self-dual:

$$\vec{\mathbf{w}}_{\mu} = \vec{\mathbf{e}}^{\mu} \quad \Rightarrow \quad \Lambda_{\mathrm{cubic}} = \Lambda^{\star}_{\mathrm{cubic}}. \tag{4.3}$$

The subgroup of the proper rotation group which maps the cubic lattice into itself is the octahedral group O whose order is 24. In the next subsection we recall its structure.

4.1. Structure of the Octahedral Group $\mathrm{O}_{24} \sim \mathrm{S}_4$. Abstractly the octahedral Group $\mathrm{O}_{24} \sim \mathrm{S}_4$ is isomorphic to the symmetric group of permutations of 4 objects. It is defined by the following generators and relations:

$$T,\, S: \quad T^3 = \mathbf{e}; \quad S^2 = \mathbf{e}; \quad (S\,T)^4 = \mathbf{e}. \tag{4.4}$$

On the other hand, O_{24} is a finite, discrete subgroup of the three-dimensional rotation group and any $\gamma \in \mathrm{O}_{24} \subset \mathrm{SO}(3)$ of its 24 elements can be uniquely identified by its action on the coordinates x, y, z, as it is displayed below:

e	1_1	$=$	$\{x, y, z\}$
C_3	2_1	$=$	$\{-y, -z, x\}$
	2_2	$=$	$\{-y, z, -x\}$
	2_3	$=$	$\{-z, -x, y\}$
	2_4	$=$	$\{-z, x, -y\}$
	2_5	$=$	$\{z, -x, -y\}$
	2_6	$=$	$\{z, x, y\}$
	2_7	$=$	$\{y, -z, -x\}$
	2_8	$=$	$\{y, z, x\}$
C_4^2	3_1	$=$	$\{-x, -y, z\}$
	3_2	$=$	$\{-x, y, -z\}$
	3_3	$=$	$\{x, -y, -z\}$

C_2	4_1	$=$	$\{-x, -z, -y\}$
	4_2	$=$	$\{-x, z, y\}$
	4_3	$=$	$\{-y, -x, -z\}$
	4_4	$=$	$\{-z, -y, -x\}$
	4_5	$=$	$\{z, -y, x\}$
	4_6	$=$	$\{y, x, -z\}$
C_4	5_1	$=$	$\{-y, x, z\}$
	5_2	$=$	$\{-z, y, x\}$
	5_3	$=$	$\{z, y, -x\}$
	5_4	$=$	$\{y, -x, z\}$
	5_5	$=$	$\{x, -z, y\}$
	5_6	$=$	$\{x, z, -y\}$

. (4.5)

As one sees from the above list, the 24 elements are distributed into 5 conjugacy classes mentioned in the first column of the table, according to a nomenclature which is standard in the chemical literature on crystallography. The relation between the abstract and concrete presentation of the octahedral group is obtained by identifying in the list (4.5) the generators T and S mentioned in Eq. (4.4). Explicitly, we have

$$T = 2_8 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad S = 4_6 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.6}$$

All other elements are reconstructed from the above two using the multiplication table of the group which is displayed below:

	1_1	2_1	2_2	2_3	2_4	2_5	2_6	2_7	2_8	3_1	3_2	3_3	4_1	4_2	4_3	4_4	4_5	4_6	5_1	5_2	5_3	5_4	5_5	5_6
1_1	1_1	2_1	2_2	2_3	2_4	2_5	2_6	2_7	2_8	3_1	3_2	3_3	4_1	4_2	4_3	4_4	4_5	4_6	5_1	5_2	5_3	5_4	5_5	5_6
2_1	2_1	2_5	2_4	3_3	3_2	1_1	3_1	2_6	2_3	2_7	2_2	2_8	5_3	4_4	5_6	4_6	5_4	4_2	4_1	4_3	5_1	5_5	4_5	5_2
2_2	2_2	2_6	2_3	1_1	3_1	3_3	3_2	2_5	2_4	2_8	2_1	2_7	4_5	5_2	5_5	5_4	4_6	4_1	4_2	5_1	4_3	5_6	5_3	4_4
2_3	2_3	3_2	1_1	2_2	2_8	2_7	2_1	3_3	3_1	2_4	2_6	2_5	4_6	5_1	5_3	5_6	4_1	4_5	5_2	4_2	5_5	4_4	4_3	5_4
2_4	2_4	3_1	3_3	2_1	2_7	2_8	2_2	1_1	3_2	2_3	2_5	2_6	5_4	4_3	4_5	5_5	4_2	5_3	4_4	4_1	5_6	5_2	5_1	4_6
2_5	2_5	1_1	3_2	2_8	2_2	2_1	2_7	3_1	3_3	2_6	2_4	2_3	5_1	4_6	5_2	4_2	5_5	4_4	5_3	5_6	4_1	4_5	5_4	4_3
2_6	2_6	3_3	3_1	2_7	2_1	2_2	2_8	3_2	1_1	2_5	2_3	2_4	4_3	5_4	4_4	4_1	5_6	5_2	4_5	5_5	4_2	5_3	4_6	5_1
2_7	2_7	2_3	2_6	3_1	1_1	3_2	3_3	2_4	2_5	2_1	2_8	2_2	5_2	4_5	4_2	5_1	4_3	5_6	5_5	5_4	4_6	4_1	4_4	5_3
2_8	2_8	2_4	2_5	3_2	3_3	3_1	1_1	2_3	2_6	2_2	2_7	2_1	4_4	5_3	4_1	4_3	5_1	5_5	5_6	4_6	5_4	4_2	5_2	4_5
3_1	3_1	2_8	2_7	2_6	2_5	2_4	2_3	2_2	2_1	1_1	3_3	3_2	5_6	5_5	4_6	5_3	5_2	4_3	5_4	4_5	4_4	5_1	4_2	4_1
3_2	3_2	2_7	2_8	2_5	2_6	2_3	2_4	2_1	2_2	3_3	1_1	3_1	5_5	5_6	5_4	4_5	4_4	5_1	4_6	5_3	5_2	4_3	4_1	4_2
3_3	3_3	2_2	2_1	2_4	2_3	2_6	2_5	2_8	2_7	3_2	3_1	1_1	4_2	4_1	5_1	5_2	5_3	5_4	4_3	4_4	4_5	4_6	5_6	5_5
4_1	4_1	5_4	4_6	4_5	5_3	5_2	4_4	5_1	4_3	5_5	5_6	4_2	1_1	3_3	2_8	2_6	2_3	2_2	2_7	2_5	2_4	2_1	3_1	3_2
4_2	4_2	4_6	5_4	5_3	4_5	4_4	5_2	4_3	5_1	5_6	5_5	4_1	3_3	1_1	2_7	2_5	2_4	2_1	2_8	2_6	2_3	2_2	3_2	3_1
4_3	4_3	5_3	5_2	5_6	4_2	5_5	4_1	4_5	4_4	4_6	5_1	5_4	2_6	2_4	1_1	2_8	2_7	3_1	3_2	2_2	2_1	3_3	2_5	2_3
4_4	4_4	4_2	5_5	5_1	5_4	4_6	4_3	5_6	4_1	5_2	4_5	5_3	2_8	2_1	2_6	1_1	3_2	2_5	2_3	3_1	3_3	2_4	2_2	2_7
4_5	4_5	5_6	4_1	4_6	4_3	5_1	5_4	4_2	5_5	5_3	4_4	5_2	2_2	2_7	2_4	3_2	1_1	2_3	2_5	3_3	3_1	2_6	2_8	2_1
4_6	4_6	4_4	4_5	4_1	5_5	4_2	5_6	5_2	5_3	4_3	5_4	5_1	2_3	2_5	3_1	2_1	2_2	1_1	3_3	2_7	2_8	3_2	2_4	2_6
5_1	5_1	4_5	4_4	5_5	4_1	5_6	4_2	5_3	5_2	5_4	4_3	4_6	2_5	2_3	3_3	2_7	2_8	3_2	3_1	2_1	2_2	1_1	2_6	2_4
5_2	5_2	4_1	5_6	4_3	4_6	5_4	5_1	5_5	4_2	4_4	5_3	4_5	2_7	2_2	2_5	3_3	3_1	2_6	2_4	3_2	1_1	2_3	2_1	2_8
5_3	5_3	5_5	4_2	5_4	5_1	4_3	4_6	4_1	5_6	4_5	5_2	4_4	2_1	2_8	2_3	3_1	3_3	2_4	2_6	1_1	3_2	2_5	2_7	2_2
5_4	5_4	5_2	5_3	4_2	5_6	4_1	5_5	4_4	4_5	5_1	4_6	4_3	2_4	2_6	3_2	2_2	2_1	3_3	1_1	2_8	2_7	3_1	2_3	2_5
5_5	5_5	4_3	5_1	4_4	5_2	5_3	4_5	4_6	5_4	4_1	4_2	5_6	3_2	3_1	2_2	2_4	2_5	2_8	2_1	2_3	2_6	2_7	3_3	1_1
5_6	5_6	5_1	4_3	5_2	4_4	4_5	5_3	5_4	4_6	4_2	4_1	5_5	3_1	3_2	2_1	2_3	2_6	2_7	2_2	2_4	2_5	2_8	1_1	3_3

(4.7)

This observation is important in relation with representation theory. Any linear representation of the group is uniquely specified by giving the matrix representation of the two generators $T = 2_8$ and $S = 4_6$. In the sequel this will be extensively utilized in the compact codification of the reducible representations that emerge in our calculations.

4.2. Irreducible Representations of the Octahedral Group. There are five conjugacy classes in O_{24} and therefore, according to theory, there are five irreducible representations of the same group, that we name D_i, $i = 1, \ldots, 5$. Let us briefly describe them.

4.2.1. D_1: *the Identity Representation.* The identity representation which exists for all groups is that one, where to each element of O we associate the number 1

$$\forall\,\gamma \in \mathrm{O}_{24} : \quad D_1(\gamma) = 1. \tag{4.8}$$

Obviously the character of such a representation is

$$\chi_1 = \{1, 1, 1, 1, 1\}. \tag{4.9}$$

4.2.2. D_2: the Quadratic Vandermonde Representation. The representation D_2 is also one-dimensional. It is constructed as follows. Consider the following polynomial of order six in the coordinates of a point in $\mathbb{R}^3$ or T^3:

$$\mathfrak{V}(x, y, z) = (x^2 - y^2)\,(x^2 - z^2)\,(y^2 - z^2). \tag{4.10}$$

As one can explicitly check under the transformations of the octahedral group listed in Eq. (4.5), the polynomial $\mathfrak{V}(x, y, z)$ is always mapped into itself modulo an overall sign. Keeping track of such a sign provides the form of the second one-dimensional representation whose character is explicitly calculated to be the following one:

$$\chi_1 = \{1, 1, 1, -1, -1\}. \tag{4.11}$$

4.2.3. D_3: the Two-Dimensional Representation. The representation D_3 is two-dimensional and it corresponds to a homomorphism

$$D_3 : \quad \mathrm{O}_{24} \to \mathrm{SL}(2, \mathbb{Z}), \tag{4.12}$$

which associates to each element of the octahedral group a 2×2 integer valued matrix of determinant one. The homomorphism is completely specified by giving the two matrices representing the two generators

$$D_3(T) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}; \quad D_3(S) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.13}$$

The character vector of D_3 is easily calculated from the above information and we have

$$\chi_3 = \{2, -1, 2, 0, 0\}. \tag{4.14}$$

4.2.4. D_4: the Three-Dimensional Defining Representation. The three-dimensional representation D_4 is simply the defining representation, where the generators T and S are given by the matrices in Eq. (4.6):

$$D_4(T) = T; \quad D_4(S) = S. \tag{4.15}$$

From this information the characters are immediately calculated, and we get

$$\chi_3 = \{3, 0, -1, -1, 1\}. \tag{4.16}$$

4.2.5. D_5: the Three-Dimensional Unoriented Representation. The three-dimensional representation D_5 is simply that where the generators T and S are given by the following matrices:

$$D_5(T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad D_5(S) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.17}$$

From this information, the characters are immediately calculated, and we get

$$\chi_5 = \{3, 0, -1, 1, -1\}. \tag{4.18}$$

The list of characters is summarized in Table 1.

Table 1. **Character table of the proper octahedral group**

Irrep \ Class	$\{\mathbf{e}, 1\}$	$\{C_3, 8\}$	$\{C_4^2, 3\}$	$\{C_2, 6\}$	$\{C_4, 6\}$
$D_1, \quad \chi_1 =$	1	1	1	1	1
$D_2, \quad \chi_2 =$	1	1	1	-1	-1
$D_3, \quad \chi_3 =$	2	-1	2	0	0
$D_4, \quad \chi_4 =$	3	0	-1	-1	1
$D_5, \quad \chi_5 =$	3	0	-1	1	-1

5. EXTENSION OF THE POINT GROUP WITH TRANSLATIONS AND MORE GROUP THEORY

We come now to what constitutes the main mathematical point of the present paper, namely, the extension of the point group with appropriate discrete subgroups of the compactified translation group $\mathrm{U}(1)^3$. This issue bears on a classical topic dating back to the XIX century, which was developed by crystallographers and, in particular, by the great Russian mathematician Fyodorov [22]. We refer here to the issue of space groups which historically resulted in the classification of the 230 crystallographic groups, well known in the chemical literature, for which an international system of notations and conventions has been established that is available in numerous encyclopedic tables and books [20]. Although we will utilize one key-point of the logic that leads to the classification of space groups, yet our goal happens to be slightly different and what we aim at is not the identification of space groups, rather the construction of what we name a *Universal Classifying Group*, namely of a group which contains all existing space groups as subgroups. We advocate that such a Universal Classifying Group is the one appropriate to organize the eigenfunctions of the $\star_g \mathrm{d}$ operator into irreducible representations and eventually to uncover the available hidden symmetries of all Arnold–Beltrami flows.

5.1. The Idea of Space Groups and Frobenius Congruences. The idea of space groups arises naturally in the following way. The covering manifold of the T^3 torus is $\mathbb{R}^3$ which can be regarded as the following coset manifold:

$$\mathbb{R}^3 \simeq \frac{\mathbb{E}^3}{\mathrm{SO}(3)}; \quad \mathbb{E}^3 \equiv \mathrm{ISO}(3) \doteq \mathrm{SO}(3) \ltimes \mathcal{T}^3, \tag{5.1}$$

where $\mathcal{T}^3$ is the three-dimensional translation group acting on $\mathbb{R}^3$ in the standard way

$$\forall \mathbf{t} \in \mathcal{T}^3, \forall \mathbf{x} \in \mathbb{R}^3 \quad | \quad \mathbf{t} : \mathbf{x} \to \mathbf{x} + \mathbf{t}, \tag{5.2}$$

and the Euclidean group $\mathbb{E}^3$ is the semidirect product of the proper rotation group $\mathrm{SO}(3)$ with the translation group $\mathcal{T}^3$. Harmonic analysis on $\mathbb{R}^3$ is a complicated matter of functional analysis since $\mathcal{T}^3$ is a noncompact group and its unitary irreducible representations are infinite-dimensional. The landscape changes drastically when we compactify our manifold from $\mathbb{R}^3$ to the three-torus T^3. Compactification is obtained taking the quotient of $\mathbb{R}^3$ with respect to the lattice $\Lambda \subset \mathcal{T}^3$. As a result of this quotient, the manifold becomes $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ but also the isometry group is reduced. Instead of $\mathrm{SO}(3)$ as rotation group we are left with its discrete subgroup $\mathfrak{P}_\Lambda \subset \mathrm{SO}(3)$ which maps the lattice Λ into itself (the point group) and instead of the translation subgroup $\mathcal{T}^3$ we are left with the quotient group:

$$\mathfrak{T}^3_\Lambda \equiv \frac{\mathcal{T}^3}{\Lambda} \simeq \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1). \tag{5.3}$$

In this way we obtain a new group which replaces the Euclidean group and which is the semidirect product of the point group $\mathfrak{P}_\Lambda$ with $\mathfrak{T}^3_\Lambda$

$$\mathfrak{G}_\Lambda \equiv \mathfrak{P}_\Lambda \ltimes \mathfrak{T}^3_\Lambda. \tag{5.4}$$

The group $\mathfrak{G}_\Lambda$ is an exact symmetry of the Beltrami equation (3.1) and its action is naturally defined on the parameter space of any of its solutions $\mathbf{V}(\mathbf{x}|\mathbf{F})$ that we can obtain by means of the algorithm described in Subsec. 3.1. To appreciate this point, let us recall that every component of the vector field $\mathbf{V}(\mathbf{x}|\mathbf{F})$ associated with a $\mathfrak{P}_\Lambda$ point–orbit $\mathcal{O}$ is a linear combination of the functions $\cos[2\pi\,\mathbf{k}_i \cdot \mathbf{x}]$ and $\sin[2\pi\,\mathbf{k}_i \cdot \mathbf{x}]$, where $\mathbf{k}_i \in \mathcal{O}$ are all the momentum vectors contained in the orbit. Consider next the same functions in a translated point of the three-torus $\mathbf{x}' = \mathbf{x} + \mathbf{c}$, where $\mathbf{c} = \{\xi_1, \xi_2, \xi_3\}$ is a representative of an equivalence class $\mathfrak{c}$ of constant vectors defined modulo the lattice

$$\mathfrak{c} = \mathbf{c} + \mathbf{y}; \quad \forall \mathbf{y} \in \Lambda. \tag{5.5}$$

The above equivalence classes are the elements of the quotient group $\mathfrak{T}^3_\Lambda$. Using standard trigonometric identities, $\cos\left[2\pi\,\mathbf{k}_i\cdot\mathbf{x}+2\pi\,\mathbf{k}_i\cdot\mathbf{c}\right]$ can be re-expressed as a linear combination of the $\cos\left[2\pi\,\mathbf{k}_i\cdot\mathbf{x}\right]$ and $\sin\left[2\pi\,\mathbf{k}_i\cdot\mathbf{x}\right]$ functions with coefficients that depend on trigonometric functions of c. The same is true for $\sin\left[2\pi\,\mathbf{k}_i\cdot\mathbf{x}+2\pi\,\mathbf{k}_i\cdot\mathbf{c}\right]$. Note also that because of the periodicity of the trigonometric functions, the shift in their argument by a lattice translation is not effective, so that one deals only with the equivalence classes (5.5). It follows that for each element $\mathfrak{c}\in\mathfrak{T}^3_\Lambda$, we obtain a matrix representation $\mathcal{M}_\mathfrak{c}$ realized on the F parameters and defined by the following equation:

$$\mathbf{V}\left(\mathbf{x}+\mathbf{c}|\mathbf{F}\right)=\mathbf{V}\left(\mathbf{x}|\mathcal{M}_\mathfrak{c}\mathbf{F}\right). \tag{5.6}$$

As we already noted in Eq. (3.23), for any group element $\gamma\in\mathfrak{P}_\Lambda$ we also have a matrix representation induced on the parameter space by the same mechanism:

$$\forall\,\gamma\in\mathfrak{P}_\Lambda:\quad \gamma^{-1}\cdot\mathbf{V}\left(\gamma\cdot\mathbf{x}\,|\,\mathbf{F}\right)=\mathbf{V}\left(\mathbf{x}\,|\,\mathfrak{R}[\gamma]\cdot\mathbf{F}\right). \tag{5.7}$$

Combining Eqs. (5.6) and (5.7), we obtain a matrix realization of the entire group $\mathfrak{G}_\Lambda$ in the following way:

$$\mathbf{V}\left(\gamma\cdot\mathbf{x}+\mathbf{c}|\mathbf{F}\right)=\gamma\cdot\mathbf{V}\left(\mathbf{x}\,|\,\mathfrak{R}[\gamma]\cdot\mathcal{M}_\mathfrak{c}\cdot\mathbf{F}\right), \tag{5.8}$$

$$\Downarrow$$

$$\forall\,(\gamma,\,\mathfrak{c})\in\mathfrak{G}_\Lambda\to D\left[(\gamma,\,\mathfrak{c})\right]=\mathfrak{R}[\gamma]\cdot\mathcal{M}_\mathfrak{c}. \tag{5.9}$$

Actually the construction of the Beltrami vector fields in the lowest-lying point-orbit, which usually yields a faithful matrix representation of all group elements, can be regarded as an automatic way of taking the quotient (5.3), and the resulting representation can be regarded as the defining representation of the group $\mathfrak{G}_\Lambda$.

The next point in the logic which leads to space groups is the following observation. $\mathfrak{G}_\Lambda$ is an unusual mixture of a discrete group (the point group) with a continuous one (the translation subgroup $\mathfrak{T}^3_\Lambda$). This latter is rather trivial, since its action corresponds to shifting the origin of coordinates in three-dimensional space and, from the point of view of the first-order differential system that defines trajectories (see Eq. (1.4)), it simply corresponds to varying the integration constants. Yet, in $\mathfrak{G}_\Lambda$ there are some discrete subgroups which can be isomorphic to the point group $\mathfrak{P}_\Lambda$, or to one of its subgroups $H_\Lambda\subset\mathfrak{P}_\Lambda$, without being their conjugate. Such subgroups cannot be disposed of by shifting the origin of coordinates and, consequently, they can encode nontrivial hidden symmetries of the dynamical system (1.4). The search for such nontrivial discrete subgroups of $\mathfrak{G}_\Lambda$ is the mission accomplished by crystallographers, the result of the mission being the classification of space groups.

The simplest and most intuitive way of constructing space-groups relies on the so-called Frobenius congruences [23, 24]. Let us outline this construction.

Following classical approaches, we use a 4×4 matrix representation of the group $\mathfrak{G}_\Lambda$

$$\forall\,(\gamma,\,\mathfrak{c}) \in \mathfrak{G}_\Lambda \;\rightarrow\; \hat{D}\,[(\gamma,\,\mathfrak{c})] = \left(\begin{array}{c|c} \gamma & \mathfrak{c} \\ \hline 0 & 1 \end{array}\right). \tag{5.10}$$

Performing the matrix product of two elements, in the translation block one has to take into account equivalence modulo lattice Λ, namely,

$$\left(\begin{array}{c|c} \gamma_1 & \mathfrak{c}_1 \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \gamma_2 & \mathfrak{c}_2 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} \gamma_1 \cdot \gamma_2 & \gamma_1\,\mathfrak{c}_2 + \mathfrak{c}_1 + \Lambda \\ \hline 0 & 1 \end{array}\right). \tag{5.11}$$

Utilizing this notation, the next step consists of introducing translation deformations of the generators of the point group searching for deformations that cannot be eliminated by conjugation. We illustrate the steps of such a construction utilizing the example of the cubic lattice and of the octahedral point group.

5.1.1. Frobenius Congruences for the Octahedral Group O_{24}. The octahedral group is abstractly defined by the presentation displayed in Eq. (4.4). As a first step we parameterize the candidate deformations of the two generators T and S in the following way:

$$\hat{T} = \left(\begin{array}{ccc|c} 0 & 1 & 0 & \tau_1 \\ 0 & 0 & 1 & \tau_2 \\ 1 & 0 & 0 & \tau_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right); \quad \hat{S} = \left(\begin{array}{ccc|c} 0 & 0 & 1 & \sigma_1 \\ 0 & -1 & 0 & \sigma_2 \\ 1 & 0 & 0 & \sigma_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \tag{5.12}$$

which should be compared with Eq. (4.6). Next we try to impose on the deformed generators the defining relations of O_{24}. By explicit calculation we find:

$$\hat{T}^3 = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \tau_1+\tau_2+\tau_3 \\ 0 & 1 & 0 & \tau_1+\tau_2+\tau_3 \\ 0 & 0 & 1 & \tau_1+\tau_2+\tau_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right); \quad \hat{S}^2 = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \sigma_1+\sigma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sigma_1+\sigma_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right);$$

$$\left(\hat{S}\hat{T}\right)^4 = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4\sigma_1+4\tau_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \tag{5.13}$$

so that we obtain the conditions

$$\tau_1+\tau_2+\tau_3 \in \mathbb{Z}; \quad \sigma_1+\sigma_3 \in \mathbb{Z}; \quad 4\sigma_1+4\tau_3 \in \mathbb{Z}, \tag{5.14}$$

which are the Frobenius congruences for the present case. Next we consider the effect of conjugation with the most general translation element of the group $\mathfrak{G}_{\mathrm{cubic}}$. Just for convenience, we parameterize the translation subgroup as follows:

$$\mathfrak{t} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & a+c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a-c \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \tag{5.15}$$

and we get

$$\mathfrak{t}\,\hat{T}\,\mathfrak{t}^{-1} = \left(\begin{array}{ccc|c} 0 & 1 & 0 & a-b+c+\tau_1 \\ 0 & 0 & 1 & -a+b+c+\tau_2 \\ 1 & 0 & 0 & \tau_3 - 2c \\ \hline 0 & 0 & 0 & 1 \end{array}\right) ; \mathfrak{t}\,\hat{S}\,\mathfrak{t}^{-1} = \left(\begin{array}{ccc|c} 0 & 0 & 1 & 2c+\sigma_1 \\ 0 & -1 & 0 & 2b+\sigma_2 \\ 1 & 0 & 0 & \sigma_3 - 2c \\ \hline 0 & 0 & 0 & 1 \end{array}\right). \tag{5.16}$$

This shows that by using the parameters b, c, we can always put $\sigma_1 = \sigma_2 = 0$, while using the parameter a we can put $\tau_1 = 0$ (this is obviously, not the only possible gauge choice, but it is the most convenient) so that Frobenius congruences reduce to

$$\tau_2 + \tau_3 \in \mathbb{Z}; \quad \sigma_3 \in \mathbb{Z}; \quad 4\tau_3 \in \mathbb{Z}. \tag{5.17}$$

Equation (5.17) is of great momentum. It tells us that any nontrivial subgroup of $\mathfrak{G}_{\mathrm{cubic}}$, which is not conjugate to the point group, contains point group elements extended with rational translations of the form $\mathfrak{c} = \left\{\frac{n_1}{4}, \frac{n_2}{4}, \frac{n_3}{4}\right\}$. Up to this point, our way and that of crystallographers was the same: hereafter our paths separate. The crystallographers classify all possible nontrivial groups that extend the point group with such translation deformations: indeed looking at the crystallographic tables one realizes that all known space groups for the cubic lattice have translation components of the form $\mathfrak{c} = \left\{\frac{n_1}{4}, \frac{n_2}{4}, \frac{n_3}{4}\right\}$. On the other hand, we do something much simpler which leads to a quite big group containing all possible space groups as subgroups, together with other subgroups that are not space groups in the crystallographic sense.

5.2. The Universal Classifying Group for the Cubic Lattice: G_{1536}**.** Inspired by the space-group construction and by Frobenius congruences we just consider the subgroup of $\mathfrak{G}_{\mathrm{cubic}}$, where translations are quantized in units of $\frac{1}{4}$. In each direction and modulo integers there are just four translations $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, so that the translation subgroup reduces to $\mathbb{Z}_4 \otimes \mathbb{Z}_4 \otimes \mathbb{Z}_4$ that has a total of 64 elements. In this way we single out a discrete subgroup $\mathrm{G}_{1536} \subset \mathfrak{G}_{\mathrm{cubic}}$ of order $24 \times 64 = 1536$, which is simply the semidirect product of the point group O_{24}

with $\mathbb{Z}_4 \otimes \mathbb{Z}_4 \otimes \mathbb{Z}_4$:

$$\mathfrak{G}_{\text{cubic}} \supset \mathrm{G}_{1536} \simeq \mathrm{O}_{24} \ltimes (\mathbb{Z}_4 \otimes \mathbb{Z}_4 \otimes \mathbb{Z}_4)\,. \tag{5.18}$$

We name G_{1536} the universal classifying group of the cubic lattice, and its elements can be labeled as follows:

$$\mathrm{G}_{1536} \in \left\{p_q,\, \frac{2n_1}{4},\, \frac{2n_2}{4},\, \frac{2n_3}{4}\right\} \Rightarrow \begin{cases} p_q \in \mathrm{O}_{24} \\ \left\{\dfrac{n_1}{4},\, \dfrac{n_2}{4},\, \dfrac{n_3}{4}\right\} \in \mathbb{Z}_4 \otimes \mathbb{Z}_4 \otimes \mathbb{Z}_4 \end{cases}, \tag{5.19}$$

where for the elements of the point group we use the labels p_q established in Eq. (4.5), while for the translation part our notation encodes an equivalence class of translation vectors $\mathfrak{c} = \left\{\frac{n_1}{4},\, \frac{n_2}{4},\, \frac{n_3}{4}\right\}$. The reason why we use $\left\{\frac{2n_1}{4},\, \frac{2n_2}{4},\, \frac{2n_3}{4}\right\}$ is simply due to computer convenience. In the quite elaborate MATHEMATICA codes that we have utilized to derive all our results, we internally used such a notation, and the automatic LaTeX Export of the outputs is provided in this way. In view of Eq. (5.9), we can associate an explicit matrix to each group element of G_{1536}, starting from the construction of the Beltrami vector field associated with one-point orbit of the octahedral group. We can consider such matrices as the defining representation of the group if the representation is faithful. We used the lowest-lying 6-dimensional orbit to be discussed in Sec. 7, which we verified to be indeed faithful. Three matrices are sufficient to characterize completely the defining representation just as any other representation: the matrix representing the generator T, the matrix representing the generator S, and the matrix representing the translation $\left\{\frac{n_1}{4},\, \frac{n_2}{4},\, \frac{n_3}{4}\right\}$. We have found

$$\begin{aligned} \mathfrak{R}^{\text{defi}}[T] &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}; \\ \mathfrak{R}^{\text{defi}}[S] &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \end{aligned} \tag{5.20}$$

$$\mathcal{M}^{\text{defi}}_{\{\frac{2n_1}{4},\frac{2n_2}{4},\frac{2n_3}{4}\}} = \\ = \begin{pmatrix} \cos\left(\frac{\pi}{2}n_3\right) & 0 & \sin\left(\frac{\pi}{2}n_3\right) & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\pi}{2}n_2\right) & 0 & 0 & -\sin\left(\frac{\pi}{2}n_2\right) & 0 \\ -\sin\left(\frac{\pi}{2}n_3\right) & 0 & \cos\left(\frac{\pi}{2}n_3\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\left(\frac{\pi}{2}n_1\right) & 0 & \sin\left(\frac{\pi}{2}n_1\right) \\ 0 & \sin\left(\frac{\pi}{2}n_2\right) & 0 & 0 & \cos\left(\frac{\pi}{2}n_2\right) & 0 \\ 0 & 0 & 0 & -\sin\left(\frac{\pi}{2}n_1\right) & 0 & \cos\left(\frac{\pi}{2}n_1\right) \end{pmatrix}. \tag{5.21}$$

Relying on the above matrices, any of the 1536 group elements obtains an explicit 6×6 matrix representation upon the use of formula (5.9). As is already stressed, we can regard that above as the actual definition of the group G_{1536} which from this point on can be studied intrinsically in terms of pure group theory without any further reference to lattices, Beltrami flows, or dynamical systems.

5.3. Structure of the G_{1536} Group and Derivation of Its Irreps. The identity card of a finite group is given by the organization of its elements into conjugacy classes, the list of its irreducible representation and finally its character table. Since ours is not any of the crystallographic groups, no explicit information is available in the literature about its conjugacy classes, its irreps, and its character table. We were forced to do everything from scratch by ourselves and we could accomplish the task by means of purposely written MATHEMATICA codes. Most of our results are presented in the form of tables in Appendices. Since this is a purely mathematical information, we think that it might be useful also in other contexts different from the present context that has motivated their derivation.

Conjugacy Classes. The conjugacy classes of G_{1536} are presented in Appendix A.1. There are 37 conjugacy classes whose populations are distributed as follows:

1) 2 classes of length 1,
2) 2 classes of length 3,
3) 2 classes of length 6,
4) 1 class of length 8,
5) 7 classes of length 12,
6) 4 classes of length 24,
7) 13 classes of length 48,
8) 2 classes of length 96,
9) 4 classes of length 128.

It follows that there must be 37 irreducible representations whose construction is a task to be solved.

5.3.1. Strategy to Construct the Irreducible Representations of a Solvable Group. In general, the derivation of the irreps and of the ensuing character table of a finite group G is a quite hard task. Yet a definite constructive algorithm can be devised if G is solvable and if one can establish a chain of normal subgroups ending with an Abelian one, whose index is, at each step, a prime number q_i, namely, if we have the following situation:

$$\begin{aligned} G = \mathrm{G}_{\mathrm{N_p}} \rhd \mathrm{G}_{\mathrm{N_{p-1}}} \rhd \ldots \rhd \mathrm{G}_{\mathrm{N_1}} \rhd \mathrm{G}_{\mathrm{N_0}} = \text{Abelian group}, \\ \left| \frac{\mathrm{G}_{\mathrm{N_i}}}{\mathrm{G}_{\mathrm{N_{i-1}}}} \right| = \frac{\mathrm{N_i}}{\mathrm{N_{i-1}}} \equiv q_i = \text{prime integer number}. \end{aligned} \tag{5.22}$$

The algorithm for the construction of the irreducible representations is based on an inductive procedure [23] that allows one to derive the irreps of the group $\mathrm{G}_{\mathrm{N_i}}$, if we know those of the group $\mathrm{G}_{\mathrm{N_{i-1}}}$ and if the index q_i is a prime number. The first step of the induction is immediately solved because any Abelian finite group is necessarily a direct product of cyclic groups $\mathbb{Z}_k$, whose irreps are all one-dimensional and obtained by assigning to their generator one of the kth roots of unity. In our case the index q_i is always either 2 or 3 which, to the none's wonder, is the same situation met in the construction of crystallographic group irreps. Hence we sketch the inductive algorithms with particular reference to the two cases of $q = 2$ and $q = 3$.

5.3.2. The Inductive Algorithm for Irreps. To simplify notation, we name $\mathcal{G} = \mathrm{G}_{\mathrm{N_i}}$ and $\mathcal{H} = \mathrm{G}_{\mathrm{N_{i-1}}}$. By hypothesis, $\mathcal{H} \lhd \mathcal{G}$ is a normal subgroup. Furthermore, $q \equiv |\mathcal{G}/\mathcal{H}| = \textit{prime number}$ (in particular, $q = 2, \text{or}\, 3$). Let us name $D_\alpha\,[\mathcal{H}, d_\alpha]$ the irreducible representations of the subgroup. The index α (with $\alpha = 1, \ldots, r_H \equiv \#$ of conj. classes of $\mathcal{H}$) enumerates them. In each case d_α denotes the dimension of the corresponding carrying vector space or, in mathematical jargon, of the corresponding module.

The first step to be taken is to distribute the $\mathcal{H}$ irreps into conjugation classes with respect to the bigger group. Conjugation classes of irreps are defined as follows. First, one observes that, given an irreducible representation $D_\alpha\,[\mathcal{H}, d_\alpha]$, for every $g \in \mathcal{G}$ we can create another irreducible representation $D^{(g)}_\alpha\,[\mathcal{H}, d_\alpha]$, named the conjugate of $D_\alpha\,[\mathcal{H}, d_\alpha]$ with respect to g. The new representation is as follows:

$$\forall\, h \in \mathcal{H} : \quad D^{(g)}_\alpha\,[\mathcal{H}, d_\alpha]\,(h) = D_\alpha\,[\mathcal{H}, d_\alpha]\,(g^{-1}\,h\,g). \tag{5.23}$$

That the one defined above is a homomorphism of $\mathcal{H}$ onto $\mathrm{GL}(\mathrm{d}_\alpha, \mathbb{R})$ is obvious and, as a consequence, it is also obvious that the new representation has the same dimension as the first one. Secondly, if $g = \widetilde{h} \in \mathcal{H}$ is an element of the subgroup, we get

$$D_\alpha^{(\widetilde{h})}\left[\mathcal{H}, d_\alpha\right](h) = A^{-1}\, D_\alpha\left[\mathcal{H}, d_\alpha\right](h)\, A, \quad \text{where} \quad A = D_\alpha\left[\mathcal{H}, d_\alpha\right](\widetilde{h}), \tag{5.24}$$

so that conjugation amounts simply to a change of basis (a similarity transformation) inside the same representation. This does not alter the character vector, and the new representation is equivalent to the old one. Hence the only nontrivial conjugations to be considered are those with respect to representatives of the different equivalence classes in $\mathcal{G}/\mathcal{H}$. Let us name γ_i, $(i = 0, \ldots, q-1)$ a set of representatives of such equivalence classes and define the orbit of each irrep $D_\alpha\left[\mathcal{H}, d_\alpha\right]$ as follows:

$$\text{Orbit}_\alpha \equiv \left\{ D_\alpha^{(\gamma_0)}\left[\mathcal{H}, d_\alpha\right],\, D_\alpha^{(\gamma_1)}\left[\mathcal{H}, d_\alpha\right],\, \ldots,\, D_\alpha^{(\gamma_{q-1})}\left[\mathcal{H}, d_\alpha\right] \right\}. \tag{5.25}$$

Since the available irreducible representations are a finite set, every $D_\alpha^{(\gamma_i)}\left[\mathcal{H}, d_\alpha\right]$ necessarily is identified with one of the existing $D_\beta\left[\mathcal{H}, d_\beta\right]$. Furthermore, since conjugation preserves the dimension, it follows that $d_\alpha = d_\beta$. It follows that $\mathcal{H}$-irreps of the same dimensions d arrange themselves into $\mathcal{G}$-orbits:

$$\text{Orbit}_\alpha[d] = \left\{ D_{\alpha_1}\left[\mathcal{H}, d\right],\, D_{\alpha_2}\left[\mathcal{H}, d\right],\, \ldots,\, D_{\alpha_q}\left[\mathcal{H}, d\right] \right\}, \tag{5.26}$$

and there are only two possibilities, either all $\alpha_i = \alpha$ are equal (self-conjugate representations) or they are all different (nonconjugate representations).

Once the irreps of $\mathcal{H}$ have been organized into conjugation orbits, we can proceed to promote them to irreps of the big group $\mathcal{G}$ according to the following scheme:

A) Each self-conjugate $\mathcal{H}$-irrep $D_\alpha\left[\mathcal{H}, d\right]$ is uplifted to q distinct irreducible $\mathcal{G}$-representations of the same dimension d, namely $D_{\alpha_i}\left[\mathcal{G}, d\right]$, where $i = 1, \ldots, q$.

B) From each orbit β of q distinct but conjugate $\mathcal{H}$-irreps $\left\{ D_{\alpha_1}\left[\mathcal{H}, d\right],\, D_{\alpha_2}\left[\mathcal{H}, d\right],\, \ldots,\, D_{\alpha_q}\left[\mathcal{H}, d\right] \right\}$, one extracts a single $(q \times d)$-dimensional $\mathcal{G}$-representation.

A) Uplifting of Self-Conjugate Representations. Let $D_\alpha\left[\mathcal{H}, d\right]$ be a self-conjugate irrep. If the index q of the normal subgroup is a prime number, this means that $\mathcal{G}/\mathcal{H} \simeq \mathbb{Z}_q$. In this case the representatives γ_j of the q equivalences classes that form the quotient group can be chosen in the following way:

$$\gamma_1 = \mathbf{e},\ \gamma_2 = g,\ \gamma_3 = g^2,\ \ldots,\ \gamma_q = g^{q-1}, \tag{5.27}$$

where $g \in \mathcal{G}$ is a single group element satisfying $g^q = \mathbf{e}$. The key-point in uplifting the representation $D_\alpha\left[\mathcal{H}, d\right]$ to the bigger group resides in the determination of a $d \times d$ matrix U that should satisfy the following constraints:

$$U^q = \mathbf{1}, \tag{5.28}$$

$$\forall\, h \in \mathcal{H} : \quad D_\alpha\left[\mathcal{H}, d\right]\left(g^{-1}\, h\, g\right) = U^{-1}\, D_\alpha\left[\mathcal{H}, d\right](h)\ U. \tag{5.29}$$

These algebraic equations have exactly q distinct solutions $U_{[j]}$, and each of the solutions leads to one of the irreducible $\mathcal{G}$-representations induced by $D_\alpha\,[\mathcal{H},d]$. Any element $\gamma\in\mathcal{G}$ can be written as $\gamma = g^p\,h$, with $p=0,1,\ldots,q-1$ and $h\in\mathcal{H}$. Then it suffices to write

$$D_{a_j}\,[\mathcal{G},d]\,(\gamma) = D_{a_j}\,[\mathcal{G},d]\,(g^p\,h) = U_{[j]}^p\,D_\alpha\,[\mathcal{H},d]\,(h). \tag{5.30}$$

B) Uplifting of Not Self-Conjugate Representations. In the case of not self-conjugate representations, the induced representation of dimensions $q\times d$ is constructed relying once again on the possibility to write all group elements in the form $\gamma = g^p\,h$, with $p=0,1,\ldots,q-1$ and $h\in\mathcal{H}$. Furthermore chosen one representation $D_\alpha\,[\mathcal{H},d]$ in the q-orbit (5.25), the other members of the orbit can be represented as $D_\alpha^{(g^j)}\,[\mathcal{H},d_\alpha]$, with $j=1,\ldots,q-1$. In view of this, one writes:

$$\forall\,h\in\mathcal{H}:\;D_\alpha\,[\mathcal{G},d]\,(h) =$$

$$= \left(\begin{array}{c|c|c|c|c}
D_\alpha\,[\mathcal{H},d]\,(h) & 0 & 0 & \ldots & 0\\ \hline
0 & D_\alpha^{(g)}\,[\mathcal{H},d]\,(h) & 0 & \ldots & 0\\ \hline
0 & 0 & D_\alpha^{(g^2)}\,[\mathcal{H},d]\,(h) & \ldots & 0\\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots\\ \hline
0 & 0 & \ldots & 0 & D_\alpha^{(g^{q-1})}\,[\mathcal{H},d]\,(h)
\end{array}\right),$$

$$g:\;D_\alpha\,[\mathcal{G},d]\,(g) = \left(\begin{array}{c|c|c|c|c}
0 & 0 & 0 & \ldots & \mathbf{1}\\ \hline
\mathbf{1} & 0 & 0 & \ldots & 0\\ \hline
0 & \mathbf{1} & 0 & \ldots & 0\\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots\\ \hline
0 & 0 & \ldots & \mathbf{1} & 0
\end{array}\right), \tag{5.31}$$

$$\gamma = g^p\,h:\;D_\alpha\,[\mathcal{G},d]\,(g) = \left(D_\alpha\,[\mathcal{G},d]\,(g)\right)^p\,D_\alpha\,[\mathcal{G},d]\,(h)\,.$$

5.3.3. Derivation of G_{1536} *Irreps.* Utilizing the above-described algorithm, implemented by means of purposely written MATHEMATICA codes, we were able to derive the explicit form of the 37 irreducible representations of G_{1536} and its character table. The essential tool is the following chain of normal subgroups:

$$\mathrm{G}_{1536}\,\triangleright\,\mathrm{G}_{768}\,\triangleright\,\mathrm{G}_{256}\,\triangleright\,\mathrm{G}_{128}\,\triangleright\,\mathrm{G}_{64}, \tag{5.32}$$

where $\mathrm{G}_{64}\sim\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_4$ is Abelian and corresponds to the compactified translation group. The above chain leads to the following quotient groups:

$$\frac{\mathrm{G}_{1536}}{\mathrm{G}_{768}}\sim\mathbb{Z}_2;\quad \frac{\mathrm{G}_{768}}{\mathrm{G}_{256}}\sim\mathbb{Z}_3;\quad \frac{\mathrm{G}_{256}}{\mathrm{G}_{128}}\sim\mathbb{Z}_2;\quad \frac{\mathrm{G}_{128}}{\mathrm{G}_{64}}\sim\mathbb{Z}_2. \tag{5.33}$$

The description of the normal subgroups is given in Appendix in Secs. A.2, A.3, A.4, A.5.

The result for the irreducible representations, thoroughly described in Appendix B.1, is summarized here. The 37 irreps are distributed according to the following pattern:

a) 4 irreps of dimension 1, namely $D_1, \ldots, D_4$,
b) 2 irreps of dimension 2, namely $D_5, \ldots, D_6$,
c) 12 irreps of dimension 3, namely $D_6, \ldots, D_{18}$,
d) 10 irreps of dimension 6, namely $D_7, \ldots, D_{28}$,
e) 3 irreps of dimension 8, namely $D_{29}, \ldots, D_{31}$,
f) 6 irreps of dimension 12, namely $D_{32}, \ldots, D_{37}$.

The character table is displayed in Eqs. (B.1), (B.2).

The irreducible representations of the universal classifying group are a fundamental tool in our classification of the Arnold–Beltrami vector fields. Indeed, by choosing the various point group orbits of momentum vectors in the cubic lattice, according to their classification presented in the next Sec. 6, and constructing the corresponding Arnold–Beltrami fields, we obtain all of the 37 irreducible representations of G_{1536}. Each representation appears at least once and some of them appear several times. Considering next the subgroups $\mathcal{H}_i$ of G_{1536} and the branching rules of G_{1536} irreps with respect to H_i, we obtain an explicit algorithm to construct the Arnold–Beltrami vector fields with prescribed invariance groups H_i. It suffices to select the identity representation of the subgroup in the branching rules. *These are the hidden symmetries advocated in our title.* We come back to the issue of subgroups in the next and following sections.

6. THE SPHERICAL LAYERS AND THE OCTAHEDRAL LATTICE ORBITS

Let us now analyze the action of the octahedral group on the cubic lattice. We define the orbits as the sets of vectors $\mathbf{k} \in \Lambda$ that can be mapped one into the other by the action of some element of the point group, namely of O_{24}, in the case of the cubic lattice:

$$\mathbf{k}_1 \in \mathcal{O} \quad \text{and} \quad \mathbf{k}_2 \in \mathcal{O} \quad \Rightarrow \quad \exists\, \gamma \in \mathrm{O}_{24} \,/\, \gamma \cdot \mathbf{k}_1 = \mathbf{k}_2. \tag{6.1}$$

There are four types of orbits on the cubic lattice:

Orbits of Length 6. Each of these orbits is of the following form:

$$\mathcal{O}_6 = \Big\{ \{0,0,-n\},\ \{0,0,n\},\ \{0,-n,0\},\ \{0,n,0\},\ \{-n,0,0\},\ \{n,0,0\} \Big\}\,, \tag{6.2}$$

where $n \in \mathbb{Z}$ is any integer number.

Orbits of Length 8.

$$\mathcal{O}_8 = \left\{ \begin{array}{llll} \{-n,-n,-n\}, & \{-n,-n,n\}, & \{-n,n,-n\}, & \{-n,n,n\}, \\ \{n,-n,-n\}, & \{n,-n,n\}, & \{n,n,-n\}, & \{n,n,n\} \end{array} \right\}, \quad (6.3)$$

where $n \in \mathbb{Z}$ is any integer number.

Orbits of Length 12.

$$\mathcal{O}_{12} = \left\{ \begin{array}{llll} \{0,-n,-n\}, & \{0,-n,n\}, & \{0,n,-n\}, & \{0,n,n\}, \\ \{-n,0,-n\}, & \{-n,0,n\}, & \{-n,-n,0\}, & \{-n,n,0\}, \\ \{n,0,-n\}, & \{n,0,n\}, & \{n,-n,0\}, & \{n,n,0\} \end{array} \right\}, \quad (6.4)$$

where $n \in \mathbb{Z}$ is any integer number.

Orbits of Length 24.

$$\mathcal{O}_{24} = \left\{ \begin{array}{llll} \{-p,-q,r\}, & \{-p,q,-r\}, & \{-p,-r,-q\}, & \{-p,r,q\}, \\ \{p,-q,-r\}, & \{p,q,r\}, & \{p,-r,q\}, & \{p,r,-q\}, \\ \{-q,-p,-r\}, & \{-q,p,r\}, & \{-q,-r,p\}, & \{-q,r,-p\}, \\ \{q,-p,r\}, & \{q,p,-r\}, & \{q,-r,-p\}, & \{q,r,p\}, \\ \{-r,-p,q\}, & \{-r,p,-q\}, & \{-r,-q,-p\}, & \{-r,q,p\}, \\ \{r,-p,-q\}, & \{r,p,q\}, & \{r,-q,p\}, & \{r,q,-p\}, \end{array} \right\}, \quad (6.5)$$

where $\{p,\, q,\, r\} \in \mathbb{Z}$ is any triplet of integer numbers that are not all three-equal in absolute value. Considering the spherical layers of increasing quantized squared radius r^2, we discover that for the first low-lying layers, the points lying on the surface arrange themselves into just one orbit. At $r^2 = 9$, we observe the first splitting of the layer into two orbits, one of length 6, the other of length 24. Such splittings occur again and again with more and more orbits populating the same spherical surface. Yet single orbits appear also at higher values of r^2 as is shown in Table 2. The notation adopted in the figure is the following one: $\{O_i^{r^2}, \ell\}$ denotes the orbit of length $\ell = 6, 8, 12,$ or 24, of momentum lattice points whose norm is $\mathbf{k}^2 = r^2$. The index i enumerates the individual orbits placed on the sphere of radius r. Predictions of splittings and of orbit degeneracies require investigations in number theory and diophantine equations that we have not addressed within the scope of the present paper. A visual image of the complicated pattern produced by the distribution of orbits with respect to the quantized radius is provided in Fig. 4.

6.1. Classification of the 48 **Types of Orbits.** Notwithstanding the number theory complicacies mentioned above, the notion of Universal Classifying Group introduces a very effective guide-line to tame the zoo of point orbits displayed

Table 2. **Table of the first spherical layers of momentum vectors in the cubic self-dual lattice**

r^2	Number of Points	Octahedral Point Group Orbits
0	1	$\{\{1,1\}\}$
1	6	$\{\{O_1,6\}\}$
2	12	$\left\{\left\{O_1^2,12\right\}\right\}$
3	8	$\left\{\left\{O_1^3,8\right\}\right\}$
4	6	$\left\{\left\{O_1^4,6\right\}\right\}$
5	24	$\left\{\left\{O_1^5,24\right\}\right\}$
6	24	$\left\{\left\{O_1^6,24\right\}\right\}$
8	12	$\left\{\left\{O_1^8,12\right\}\right\}$
9	30	$\left\{\left\{O_1^9,6\right\}\oplus\left\{O_2^9,24\right\}\right\}$
10	24	$\left\{\left\{O_1^{10},24\right\}\right\}$
11	24	$\left\{\left\{O_1^{11},24\right\}\right\}$
12	8	$\left\{\left\{O_1^{12},8\right\}\right\}$
13	24	$\left\{\left\{O_1^{13},24\right\}\right\}$
14	48	$\left\{\left\{O_1^{14},24\right\}\oplus\left\{O_2^{14},24\right\}\right\}$
16	6	$\left\{\left\{O_1^{16},6\right\}\right\}$
17	48	$\left\{\left\{O_1^{17},24\right\}\oplus\left\{O_2^{17},24\right\}\right\}$
18	36	$\left\{\left\{O_1^{18},24\right\}\oplus\left\{O_2^{18},12\right\}\right\}$
19	24	$\left\{\left\{O_1^{19},24\right\}\right\}$
20	24	$\left\{\left\{O_1^{20},24\right\}\right\}$
21	48	$\left\{\left\{O_1^{21},24\right\}\oplus\left\{O_2^{21},24\right\}\right\}$
22	24	$\left\{\left\{O_1^{22},24\right\}\right\}$
24	24	$\left\{\left\{O_1^{24},24\right\}\right\}$
25	30	$\left\{\left\{O_1^{25},6\right\}\oplus\left\{O_2^{25},24\right\}\right\}$
26	72	$\left\{\left\{O_1^{26},24\right\}\oplus\left\{O_2^{26},24\right\}\oplus\left\{O_3^{26},24\right\}\right\}$
27	32	$\left\{\left\{O_1^{27},24\right\}\oplus\left\{O_2^{27},8\right\}\right\}$
29	72	$\left\{\left\{O_1^{29},24\right\}\oplus\left\{O_2^{29},24\right\}\oplus\left\{O_3^{29},24\right\}\right\}$
30	48	$\left\{\left\{O_1^{30},24\right\}\oplus\left\{O_2^{30},24\right\}\right\}$
32	12	$\left\{\left\{O_1^{32},12\right\}\right\}$
33	48	$\left\{\left\{O_1^{33},24\right\}\oplus\left\{O_2^{33},24\right\}\right\}$
34	48	$\left\{\left\{O_1^{34},24\right\}\oplus\left\{O_2^{34},24\right\}\right\}$
35	48	$\left\{\left\{O_1^{35},24\right\}\oplus\left\{O_2^{35},24\right\}\right\}$
36	30	$\left\{\left\{O_1^{36},6\right\}\oplus\left\{O_2^{36},24\right\}\right\}$

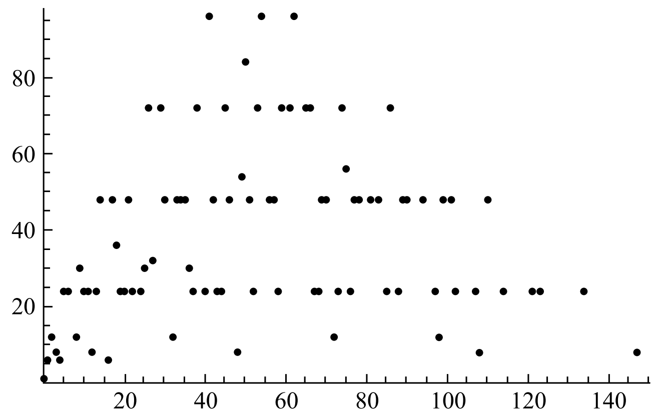

Fig. 4. A view of the distribution of the number of lattice points lying on a surface of squared radius r^2. On the horizontal axis we have r^2, on the vertical axis we have the number of points lying on that sphere. As one sees, the distribution is very much irregular and follows a complicated pattern

in Fig. 4. The first observation is that the group G_{1536} has a finite number of irreducible representations so that, irrespectively of the above-complicated pattern, the number of different types of the Arnold–Beltrami vector fields has also got to be finite, namely, as many as the 37 irreps, times the number of different ways to obtain them from orbits of length $6, 8, 12$, or 24. The second observation is the key role of the number 4 introduced by Frobenius congruences which was already the clue to the definition of G_{1536}. What we should expect is that the various orbits should be defined with integers modulo 4, in other words, that we should just consider the possible octahedral orbits on a lattice with coefficients in $\mathbb{Z}_4$ rather than $\mathbb{Z}$. The easy guess, which is confirmed by computer calculations, is that the pattern of G_{1536} representations obtained from the construction of the Arnold–Beltrami vector fields according to the algorithm of Subsec. 3.1 depends only on the equivalence classes of momentum orbits modulo 4. Hence we have a finite number of such orbits and a finite number of the Arnold–Beltrami vector fields which we presently described. Let us stress that an embryo of the exhaustive classification of orbits we are going to present was introduced by Arnold in his paper [8], Arnold's was only an embryo of the complete classification for the following two reasons:

1. The types of momenta orbits were partitioned according to *odd* and *even* (namely according to $\mathbb{Z}_2$, rather than $\mathbb{Z}_4$).

2. The classifying group was taken to be the crystallographic GS_{24}, as defined by us in Appendices (see Sec. A.9), which is too small in comparison with the universal classifying group identified by us in G_{1536}.

Let us then present the complete classification of point orbits in the momentum lattice. First, we subdivide the momenta into five groups:

A) Momenta of type $\{a, 0, 0\}$ which generate O_{24} orbits of length 6 and representations of the universal group G_{1536} also of dimensions 6.

B) Momenta of type $\{a, a, a\}$ which generate O_{24} orbits of length 8 and representations of the universal group G_{1536} also of dimensions 8.

C) Momenta of type $\{a, a, 0\}$ which generate O_{24} orbits of length 12 and representations of the universal group G_{1536} also of dimensions 12.

D) Momenta of type $\{a, a, b\}$ which generate O_{24} orbits of length 24 and representations of the universal group G_{1536} also of dimensions 24.

E) Momenta of type $\{a, b, c\}$ which generate O_{24} orbits of length 24 and representations of the universal group G_{1536} of dimensions 48.

The reason why in the cases A)... D) the dimension of the representation $\mathfrak{R}(\mathrm{G}_{1536})$ coincides with the dimension $|\mathcal{O}|$ of the orbit is simple. For each momentum in the orbit ($\forall \mathbf{k}_i \in \mathcal{O}$) its negative is also in the same orbit ($-\mathbf{k}_i \in \mathcal{O}$), hence the number of arguments $\Theta_i \equiv 2\pi\,\mathbf{k}_i \cdot \mathbf{x}$ of the independent trigonometric functions $\sin(\Theta_i)$ and $\cos(\Theta_i)$ is $\frac{1}{2} \times 2|\mathcal{O}| = |\mathcal{O}|$ since $\sin(\pm\Theta_i) = \pm\sin(\Theta_i)$ and $\cos(\pm\Theta_i) = \cos(\Theta_i)$.

In case E), instead, the negatives of all the members of the orbit $\mathcal{O}$ are not in $\mathcal{O}$. The number of independent trigonometric functions is therefore 48 and such is the dimension of the representation $\mathfrak{R}(\mathrm{G}_{1536})$.

In each of the five groups, one still has to reduce the entries to $\mathbb{Z}_4$, namely, to consider their equivalence class $\operatorname{mod} 4$. Each different choice of the pattern of $\mathbb{Z}_4$ classes appearing in an orbit leads to a different decomposition of the representation into irreducible representation of G_{1536}. A simple consideration of the combinatorics leads to the conclusion that there are in total 48 cases to be considered. The very significant result is that all of the 37 irreducible representations of G_{1536} appear at least once in the list of these decompositions. Hence for all the *irrepses* of this group, one can find a corresponding Beltrami field and for some *irrepses* such a Beltrami field admits a few inequivalent realizations. The list of the 48 distinct types of momenta is the following one:

1. $\mathbf{k} = \{0, 0, 1 + 4\rho\}$,
2. $\mathbf{k} = \{0, 0, 2 + 4\rho\}$,
3. $\mathbf{k} = \{0, 0, 3 + 4\rho\}$,
4. $\mathbf{k} = \{0, 0, 4 + 4\rho\}$,
5. $\mathbf{k} = \{1 + 4\mu, 1 + 4\mu, 1 + 4\mu\}$,
6. $\mathbf{k} = \{2 + 4\mu, 2 + 4\mu, 2 + 4\mu\}$,
7. $\mathbf{k} = \{3 + 4\mu, 3 + 4\mu, 3 + 4\mu\}$,
8. $\mathbf{k} = \{4 + 4\mu, 4 + 4\mu, 4 + 4\mu\}$,
9. $\mathbf{k} = \{0, 1 + 4\nu, 1 + 4\nu\}$,

10. $\mathbf{k} = \{0, 2+4\nu, 2+4\nu\}$,
11. $\mathbf{k} = \{0, 3+4\nu, 3+4\nu\}$,
12. $\mathbf{k} = \{0, 4+4\nu, 4+4\nu\}$,
13. $\mathbf{k} = \{1+4\mu, 1+4\mu, 2+4\rho\}$,
14. $\mathbf{k} = \{1+4\mu, 1+4\mu, 3+4\rho\}$,
15. $\mathbf{k} = \{1+4\mu, 1+4\mu, 4+4\rho\}$,
16. $\mathbf{k} = \{1+4\mu, 1+4\mu, 5+4\rho\}$,
17. $\mathbf{k} = \{1+4\mu, 2+4\mu, 2+4\rho\}$,
18. $\mathbf{k} = \{2+4\mu, 2+4\mu, 6+4\rho\}$,
19. $\mathbf{k} = \{2+4\mu, 2+4\mu, 3+4\rho\}$,
20. $\mathbf{k} = \{2+4\mu, 2+4\mu, 4+4\rho\}$,
21. $\mathbf{k} = \{1+4\mu, 3+4\mu, 3+4\rho\}$,
22. $\mathbf{k} = \{2+4\mu, 3+4\mu, 3+4\rho\}$,
23. $\mathbf{k} = \{3+4\mu, 3+4\mu, 7+4\rho\}$,
24. $\mathbf{k} = \{1+4\mu, 4+4\mu, 4+4\rho\}$,
25. $\mathbf{k} = \{2+4\mu, 4+4\mu, 4+4\rho\}$,
26. $\mathbf{k} = \{3+4\mu, 4+4\mu, 4+4\rho\}$,
27. $\mathbf{k} = \{4+4\mu, 4+4\mu, 8+4\rho\}$,
28. $\mathbf{k} = \{3+4\mu, 3+4\mu, 4+4\rho\}$,
29. $\mathbf{k} = \{4+4\mu, 8+4\nu, 12+4\rho\}$,
30. $\mathbf{k} = \{1+4\mu, 4+4\nu, 8+4\rho\}$,
31. $\mathbf{k} = \{2+4\mu, 4+4\nu, 8+4\rho\}$,
32. $\mathbf{k} = \{3+4\mu, 4+4\nu, 8+4\rho\}$,
33. $\mathbf{k} = \{1+4\mu, 2+4\nu, 4+4\rho\}$,
34. $\mathbf{k} = \{1+4\mu, 3+4\nu, 4+4\rho\}$,
35. $\mathbf{k} = \{2+4\mu, 4+4\nu, 6+4\rho\}$,
36. $\mathbf{k} = \{2+4\mu, 3+4\nu, 4+4\rho\}$,
37. $\mathbf{k} = \{1+4\mu, 5+4\nu, 9+4\rho\}$,
38. $\mathbf{k} = \{1+4\mu, 2+4\nu, 5+4\rho\}$,
39. $\mathbf{k} = \{1+4\mu, 3+4\nu, 5+4\rho\}$,
40. $\mathbf{k} = \{1+4\mu, 2+4\nu, 6+4\rho\}$,
41. $\mathbf{k} = \{1+4\mu, 2+4\nu, 3+4\rho\}$,
42. $\mathbf{k} = \{1+4\mu, 3+4\nu, 7+4\rho\}$,
43. $\mathbf{k} = \{2+4\mu, 6+4\nu, 10+4\rho\}$,
44. $\mathbf{k} = \{2+4\mu, 3+4\nu, 6+4\rho\}$,
45. $\mathbf{k} = \{2+4\mu, 3+4\nu, 7+4\rho\}$,

46. $\mathbf{k} = \{3 + 4\mu, 7 + 4\nu, 11 + 4\rho\}$,
47. $\mathbf{k} = \{1 + 4\mu, 4 + 4\nu, 5 + 4\rho\}$,
48. $\mathbf{k} = \{3 + 4\mu, 4 + 4\nu, 7 + 4\rho\}$,

where $\mu, \nu, \rho \in \mathbb{Z}$. The simplest and lowest-lying representative of each of the 48 classes of equivalent momenta is obtained choosing $\mu = \nu = 0$. In Appendix C, for each of the 48 classes enumerated above, we provide the decomposition of the corresponding Beltrami vector field parameter space into G_{1536} irreducible representations. These results are the outcome of extensive MATHEMATICA calculations that were performed with purposely written codes. As is already stressed, the most relevant point is that all the 37 irreps of the Classifying Group are reproduced: this is the main reason for its name.

7. DISCUSSION OF EXPLICIT EXAMPLES OF ARNOLD–BELTRAMI FLOWS FROM OCTAHEDRAL POINT ORBITS

In this section, utilizing the algorithm outlined in Subsec. 3.1, we construct the Arnold–Beltrami flows associated with some of the 48 types of octahedral point orbits in the cubic lattice that have been classified in Sec. 6. We consider examples with orbits of length $6, 8, 12$, and 24. In all cases we devote attention to the group structure and we exhibit the explicit form of the Arnold–Beltrami flows that have the largest possible group of symmetries available in that orbit. For some of these examples, we also exhibit computer generated plots of the vector field and of its associated streamlines.

In view of the popularity of the ABC-flows in the hydrodynamical literature, particularly in depth analysis, the lowest-lying octahedral orbit of length 6 is presented in which these models are embedded. Our main concern is to unveil the group theoretical structure behind the celebrated simple form of Eq. (1.17), which so far seemed to be a sort of miraculous arbitrary invention. In particular, we spot the subgroup of the Universal Classifying Group with respect to which the three parameters (A, B, C) form an irreducible representation. Similarly, we exhibit the groups and subgroups associated with the various popularly studied subcases (A, A, A), $(A, B, 0)$, $(A, A, 0)$, and $(A, 0, 0)$.

For the other considered orbits of length 8, 12, 24, we provide a similar group theoretical analysis although less detailed, since, as we already stated above, we mainly confine ourselves to the construction of the Arnold–Beltrami flows with the largest group of hidden symmetries.

7.1. The Lowest-Lying Octahedral Orbit of Length 6 in the Cubic Lattice and the ABC-Flows. Let us now consider case 1 in our list of 48 classes of momentum vectors. If we take the representative $\rho = 0$ of the class, we obtain the lowest-lying orbit of length 6 of the cubic lattice. Under the action of the

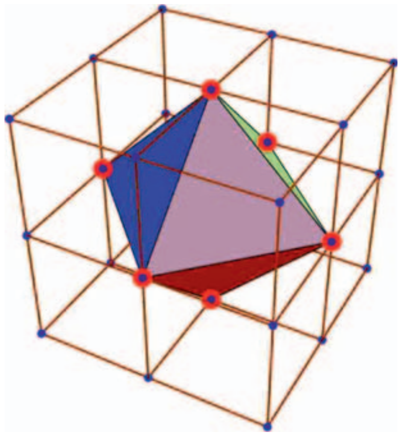

Fig. 5. A view of the octahedral 6-orbit in the cubic lattice, corresponding to the vertices of a regular octahedron

octahedral point group O_{24}, the vector $\mathbf{k} = \{0,0,1\}$ is mapped into its other five copies that together with it constitute the vertices

$$\begin{aligned} p_1 &= \{-1,0,0\}; \quad p_4 = \{0,0,1\}, \\ p_2 &= \{0,-1,0\}; \quad p_5 = \{0,1,0\}, \\ p_3 &= \{0,0,-1\}; \quad p_6 = \{1,0,0\} \end{aligned} \tag{7.1}$$

of a regular octahedron inscribed in the sphere of radius $r^2 = 1$, as is depicted in Fig. 5.

Applying the construction algorithm (see Subsec. 3.1) of sum-over-lattice points that belong to a point group orbit, we obtain the following six-parameter vector field:

$$\mathbf{V}^{(6)}(\mathbf{r}|\mathbf{F}) =$$
$$= \begin{pmatrix} 2\cos(2\pi\Theta_3)\,F_1 + 2\cos(2\pi\Theta_2)\,F_2 + 2\sin(2\pi\Theta_3)\,F_3 - 2\sin(2\pi\Theta_2)\,F_5 \\ -2\sin(2\pi\Theta_3)\,F_1 + 2\cos(2\pi\Theta_3)\,F_3 + 2\cos(2\pi\Theta_1)\,F_4 + 2\sin(2\pi\Theta_1)\,F_6 \\ 2\sin(2\pi\Theta_2)\,F_2 - 2\sin(2\pi\Theta_1)\,F_4 + 2\cos(2\pi\Theta_2)\,F_5 + 2\cos(2\pi\Theta_1)\,F_6 \end{pmatrix}, \tag{7.2}$$

where F_i $(i = 1,\ldots,6)$ are real numbers, and where the arguments of the trigonometric functions are the following ones:

$$\Theta_1 = x, \;\; \Theta_2 = y, \;\; \Theta_3 = z. \tag{7.3}$$

The action of the octahedral group O_{24} on this Beltrami vector field is that presented in Eq. (3.23), where γ are the 3×3 matrices of the fundamental defining representation, while the 6×6 matrices $\mathfrak{R}^{(6)}[\gamma]$ acting on the parameter vector $\mathbf{F}$ and defining a reducible representation of O_{24} are those determined from the explicit form of the two generators T, S (see Eq. (4.6)), displayed in Eq. (5.20). Retrieving from the above generators all the group elements and, in particular, a representative for each of the conjugacy classes, we can easily compute their traces and in this way establish the character vector of such a representation. We get

$$\chi\,[\mathbf{6}] = \{6, 0, -2, 0, 0\}. \tag{7.4}$$

The multiplicity vector is

$$\mathrm{m}\,[\mathbf{6}] = \{0, 0, 0, 1, 1\} \tag{7.5}$$

implying that the six-dimensional parameter space decomposes into a $D_4\,[\mathrm{O}_{24}, 3]$ plus a $D_5\,[\mathrm{O}_{24}, 3]$ representations. Utilizing Eq. (2.24) we find

$$\begin{aligned}
\Pi^4\,[\mathrm{O}_{24}, 3]\;\mathbf{F} &= \frac{1}{2}\,\{F_1 + F_2, F_1 + F_2, F_3 + F_4, F_3 + F_4, F_5 + F_6, F_5 + F_6\}\,,\\
\Pi^5\,[\mathrm{O}_{24}, 3]\;\mathbf{F} &= \frac{1}{2}\,\{F_1 - F_2, F_2 - F_1, F_3 - F_4, F_4 - F_3, F_5 - F_6, F_6 - F_5\}\,,
\end{aligned} \tag{7.6}$$

which tells us that the two irreducible three-dimensional representations are obtained by identifying pairwise or antipairwise the six coefficients.

Let us now uplift the point group representation to a representation of the Universal Classifying Group G_{1536}. As explained in Subsec. 5.2, this is done by including the elements of the quantized translation group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ via their 6-dimensional representation anticipated in Eq. (5.21). Indeed, as we already stressed, the representation of G_{1536} obtained from this fundamental orbit can be regarded as the very definition of the Universal Classifying Group.

What is then this fundamental representation of G_{1536} that we have explicitly derived from the constructed Beltrami vector field? It is an *irreducible, faithful representation* and, from the point of view of the abstract irreps defined by the method of induction (see Subsecs. 5.3.2 and 5.3.3), it is the representation $D_{23}\,[\mathrm{G}_{1536}, 6]$. This is the result mentioned in Appendix C.1 at the top of the list. In other words, under the action of quantized translations, the two irreducible representations of the point group described in Eq. (7.6) mix up and coalesce into an irreducible 6-dimensional representation of G_{1536}. Our result can be summarized as the branching rule of the representation $D_{23}\,[\mathrm{G}_{1536}, 6]$ with respect to the point group:

$$D_{23}\,[\mathrm{G}_{1536}, 6] = D_4\,[\mathrm{O}_{24}, 3] \,\oplus\, D_5\,[\mathrm{O}_{24}, 3]\,. \tag{7.7}$$

There are, however, other subgroups of the Universal Classifying Group, different from the point group, with respect to which we can branch the fundamental

$D_{23}\,[\mathrm{G}_{1536}, 6]$ representation. Some special choices of such subgroups are at the origin of the ABC-flows. In the spirit of the Frobenius congruence classes the most intriguing subgroups $\mathrm{H} \subset \mathrm{G}_{1536}$ are those of the space-group-type, namely, those that contain isomorphic but not conjugate copies of the point group. We focus, in particular, on the group GF_{192}, described in Appendix A.7, which contains the subgroup GS_{24}, described in Appendix A.9. This latter is isomorphic to the point group: $\mathrm{GS}_{24} \sim \mathrm{O}_{24}$, yet it is not conjugate to it inside G_{1536}. This means that $\nexists\,\gamma \in \mathrm{G}_{1536}$ such that $\mathrm{GS}_{24} = \gamma^{-1}\,\mathrm{O}_{24}\,\gamma$. We have the following chain of inclusions:

$$\mathrm{G}_{1536} \supset \mathrm{GF}_{192} \supset \mathrm{GS}_{24} \tag{7.8}$$

that is parallel to the other one:

$$\mathrm{G}_{1536} \supset \mathrm{G}_{192} \supset \mathrm{O}_{24}, \tag{7.9}$$

G_{192} being another subgroup, isomorphic to GF_{192}, but not conjugate to it in G_{1536}: it is true that $\mathrm{G}_{192} \sim \mathrm{GF}_{192}$, yet $\nexists\,\gamma \in \mathrm{G}_{1536}$ such that $\mathrm{GF}_{192} = \gamma^{-1}\,\mathrm{G}_{192}\,\gamma$ (see Appendix A.6 for the description of G_{192}). Since G_{192} and GF_{192} are isomorphic, they have the same irreps and the same character table. Yet, since they are not conjugate, the branching rules of the same G_{1536} irrep with respect to the former or the latter subgroup can be different. In the case of the representation $D_{23}\,[\mathrm{G}_{1536}, 6]$, which is that produced by the fundamental orbit of order six, we have (see Appendix D)

$$\begin{aligned} &D_{23}\,[\mathrm{G}_{1536}, 6] = \\ &= \begin{cases} D_{20}\,[\mathrm{G}_{192}, 6] = D_4\,[\mathrm{O}_{24}, 3] \oplus D_5\,[\mathrm{O}_{24}, 3] \\ D_{12}[\mathrm{GF}_{192}, 3] \oplus D_{15}[\mathrm{GF}_{192}, 3] = D_1[\mathrm{GS}_{24}, 1] \oplus D_3[\mathrm{GS}_{24}, 2] \oplus D_4[\mathrm{GS}_{24}, 3], \end{cases} \end{aligned} \tag{7.10}$$

where in the second line we have used the branching rules

$$D_{12}\,[\mathrm{GF}_{192}, 3] = D_1\,[\mathrm{GS}_{24}, 1] \oplus D_3\,[\mathrm{GS}_{24}, 2]\,, \tag{7.11}$$

$$D_{15}\,[\mathrm{GF}_{192}, 3] = D_4\,[\mathrm{GS}_{24}, 3]\,, \tag{7.12}$$

that, in view of the isomorphism, are identical with

$$D_{12}\,[\mathrm{G}_{192}, 3] = D_1\,[\mathrm{O}_{24}, 1] \oplus D_3\,[\mathrm{O}_{24}, 2]\,, \tag{7.13}$$

$$D_{15}\,[\mathrm{G}_{192}, 3] = D_4\,[\mathrm{O}_{24}, 3]\,. \tag{7.14}$$

Equation (7.10) has far reaching consequences. While there are no Beltrami vector fields obtained from this orbit that are invariant with respect to the octahedral point group O_{24}, there exists such an invariant Beltrami flow with respect to the isomorphic GS_{24}: it corresponds to the $D_1\,[\mathrm{GS}_{24},1]$ irrep in the second line of (7.10). Furthermore, while the six-parameter space $\mathbf{F}$ is irreducible with respect to the action of the group G_{192} (the irrep $D_{20}\,[\mathrm{G}_{192},6]$), it splits into two three-dimensional subspaces with respect to GF_{192}. This is the origin of the ABC-flows. Indeed the ABC Beltrami flows can be identified with the irreducible representation $D_{12}\,[\mathrm{GF}_{192},3]$. Let us see how. Explicitly we have the following projection operators on the two irreducible representations, D_{12} and D_{15}:

$$\Pi^{(12)}\,[\mathrm{GF}_{192},3]\ \mathbf{F} = \{F_1,F_2,0,F_4,0,0\}\,, \tag{7.15}$$

$$\Pi^{(15)}\,[\mathrm{GF}_{192},3]\ \mathbf{F} = \{0,0,F_3,0,F_5,F_6\}\,. \tag{7.16}$$

If we set $F_3 = F_5 = F_6 = 0$, we kill the irreducible representation $D_{15}\,[\mathrm{GF}_{192},3]$ and the residual Beltrami vector field, upon the following identifications:

$$A = F_1; \quad B = F_4; \quad C = F_2, \tag{7.17}$$

coincide with the time honored ABC-flow of Eq. (1.17). Indeed inserting the special parameter vector $\mathbf{F} = \{A,C,0,B,0,0\}$ in Eq. (7.2), we obtain

$$\mathbf{V}^{(6)}\left(\{x,y,z\} + \{\frac{3}{4},0,-\frac{1}{4}\} \mid \{A,C,0,B,0,0\}\right) = \mathbf{V}^{(ABC)}(x,y,z), \tag{7.18}$$

the vector field $\mathbf{V}^{(ABC)}(x,y,z)$ being that defined by Eq. (1.17).

The next step is provided by considering the explicit form of the decomposition of the $D_{12}\,[\mathrm{GF}_{192},3]$ irrep, *i.e.*, the ABC-flow, into irreducible representations of the subgroup GS_{24}. The two invariant subspaces are immediately characterized in terms of the parameters A,B,C as follows:

$$D_1\,[\mathrm{GS}_{24},1] \Leftrightarrow A = B = C \neq 0, \tag{7.19}$$

$$D_3\,[\mathrm{GS}_{24},2] \Leftrightarrow A + B + C = 0. \tag{7.20}$$

7.1.1. The $(\mathrm{A},\mathrm{A},\mathrm{A})$*-Flow Invariant under* GS_{24}. This information suffices to understand the role of the $A:A:A=1$ Beltrami vector field often considered in the literature. It is the unique one invariant under the order 24 group GS_{24} isomorphic to the octahedral point group . Explicitly, in our notations it takes the following form*:

$$\mathbf{V}^{(A,A,A)}(\mathbf{r}) = \mathbf{V}^{(A,A,A)}(x,y,z) \equiv 2A \begin{pmatrix} (\cos(2\pi y) + \cos(2\pi z)) \\ (\cos(2\pi x) - \sin(2\pi z)) \\ (\sin(2\pi y) - \sin(2\pi x)) \end{pmatrix}. \tag{7.21}$$

*Observe that here and in the sequel we stick to our conventions for x,y,z, which differ from those of Eq. (1.17) by the already mentioned shift $\{\frac{3}{4},0,-\frac{1}{4}\}$).

This vector field $\mathbf{V}^{(A,A,A)}(x,y,z)$ is everywhere nonsingular in the fundamental unit cube (the torus T^3) apart from eight isolated *stagnation points* where it vanishes. They are listed below:

$$\begin{aligned} s_1 &= \left\{\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{3}{8}\right\}; \; s_2 = \left\{\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{1}{8}\right\}; \\ s_3 &= \left\{\tfrac{3}{8}, \tfrac{1}{8}, \tfrac{5}{8}\right\}; \; s_4 = \left\{\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{7}{8}\right\}; \\ s_5 &= \left\{\tfrac{5}{8}, \tfrac{5}{8}, \tfrac{7}{8}\right\}; \; s_6 = \left\{\tfrac{5}{8}, \tfrac{7}{8}, \tfrac{5}{8}\right\}; \\ s_7 &= \left\{\tfrac{7}{8}, \tfrac{5}{8}, \tfrac{1}{8}\right\}; \; s_8 = \left\{\tfrac{7}{8}, \tfrac{7}{8}, \tfrac{3}{8}\right\}. \end{aligned} \tag{7.22}$$

A numerical plot of this vector field is displayed in Fig. 6.

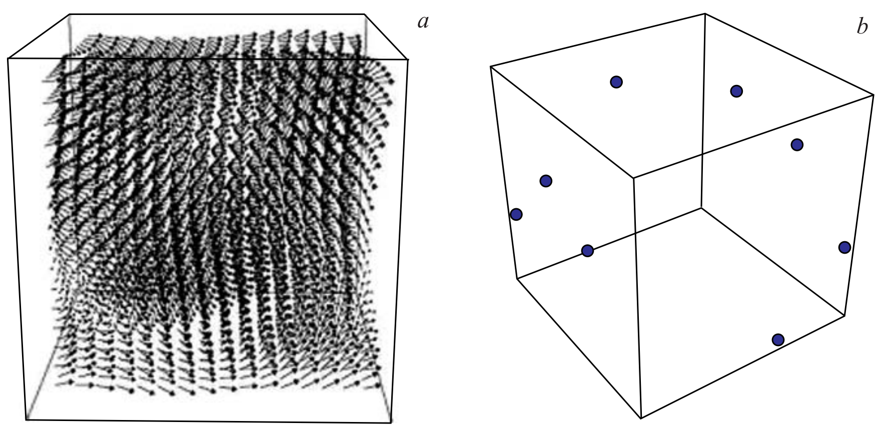

Fig. 6. *a*) A plot of the $A:A:A=1$ Beltrami vector field invariant under the group GS_{24} and *b*) a view of its eight stagnation points of Eq. (7.22)

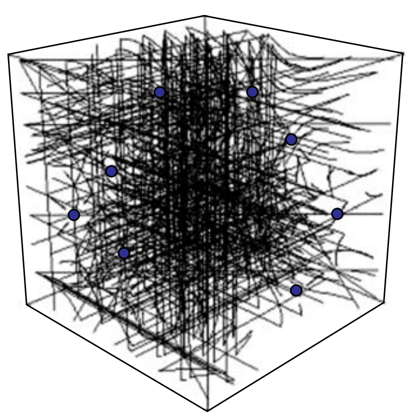

Fig. 7. A plot of 125 streamlines of the $A:A:A=1$ Beltrami vector field with equally spaced initial conditions. The numerical solutions are smooth in $\mathbb{R}^3$. When a branch reaches a boundary of the unit cube, it is continued with its image in the cube modulo the appropriate lattice translation. The circles in this figure are the eight stagnation points

In order to provide the reader with a visual impression of the dynamics of this flow, in Fig. 7 we display a set of $5 \times 5 \times 5 = 125$ streamlines, namely of numerical integrations of the differential system

$$\frac{dr}{dt} = \mathbf{V}^{(A,A,A)}(\mathbf{r}), \tag{7.23}$$

with initial conditions

$$\mathbf{r}(0) = \mathbf{r}_0 = \left\{ \frac{n_1}{5}, \frac{n_2}{5}, \frac{n_3}{5} \right\}; \quad n_{1,2,3} = 0, 1, 2, 3, 4. \tag{7.24}$$

7.1.2. Chains of Subgroups and the Flows $(A, B, 0)$*,* $(A, A, 0)$*, and* $(A, 0, 0)$*.* In the literature a lot of attention has been given to the special subcases of the ABC-flows, where one or two of the parameters vanish or two are equal among themselves and one vanishes. Also these cases can be thoroughly characterized in group theoretical terms and their special features can be traced back to the hidden subgroup structure associated with them.

The $(A, B, 0)$ *Case and Its Associated Chain of Subgroups.* When we put $C = 0$, we define a two-dimensional subspace of the representation $D_{12}\,[\mathrm{GF}_{192}, 3]$, which is invariant under some proper subgroup $\mathrm{H}^{(\mathrm{A,B,0})} \subset \mathrm{GF}_{192}$. This group $\mathrm{H}^{(\mathrm{A,B,0})}$ can be calculated and found to be of order 64, yet we do not dwell on it because the subgroup of the classifying group G_{1536}, which leaves the subspace $(A, 0, 0, B, 0, 0)$ invariant, is larger than $\mathrm{H}^{(\mathrm{A,B,0})}$ and it is not contained in GF_{192}. It has order 128 and we name it $\mathrm{G}_{128}^{(\mathrm{A,B,0})}$. This short discussion is important because it implies the following: the flows $(\mathrm{A}, \mathrm{B}, 0)$ should not be considered just as a particular case of the ABC-flows rather as a different set of flows, whose properties are encoded in the group $\mathrm{G}_{128}^{(\mathrm{A,B,0})}$.

The group $\mathrm{G}_{128}^{(\mathrm{A,B,0})}$ is solvable and a chain of normal subgroups can be found, all of index 2 which ends with the Abelian $\mathrm{G}_{4}^{(\mathrm{A,B,0})}$ isomorphic to $\mathbb{Z}_4$. This latter is nothing else than the group of quantized translation in the y-direction and its inclusion in the group leaving the space $(A, 0, 0, B, 0, 0)$ invariant actually means that the differential system must be y-independent and hence two-dimensional. The chain of normal subgroups is displayed here below:

$$\mathbb{Z}_4 \sim \mathrm{G}_{4}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{8}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{16}^{(\mathrm{A,B,0})} \lhd \begin{cases} \lhd\, \mathrm{G}_{32}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{64}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{128}^{(\mathrm{A,B,0})} \\ \lhd\, \mathrm{G}_{32}^{(\mathrm{A,A,0})} \end{cases} \tag{7.25}$$

and it allows for the construction of irreducible representations of $\mathrm{G}_{128}^{(\mathrm{A,B,0})}$ and all other members of the chain, by means of the induction algorithm. Such a construction we have not done, but all the groups of the chain are listed with their conjugacy classes in Appendix E. The group $\mathrm{G}_{128}^{(\mathrm{A,B,0})}$ leaves the subspace $(A, 0, 0, B, 0, 0)$ invariant but still mixes the parameters A and B among themselves. The subgroup $\mathrm{G}_{16}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{128}^{(\mathrm{A,B,0})}$ instead stabilizes the very vector $(A,0,0,B,0,0)$. This means that any $(\mathrm{A},\mathrm{B},0)$-flow has a hidden symmetry of order 16 provided by the group $\mathrm{G}_{16}^{(\mathrm{A,B,0})}$. The general form of these Beltrami fields is the following one:

$$\mathbf{V}^{(A,B,0)}(\mathbf{r}) = \mathbf{V}^{(A,B,0)}(x,y,z) \equiv \left(\begin{array}{l} A\cos(2\pi z) \\ B\cos(2\pi x) - A\sin(2\pi z) \\ -B\sin(2\pi x) \end{array}\right). \tag{7.26}$$

Looking at Eq. (7.25), we notice that there is another group of order 32, namely $\mathrm{G}_{32}^{(\mathrm{A,A,0})}$ which contains $\mathrm{G}_{16}^{(\mathrm{A,B,0})}$, but it is not contained neither in $\mathrm{G}_{128}^{(\mathrm{A,B,0})}$ nor in GF_{192}. This group is the stabilizer of the vector $(A,0,0,A,0,0)$ and hence it is the hidden symmetry group of the flows of type $(A,A,0)$. Once again the very fact that $\mathrm{G}_{32}^{(\mathrm{A,A,0})}$ is not contained in $\mathrm{G}_{128}^{(\mathrm{A,B,0})}$ shows that the $(A,A,0)$-flow should not be considered as a particular case of the $(A,B,0)$-flows rather as a new type of its own. Let us also stress the difference with the case of the (A,A,A)-flow. Here the hidden symmetry group GS_{24} is contained in GF_{192} and the interpretation of the (A,A,A)-flow as a particular case of the (A,B,C)-flows is permitted. Having set

$$\mathbf{V}^{(A,A,0)}(\mathbf{r}) = \mathbf{V}^{(A,A,0)}(x,y,z) \equiv A\left(\begin{array}{l} \cos(2\pi z) \\ \cos(2\pi x) - \sin(2\pi z) \\ -\sin(2\pi x) \end{array}\right), \tag{7.27}$$

in Fig. 8, we display a plot of the vector field $\mathbf{V}^{(A,A,0)}(\mathbf{r})$ and a family of its streamlines. In the case of this flow, there are not isolated stagnation points, rather, because of the y-independence of the Beltrami vector field, there are two entire stagnation lines explicitly given below:

$$\mathrm{sl}_1 = \left\{\frac{1}{2}, y, \frac{3}{4}\right\}; \quad \mathrm{sl}_2 = \left\{1, y, \frac{1}{4}\right\}. \tag{7.28}$$

Let us finally come to the case of the flow $(A,0,0)$. The one-dimensional subspace of vectors of the form $(A,0,0,0,0,0)$ is left-invariant by a rather big subgroup of the classifying group which is of order 256. We name it $\mathrm{G}_{256}^{(\mathrm{A,0,0})}$, and its description is given in Appendix E. It is a solvable group with a chain of normal subgroups of index 2 which ends into a subgroup of order 16 isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$. This information is summarized in the equation below:

$$\begin{array}{l} \mathbb{Z}_4 \times \mathbb{Z}_4 \sim \mathrm{G}_{16}^{(\mathrm{A,0,0})} \lhd \mathrm{G}_{32}^{(\mathrm{A,0,0})} \lhd \mathrm{G}_{64}^{(\mathrm{A,0,0})} \lhd \\ \mathbb{Z}_4 \sim \mathrm{G}_{4}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{8}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{16}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{32}^{(\mathrm{A,B,0})} \lhd \mathrm{G}_{64}^{(\mathrm{A,B,0})} \subset \end{array} \Big\rangle \circ\ \mathrm{G}_{128}^{(\mathrm{A,0,0})} \lhd \mathrm{G}_{256}^{(\mathrm{A,0,0})}. \tag{7.29}$$

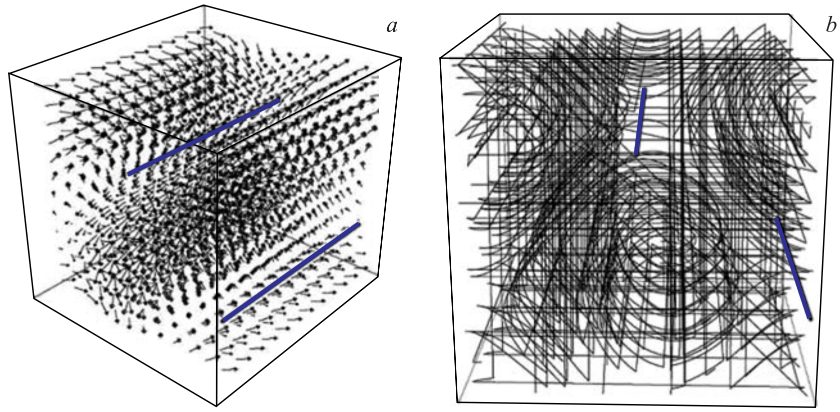

Fig. 8. *a*) A plot of the Betrami vector field $\mathbf{V}^{(A,A,0)}(\mathbf{r})$, where the two stagnation lines of the flow are visible (fat lines). *b*) A family of streamlines with equally spaced initial conditions is displayed

The group $\mathrm{G}_{256}^{(\mathrm{A},0,0)}$ leaves the subspace $(A,0,0,0,0,0)$ invariant but occasionally changes the sign of A. The subgroup $\mathrm{G}_{128}^{(\mathrm{A},0,0)} \subset \mathrm{G}_{256}^{(\mathrm{A},0,0)}$ stabilizes the very vector $(A,0,0,0,0,0)$ and therefore it is the hidden symmetry of the $(A,0,0)$ flows encoded in the planar vector field

$$\mathbf{V}^{(A,0,0)}(\mathbf{r}) = \mathbf{V}^{(A,0,0)}(x,y,z) \equiv A \begin{pmatrix} \cos(2\pi z) \\ -\sin(2\pi z) \\ 0 \end{pmatrix}. \tag{7.30}$$

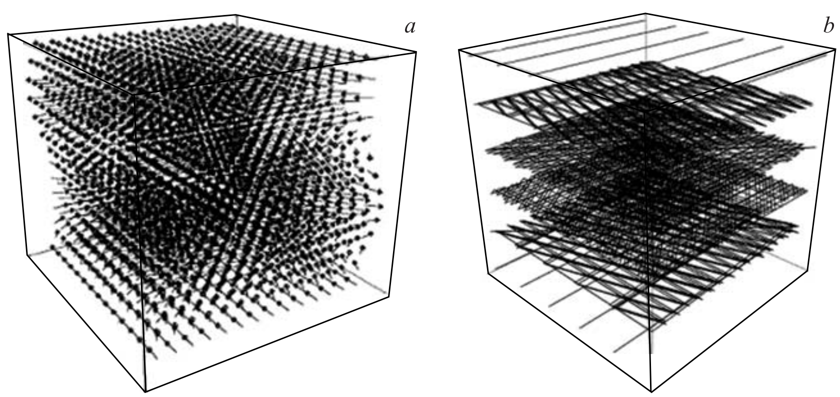

Fig. 9. *a*) A plot of the Betrami vector field $\mathbf{V}^{(A,0,0)}(\mathbf{r})$. *b*) A family of streamlines with equally spaced initial conditions is displayed. The planar structure of the streamlines is quite visible

Looking back at Eq. (7.29) it is important to note that the group $\mathrm{G}_{128}^{(\mathrm{A},0,0)} \neq \mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}$ is different from the homologous group appearing in the group-chain of the $(\mathrm{A}, \mathrm{B}, 0)$-flows. So, once again the $(\mathrm{A}, 0, 0)$-flows cannot be regarded as particular cases of the $(\mathrm{A}, \mathrm{B}, 0)$-flows. Yet, the group $\mathrm{G}_{128}^{(\mathrm{A},0,0)}$ contains the entire chain of normal subgroups $\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}$ starting from $\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}$. There is however a very relevant proviso that $\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}$ is a subgroup of $\mathrm{G}_{128}^{(\mathrm{A},0,0)}$, but it is not normal. In Fig. 9, we show a plot of the vector field $\mathbf{V}^{(A,0,0)}(\mathbf{r})$ and a family of its streamlines.

7.2. The Lowest-Lying Octahedral Orbit of Length 12 in the Cubic Lattice and the Beltrami Flows Respectively Invariant under GP_{24} and GK_{24}. The next example that we consider corresponds to the length 12 octahedral orbit of momentum vectors in the class numbered 9) in our list of 48 classes, namely:

$$\mathbf{k} = \{0, 1 + 4\nu, 1 + 4\nu\}. \tag{7.31}$$

Choosing the representative $\nu = 0$, we obtain the following lowest-lying orbit of 12 points:

$$\begin{aligned} p_1 &= \{-1, -1, 0\}; & p_7 &= \{0, 1, -1\}; \\ p_2 &= \{-1, 0, -1\}; & p_8 &= \{0, 1, 1\}; \\ p_3 &= \{-1, 0, 1\}; & p_9 &= \{1, -1, 0\}; \\ p_4 &= \{-1, 1, 0\}; & p_{10} &= \{1, 0, -1\}; \\ p_5 &= \{0, -1, -1\}; & p_{11} &= \{1, 0, 1\}; \\ p_6 &= \{0, -1, 1\}; & p_{12} &= \{1, 1, 0\}. \end{aligned} \tag{7.32}$$

These are the lattice points displayed in Fig. 10.

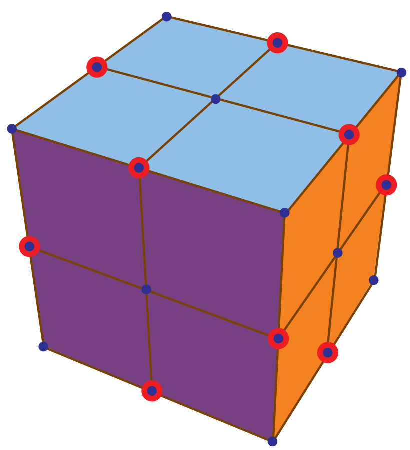

Fig. 10. A view of the octahedral 12-orbit in the cubic lattice, corresponding to the midpoints of the edges of a regular cube

Applying the strategy of sum over lattice points that belong to a point group orbit, we obtain the following 12-parameter vector field:

$$\mathbf{V}^{(12)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\}\,,$$

$$\begin{aligned}
V_x = {} & 2\cos(2\pi\Theta_6)\,F_1 + 2\cos(2\pi\Theta_5)\,F_2 + 2\cos(2\pi\Theta_4)\,F_3 - \\
& -\sqrt{2}\sin(2\pi\Theta_3)\,F_4 + \sqrt{2}\sin(2\pi\Theta_2)\,F_5 - 2\cos(2\pi\Theta_1)\,F_6 - \\
& -2\sqrt{2}\sin(2\pi\Theta_6)\,F_7 - 2\sqrt{2}\sin(2\pi\Theta_5)\,F_8 + \sqrt{2}\sin(2\pi\Theta_4)\,F_9 + \\
& +2\cos(2\pi\Theta_3)\,F_{10} - 2\cos(2\pi\Theta_2)\,F_{11} - \sqrt{2}\sin(2\pi\Theta_1)\,F_{12}, \\
V_y = {} & \sqrt{2}\sin(2\pi\Theta_6)\,F_1 - \sqrt{2}\sin(2\pi\Theta_5)\,F_2 + 2\cos(2\pi\Theta_4)\,F_3 + \\
& +2\cos(2\pi\Theta_3)\,F_4 + 2\cos(2\pi\Theta_2)\,F_5 + 2\cos(2\pi\Theta_1)\,F_6 + \\
& +2\cos(2\pi\Theta_6)\,F_7 - 2\cos(2\pi\Theta_5)\,F_8 + \sqrt{2}\sin(2\pi\Theta_4)\,F_9 + \\
& +2\sqrt{2}\sin(2\pi\Theta_3)\,F_{10} + 2\sqrt{2}\sin(2\pi\Theta_2)\,F_{11} + \\
& -\sqrt{2}\sin(2\pi\Theta_1)\,F_{12}, \\
V_z = {} & \sqrt{2}\sin(2\pi\Theta_6)\,F_1 + \sqrt{2}\sin(2\pi\Theta_5)\,F_2 - 2\sqrt{2}\sin(2\pi\Theta_4)\,F_3 - \\
& -\sqrt{2}\sin(2\pi\Theta_3)\,F_4 - \sqrt{2}\sin(2\pi\Theta_2)\,F_5 - 2\sqrt{2}\sin(2\pi\Theta_1)\,F_6 + \\
& +2\cos(2\pi\Theta_6)\,F_7 + 2\cos(2\pi\Theta_5)\,F_8 + 2\cos(2\pi\Theta_4)\,F_9 + \\
& +2\cos(2\pi\Theta_3)\,F_{10} + 2\cos(2\pi\Theta_2)\,F_{11} + 2\cos(2\pi\Theta_1)\,F_{12},
\end{aligned} \tag{7.33}$$

where F_i $(i = 1, \ldots, 12)$ are real numbers, and the angles Θ_i are defined as follows:

$$\begin{aligned}
&\Theta_1 = x + y; \qquad \Theta_2 = x + z; \qquad \Theta_3 = x - z; \\
&\Theta_4 = x - y; \qquad \Theta_5 = y + z; \qquad \Theta_6 = y - z.
\end{aligned} \tag{7.34}$$

Decomposition of This Orbit with Respect to the Point Group O_{24}. The action of the octahedral point group O_{24} is easily determined on such a vector field by the standard procedure illustrated above. The form of the two O_{24} generators is displayed below

$$
\mathfrak{R}^{(12)}[T] = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\mathfrak{R}^{(12)}[S] = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}. \tag{7.35}
$$

Retrieving from the above generators all the group elements and, in particular, a representative for each of the conjugacy classes, we easily compute the character vector of this representation. Explicitly we get

$$\chi\,[\mathbf{12}] = \{12, 0, 0, -2, 0\}. \tag{7.36}$$

The multiplicity vector is

$$\mathrm{m}\,[\mathbf{12}] = \{0, 1, 1, 2, 1\}. \tag{7.37}$$

This implies that we have a one-dimensional nonsinglet representation $D_2\,[\mathrm{O}_{24},1]$, a two-dimensional representation $D_3\,[\mathrm{O}_{24},2]$, two three-dimensional representations $D_4\,[\mathrm{O}_{24},3]$ and one three-dimensional representation $D_5\,[\mathrm{O}_{24},3]$.

The splitting of the parameter space into these five invariant subspaces requires a little bit of algebraic work that we omit, quoting only the result.

We name A the coefficient corresponding to the one-dimensional D_2 representation; $\Gamma_{1,2}$, the two coefficients corresponding to the two-dimensional D_3 representation; $K_{1,2,3}$, the three coefficients corresponding to the three-dimensional D_{4a} representation; $\Delta_{1,2,3}$, the three coefficients corresponding to the three-dimensional D_{4b} representation; and $\Lambda_{1,2,3}$, the three coefficients corresponding to the three-dimensional D_5 representation. Expressed in terms of these variables, the parameter vector $\mathbf{F}$ has the following form:

$$\mathbf{F} = \begin{pmatrix} -A + K_2 + \Gamma_1 + \Gamma_2 \\ A + K_2 - \Gamma_1 - \Gamma_2 \\ -\Delta_1 + \Delta_3 + \Lambda_1 + \Lambda_2 \\ -A + K_3 - \Gamma_1 \\ A + K_3 + \Gamma_1 \\ -\Delta_1 + \Delta_2 + \Lambda_1 + \Lambda_3 \\ -\Delta_1 + \Delta_2 + \Delta_3 - \Lambda_1 - \Lambda_2 - \Lambda_3 \\ \Delta_1 + \Lambda_1 \\ -A + K_1 - \Gamma_2 \\ \Delta_3 + \Lambda_3 \\ \Delta_2 + \Lambda_2 \\ A + K_1 + \Gamma_2 \end{pmatrix}. \tag{7.38}$$

In Eq. (7.38), the decomposition into O_{24} irreducible representations is fully encoded.

Uplifting to the Universal Classifying Group G_{1536}. As in all other cases this reducible representation of the point group can be uplifted to a representation of the entire classifying group G_{1536} including the representation of the translation generators which is calculated to be that displayed in Eq. (F.1) (see Appendix F.1). The result of this uplifting is that the 12-parameter vector field (7.33) corresponds to an irreducible 12-dimensional representation of the classifying group, precisely to $D_{32}\,[\mathrm{G}_{1536},12]$, and the previous results correspond to the following branching rule:

$$D_{32}\,[\mathrm{G}_{1536},12] = D_2\,[\mathrm{O}_{24},1] + D_3\,[\mathrm{O}_{24},2] + 2D_4\,[\mathrm{O}_{24},3] + D_5\,[\mathrm{O}_{24},3]\,. \tag{7.39}$$

It is interesting to consider the branching rule of the same representation with respect to the two subgroups $\mathrm{G}_{192} \subset \mathrm{G}_{1536}$ and $\mathrm{GF}_{192} \subset \mathrm{G}_{1536}$ that, as we have shown in the previous section, play such an important role in understanding the ABC-flows and their hidden symmetries.

Decomposition of the Orbit with Respect to the Groups G_{192} *and* GF_{192}. With the help of the character tables we easily obtain

$$D_{32}\left[\mathrm{G}_{1536}, 12\right] = D_9\left[\mathrm{G}_{192}, 3\right] + D_{13}\left[\mathrm{G}_{192}, 3\right] + D_{19}\left[\mathrm{G}_{192}, 6\right] \tag{7.40}$$

and

$$D_{32}\left[\mathrm{G}_{1536}, 12\right] = D_9\left[\mathrm{GF}_{192}, 3\right] + D_{13}\left[\mathrm{GF}_{192}, 3\right] + D_{19}\left[\mathrm{GF}_{192}, 6\right]. \tag{7.41}$$

This result is very interesting. The decomposition of the irreducible representation $D_{32}\left[\mathrm{G}_{1536}, 12\right]$ with respect to the two isomorphic but not conjugate subgroups G_{192} and GF_{192} is identical. It follows that it is identical also with respect to any of their homologous subgroups. However, the G_{192} and GF_{192} invariant subspaces are far from being the same, and the corresponding Beltrami vector fields are different. In the case of the group G_{192}, which contains the point group as a subgroup $\mathrm{O}_{24} \subset \mathrm{G}_{192}$, the three representations $D_9\left[\mathrm{G}_{192}, 3\right]$, $D_{13}\left[\mathrm{G}_{192}, 3\right]$, and $D_{19}\left[\mathrm{G}_{192}, 6\right]$ simply join together the point group representations in the following way:

$$\begin{aligned} D_9\left[\mathrm{G}_{192}, 3\right] &= D_{4a}\left[\mathrm{O}_{24}, 3\right], \\ D_{13}\left[\mathrm{G}_{192}, 3\right] &= D_2\left[\mathrm{O}_{24}, 1\right] \oplus D_3\left[\mathrm{O}_{24}, 2\right], \\ D_{19}\left[\mathrm{G}_{192}, 6\right] &= D_{4b}\left[\mathrm{O}_{24}, 3\right] \oplus D_5\left[\mathrm{O}_{24}, 3\right]. \end{aligned} \tag{7.42}$$

In the case of the group GF_{192}, the invariant subspaces mix the point group representations in a capricious way and the right-hand side of Eqs. (7.42) is not true if in the left-hand side we replace G_{192} with GF_{192}. Yet, if we consistently replace O_{24} with its homologous GS_{24} also in the right-hand side, then we obtain a true set of equations

$$\begin{aligned} D_9\left[\mathrm{GF}_{192}, 3\right] &= D_{4a}\left[\mathrm{GS}_{24}, 3\right], \\ D_{13}\left[\mathrm{GF}_{192}, 3\right] &= D_2\left[\mathrm{GS}_{24}, 1\right] \oplus D_3\left[\mathrm{GS}_{24}, 2\right], \\ D_{19}\left[\mathrm{GF}_{192}, 6\right] &= D_{4b}\left[\mathrm{GS}_{24}, 3\right] \oplus D_5\left[\mathrm{GS}_{24}, 3\right]. \end{aligned} \tag{7.43}$$

Observing Eqs. (7.42) and (7.43), we conclude that from the 12-parameter vector field (7.33) we cannot extract any one that is invariant under either O_{24} or GS_{24} since no D_1 representation emerges. This implies that from this orbit we cannot extract any Beltrami vector field with a hidden symmetry isomorphic to that of the proper octahedral group $\mathrm{O}_{24} \sim \mathrm{S}_4$. Yet there is another abstract group of order 24 which has isomorphic subgroup copies in G_{192} and GF_{192}. This is the abstarct group $\mathrm{A}_4 \otimes \mathbb{Z}_2$, and it happens to be the group number 13 in the list of the 15 groups of order 24 (see [25]). It is the unique group of such an order that has 8 conjugacy classes: its not conjugate copies in the classifying group were respectively named by us $\mathrm{GP}_{24} \subset \mathrm{G}_{192}$ and $\mathrm{GK}_{24} \subset \mathrm{GF}_{192}$ (see Appendices A.10 and A.11 for their description). As we show in the next subsection, there are Beltrami vector fields that have hidden symmetry either GP_{24} or GK_{24}.

7.2.1. Beltrami Flows Invariant with Respect to the Subgroups GP_{24} *and* GK_{24}. As stated above, the subgroups $\mathrm{GP}_{24} \subset \mathrm{G}_{192} \subset \mathrm{G}_{1536}$ and $\mathrm{GK}_{24} \subset \mathrm{GF}_{192} \subset \mathrm{G}_{1536}$ are isomorphic among themselves and to the abstract group $\mathrm{A}_4 \otimes \mathbb{Z}_2$. This latter, which has order 24, can be defined by the following generators and relations:

$$\mathrm{A}_4 \otimes \mathbb{Z}_2 \equiv \Big(\mathcal{T},\, \mathcal{P} \ \mid\ \mathcal{T}^6 = \mathrm{e},\, \mathcal{P}^2 = \mathrm{e},\, (\mathcal{P} \cdot \mathcal{T})^2 = \mathrm{e}\Big)\,. \tag{7.44}$$

In the case of GP_{24}, we have

$$\mathcal{T} = \{2_8, 1, 1, 1\}\,; \quad \mathcal{P} = \{3_3, 1, 0, 0\} \tag{7.45}$$

and the resulting group is that described in Sec. A.10.

In the case of GK_{24}, we have instead

$$\mathcal{T} = \left\{2_1, \frac{1}{2}, \frac{3}{2}, 0\right\}\,; \quad \mathcal{P} = \{3_3, 1, 1, 0\}\,, \tag{7.46}$$

and the resulting group is that described in Sec. A.11.

We consider next the decomposition of the irreducible representations D_9, D_{13} and D_{19} of either G_{192} or GF_{192} with respect to either GP_{24} or GK_{24} and we get

$$\begin{aligned} D_9\,[\mathrm{G}_{192}, 3] &= D_1\,[\mathrm{GP}_{24}, 1] \oplus D_2\,[\mathrm{GP}_{24}, 1] \oplus D_3\,[\mathrm{GP}_{24}, 1]\,,\\ D_{13}\,[\mathrm{G}_{192}, 3] &= D_{7a}\,[\mathrm{GP}_{24}, 3]\,,\\ D_{19}\,[\mathrm{G}_{192}, 6] &= D_{7b}\,[\mathrm{GP}_{24}, 3] \oplus D_{7c}\,[\mathrm{GP}_{24}, 3] \end{aligned} \tag{7.47}$$

and

$$\begin{aligned} D_9\,[\mathrm{GF}_{192}, 3] &= D_1\,[\mathrm{GK}_{24}, 1] \oplus D_2\,[\mathrm{GK}_{24}, 1] \oplus D_3\,[\mathrm{GK}_{24}, 1]\,,\\ D_{13}\,[\mathrm{GF}_{192}, 3] &= D_{7a}\,[\mathrm{GK}_{24}, 3]\,,\\ D_{19}\,[\mathrm{GF}_{192}, 6] &= D_{7b}\,[\mathrm{GK}_{24}, 3] \oplus D_{7c}\,[\mathrm{GK}_{24}, 3]\,. \end{aligned} \tag{7.48}$$

This implies that there are both the Beltrami vector field invariant under GP_{24} and another one invariant under GK_{24}.

The Beltrami Vector Field Invariant under GP_{24}. Applying to the vector field of Eq. (7.33) the projector onto the singlet representation $D_1\,[\mathrm{GP}_{24}, 1]$, we obtain the following Beltrami vector field:

$$\mathbf{V}^{(\mathrm{GP}_{24}|D_1)}(\mathbf{r}) = \{V_x,\, V_y,\, V_z\}\,,$$

$$\begin{aligned}
V_x &= \cos\left(2\pi(y-z)\right) + 2\cos\left(2\pi(y+z)\right) + \sqrt{2}\sin\left(2\pi(x-y)\right) - \\
&\quad - \sqrt{2}\sin\left(2\pi(x+y)\right) - \sqrt{2}\sin\left(2\pi(x-z)\right) + \sqrt{2}\sin\left(2\pi(x+z)\right), \\
V_y &= 2\cos\left(2\pi(x-z)\right) + 2\cos\left(2\pi(x+z)\right) + \sqrt{2}\sin\left(2\pi(x-y)\right) + \\
&\quad + \sqrt{2}\sin\left(2\pi(x+y)\right) + \sqrt{2}\sin\left(2\pi(y-z)\right) - \sqrt{2}\sin\left(2\pi(y+z)\right), \\
V_z &= 2\cos\left(2\pi(x-y)\right) + 2\cos\left(2\pi(x+y)\right) - \sqrt{2}\sin\left(2\pi(x-z)\right) + \\
&\quad + \sqrt{2}\sin\left(2\pi(y-z)\right) - \sqrt{2}\sin\left(2\pi(x+z)\right) + \sqrt{2}\sin\left(2\pi(y+z)\right).
\end{aligned} \tag{7.49}$$

In the unit cube this vector field has 26 stagnation points:

$$\mathbf{V}^{(\mathrm{GP}_{24}|D_1)}(\mathbf{s}_i), \quad i = 1, \ldots, 26, \tag{7.50}$$

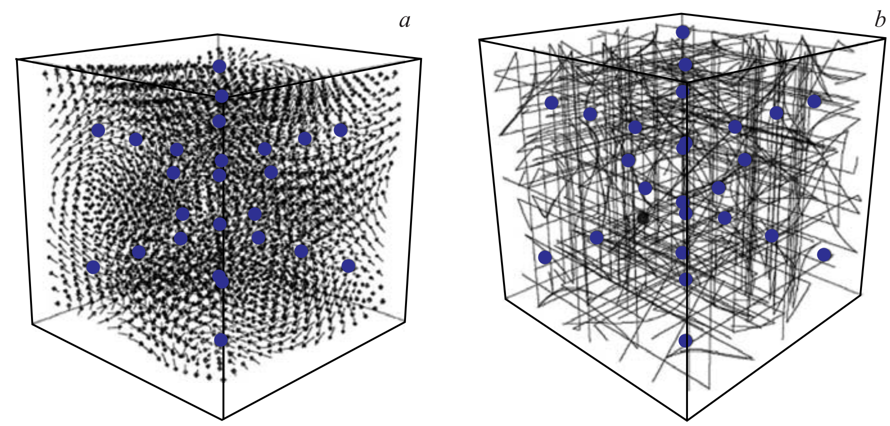

Fig. 11. *a*) A plot of the GP_{24} invariant Beltrami vector field $\mathbf{V}^{(\mathrm{GP}_{24}|D_1)}(\mathbf{r})$ obtained from the octahedral 12-point orbit in the cubic lattice (midpoints of the edges of a regular cube). The field is analytically defined in Eq. (7.49). *b*) A family of streamlines of this vector field with equally spaced initial conditions. In both pictures the circles denote the 26 isolated stagnation points of this flow

whose coordinates are explicitly given below:

$$
\begin{aligned}
s_1 &= \left\{0, \tfrac{1}{4}, \tfrac{1}{4}\right\}; & s_2 &= \left\{0, \tfrac{3}{4}, \tfrac{3}{4}\right\}; & s_3 &= \left\{\tfrac{1}{4}, 0, \tfrac{1}{4}\right\};\\
s_4 &= \left\{\tfrac{1}{4}, \tfrac{1}{4}, 0\right\}; & s_5 &= \left\{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\}; & s_6 &= \left\{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\right\};\\
s_7 &= \left\{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{3}{4}\right\}; & s_8 &= \left\{\tfrac{1}{4}, \tfrac{1}{4}, 1\right\}; & s_9 &= \left\{\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}\right\};\\
s_{10} &= \left\{\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}\right\}; & s_{11} &= \left\{\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}\right\}; & s_{12} &= \left\{\tfrac{1}{4}, 1, \tfrac{1}{4}\right\};\\
s_{13} &= \left\{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}\right\}; & s_{14} &= \left\{\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{3}{4}\right\}; & s_{15} &= \left\{\tfrac{3}{4}, 0, \tfrac{3}{4}\right\};\\
s_{16} &= \left\{\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\}; & s_{17} &= \left\{\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{4}\right\}; & s_{18} &= \left\{\tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}\right\};\\
s_{19} &= \left\{\tfrac{3}{4}, \tfrac{3}{4}, 0\right\}; & s_{20} &= \left\{\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}\right\}; & s_{21} &= \left\{\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{2}\right\};\\
s_{22} &= \left\{\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}\right\}; & s_{23} &= \left\{\tfrac{3}{4}, \tfrac{3}{4}, 1\right\}; & s_{24} &= \left\{\tfrac{3}{4}, 1, \tfrac{3}{4}\right\};\\
s_{25} &= \left\{1, \tfrac{1}{4}, \tfrac{1}{4}\right\}; & s_{26} &= \left\{1, \tfrac{3}{4}, \tfrac{3}{4}\right\}.
\end{aligned}
\tag{7.51}
$$

A plot of this vector field and of a family of its streamlines is shown in Fig. 11.

The Beltrami Vector Field Invariant under GK_{24}. Applying to the vector field of Eq. (7.33) the projector onto the singlet representation $D_1\,[\mathrm{GK}_{24}, 1]$, we obtain the following Beltrami vector field:

$$
\begin{aligned}
&\mathbf{V}^{(\mathrm{GK}_{24}|D_1)}(\mathbf{r}) = \{V_x,\, V_y,\, V_z\},\\
V_x &= 2\cos(2\pi(x-y)) + 2\cos(2\pi(x+y)) + \sin(2\pi(x-z)) -\\
&\quad - 2\sqrt{2}\sin(2\pi(y-z)) - \sin(2\pi(x+z)) - 2\sqrt{2}\sin(2\pi(y+z)),\\
V_y &= 2\cos(2\pi(x-y)) - 2\cos(2\pi(x+y)) - \sqrt{2}\cos(2\pi(x-z)) -\\
&\quad - \sqrt{2}\cos(2\pi(x+z)) + 4\sin(2\pi y)\sin(2\pi z),\\
V_z &= 2\cos(2\pi(y-z)) + 2\cos(2\pi(y+z)) - 2\sqrt{2}\sin(2\pi(x-y)) +\\
&\quad + 2\sqrt{2}\sin(2\pi(x+y)) + \sin(2\pi(x-z)) + \sin(2\pi(x+z)).
\end{aligned}
\tag{7.52}
$$

This vector field has 16 stagnation points

$$
\mathbf{V}^{(\mathrm{GK}_{24}|D_1)}(\mathbf{s}_i), \quad i = 1, \dots, 16,
\tag{7.53}
$$

whose coordinates are explicitly given below:

$$
\begin{aligned}
s_1 &= \left\{\tfrac{1}{4}, 0, \tfrac{1}{4}\right\}; & s_2 &= \left\{\tfrac{1}{4}, 0, \tfrac{3}{4}\right\}; & s_3 &= \left\{\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\};\\
s_4 &= \left\{\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}\right\}; & s_5 &= \left\{\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\right\}; & s_6 &= \left\{\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\};\\
s_7 &= \left\{\tfrac{1}{4}, 1, \tfrac{1}{4}\right\}; & s_8 &= \left\{\tfrac{1}{4}, 1, \tfrac{3}{4}\right\}; & s_9 &= \left\{\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\};\\
s_{10} &= \left\{\tfrac{3}{4}, 0, \tfrac{3}{4}\right\}; & s_{11} &= \left\{\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\}; & s_{12} &= \left\{\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\};\\
s_{13} &= \left\{\tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}\right\}; & s_{14} &= \left\{\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}\right\}; & s_{15} &= \left\{\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\};\\
s_{16} &= \left\{\tfrac{3}{4}, 1, \tfrac{3}{4}\right\}.
\end{aligned}
\tag{7.54}
$$

A plot of this vector field and of a family of its streamlines is shown in Fig. 12.

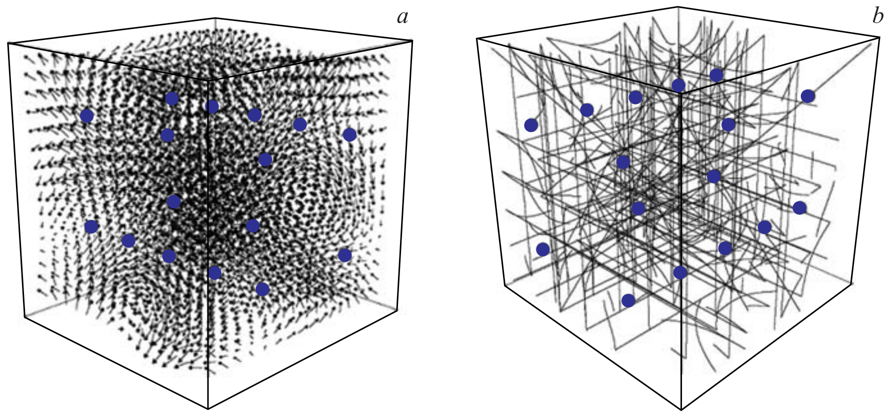

Fig. 12. *a*) A plot of the GK_{24} invariant Beltrami vector field $\mathbf{V}^{(\mathrm{GK}_{24}|D_1)}(\mathbf{r})$ obtained from the octahedral 12-point orbit in the cubic lattice (midpoints of the edges of a regular cube). The field is analytically defined in Eq. (7.52). *b*) A family of streamlines of this vector field with equally spaced initial conditions. In both pictures the circles denote the 16 isolated stagnation points of this flow

As is made manifest by the difference in the number of stagnation points and as can be visually appreciated by comparing Figs. 11 and 12, although their invariance groups are isomorphic, the Beltrami fields $\mathbf{V}^{(\mathrm{GP}_{24}|D_1)}(\mathbf{r})$ and $\mathbf{V}^{(\mathrm{GK}_{24}|D_1)}(\mathbf{r})$ are genuinely different. This is an important lesson to be remembered. The complete classification of all invariant Beltrami vector fields requires a complete classification of all subgroups of G_{1536} up to conjugation and not simply up to isomorphism. Furthermore, for all such subgroups one needs to find all singlet representations D_1.

7.3. The Lowest-Lying Octahedral Orbit of Length 8 in the Cubic Lattice. The next case we consider is the class of momentum vectors number 5 in our list of 48, namely:

$$\mathbf{k} = \{1 + 4\mu,\, 1 + 4\mu,\, 1 + 4\mu\}\,. \tag{7.55}$$

If we choose the lowest-lying representative of the class ($\mu = 0$), we obtain the following octahedral orbit of 8 points:

$$\begin{aligned} p_1 &= \{-1,-1,-1\}; & p_5 &= \{1,-1,-1\};\\ p_2 &= \{-1,-1,1\}; & p_6 &= \{1,-1,1\};\\ p_3 &= \{-1,1,-1\}; & p_7 &= \{1,1,-1\};\\ p_4 &= \{-1,1,1\}; & p_8 &= \{1,1,1\}. \end{aligned} \tag{7.56}$$

These 8 points are the vertices of a regular cube inscribed in the sphere of radius $r^2 = 3$, as is displayed in Fig. 13.

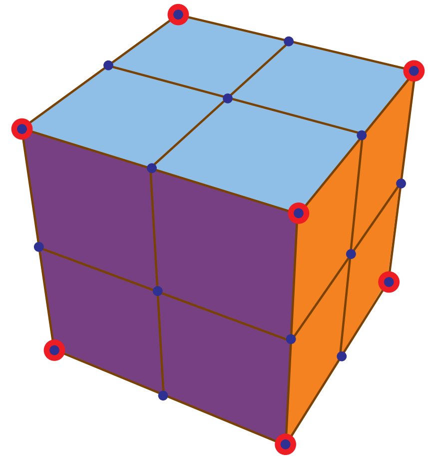

Fig. 13. A view of the octahedral 12-orbit in the cubic lattice, corresponding to the midpoints of the edges of a regular cube

Applying the strategy of sum-over-lattice points that belong to a point group orbit, we obtain the following 8-parameter vector field:

$$\begin{aligned}
&\mathbf{V}^{(8)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\}\,,\\
V_x &= \frac{2}{3}\Big(\sqrt{3}\sin(2\pi\Theta_4)\,(F_5 - F_1) + 3\cos(2\pi\Theta_4)\,(F_1 + F_5) +\\
&\quad + 3\cos(2\pi\Theta_3)\,(F_2 - F_6) + \sqrt{3}\sin(2\pi\Theta_3)\,(F_2 + F_6) +\\
&\quad + 3\cos(2\pi\Theta_2)\,(F_7 - F_3) - \sqrt{3}\sin(2\pi\Theta_2)\,(F_3 + F_7) +\\
&\quad + \sqrt{3}\sin(2\pi\Theta_1)\,(F_4 - F_8) - 3\cos(2\pi\Theta_1)\,(F_4 + F_8)\Big),\\
V_y &= \frac{2}{3}\Big(3\cos(2\pi\Theta_4)\,F_1 + 3\cos(2\pi\Theta_3)\,F_2 + 3\cos(2\pi\Theta_2)\,F_3 +\\
&\quad + 3\cos(2\pi\Theta_1)\,F_4 + \sqrt{3}\sin(2\pi\Theta_4)\,(F_1 + 2F_5) -\\
&\quad - \sqrt{3}\sin(2\pi\Theta_3)\,(F_2 - 2F_6) - \sqrt{3}\sin(2\pi\Theta_2)\,(F_3 - 2F_7) +\\
&\quad + \sqrt{3}\sin(2\pi\Theta_1)\,(F_4 + 2F_8)\Big),\\
V_z &= \frac{2}{3}\Big(-\sqrt{3}\sin(2\pi\Theta_4)\,(2F_1 + F_5) + \sqrt{3}\sin(2\pi\Theta_3)\,(F_6 - 2F_2) +\\
&\quad + \sqrt{3}\sin(2\pi\Theta_2)\,(F_7 - 2F_3) - \sqrt{3}\sin(2\pi\Theta_1)\,(2F_4 + F_8) +\\
&\quad + 3\,(\cos(2\pi\Theta_4)\,F_5 + \cos(2\pi\Theta_3)\,F_6 + \cos(2\pi\Theta_2)\,F_7 +\\
&\quad + \cos(2\pi\Theta_1)\,F_8)\Big),
\end{aligned} \tag{7.57}$$

where F_i $(i = 1, \ldots, 8)$ are real numbers, and the angles Θ_i are the following ones:

$$\Theta_1 = x + y + z;\ \Theta_2 = x + y - z;\ \Theta_3 = x - y + z;\ \Theta_4 = x - y - z. \tag{7.58}$$

The action of the octahedral group O_{24} is easily determined on such a vector field. We have

$$\forall\,\gamma \in \mathrm{O}_{24}\ :\quad \gamma^{-1}\cdot \mathbf{V}^{(8)}\left(\gamma\cdot\mathbf{r}|\mathbf{F}\right) = \mathbf{V}^{(8)}\left(\mathbf{r}|\,\mathfrak{R}^{(8)}[\gamma]\cdot\mathbf{F}\right), \tag{7.59}$$

where, as before, γ are the 3×3 matrices of the fundamental defining representation, while $\mathfrak{R}^{(8)}[\gamma]$ are 8×8 matrices acting on the parameter vector $\mathbf{F}$ that defines a reducible representation of O_{24}. The matrix representation of the two generators (4.6) is explicitly given by

$$\begin{aligned}
\mathfrak{R}^{(8)}[T] &= \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix},\\
\mathfrak{R}^{(8)}[S] &= \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1\\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}.
\end{aligned} \tag{7.60}$$

Retrieving from the above generators all the group elements and, in particular, a representative for each of the conjugacy classes, we easily compute the character vector of this representation. Explicitly we get

$$\chi\,[\mathbf{8}] = \{8, -1, 0, 0, 0\}. \tag{7.61}$$

The multiplicity vector is

$$\mathrm{m}\,[\mathbf{8}] = \{0, 0, 1, 1, 1\}, \tag{7.62}$$

implying that the eight-dimensional parameter space decomposes into a $D_3\,[\mathrm{O}_{24}, 2]$, plus a $D_4\,[\mathrm{O}_{24}, 3]$, plus a $D_5\,[\mathrm{O}_{24}, 3]$ representation.

Uplifting to the Universal Classifying Group. As in all the other cases, shifting the coordinate vector $\mathbf{r}$ by means of the constant vector $\frac{1}{4}\,\mathbf{n} = \frac{1}{4}\{n_1\,,\, n_2,\, n_3\,\}$ induces a rotation on the parameter-vector

$$\mathbf{V}^{(8)}\left(\mathbf{r} + \frac{1}{4}\,\mathbf{n}|\mathbf{F}\right) = \mathbf{V}^{(8)}\,(\mathbf{r}|\mathcal{M}_{\mathbf{n}}\mathbf{F})\,. \tag{7.63}$$

The explicit form of the matrix $\mathcal{M}_{\mathbf{n}}$ is given in Appendix F (see Eq. (F.2)). This information is sufficient to complete the uplifting of the 8-dimensional representation of the point group to a representation of the Universal Classifying Group G_{1536}. It turns out that with respect to this latter, the 8-dimensional representation is an irreducible one, precisely the $D_{30}\,[\mathrm{G}_{1536}, 8]$. The previous results are summarized in the following branching rule:

$$D_{30}\,[\mathrm{G}_{1536}, 8] = D_3\,[\mathrm{O}_{24}, 2] \oplus D_4\,[\mathrm{O}_{24}, 3] \oplus D_5\,[\mathrm{O}_{24}, 3]\,. \tag{7.64}$$

As it happened for the orbit of length 12, the branching rule (7.64) is uplifted to the subgroup G_{192}:

$$D_{30}\,[\mathrm{G}_{1536}, 8] = D_{18}\,[\mathrm{G}_{192}, 2] \oplus D_5\,[\mathrm{G}_{192}, 3] \oplus D_6\,[\mathrm{G}_{192}, 3]\,, \tag{7.65}$$

since we have

$$\begin{aligned} D_{18}\,[\mathrm{G}_{192}, 2] &= D_3\,[\mathrm{O}_{24}, 2]\,,\\ D_5\,[\mathrm{G}_{192}, 3] &= D_4\,[\mathrm{O}_{24}, 3]\,,\\ D_6\,[\mathrm{G}_{192}, 3] &= D_5\,[\mathrm{O}_{24}, 3]\,. \end{aligned} \tag{7.66}$$

Utilizing our character tables, we also verify that the decomposition of the representation $D_{30}\,[\mathrm{G}_{1536}, 8]$, with respect to the group GF_{192}, is identical to its decomposition with respect to the isomorphic (but not conjugate) group G_{192}, namely:

$$D_{30}\,[\mathrm{G}_{1536}, 8] = D_{18}\,[\mathrm{GF}_{192}, 2] \oplus D_5\,[\mathrm{GF}_{192}, 3] \oplus D_6\,[\mathrm{GF}_{192}, 3]\,, \tag{7.67}$$

and we obviously have

$$\begin{aligned} D_{18}\,[\mathrm{GF}_{192}, 2] &= D_3\,[\mathrm{GS}_{24}, 2]\,,\\ D_5\,[\mathrm{GF}_{192}, 3] &= D_4\,[\mathrm{GS}_{24}, 3]\,,\\ D_6\,[\mathrm{GF}_{192}, 3] &= D_5\,[\mathrm{GS}_{24}, 3]\,. \end{aligned} \tag{7.68}$$

From the above decompositions it appears that from the 8-parameter vector field (7.57) no instance can be extracted of the Beltrami vector field that is invariant either under O_{24} or under its homologous GS_{24}. Similarly by explicit decomposition (see the branching rules in Appendix D.4) one reaches the conclusion that no singlet D_1 representation emerges with respect to the groups GP_{24} or GK_{24}. There is, however, another pair of isomorphic (but not conjugate) subgroups $\mathrm{GS}_{32} \subset \mathrm{G}_{192} \subset \mathrm{G}_{1536}$ and $\mathrm{GK}_{32} \subset \mathrm{GF}_{192} \subset \mathrm{G}_{1536}$, both of order 32, with respect to which singlet invariant vector fields do exist. They are described in the next section.

7.3.1. Beltrami Vector Fields with Hidden Symmetry GS_{32} *and* GK_{32}*, Respectively.* The subgroups GS_{32} and GK_{32} are explicitly described in Secs. A.12 and A.13. Their structure is very simple. They share a normal Abelian subgroup of order 16, named G_{16}, which is isomorphic to $\mathbb{Z}_2^4$:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \sim \mathrm{G}_{16} \lhd \begin{cases} \lhd\, \mathrm{GS}_{32} \subset \mathrm{G}_{192} \\ \lhd\, \mathrm{GK}_{32} \subset \mathrm{GF}_{192}\,. \end{cases} \tag{7.69}$$

The branching rules of the G_{192} representations (7.66) with respect to GS_{32} are the following ones:

$$\begin{aligned} D_{18}\left[\mathrm{G}_{192}, 2\right] &= D_1\left[\mathrm{GS}_{32}, 1\right] \oplus D_2\left[\mathrm{GS}_{32}, 1\right], \\ D_5\left[\mathrm{G}_{192}, 3\right] &= D_3\left[\mathrm{GS}_{32}, 1\right] \oplus D_9\left[\mathrm{GS}_{32}, 2\right], \\ D_6\left[\mathrm{G}_{192}, 3\right] &= D_4\left[\mathrm{GS}_{32}, 1\right] \oplus D_9\left[\mathrm{GS}_{32}, 2\right], \end{aligned} \tag{7.70}$$

and we similarly have

$$\begin{aligned} D_{18}\left[\mathrm{GF}_{192}, 2\right] &= D_1\left[\mathrm{GK}_{32}, 1\right] \oplus D_2\left[\mathrm{GK}_{32}, 1\right], \\ D_5\left[\mathrm{GF}_{192}, 3\right] &= D_3\left[\mathrm{GK}_{32}, 1\right] \oplus D_9\left[\mathrm{GK}_{32}, 2\right], \\ D_6\left[\mathrm{GF}_{192}, 3\right] &= D_4\left[\mathrm{GK}_{32}, 1\right] \oplus D_9\left[\mathrm{GK}_{32}, 2\right]. \end{aligned} \tag{7.71}$$

The two identity D_1 representations appearing in Eqs. (7.70) and (7.71) signalize that from this orbit we can construct the Beltrami vector fields invariant with respect either to GS_{32} or to GK_{32}. Previous experience tells us that they should be physically different flows although they have isomorphic hidden symmetries. We construct them in the next two subsections.

The Beltrami Vector Field Invariant under GS_{32}. Performing the projection onto the $D_1\left[\mathrm{GS}_{32}, 1\right]$ representation, we get the following Beltrami vector field:

$$\mathbf{V}^{(\mathrm{GS}_{32})}(\mathbf{r}) = \{V_x,\, V_y,\, V_z\}\,,$$

$$V_x = 8\cos(2\pi x)\sin(2\pi y)\sin(2\pi z),$$

$$
\begin{aligned}
V_y =& -\cos(2\pi(x-y-z)) - \cos(2\pi(x+y-z)) + \cos(2\pi(x-y+z)) + \\
& + \cos(2\pi(x+y+z)) + \sqrt{3}(-\sin(2\pi(x-y-z)) + \\
& + \sin(2\pi(x+y-z)) - \sin(2\pi(x-y+z)) + \sin(2\pi(x+y+z))), \\
V_z =& -\cos(2\pi(x-y-z)) + \cos(2\pi(x+y-z)) - \cos(2\pi(x-y+z)) + \\
& + \cos(2\pi(x+y+z)) + \sqrt{3}(\sin(2\pi(x-y-z)) + \\
& + \sin(2\pi(x+y-z)) - \sin(2\pi(x-y+z)) - \sin(2\pi(x+y+z))).
\end{aligned}
\tag{7.72}
$$

The vector field (7.72) has 53 stagnation points in the unit cube:

$$
\mathbf{V}^{(\mathrm{GS}_{32}|D_1)}(\mathbf{s}_i), \quad i = 1, \dots, 53, \tag{7.73}
$$

whose explicit form is given here below

$$
\begin{aligned}
s_1 &= \{0,0,0\}; & s_2 &= \left\{0,0,\tfrac{1}{2}\right\}; & s_3 &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_4 &= \left\{0,\tfrac{1}{2},0\right\}; & s_5 &= \left\{0,\tfrac{1}{2},\tfrac{1}{2}\right\}; & s_6 &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_7 &= \{0,1,0\}; & s_8 &= \left\{0,1,\tfrac{1}{2}\right\}; & s_9 &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{10} &= \left\{\tfrac{1}{4},0,0\right\}; & s_{11} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; & s_{12} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{13} &= \left\{\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}\right\}; & s_{14} &= \left\{\tfrac{1}{4},\tfrac{1}{4},\tfrac{3}{4}\right\}; & s_{15} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{16} &= \left\{\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{2}\right\}; & s_{17} &= \left\{\tfrac{1}{4},\tfrac{1}{2},1\right\}; & s_{18} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{19} &= \left\{\tfrac{1}{4},\tfrac{3}{4},\tfrac{3}{4}\right\}; & s_{20} &= \left\{\tfrac{1}{4},1,0\right\}; & s_{21} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{22} &= \left\{\tfrac{1}{4},1,1\right\}; & s_{23} &= \left\{\tfrac{1}{2},0,0\right\}; & s_{24} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{25} &= \left\{\tfrac{1}{2},0,1\right\}; & s_{26} &= \left\{\tfrac{1}{2},\tfrac{1}{2},0\right\}; & s_{27} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{28} &= \left\{\tfrac{1}{2},\tfrac{1}{2},1\right\}; & s_{29} &= \left\{\tfrac{1}{2},1,0\right\}; & s_{30} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{31} &= \left\{\tfrac{1}{2},1,1\right\}; & s_{32} &= \left\{\tfrac{3}{4},0,0\right\}; & s_{33} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{34} &= \left\{\tfrac{3}{4},0,1\right\}; & s_{35} &= \left\{\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4}\right\}; & s_{36} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{37} &= \left\{\tfrac{3}{4},\tfrac{1}{2},0\right\}; & s_{38} &= \left\{\tfrac{3}{4},\tfrac{1}{2},\tfrac{1}{2}\right\}; & s_{39} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{40} &= \left\{\tfrac{3}{4},\tfrac{3}{4},\tfrac{1}{4}\right\}; & s_{41} &= \left\{\tfrac{3}{4},\tfrac{3}{4},\tfrac{3}{4}\right\}; & s_{42} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{43} &= \left\{\tfrac{3}{4},1,\tfrac{1}{2}\right\}; & s_{44} &= \left\{\tfrac{3}{4},1,1\right\}; & s_{45} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{46} &= \left\{1,0,\tfrac{1}{2}\right\}; & s_{47} &= \{1,0,1\}; & s_{48} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}; \\
s_{49} &= \left\{1,\tfrac{1}{2},\tfrac{1}{2}\right\}; & s_{50} &= \left\{1,\tfrac{1}{2},1\right\}; & s_{51} &= \left\{\tfrac{1}{4},0,\tfrac{1}{2}\right\}.
\end{aligned}
\tag{7.74}
$$

A plot of this vector field with a family of its streamlines is displayed in Fig. 14.

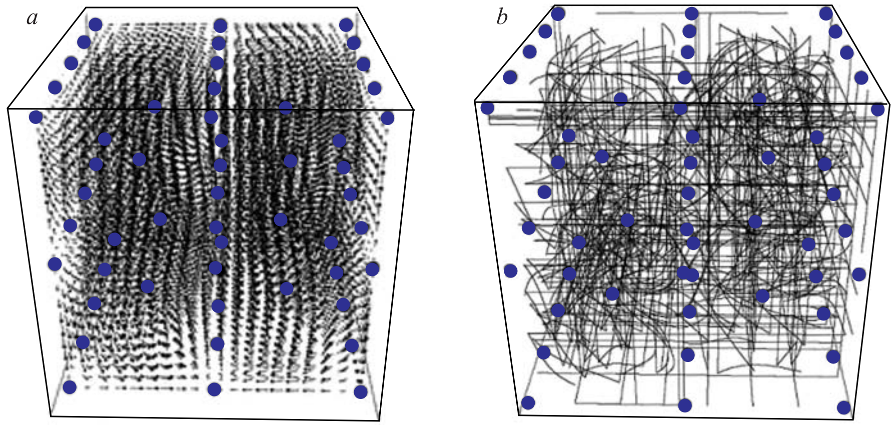

Fig. 14. *a*) A plot of the vector field $\mathbf{V}^{(\mathrm{GS}_{32})}(\mathbf{r})$ defined in Eq. (7.72) which is invariant under the group GS_{32}. *b*) A family of streamlines of such a flow is displayed. As in the previous figures the circles denote the 53 stagnation points

The Beltrami Vector Field Invariant under GK_{32}. Performing the projection onto the $D_1\,[\mathrm{GK}_{32}, 1]$ representation, we get the following Beltrami vector field:

$$\begin{aligned}
&\mathbf{V}^{(\mathrm{GK}_{32})}(\mathbf{r}) = \{V_x,\, V_y,\, V_z\}\,,\\
V_x &= (-9+\sqrt{3})\cos(2\pi(x-y-z)) - (-9+\sqrt{3})\cos(2\pi(x+y-z)) -\\
&\quad - (-9+\sqrt{3})\cos(2\pi(x-y+z)) + (-9+\sqrt{3})\cos(2\pi(x+y+z)) -\\
&\quad - \frac{1}{3}(15+7\sqrt{3})(\sin(2\pi(x-y-z)) +\\
&\quad + \sin(2\pi(x+y-z)) + \sin(2\pi(x-y+z)) + \sin(2\pi(x+y+z))),\\
V_y &= (-1+3\sqrt{3})\cos(2\pi(x-y-z)) + (-1+3\sqrt{3})\cos(2\pi(x+y-z)) +\\
&\quad + (1-3\sqrt{3})\cos(2\pi(x-y+z)) + (1-3\sqrt{3})\cos(2\pi(x+y+z)) +\\
&\quad + \frac{1}{3}(3+17\sqrt{3})(-\sin(2\pi(x-y-z)) + \\
&\quad + \sin(2\pi(x+y-z)) - \sin(2\pi(x-y+z)) + \sin(2\pi(x+y+z))),\\
V_z &= \frac{2}{3}\Big(-3(4+\sqrt{3})\cos(2\pi(x-y-z)) + 3(4+\sqrt{3})\cos(2\pi(x+y-z)) -\\
&\quad - 3(4+\sqrt{3})\cos(2\pi(x-y+z)) + 3(4+\sqrt{3})\cos(2\pi(x+y+z)) +\\
&\quad + (-6+5\sqrt{3})\big(\sin(2\pi(x-y-z)) + \sin(2\pi(x+y-z)) -\\
&\quad - \sin(2\pi(x-y+z)) - \sin(2\pi(x+y+z))\big)\Big).
\end{aligned} \tag{7.75}$$

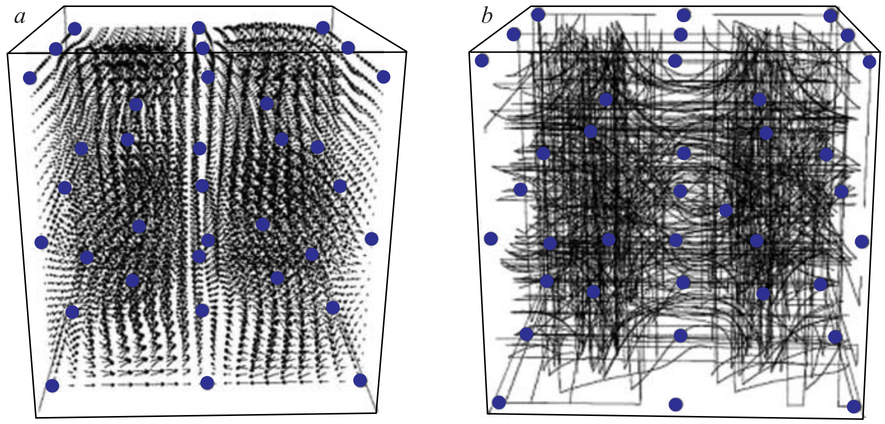

Fig. 15. *a*) A plot of the vector field $\mathbf{V}^{(\mathrm{GK}_{32})}(\mathbf{r})$ defined in Eq. (7.75) which is invariant under the group GK_{32}. *b*) A family of streamlines of such a flow is displayed. The circles denote the 35 stagnation points of this flow

This vector field has 35 stagnation points, namely, all those listed in Eq. (7.74) with the exception of the 18 listed below:

$$\text{not stagnation} = \big\{s_{10}, s_{11}, s_{12}, s_{15}, s_{16}, s_{17}, \\ s_{20}, s_{21}, s_{22}, s_{32}, s_{33}, s_{34}, s_{37}, s_{38}, s_{39}, s_{42}, s_{43}, s_{44}\big\}. \quad (7.76)$$

A plot of this vector field with a family of its streamlines is displayed in Fig. 15

7.4. Example of an Octahedral Orbit of Length 24 in the Cubic Lattice. As the last example in the present discussion we consider the case numbered 13 in our list of 48 momentum vector classes, namely:

$$\mathbf{k} = \{1 + 4\,\mu,\, 1 + 4\,\mu, 2 + 4\,\rho\}. \quad (7.77)$$

Choosing the lowest-lying representative in the class ($\mu = \rho = 0$), we obtain the octahedral point orbit of order 24 listed below:

$$\mathcal{O}^{(24)}_{1,1,2} = \left\{ \begin{array}{cccc} \{-2,-1,-1\}, & \{-2,-1,1\}, & \{-2,1,-1\}, & \{-2,1,1\}, \\ \{-1,-2,-1\}, & \{-1,-2,1\}, & \{-1,-1,-2\}, & \{-1,-1,2\}, \\ \{-1,1,-2\}, & \{-1,1,2\}, & \{-1,2,-1\}, & \{-1,2,1\}, \\ \{1,-2,-1\}, & \{1,-2,1\}, & \{1,-1,-2\}, & \{1,-1,2\}, \\ \{1,1,-2\}, & \{1,1,2\}, & \{1,2,-1\}, & \{1,2,1\}, \\ \{2,-1,-1\}, & \{2,-1,1\}, & \{2,1,-1\}, & \{2,1,1\} \end{array} \right\}. \quad (7.78)$$

A geometrical picture of these points in the lattice is displayed in Fig. 16.

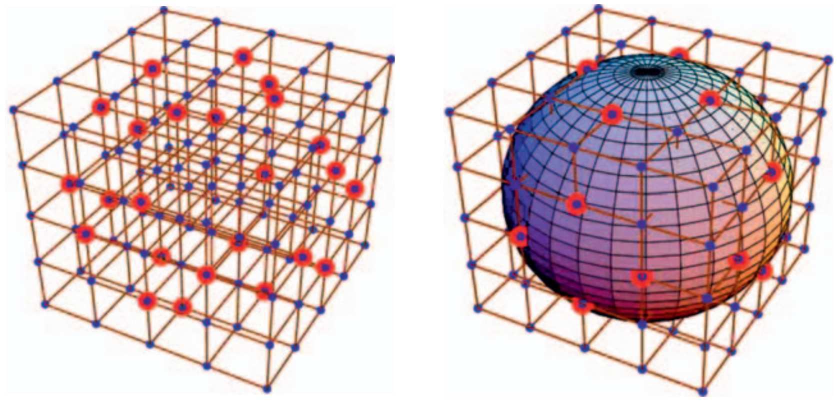

Fig. 16. A view of the considered orbit of length 24 in the cubic lattice: the lattice points intersect the sphere of radius $r^2 = 6$

Starting from the above point orbit, the construction algorithm (3.1) produces a vector field containing 24 parameters which has the following form:

$$\mathbf{V}^{(24)}(\mathbf{r}|\mathbf{F}) = \bigoplus_{i=1}^{6} \Delta V_i^{(24)}(\mathbf{r}|\mathbf{F}), \tag{7.79}$$

where

$$\Delta V_1^{(24)}(\mathbf{r}|\mathbf{F}) = \\ = \begin{pmatrix} -(F_{12}+F_{24})\,\Omega_1 + (F_{23}-F_{11})\,\Omega_2 + (F_{10}-F_{22})\,\Omega_3 + (F_9+F_{21})\,\Omega_4 \\ 2\,(F_{12}\Omega_1 + F_{11}\Omega_2 + F_{10}\Omega_3 + F_9\Omega_4) \\ 2\,(F_{24}\Omega_1 + F_{23}\Omega_2 + F_{22}\Omega_3 + F_{21}\Omega_4) \end{pmatrix}, \tag{7.80}$$

$$\Delta V_2^{(24)}(\mathbf{r}|\mathbf{F}) = \begin{pmatrix} -2\,(2F_8+F_{20})\,\Omega_5 + (2F_{19}-4F_7)\,\Omega_6 - \\ \quad -2\,(F_6+2F_{18})\,\Omega_7 - 2\,(F_5-2F_{17})\,\Omega_8 \\ 2\,(F_8\Omega_5 + F_7\Omega_6 + F_6\Omega_7 + F_5\Omega_8) \\ 2\,(F_{20}\Omega_5 + F_{19}\Omega_6 + F_{18}\Omega_7 + F_{17}\Omega_8) \end{pmatrix}, \tag{7.81}$$

$$\Delta V_3^{(24)}(\mathbf{r}|\mathbf{F}) = \begin{pmatrix} 2\,(F_4-2F_{16})\,\Omega_9 + 2\,(F_3+2F_{15})\,\Omega_{10} + \\ \quad +(4F_2-2F_{14})\,\Omega_{11} + 2\,(2F_1+F_{13})\,\Omega_{12} \\ 2\,(F_4\Omega_9 + F_3\Omega_{10} + F_2\Omega_{11} + F_1\Omega_{12}) \\ 2\,(F_{16}\Omega_9 + F_{15}\Omega_{10} + F_{14}\Omega_{11} + F_{13}\Omega_{12}) \end{pmatrix}, \tag{7.82}$$

$$\Delta V_4^{(24)}(\mathbf{r}|\mathbf{F}) = $$
$$= \begin{pmatrix} \sqrt{\frac{2}{3}}\big((F_{12}-F_{24})\,\Omega_{13} - (F_{11}+F_{23})\,\Omega_{14} + (F_{10}+F_{22})\,\Omega_{15} + \\ \quad + (F_{21}-F_9)\,\Omega_{16}\big) \\ \frac{(F_{12}+5F_{24})\Omega_{13}-(F_{11}-5F_{23})\Omega_{14}-(F_{10}-5F_{22})\Omega_{15}+(F_9+5F_{21})\Omega_{16}}{\sqrt{6}} \\ \frac{-(5F_{12}+F_{24})\Omega_{13}+(F_{23}-5F_{11})\Omega_{14}+(F_{22}-5F_{10})\Omega_{15}-(5F_9+F_{21})\Omega_{16}}{\sqrt{6}} \end{pmatrix}, \quad (7.83)$$

$$\Delta V_5^{(24)}(\mathbf{r}|\mathbf{F}) = \begin{pmatrix} \sqrt{\frac{2}{3}}\big((F_8-2F_{20})\,\Omega_{17} - (F_7+2F_{19})\,\Omega_{18} + \\ \quad + (2F_6-F_{18})\,\Omega_{19} - (2F_5+F_{17})\,\Omega_{20}\big) \\ \sqrt{\frac{2}{3}}\big(2\,(F_8+F_{20})\,\Omega_{17} + 2\,(F_{19}-F_7)\,\Omega_{18} + \\ \quad + (2F_6+5F_{18})\,\Omega_{19} + (5F_{17}-2F_5)\,\Omega_{20}\big) \\ \sqrt{\frac{2}{3}}\big(-(5F_8+2F_{20})\,\Omega_{17} + (2F_{19}-5F_7)\,\Omega_{18} - \\ \quad -2\,(F_6+F_{18})\,\Omega_{19} + 2\,(F_{17}-F_5)\,\Omega_{20}\big) \end{pmatrix}, \quad (7.84)$$

$$\Delta V_6^{(24)}(\mathbf{r}|\mathbf{F}) = \begin{pmatrix} \sqrt{\frac{2}{3}}\big((2F_4+F_{16})\,\Omega_{21} + (F_{15}-2F_3)\,\Omega_{22} + \\ \quad + (F_2+2F_{14})\,\Omega_{23} - (F_1-2F_{13})\,\Omega_{24}\big) \\ \sqrt{\frac{2}{3}}\big((5F_{16}-2F_4)\,\Omega_{21} + (2F_3+5F_{15})\,\Omega_{22} + \\ \quad +2\,(F_{14}-F_2)\,\Omega_{23} + 2\,(F_1+F_{13})\,\Omega_{24}\big) \\ \sqrt{\frac{2}{3}}\big(2\,(F_{16}-F_4)\,\Omega_{21} - 2\,(F_3+F_{15})\,\Omega_{22} + \\ \quad + (2F_{14}-5F_2)\,\Omega_{23} - (5F_1+2F_{13})\,\Omega_{24}\big) \end{pmatrix}, \quad (7.85)$$

F_i being the 24 real parameters, Ω_i denoting the 24 periodic basis functions:

$$\begin{aligned} &\Omega_1 = \cos(2\pi\Theta_1)\,; \ \Omega_2 = \cos(2\pi\Theta_2)\,; \quad \Omega_3 = \cos(2\pi\Theta_3)\,; \quad \Omega_4 = \cos(2\pi\Theta_4)\,; \\ &\Omega_5 = \cos(2\pi\Theta_5)\,; \ \Omega_6 = \cos(2\pi\Theta_6)\,; \quad \Omega_7 = \cos(2\pi\Theta_7)\,; \quad \Omega_8 = \cos(2\pi\Theta_8)\,; \\ &\Omega_9 = \cos(2\pi\Theta_9)\,; \ \Omega_{10} = \cos(2\pi\Theta_{10})\,; \ \Omega_{11} = \cos(2\pi\Theta_{11})\,; \ \Omega_{12} = \cos(2\pi\Theta_{12})\,; \\ &\Omega_{13} = \sin(2\pi\Theta_1)\,; \ \Omega_{14} = \sin(2\pi\Theta_2)\,; \quad \Omega_{15} = \sin(2\pi\Theta_3)\,; \quad \Omega_{16} = \sin(2\pi\Theta_4)\,; \\ &\Omega_{17} = \sin(2\pi\Theta_5)\,; \ \Omega_{18} = \sin(2\pi\Theta_6)\,; \quad \Omega_{19} = \sin(2\pi\Theta_7)\,; \quad \Omega_{20} = \sin(2\pi\Theta_8)\,; \\ &\Omega_{21} = \sin(2\pi\Theta_9)\,; \ \Omega_{22} = \sin(2\pi\Theta_{10})\,; \ \Omega_{23} = \sin(2\pi\Theta_{11})\,; \ \Omega_{24} = \sin(2\pi\Theta_{12})\,, \end{aligned} \quad (7.86)$$

and the 12 independent arguments of the trigonometric functions being those listed below:

$$\begin{aligned}
\Theta_1 &= 2x + y + z, \\
\Theta_2 &= 2x + y - z, \\
\Theta_3 &= 2x - y + z, \\
\Theta_4 &= 2x - y - z, \\
\Theta_5 &= x + 2y + z, \\
\Theta_6 &= x + 2y - z, \\
\Theta_7 &= x + y + 2z, \\
\Theta_8 &= x + y - 2z, \\
\Theta_9 &= x - y + 2z, \\
\Theta_{10} &= x - y - 2z, \\
\Theta_{11} &= x - 2y + z, \\
\Theta_{12} &= x - 2y - z.
\end{aligned} \tag{7.87}$$

7.4.1. Point Group Irreps and Uplifting to the Universal Classifying Group. As in all previous cases we can easily derive the representation of the point group O_{24} and of the quantized translations on the parameter space provided by 24×24 matrices. We do not display the explicit form of the generators of G_{1536} since they are too large to fit on paper. We just encode the relevant information in the form of the splitting of the 24-parameter space in irreducible representation of G_{1536} and of its relevant subgroups.

Firstly, we find that the 24-dimensional representation of G_{1536} is reducible and splits as follows:

$$\mathfrak{R}_{(1,1,2)}\left[\mathrm{G}_{1536}, 24\right] = D_{34}\left[\mathrm{G}_{1536}, 12\right] \oplus D_{35}\left[\mathrm{G}_{1536}, 12\right]. \tag{7.88}$$

Secondly, we recall from Appendix D the following branching rules with respect to the subgroups G_{192} and GF_{192}:

$$\begin{aligned}
D_{34}\left[\mathrm{G}_{1536}, 12\right] &= \begin{cases} D_{10}\left[\mathrm{G}_{192}, 3\right] \oplus D_{14}\left[\mathrm{G}_{192}, 3\right] \oplus D_{19}\left[\mathrm{G}_{192}, 6\right] \\ D_{9}\left[\mathrm{GF}_{192}, 3\right] \oplus D_{13}\left[\mathrm{GF}_{192}, 3\right] \oplus D_{19}\left[\mathrm{GF}_{192}, 6\right] \end{cases}, \\
D_{35}\left[\mathrm{G}_{1536}, 12\right] &= \begin{cases} D_{9}\left[\mathrm{G}_{192}, 3\right] \oplus D_{13}\left[\mathrm{G}_{192}, 3\right] \oplus D_{19}\left[\mathrm{G}_{192}, 6\right] \\ D_{10}\left[\mathrm{GF}_{192}, 3\right] \oplus D_{14}\left[\mathrm{GF}_{192}, 3\right] \oplus D_{19}\left[\mathrm{GF}_{192}, 6\right] \end{cases}.
\end{aligned} \tag{7.89}$$

Thirdly, we consider the following branching rules of the considered irreps of the groups G_{192} and GF_{192} with respect to their subgroups O_{24} and GS_{24}:

$$\begin{aligned}
D_{19}\left[\mathrm{G}_{192},6\right] &= D_4\left[\mathrm{O}_{24},3\right] \oplus D_5\left[\mathrm{O}_{24},3\right],\\
D_{14}\left[\mathrm{G}_{192},3\right] &= D_1\left[\mathrm{O}_{24},1\right] \oplus D_3\left[\mathrm{O}_{24},2\right],\\
D_{13}\left[\mathrm{G}_{192},3\right] &= D_2\left[\mathrm{O}_{24},1\right] \oplus D_3\left[\mathrm{O}_{24},2\right],\\
D_{10}\left[\mathrm{G}_{192},3\right] &= D_5\left[\mathrm{O}_{24},3\right],\\
D_{9}\left[\mathrm{G}_{192},3\right] &= D_5\left[\mathrm{O}_{24},3\right],
\end{aligned} \tag{7.90}$$

$$\begin{aligned}
D_{19}\left[\mathrm{GF}_{192},6\right] &= D_4\left[\mathrm{GS}_{24},3\right] \oplus D_5\left[\mathrm{GS}_{24},3\right],\\
D_{14}\left[\mathrm{GF}_{192},3\right] &= D_1\left[\mathrm{GS}_{24},1\right] \oplus D_3\left[\mathrm{GS}_{24},2\right],\\
D_{13}\left[\mathrm{GF}_{192},3\right] &= D_2\left[\mathrm{GS}_{24},1\right] \oplus D_3\left[\mathrm{GS}_{24},2\right],\\
D_{10}\left[\mathrm{GF}_{192},3\right] &= D_5\left[\mathrm{GS}_{24},3\right],\\
D_{9}\left[\mathrm{GF}_{192},3\right] &= D_5\left[\mathrm{GS}_{24},3\right].
\end{aligned} \tag{7.91}$$

From inspection of Eqs. (7.90) and (7.91) we conclude that there is a singlet representation both of the point group O_{24} and of its homologous nonconjugate copy GS_{24}. Hence from this orbit we can construct the Beltrami vector fields with O_{24} or GS_{24} hidden symmetry. This is certainly true, but the situation is even better. There exists a subgroup named by us Oh_{48} (see Appendix A.8 for its description), which is isomorphic to the extended octahedral group, and it is embedded in G_{1536} in the following way:

$$\mathrm{O}_{24} \subset \mathrm{Oh}_{48} \subset \mathrm{G}_{192} \subset \mathrm{G}_{1536}. \tag{7.92}$$

The branching rule of the entire 24-dimensional representation of the classifying group with respect to Oh_{48} is the following one:

$$\begin{aligned}
\mathfrak{R}_{(1,1,2)}\left[\mathrm{G}_{1536},24\right] = D_1\left[\mathrm{Oh}_{48},1\right] \oplus D_3\left[\mathrm{Oh}_{48},1\right] \oplus 2\,D_5\left[\mathrm{Oh}_{48},2\right] \oplus \\
\oplus\ 3\,D_7\left[\mathrm{Oh}_{48},3\right] \oplus 3\,D_9\left[\mathrm{Oh}_{48},3\right].
\end{aligned} \tag{7.93}$$

Hence there exists an invariant vector field with respect to the order 48 subgroup Oh_{48}. This is certainly invariant with respect to all subgroups of the same, in particular, O_{24}. As we have only one O_{24} singlet, it means that the unique vector field invariant under O_{24} has actually an enhanced symmetry Oh_{48}. Furthermore, the isomorphism of G_{192} and GF_{192} implies that there must exist another subgroup $\mathrm{OKh}_{48} \sim \mathrm{Oh}_{48}$ also isomorphic to the extended octahedral group and satisfying the inclusion relations homologous to those displayed in Eq. (7.92), namely:

$$\mathrm{GS}_{24} \subset \mathrm{OKh}_{48} \subset \mathrm{GF}_{192} \subset \mathrm{G}_{1536}. \tag{7.94}$$

The branching rules in Eqs. (7.90) and (7.91) imply that the singlet vector field with respect to GS_{24} is actually invariant with respect to the order 48 group OKh_{48}. We have not constructed the vector field invariant with respect to OKh_{48} and we just constructed the Oh_{48}-invariant one.

7.4.2. The Beltrami Flow Invariant under Oh_{48}. Applying the projector onto the irrep $D_1\,[\mathrm{Oh}_{48}, 1]$, we obtain the following Beltrami vector field:

$$\begin{aligned}
&\mathbf{V}^{(\mathrm{Oh}_{48}|D_1)}(\mathbf{r}) = \{V_x,\, V_y,\, V_z\}\,,\\
V_x &= 18\Omega_5 - 18\Omega_6 - 18\Omega_7 + 18\Omega_8 + 18\Omega_9 - 18\Omega_{10} - 18\Omega_{11} + 18\Omega_{12} +\\
&\quad + \sqrt{6}\big(-\Omega_{13} - \Omega_{14} - \Omega_{15} - \Omega_{16} + \Omega_{17} + \Omega_{18} + \Omega_{19} +\\
&\quad + \Omega_{20} + \Omega_{21} + \Omega_{22} + \Omega_{23} + \Omega_{24}\big), \qquad (7.95)\\
V_y &= -3\Omega_1 + 3\Omega_2 - 3\Omega_3 + 3\Omega_4 - 6\Omega_5 + 6\Omega_6 + 6\Omega_7 - 6\Omega_8 + 6\Omega_9 -\\
&\quad - 6\Omega_{10} - 6\Omega_{11} + 6\Omega_{12} + \sqrt{6}\big(\Omega_{13} + \Omega_{14} - \Omega_{15} - \Omega_{16} - 4\Omega_{17} -\\
&\quad - 4\Omega_{18} + 7\Omega_{19} + 7\Omega_{20} - 7\Omega_{21} - 7\Omega_{22} + 4\Omega_{23} + 4\Omega_{24}\big),\\
V_z &= 3\Omega_1 + 3\Omega_2 - 3\Omega_3 - 3\Omega_4 - 6\Omega_5 - 6\Omega_6 + 6\Omega_7 + 6\Omega_8 - 6\Omega_9 - 6\Omega_{10} +\\
&\quad + 6\Omega_{11} + 6\Omega_{12} + \sqrt{6}\big(\Omega_{13} - \Omega_{14} + \Omega_{15} - \Omega_{16} + 7\Omega_{17} -\\
&\quad - 7\Omega_{18} - 4\Omega_{19} + 4\Omega_{20} - 4\Omega_{21} + 4\Omega_{22} + 7\Omega_{23} - 7\Omega_{24}\big).
\end{aligned}$$

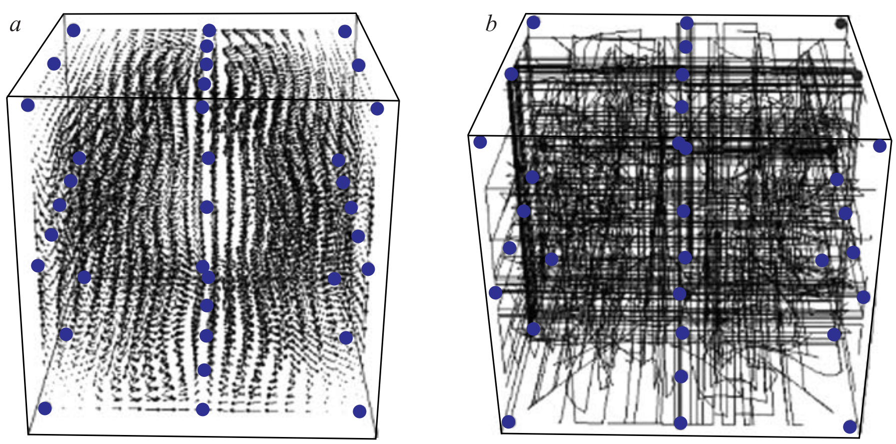

Fig. 17. *a*) A plot of the Beltrami vector field $\mathbf{V}^{(\mathrm{Oh}_{48}|D_1)}(\mathbf{r})$ defined in Eq. (7.95), which is invariant under the group Oh_{48} isomorphic to the extended octahedral group. *b*) A family of streamlines of such a flow is displayed

A plot of the vector field and a family of its streamlines are displayed in Fig. 17. Note that this vector field has 35 stagnation points whose coordinates we do not display for brevity. The stagnation points are as usual denoted by circles in Fig. 17.

8. THE HEXAGONAL LATTICE AND THE DIHEDRAL GROUP $\mathcal{D}_6$

We come next to a quite short discussion of the hexagonal lattice. In this case we do not construct the Universal Classifying Group and we limit ourselves to display some solutions of the Beltrami equation corresponding to the lowest-lying orbits of the point group of this lattice which is the dihedral group $\mathcal{D}_6$.

Our main purpose is to illustrate, by means of this example, the new features that appear when the lattice is not self-dual. Since in this section all considered representations are relative to the point group, we simplify the notation mentioning the irreps only as $D_1, \ldots, D_6$ without writing in square brackets the group.

The Hexagonal Lattice. Λ_{hex} and its dual $\Lambda^{\star}_{\text{hex}}$ are displayed in Figs. 18, 19, and 20.These lattices are not self-dual and there is a constant metric which is not diagonal.

The basis vectors of the hexagonal space lattice Λ_{hex} are the following ones:

$$\vec{\mathbf{w}}_1 = \{1,0,0\}; \quad \vec{\mathbf{w}}_2 = \left\{\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right\}; \quad \vec{\mathbf{w}}_3 = \{0,0,1\}, \tag{8.1}$$

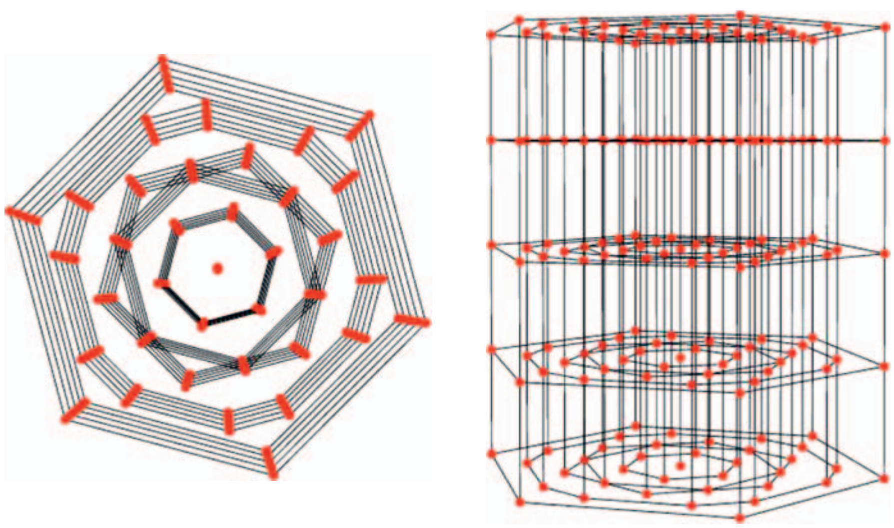

Fig. 18. A view of the hexagonal space lattice Λ_{hex}, seen from above and in a front view

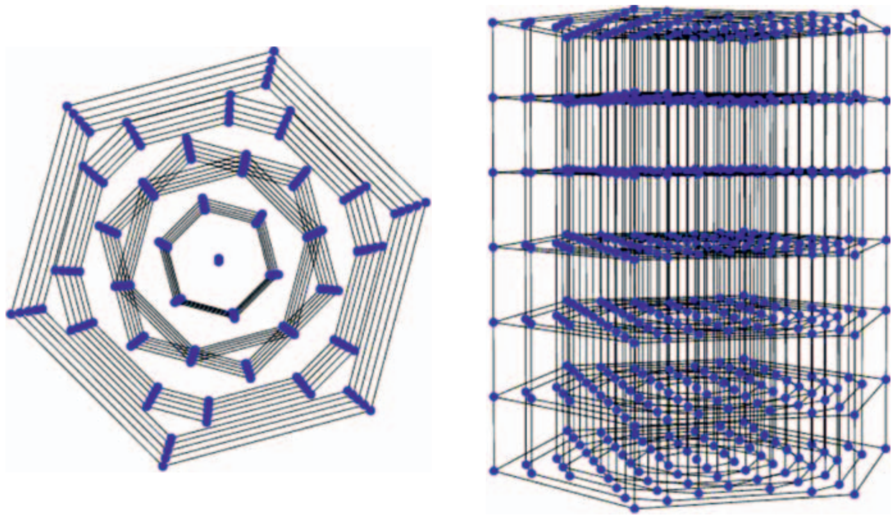

Fig. 19. A view of the hexagonal momentum lattice $\Lambda^{\star}_{\mathrm{hex}}$, seen from above and in a front view

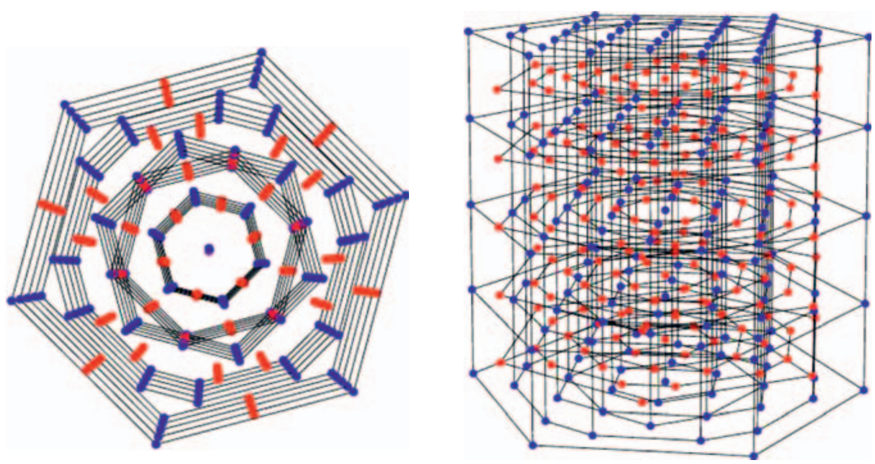

Fig. 20 (color online). A comparative view of the hexagonal space and momentum lattices, seen from above and in a front view. The blue points are momentum lattice points ($\in \Lambda^{\star}_{\mathrm{hex}}$), while the red points are space lattice points ($\in \Lambda_{\mathrm{hex}}$)

which implies that the metric is the following nondiagonal one:

$$g_{\mu\nu} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{3}{4} \end{pmatrix}. \tag{8.2}$$

The basis vectors $\vec{\mathbf{e}}^{\mu}$ of the dual momentum lattice $\Lambda^{\star}_{\text{hex}}$ do not coincide with those of the lattice Λ_{hex}. They are the following ones:

$$\vec{\mathbf{e}}^1 = \left\{1, \frac{1}{\sqrt{3}}, 0\right\}; \quad \vec{\mathbf{e}}^2 = \left\{0, -\frac{2}{\sqrt{3}}, 0\right\}; \quad \vec{\mathbf{e}}^3 = \left\{0, 0, \frac{2}{\sqrt{3}}\right\}. \tag{8.3}$$

The subgroup of the proper rotation group which maps the cubic lattice into itself is the dihedral group d_6 whose order is 12. In the next subsection, we recall its structure.

8.1. The Dihedral Group $\mathcal{D}_6$. Abstractly, the dihedral $\mathcal{D}_6$ group is defined by the following generators and relations:

$$A,\ B: \quad A^6 = \mathbf{e}; \quad B^2 = \mathbf{e}; \quad (BA)^2 = \mathbf{e}. \tag{8.4}$$

Table 3. **Conjugacy classes of the dihedral group $\mathcal{D}_6$**

e	1_1	$=$	$\{x, y, z\}$
	2_1	$=$	$\left\{\frac{1}{2}\left(x+\sqrt{3}y\right), \frac{1}{2}\left(y-\sqrt{3}x\right), z\right\}$
A	2_2	$=$	$\left\{\frac{1}{2}\left(x-\sqrt{3}y\right), \frac{1}{2}\left(\sqrt{3}x+y\right), z\right\}$
	3_1	$=$	$\left\{\frac{1}{2}\left(\sqrt{3}y-x\right), \frac{1}{2}\left(-\sqrt{3}x-y\right), z\right\}$
A^2	3_2	$=$	$\left\{\frac{1}{2}\left(-x-\sqrt{3}y\right), \frac{1}{2}\left(\sqrt{3}x-y\right), z\right\}$
A^3	4_1	$=$	$\{-x, -y, z\}$
	5_1	$=$	$\{-x, y, -z\}$
B	5_2	$=$	$\left\{\frac{1}{2}\left(x-\sqrt{3}y\right), \frac{1}{2}\left(-\sqrt{3}x-y\right), -z\right\}$
	5_3	$=$	$\left\{\frac{1}{2}\left(x+\sqrt{3}y\right), \frac{1}{2}\left(\sqrt{3}x-y\right), -z\right\}$
	6_1	$=$	$\left\{\frac{1}{2}\left(-x-\sqrt{3}y\right), \frac{1}{2}\left(y-\sqrt{3}x\right), -z\right\}$
BA	6_2	$=$	$\{x, -y, -z\}$
	6_3	$=$	$\left\{\frac{1}{2}\left(\sqrt{3}y-x\right), \frac{1}{2}\left(\sqrt{3}x+y\right), -z\right\}$

Table 4. **Multiplication table of the dihedral group $\mathcal{D}_6$**

	1_1	2_1	2_2	3_1	3_2	4_1	5_1	5_2	5_3	6_1	6_2	6_3
1_1	1_1	2_1	2_2	3_1	3_2	4_1	5_1	5_2	5_3	6_1	6_2	6_3
2_1	2_1	3_1	1_1	4_1	2_2	3_2	6_3	6_1	6_2	5_1	5_2	5_3
2_2	2_2	1_1	3_2	2_1	4_1	3_1	6_1	6_2	6_3	5_2	5_3	5_1
3_1	3_1	4_1	2_1	3_2	1_1	2_2	5_3	5_1	5_2	6_3	6_1	6_2
3_2	3_2	2_2	4_1	1_1	3_1	2_1	5_2	5_3	5_1	6_2	6_3	6_1
4_1	4_1	3_2	3_1	2_2	2_1	1_1	6_2	6_3	6_1	5_3	5_1	5_2
5_1	5_1	6_1	6_3	5_2	5_3	6_2	1_1	3_1	3_2	2_1	4_1	2_2
5_2	5_2	6_2	6_1	5_3	5_1	6_3	3_2	1_1	3_1	2_2	2_1	4_1
5_3	5_3	6_3	6_2	5_1	5_2	6_1	3_1	3_2	1_1	4_1	2_2	2_1
6_1	6_1	5_2	5_1	6_2	6_3	5_3	2_2	2_1	4_1	1_1	3_1	3_2
6_2	6_2	5_3	5_2	6_3	6_1	5_1	4_1	2_2	2_1	3_2	1_1	3_1
6_3	6_3	5_1	5_3	6_1	6_2	5_2	2_1	4_1	2_2	3_1	3_2	1_1

Explicitly, in three dimensions we can take the following matrix representation for the generators of $\mathcal{D}_6$:

$$A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{8.5}$$

The group generated by the above generators has 12 elements that can be arranged into 6 conjugacy classes, as is displayed in Table 3. In this Table, every group element is uniquely identified by its action on the three-dimensional vector $\{x, y, z\}$. The multiplication table of the group $\mathcal{D}_6$ is shown in Table 4.

8.2. Irreducible Representations of the Dihedral Group $\mathcal{D}_6$ and the Character Table. The group $\mathcal{D}_6$ has six conjugacy classes. Therefore according to the theory we expect six irreducible representations that we name D_i, $i = 1, \ldots, 6$. Let us briefly describe them. The first four representations are one-dimensional.

8.2.1. D_1: the Identity Representation. The identity representation which exists for all groups is that one where to each element of $\mathcal{D}_6$ we associate the number 1

$$\forall\,\gamma \in \mathrm{O}\,: \quad D_1(\gamma) = 1. \tag{8.6}$$

Obviously, the character of such a representation is

$$\chi_1 = \{1, 1, 1, 1, 1\}. \tag{8.7}$$

8.2.2. D_2: the Second One-Dimensional Representation. The representation D_2 is also one-dimensional. It is constructed as follows:

$$\begin{array}{lll} \forall\,\gamma \in & \{\mathbf{e}\} & : D_2(\gamma) = \;\;1, \\ \forall\,\gamma \in & \{A\} & : D_2(\gamma) = -1, \\ \forall\,\gamma \in & \left\{A^2\right\} & : D_2(\gamma) = \;\;1, \\ \forall\,\gamma \in & \left\{A^3\right\} & : D_2(\gamma) = -1, \\ \forall\,\gamma \in & \{B\} & : D_2(\gamma) = \;\;1, \\ \forall\,\gamma \in & \{BA\} & : D_2(\gamma) = -1. \end{array} \tag{8.8}$$

Clearly the corresponding character vector is the following one:

$$\chi_2 = \{1, -1, 1, -1, 1, -1\}. \tag{8.9}$$

Said in another way, this is the representation, where $A = -1$ and $B = 1$.

8.2.3. D_3: the Third One-Dimensional Representation. The representation D_3 is also one-dimensional. It is constructed as follows:

$$\begin{array}{lll} \forall\,\gamma \in & \{\mathbf{e}\} & : D_2(\gamma) = \;\;1, \\ \forall\,\gamma \in & \{A\} & : D_2(\gamma) = -1, \\ \forall\,\gamma \in & \left\{A^2\right\} & : D_2(\gamma) = \;\;1, \\ \forall\,\gamma \in & \left\{A^3\right\} & : D_2(\gamma) = -1, \\ \forall\,\gamma \in & \{B\} & : D_2(\gamma) = -1, \\ \forall\,\gamma \in & \{BA\} & : D_2(\gamma) = \;\;1. \end{array} \tag{8.10}$$

Clearly the corresponding character vector is the following one:

$$\chi_3 = \{1, -1, 1, -1, -1, 1\} \tag{8.11}$$

Said in another way, this is the representation, where $A = -1$ and $B = -1$.

8.2.4. D_4: the Fourth One-Dimensional Representation. The representation D_4 is also one-dimensional. It is constructed as follows:

$$\begin{aligned}
\forall\,\gamma \in \ \ \{\mathbf{e}\} \ \ &: D_2(\gamma) = \ \ 1,\\
\forall\,\gamma \in \ \ \{A\} \ \ &: D_2(\gamma) = \ \ 1,\\
\forall\,\gamma \in \left\{A^2\right\} &: D_2(\gamma) = \ \ 1,\\
\forall\,\gamma \in \left\{A^3\right\} &: D_2(\gamma) = \ \ 1,\\
\forall\,\gamma \in \ \ \{B\} \ \ &: D_2(\gamma) = -1,\\
\forall\,\gamma \in \{BA\} &: D_2(\gamma) = -1.
\end{aligned} \tag{8.12}$$

Clearly, the corresponding character vector is the following one:

$$\chi_4 = \{1, 1, 1, 1, -1, -1\}. \tag{8.13}$$

Said in another way, this is the representation, where $A = 1$ and $B = -1$.

8.2.5. D_5: the First Two-Dimensional Representation. The representation D_5 is two-dimensional and it corresponds to a homomorphism

$$D_5 : \quad \mathrm{D}_6 \to \mathrm{SL}(2, \mathbb{C}), \tag{8.14}$$

which associates to each element of the dihedral group a 2×2 complex valued matrix of determinant one. The homomorphism is completely specified by giving the two matrices representing the two generators:

$$D_5(A) = \begin{pmatrix} \mathrm{e}^{i\pi/3} & 0 \\ 0 & \mathrm{e}^{-i\pi/3} \end{pmatrix}; \quad D_5(B) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.15}$$

The character vector of D_5 is easily calculated from the above information and we have

$$\chi_5 = \{2,\, 1,\, -1,\, -2,\, 0,\, 0\}. \tag{8.16}$$

8.2.6. D_6: the Second Two-Dimensional Representation. The representation D_6 is also two-dimensional and it corresponds to a homomorphism

$$D_6 : \quad \mathrm{D}_6 \to \mathrm{SL}(2, \mathbb{C}), \tag{8.17}$$

which associates to each element of the dihedral group a 2×2 complex valued matrix of determinant one. The homomorphism is completely specified by giving the two matrices representing the two generators:

$$D_6(A) = \begin{pmatrix} \mathrm{e}^{2i\pi/3} & 0 \\ 0 & \mathrm{e}^{-2i\pi/3} \end{pmatrix}; \quad D_6(B) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.18}$$

The character vector of D_6 is easily calculated from the above information, and we have

$$\chi_6 = \{2,\, -1,\, -1,\, 2,\, 0,\, 0\}. \tag{8.19}$$

The character table of the $\mathcal{D}_6$ group is summarized in Table 5.

Table 5. **The character table of the dihedral group $\mathcal{D}_6$**

Class / Irrep	$\{\mathbf{e},1\}$	$\{A,2\}$	$\{A^2,2\}$	$\{A^3,1\}$	$\{B,3\}$	$\{BA,3\}$
$D_1, \quad \chi_1 =$	1	1	1	1	1	1
$D_2, \quad \chi_2 =$	1	-1	1	-1	1	-1
$D_3, \quad \chi_3 =$	1	-1	1	-1	-1	1
$D_4, \quad \chi_4 =$	1	1	1	1	-1	-1
$D_5, \quad \chi_5 =$	2	1	-1	-2	0	0
$D_6, \quad \chi_6 =$	2	-1	-1	2	0	0

8.3. Spherical Layers in the Hexagonal Lattice and $\mathcal{D}_6$ Orbits. Let us now analyze the action of the dihedral group $\mathcal{D}_6$ on the hexagonal lattice. Just as in the case of the cubic lattice, we define the orbits as the sets of vectors $\mathbf{k} \in \Lambda^\star_{\mathrm{hex}}$ that can be mapped one into the other by the action of some element of the group $\mathcal{D}_6$:

$$\mathbf{k}_1 \in \mathcal{O} \quad \text{and} \quad \mathbf{k}_2 \in \mathcal{O} \quad \Rightarrow \quad \exists\, \gamma \in \mathcal{D}_6 \;/\; \gamma \cdot \mathbf{k}_1 = \mathbf{k}_2. \tag{8.20}$$

The hexagonal lattice displays a more variegated bestiary of orbit types with respect to the case of the cubic lattice. There are orbits of length 2, 6, and 12, but those of length 6 and 12 appear in a few different types.

Orbits of Length 2. Each of these orbits is of the following form:

$$\mathcal{O}_2 = \left\{ \; \left\{0, 0, -\tfrac{2r}{\sqrt{3}}\right\}, \quad \left\{0, 0, \tfrac{2r}{\sqrt{3}}\right\} \; \right\}, \tag{8.21}$$

where $r \in \mathbb{Z}$ is any integer number.

Type-One Orbits of Length 6. Each of these orbits is of the following form:

$$\mathcal{O}_6^{(1)} = \left\{ \left\{0,-\tfrac{2p}{\sqrt{3}},0\right\}, \left\{0,\tfrac{2p}{\sqrt{3}},0\right\}, \left\{-p,-\tfrac{p}{\sqrt{3}},0\right\}, \left\{-p,\tfrac{p}{\sqrt{3}},0\right\}, \left\{p,-\tfrac{p}{\sqrt{3}},0\right\}, \left\{p,\tfrac{p}{\sqrt{3}},0\right\} \right\}, \tag{8.22}$$

where $p \in \mathbb{Z}$ is any integer number.

Type-Two Orbits of Length 6. Each of these orbits is of the following form:

$$\mathcal{O}_6^{(2)} = \left\{ \{-2p,0,0\},\ \left\{-p,-\sqrt{3}p,0\right\},\ \left\{-p,\sqrt{3}p,0\right\},\ \left\{p,-\sqrt{3}p,0\right\},\ \left\{p,\sqrt{3}p,0\right\},\ \{2p,0,0\} \right\}, \tag{8.23}$$

where $p \in \mathbb{Z}$ is any integer number.

Type-One Orbits of Length 12. Each of these orbits is of the following form:

$$\mathcal{O}_{12}^{(1)} = \left\{ \begin{array}{cccc} \left\{-p,-\frac{p-2q}{\sqrt{3}},0\right\}, & \left\{-p,\frac{p-2q}{\sqrt{3}},0\right\}, & \left\{p,-\frac{p-2q}{\sqrt{3}},0\right\}, & \left\{p,\frac{p-2q}{\sqrt{3}},0\right\}, \\ \left\{p-q,-\frac{p+q}{\sqrt{3}},0\right\}, & \left\{p-q,\frac{p+q}{\sqrt{3}},0\right\}, & \left\{-q,\frac{2p-q}{\sqrt{3}},0\right\}, & \left\{-q,\frac{q-2p}{\sqrt{3}},0\right\}, \\ \left\{q,\frac{2p-q}{\sqrt{3}},0\right\}, & \left\{q,\frac{q-2p}{\sqrt{3}},0\right\}, & \left\{q-p,-\frac{p+q}{\sqrt{3}},0\right\}, & \left\{q-p,\frac{p+q}{\sqrt{3}},0\right\} \end{array} \right\}, \tag{8.24}$$

where $p, q \in \mathbb{Z}$ and $q \neq \pm p$ and $q \neq 2\,p$.

Type-Two Orbits of Length 12. Each of these orbits is of the following form:

$$\mathcal{O}_{12}^{(2)} = \left\{ \begin{array}{cccc} \left\{-p,-\frac{p-2q}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, & \left\{-p,\frac{p-2q}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{p,-\frac{p-2q}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{p,\frac{p-2q}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, \\ \left\{p-q,-\frac{p+q}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, & \left\{p-q,\frac{p+q}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{-q,\frac{2p-q}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\} & \left\{-q,\frac{q-2p}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, \\ \left\{q,\frac{2p-q}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, & \left\{q,\frac{q-2p}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{q-p,-\frac{p+q}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{q-p,\frac{p+q}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\} \end{array} \right\}, \tag{8.25}$$

where $p, q, r \in \mathbb{Z}$ and $q \neq \pm p$ and $q \neq 2\,p$.

Type-Three Orbits of Length 12. Each of these orbits is of the following form:

$$\mathcal{O}_{12}^{(3)} = \left\{ \begin{array}{cccc} \left\{0,-\frac{2p}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{0,-\frac{2p}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, & \left\{0,\frac{2p}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{0,\frac{2p}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, \\ \left\{-p,-\frac{p}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{-p,-\frac{p}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, & \left\{-p,\frac{p}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{-p,\frac{p}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, \\ \left\{p,-\frac{p}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{p,-\frac{p}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\}, & \left\{p,\frac{p}{\sqrt{3}},-\frac{2r}{\sqrt{3}}\right\}, & \left\{p,\frac{p}{\sqrt{3}},\frac{2r}{\sqrt{3}}\right\} \end{array} \right\}, \tag{8.26}$$

where $p, q \in \mathbb{Z}$ are two arbitrary integer numbers.

Type-Four Orbits of Length 12. Each of these orbits is of the following form:

$$\mathcal{O}_{12}^{(4)} = \left\{ \begin{array}{cccc} \left\{-2p,0,-\frac{2r}{\sqrt{3}}\right\}, & \left\{-2p,0,\frac{2r}{\sqrt{3}}\right\}, & \left\{-p,-\sqrt{3}p,-\frac{2r}{\sqrt{3}}\right\}, & \left\{-p,-\sqrt{3}p,\frac{2r}{\sqrt{3}}\right\}, \\ \left\{-p,\sqrt{3}p,-\frac{2r}{\sqrt{3}}\right\}, & \left\{-p,\sqrt{3}p,\frac{2r}{\sqrt{3}}\right\}, & \left\{p,-\sqrt{3}p,-\frac{2r}{\sqrt{3}}\right\}, & \left\{p,-\sqrt{3}p,\frac{2r}{\sqrt{3}}\right\}, \\ \left\{p,\sqrt{3}p,-\frac{2r}{\sqrt{3}}\right\}, & \left\{p,\sqrt{3}p,\frac{2r}{\sqrt{3}}\right\}, & \left\{2p,0,-\frac{2r}{\sqrt{3}}\right\}, & \left\{2p,0,\frac{2r}{\sqrt{3}}\right\}, \end{array} \right\}, \tag{8.27}$$

where $p, q \in \mathbb{Z}$ are two arbitrary integer numbers. As we see, the shorter orbit of length 2 is actually vertical, namely, the associated Beltrami flows correspond to decoupled systems where only the coordinate $z(t)$ obeys a nonlinear differential equation. The other two coordinates form a free system. Similarly, the orbits of length 6 and the first orbit of length 12 are all planar. In the corresponding Beltrami flow there is no dependence on the coordinate z which forms a free system. Presumably all the Beltrami flows of this type are integrable. Only the maximal orbits of length 12 of type two, three, and four are truly three-dimensional and give rise to systems that might develop a chaos.

Table 6. **Spherical layers in the hexagonal momentum lattice**

r^2	Number of points	Dihedral $\mathcal{D}_6$ group point orbits
0	1	$\{1\}$
$\frac{4}{3}$	8	$\{6 \oplus 2\}$
$\frac{8}{3}$	12	$\{12\}$
4	6	$\{6\}$
$\frac{16}{3}$	20	$\{12 \oplus 6 \oplus 2\}$
$\frac{20}{3}$	24	$\{12 \oplus 12\}$
$\frac{28}{3}$	24	$\{12 \oplus 12\}$
$\frac{32}{3}$	36	$\{12 \oplus 12 \oplus 12\}$
12	8	$\{6 \oplus 2\}$
$\frac{40}{3}$	24	$\{12 \oplus 12\}$
$\frac{44}{3}$	24	$\{12 \oplus 12\}$
16	18	$\{6 \oplus 12\}$
$\frac{52}{3}$	48	$\{12 \oplus 12 \oplus 12 \oplus 12\}$
$\frac{56}{3}$	24	$\{12 \oplus 12\}$
$\frac{64}{3}$	36	$\{12 \oplus 12 \oplus 12\}$
$\frac{68}{3}$	24	$\{12 \oplus 12\}$
24	12	$\{12\}$

In Table 6, we have displayed the counting of lattice points on the first lying spherical layers of the hexagonal lattice and their splitting into orbits of the dihedral group $\mathcal{D}_6$. In the next section, we consider the construction of the Beltrami flows associated with the first few of such layers.

9. BELTRAMI FIELDS FROM SPHERICAL LAYERS IN THE HEXAGONAL LATTICE

In this section, as announced, by utilizing the algorithm outlined in Subsec. 3.1 we construct the Arnold-like Beltrami flows associated with the first low-lying spherical layers of the hexagonal lattice.

9.1. The Lowest-Lying Layer of Length 8 in the Hexagonal Lattice $\Lambda^\star_{\rm hex}$. In Fig. 21, we show the location of the momentum lattice points forming the lowest-lying spherical layer of length 8. Under the action of the dihedral group $\mathcal{D}_6$ these eight points are split into a 6-orbit of type one and a 2-orbit:

$$\mathcal{O}_6^{(1)} = \left\{\left\{-1,-\tfrac{1}{\sqrt{3}},0\right\}\left\{-1,\tfrac{1}{\sqrt{3}},0\right\}\left\{0,-\tfrac{2}{\sqrt{3}},0\right\}\left\{0,\tfrac{2}{\sqrt{3}},0\right\}\left\{1,-\tfrac{1}{\sqrt{3}},0\right\}\left\{1,\tfrac{1}{\sqrt{3}},0\right\}\right\}, \quad (9.1)$$

$$\mathcal{O}_2 = \left\{\left\{0,0,-\frac{2}{\sqrt{3}}\right\}\quad\left\{0,0,\frac{2}{\sqrt{3}}\right\}\right\}. \quad (9.2)$$

Implementing the construction algorithm, we find the following vector field:

$$\begin{aligned}
&\mathbf{V}^{(8)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\},\\
V_x &= 2\cos(\Theta_3)\,F_1 + \cos(\Theta_4)\,F_2 - 2\sin(\Theta_3)\,F_3 + \frac{\cos(\Theta_2)\,F_4}{\sqrt{3}} -\\
&\quad - \frac{\cos(\Theta_1)\,F_5}{\sqrt{3}} + \sin(\Theta_4)\,F_6 - \frac{1}{2}\sin(\Theta_2)\,F_7 + \frac{1}{2}\sin(\Theta_1)\,F_8,\\
V_y &= 2\sin(\Theta_3)\,F_1 + 2\cos(\Theta_3)\,F_3 + \cos(\Theta_2)\,F_4 + \cos(\Theta_1)\,F_5 - \qquad (9.3)\\
&\quad - \frac{1}{2}\sqrt{3}\sin(\Theta_2)\,F_7 - \frac{1}{2}\sqrt{3}\sin(\Theta_1)\,F_8,\\
V_z &= -\sin(\Theta_4)\,F_2 + \frac{2\sin(\Theta_2)\,F_4}{\sqrt{3}} + \frac{2\sin(\Theta_1)\,F_5}{\sqrt{3}} +\\
&\quad + \cos(\Theta_4)\,F_6 + \cos(\Theta_2)\,F_7 + \cos(\Theta_1)\,F_8,
\end{aligned}$$

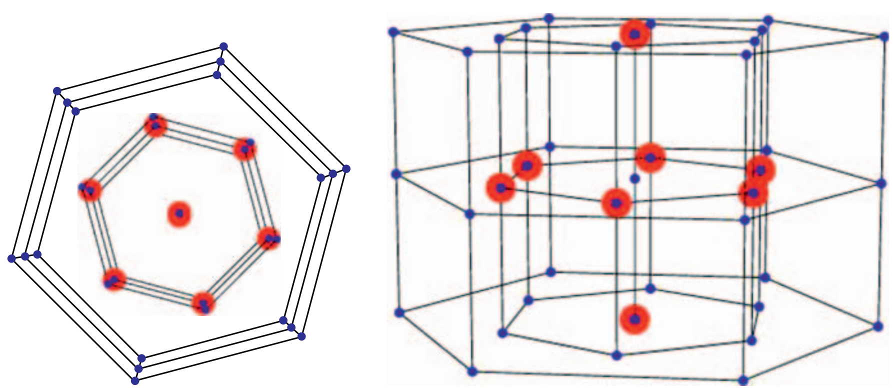

Fig. 21. In this picture, we show the location of the 8 points forming the lowest spherical layer in the momentum lattice ($\in \Lambda^\star_{\rm hex}$) whose squared radius is $r^2 = 4/3$. These points split into two orbits under the $\mathcal{D}_6$ group. The two points corresponding to the North and South Pole of the sphere form a 2-orbit. The six points on the equator of the sphere, that are the vertices of an hexagon, form a 6-orbit

where F_i $(i = 1, \ldots, 8)$ are real numbers and the angles Θ_i are the following ones:

$$\begin{aligned} \Theta_1 &= -\frac{2}{3}\pi\left(3x + \sqrt{3}y\right), \quad & \Theta_2 &= 2\pi\left(\frac{y}{\sqrt{3}} - x\right), \\ \Theta_3 &= -\frac{4\pi z}{\sqrt{3}}, & \Theta_4 &= -\frac{4\pi y}{\sqrt{3}}. \end{aligned} \tag{9.4}$$

From the explicit expression of the vector field in full analogy with Eq. (7.59), we construct the action of the dihedral group $\mathcal{D}_6$ on the parameter vector $\mathbf{F}$. Writing

$$\forall\, \gamma \in \mathcal{D}_6 : \quad \gamma^{-1} \cdot \mathbf{V}^{(8)}\left(\gamma \cdot \mathbf{r}|\mathbf{F}\right) = \mathbf{V}^{(8)}\left(\mathbf{r}|\, \mathfrak{R}^{(8)}[\gamma] \cdot \mathbf{F}\right), \tag{9.5}$$

we obtain the form of the reducible representation $\mathfrak{R}^{(8)}[\gamma]$ which is completely specified giving the images $\mathfrak{R}^{(8)}[A]$, and $\mathfrak{R}^{(8)}[B]$ of the two group generators. Explicitly, we find

$$\begin{aligned} \mathfrak{R}^{(8)}[A] &= \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}; \\ \mathfrak{R}^{(8)}[B] &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \end{aligned} \tag{9.6}$$

Retrieving from the above generators all the group elements and, in particular, a representative for each of the conjugacy classes, we can easily compute their traces, and in this way establish the character vector of this representation. We get

$$\chi\,[\mathbf{8}] = \{8, 1, -1, -2, -2, 0\}. \tag{9.7}$$

The multiplicity vector is

$$\mathrm{m}\,[\mathbf{8}] = \{0, 0, 1, 1, 2, 1\} \tag{9.8}$$

implying that the 8-dimensional parameter space decomposes into a D_3, plus a D_4, plus two D_5, and one D_6 representations. The corresponding irreducible Beltrami fields are easily constructed.

Irreducible Beltrami Field in the D_3 Representation.

$$\mathbf{V}^{(8|D_3)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\} = \begin{pmatrix} \frac{1}{2}\left(-\cos\left(\Theta_1\right) - \cos\left(\Theta_2\right) + 2\cos\left(\Theta_4\right)\right) \\ \frac{1}{2}\sqrt{3}\left(\cos\left(\Theta_1\right) - \cos\left(\Theta_2\right)\right) \\ \sin\left(\Theta_1\right) - \sin\left(\Theta_2\right) - \sin\left(\Theta_4\right) \end{pmatrix}. \tag{9.9}$$

Irreducible Beltrami Field in the D_4 Representation.

$$\mathbf{V}^{(8|D_4)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\} = \begin{pmatrix} \frac{1}{2}\left(\sin\left(\Theta_1\right) - \sin\left(\Theta_2\right) + 2\sin\left(\Theta_4\right)\right) \\ -\frac{1}{2}\sqrt{3}\left(\sin\left(\Theta_1\right) + \sin\left(\Theta_2\right)\right) \\ \cos\left(\Theta_1\right) + \cos\left(\Theta_2\right) + \cos\left(\Theta_4\right) \end{pmatrix}. \tag{9.10}$$

Irreducible Beltrami Field in the D_{5a} Representation.

$$\mathbf{V}^{(8|D_{5a})}(\mathbf{r}|\{A, B\}) = \{V_x,\, V_y,\, V_z\} = \begin{pmatrix} 2A\cos\left(\Theta_3\right) - 2B\sin\left(\Theta_3\right) \\ 2\left(B\cos\left(\Theta_3\right) + A\sin\left(\Theta_3\right)\right) \\ 0 \end{pmatrix}. \tag{9.11}$$

Irreducible Beltrami Field in the D_{5b} Representation.

$$\begin{aligned}\mathbf{V}^{(8|D_{5b})}(\mathbf{r}|\{A,B\}) &= \{V_x,\, V_y,\, V_z\} \\ &= \begin{pmatrix} -B\cos(\Theta_1) + A\cos(\Theta_2) + 2(A-B)\cos(\Theta_4) \\ \sqrt{3}\,(B\cos(\Theta_1) + A\cos(\Theta_2)) \\ 2\,(B\sin(\Theta_1) + A\sin(\Theta_2) + (B-A)\sin(\Theta_4)) \end{pmatrix}.\end{aligned} \tag{9.12}$$

Irreducible Beltrami Field in the D_6 Representation.

$$\begin{aligned}\mathbf{V}^{(8|D_6)}(\mathbf{r}|\{A,B\}) &= \{V_x,\, V_y,\, V_z\} \\ &= \begin{pmatrix} \frac{1}{2}\sqrt{3}B\,(\sin(\Theta_1) - \sin(\Theta_2) - 4\sin(\Theta_4)) \\ -\frac{3}{2}B\,(\sin(\Theta_1) + \sin(\Theta_2)) \\ \sqrt{3}B\,(\cos(\Theta_1) + \cos(\Theta_2) - 2\cos(\Theta_4)) \end{pmatrix}.\end{aligned} \tag{9.13}$$

Let us now observe that the only angle dependent on the vertical coordinate z is Θ_3 and that all the above irreducible Beltrami fields, except D_{4a}, are independent of Θ_3. Hence all these irreducible Beltrami fields reduce to two-dimensional planar systems that are presumably integrable systems. This is geometrically understandable since the direct sum of all these representations has dimension 6 and reproduces the contribution of the points in orbit $\mathcal{O}_6^{(1)}$ which is just planar. Hence we can easily conjecture that for all orbits of type $\mathcal{O}_6^{(1)}$ we have the decomposition

$$\mathcal{O}_6^{(1)} \simeq D_2 \oplus D_3 \oplus D_{5b} \oplus D_5, \tag{9.14}$$

each irreducible component being a planar system.

The representation D_{5a}, on the other hand, corresponds to the contribution of the points in the 2-orbit and leads to a trivial differential system which is immediately integrated. The coordinate $z = z_0$ is constant in time and the coordinates x, y are linear functions of time.

This example strongly indicates that in the hexagonal lattices the only non-trivial systems are those related to orbits of length 12 and type two, three, and four as we have already advocated.

9.2. The Lowest-Lying Orbit of Length 12 in the Hexagonal Lattice $\Lambda^\star_{\mathrm{hex}}$. In Fig. 22, we show the location of the momentum lattice points forming the lowest-lying spherical layer of length 12 under the action of the dihedral group $\mathcal{D}_6$, this layer is irreducible. Using the standard method, we obtain the following

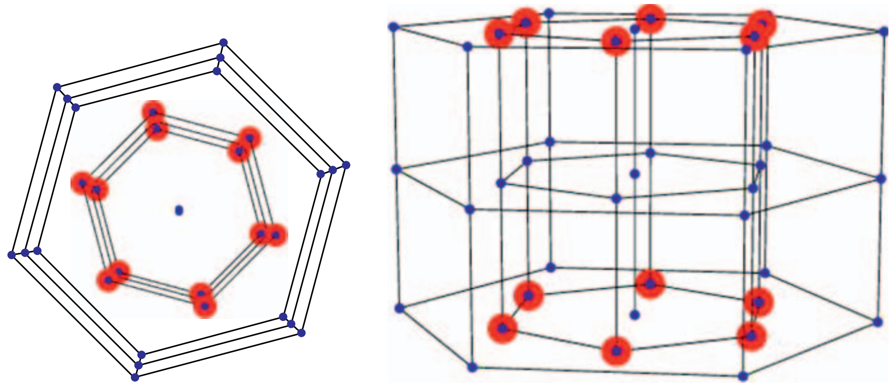

Fig. 22. In this picture, we show the location of the 12 points forming the lowest-lying $\mathcal{D}_6$ orbit in the momentum lattice ($\in \Lambda^\star_{\mathrm{hex}}$). They are located on a sphere of squared radius $r^2 = 8/3$

general solution of the Beltrami equation depending on 12 parameters:

$$\begin{aligned}
&\mathbf{V}^{(12)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\}\,,\\
&V_x = \cos(\Theta_6)\,F_1 + \cos(\Theta_5)\,F_2 + \sqrt{2}\sin(\Theta_6)\,F_7 + +\sqrt{2}\sin(\Theta_5)\,F_8 +\\
&\quad + \frac{1}{12}\sin(\Theta_4)\left(6\sqrt{2}F_3 - 3\sqrt{2}F_9\right) + \frac{1}{12}\cos(\Theta_4)\left(4\sqrt{3}F_3 + 8\sqrt{3}F_9\right) +\\
&\quad + \frac{1}{12}\sin(\Theta_3)\left(-6\sqrt{2}F_4 - 3\sqrt{2}F_{10}\right) + \frac{1}{12}\cos(\Theta_3)\left(4\sqrt{3}F_4 - 8\sqrt{3}F_{10}\right) +\\
&\quad + \frac{1}{12}\sin(\Theta_2)\left(6\sqrt{2}F_5 + 3\sqrt{2}F_{11}\right) + \frac{1}{12}\cos(\Theta_2)\left(8\sqrt{3}F_{11} - 4\sqrt{3}F_5\right) +\\
&\quad + \frac{1}{12}\sin(\Theta_1)\left(3\sqrt{2}F_{12} - 6\sqrt{2}F_6\right) + \frac{1}{12}\cos(\Theta_1)\left(-4\sqrt{3}F_6 - 8\sqrt{3}F_{12}\right),\\
&V_y = -\frac{\sin(\Theta_6)\,F_1}{\sqrt{2}} + \frac{\sin(\Theta_5)\,F_2}{\sqrt{2}} + \qquad (9.15)\\
&\quad + \cos(\Theta_4)\,F_3 + \cos(\Theta_3)\,F_4 + \cos(\Theta_2)\,F_5 + \cos(\Theta_1)\,F_6 +\\
&\quad + \cos(\Theta_6)\,F_7 - \cos(\Theta_5)\,F_8 + \sin(\Theta_4)\left(-\frac{F_3}{\sqrt{6}} - \frac{7F_9}{2\sqrt{6}}\right) +\\
&\quad + \sin(\Theta_3)\left(\frac{F_4}{\sqrt{6}} - \frac{7F_{10}}{2\sqrt{6}}\right) + \sin(\Theta_2)\left(\frac{F_5}{\sqrt{6}} - \frac{7F_{11}}{2\sqrt{6}}\right) +\\
&\quad + \sin(\Theta_1)\left(-\frac{F_6}{\sqrt{6}} - \frac{7F_{12}}{2\sqrt{6}}\right),
\end{aligned}$$

$$
\begin{aligned}
V_z = &-\frac{\sin(\Theta_6)\,F_1}{\sqrt{2}} - \frac{\sin(\Theta_5)\,F_2}{\sqrt{2}} + \cos(\Theta_6)\,F_7 + \\
&+\cos(\Theta_5)\,F_8 + \cos(\Theta_4)\,F_9 + \frac{1}{6}\sin(\Theta_4)\left(2\sqrt{6}F_3 + \sqrt{6}F_9\right) + \\
&+\cos(\Theta_3)\,F_{10} + \frac{1}{6}\sin(\Theta_3)\left(2\sqrt{6}F_4 - \sqrt{6}F_{10}\right) + \\
&+\cos(\Theta_2)\,F_{11} + \frac{1}{6}\sin(\Theta_2)\left(2\sqrt{6}F_5 - \sqrt{6}F_{11}\right) + \\
&+\cos(\Theta_1)\,F_{12} + \frac{1}{6}\sin(\Theta_1)\left(2\sqrt{6}F_6 + \sqrt{6}F_{12}\right),
\end{aligned}
$$

where F_i $(i = 1, \ldots, 8)$ are real numbers and the angles Θ_i are the following ones:

$$
\begin{array}{ll}
\Theta_1 = -\frac{2}{3}\pi\left(3x + \sqrt{3}(y + 2z)\right), & \Theta_2 = -\frac{2}{3}\pi\left(3x + \sqrt{3}(y - 2z)\right), \\
\Theta_3 = -\frac{2}{3}\pi\left(3x - \sqrt{3}(y - 2z)\right), & \Theta_4 = \frac{2}{3}\pi\left(\sqrt{3}(y + 2z) - 3x\right), \\
\Theta_5 = -\frac{4\pi(y+z)}{\sqrt{3}}, & \Theta_6 = \frac{4\pi(z-y)}{\sqrt{3}}.
\end{array}
\tag{9.16}
$$

The 12-dimensional representation of $\mathcal{D}_6$ in parameter space is also easily constructed and we find

$$
\mathfrak{R}^{(12)}[A] = \begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} \\
0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},
\tag{9.17}
$$

$$\mathfrak{R}^{(12)}[B] = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{9.18}$$

Retrieving from the above generators all the group elements and, in particular, a representative for each of the conjugacy classes, we can easily compute their traces and in this way establish the character vector of this representation. We get

$$\chi\,[\mathbf{12}] = \{12, 0, 0, 0, 0, 0\}. \tag{9.19}$$

The multiplicity vector is

$$\mathrm{m}\,[\mathbf{12}] = \{1, 1, 1, 1, 2, 2\}, \tag{9.20}$$

implying that the 12-dimensional parameter space decomposes into a D_1 (invariant Beltrami vector field) plus a D_2, plus a D_3, plus two D_5, and two D_6 representations. We present the form of the Beltrami vector fields in the various representations:

Irreducible Beltrami Field in the D_1 Representation.

$$\mathbf{V}^{(12|D_1)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\}\,,$$

$$\begin{aligned} V_x = \frac{1}{4}\big(&-2\cos(\Theta_1) + 2\cos(\Theta_2) + 2\cos(\Theta_3) - 2\cos(\Theta_4) - \\ &- 4\cos(\Theta_5) + 4\cos(\Theta_6) - \\ &- \sqrt{6}\sin(\Theta_1) - \sqrt{6}\sin(\Theta_2) - \sqrt{6}\sin(\Theta_3) - \sqrt{6}\sin(\Theta_4)\big), \end{aligned} \tag{9.21}$$

$$V_y = \frac{1}{4}\big(2\sqrt{3}\cos(\Theta_1) - 2\sqrt{3}\cos(\Theta_2) + 2\sqrt{3}\cos(\Theta_3) - 2\sqrt{3}\cos(\Theta_4) - \\ - \sqrt{2}\sin(\Theta_1) - \sqrt{2}\sin(\Theta_2) + \sqrt{2}\sin(\Theta_3) + \\ + \sqrt{2}\sin(\Theta_4) - 2\sqrt{2}\sin(\Theta_5) - 2\sqrt{2}\sin(\Theta_6)\big),$$

$$V_z = \frac{\sin(\Theta_1) - \sin(\Theta_2) + \sin(\Theta_3) - \sin(\Theta_4) + \sin(\Theta_5) - \sin(\Theta_6)}{\sqrt{2}}.$$

Irreducible Beltrami Field in the D_2 Representation.

$$\begin{aligned} &\mathbf{V}^{(12|D_2)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\}\,, \\ V_x &= \frac{1}{2}\Big(-\sqrt{3}\cos(\Theta_1) - \sqrt{3}\cos(\Theta_2) + \sqrt{3}\cos(\Theta_3) + \\ &+\sqrt{3}\cos(\Theta_4) + \sqrt{2}\sin(\Theta_1) - \sqrt{2}\sin(\Theta_2) + \\ &+\sqrt{2}\sin(\Theta_3) - \sqrt{2}\sin(\Theta_4) - 2\sqrt{2}\sin(\Theta_5) + 2\sqrt{2}\sin(\Theta_6)\Big)\,, \\ V_y &= \frac{1}{2}\,(-\cos(\Theta_1) - \cos(\Theta_2) - \cos(\Theta_3) - \cos(\Theta_4) + \\ &+2\cos(\Theta_5) + 2\cos(\Theta_6) - \sqrt{6}\sin(\Theta_1) + \\ &+\sqrt{6}\sin(\Theta_2) + \sqrt{6}\sin(\Theta_3) - \sqrt{6}\sin(\Theta_4)\Big)\,, \\ V_z &= \cos(\Theta_1) - \cos(\Theta_2) - \cos(\Theta_3) + \cos(\Theta_4) - \cos(\Theta_5) + \cos(\Theta_6)\,. \end{aligned} \tag{9.22}$$

Irreducible Beltrami Field in the D_3 Representation.

$$\begin{aligned} &\mathbf{V}^{(12|D_3)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\}\,, \\ V_x &= \frac{1}{4}\big(-2\cos(\Theta_1) - 2\cos(\Theta_2) - 2\cos(\Theta_3) - 2\cos(\Theta_4) + \\ &+ 4\cos(\Theta_5) + 4\cos(\Theta_6) - \sqrt{6}\sin(\Theta_1) + \\ &+ \sqrt{6}\sin(\Theta_2) + \sqrt{6}\sin(\Theta_3) - \sqrt{6}\sin(\Theta_4)\big), \\ V_y &= \frac{1}{4}\big(2\sqrt{3}\cos(\Theta_1) + 2\sqrt{3}\cos(\Theta_2) - 2\sqrt{3}\cos(\Theta_3) - 2\sqrt{3}\cos(\Theta_4) - \\ &- \sqrt{2}\sin(\Theta_1) + \sqrt{2}\sin(\Theta_2) - \sqrt{2}\sin(\Theta_3) + \\ &+ \sqrt{2}\sin(\Theta_4) + 2\sqrt{2}\sin(\Theta_5) - 2\sqrt{2}\sin(\Theta_6)\big), \\ V_z &= \frac{\sin(\Theta_1) + \sin(\Theta_2) - \sin(\Theta_3) - \sin(\Theta_4) - \sin(\Theta_5) - \sin(\Theta_6)}{\sqrt{2}}. \end{aligned} \tag{9.23}$$

Irreducible Beltrami Field in the D_4 Representation.

$$
\begin{aligned}
&\mathbf{V}^{(12|D_4)}(\mathbf{r}|\mathbf{F}) = \{V_x,\, V_y,\, V_z\}\,,\\
V_x &= \frac{1}{2}\big(-\sqrt{3}\cos(\Theta_1) + \sqrt{3}\cos(\Theta_2) - \sqrt{3}\cos(\Theta_3) + \\
&\quad + \sqrt{3}\cos(\Theta_4) + \sqrt{2}\sin(\Theta_1) + \sqrt{2}\sin(\Theta_2) - \\
&\quad - \sqrt{2}\sin(\Theta_3) - \sqrt{2}\sin(\Theta_4) + 2\sqrt{2}\sin(\Theta_5) + 2\sqrt{2}\sin(\Theta_6)\big),\\
V_y &= \frac{1}{2}\big(-\cos(\Theta_1) + \cos(\Theta_2) + \cos(\Theta_3) - \cos(\Theta_4) - \\
&\quad - 2\cos(\Theta_5) + 2\cos(\Theta_6) - \\
&\quad - \sqrt{6}\sin(\Theta_1) - \sqrt{6}\sin(\Theta_2) - \sqrt{6}\sin(\Theta_3) - \sqrt{6}\sin(\Theta_4)\big),\\
V_z &= \cos(\Theta_1) + \cos(\Theta_2) + \cos(\Theta_3) + \cos(\Theta_4) + \cos(\Theta_5) + \cos(\Theta_6)\,.
\end{aligned}
\tag{9.24}
$$

For the representations D_5 and D_6, we observe that the 2 irreducible representations of $\mathcal{D}_6$ are complex so that in the real field, not surprisingly can happen that we cannot separate the 4-dimensional D_5 or D_6 space into two orthogonal irreducible subspaces. What actually happens is that in these spaces there is an invariant real two-dimensional subspace, but its orthogonal complement is not invariant. So, it is better to keep four parameters in each of these spaces.

Beltrami Field in the D_5 Representation. Since the formulae in this case become very big, we just present the projection on the D_5 representation as a substitution rule of the 12 parameters F_i in Eq. (9.15) in terms of four independent parameters $(A, B, C; D)$. Explicitly, we have

$$
\begin{aligned}
&F_1 \to \frac{2A - 2B - C + 2D}{\sqrt{3}},\\
&F_2 \to \frac{2A - 2B - C + 2D}{\sqrt{3}},\\
&F_3 \to A,\\
&F_4 \to A,\\
&F_5 \to B,\\
&F_6 \to B,\\
&F_7 \to C,\\
&F_8 \to -C,\\
&F_9 \to D - C,\\
&F_{10} \to C - D,\\
&F_{11} \to D,\\
&F_{12} \to -D.
\end{aligned}
\tag{9.25}
$$

Beltrami Field in the D_6 Representation. In the same way as above, the D_6 Beltrami field can be obtained from Eq. (9.15) with the following substitution:

$$\left(\begin{array}{l} F_1 \to \dfrac{2A+2B-C-2D}{\sqrt{3}} \\ F_2 \to \dfrac{-2A-2B+C+2D}{\sqrt{3}} \\ F_3 \to A \\ F_4 \to -A \\ F_5 \to B \\ F_6 \to -B \\ F_7 \to C \\ F_8 \to C \\ F_9 \to -C-D \\ F_{10} \to -C-D \\ F_{11} \to D \\ F_{12} \to D \end{array}\right). \tag{9.26}$$

This concludes our short discussion of the hexagonal lattice case. By now it should be clear that the algorithm works for any lattice and that the properties of the corresponding Beltrami fields can be analyzed analogously to what we did in depth for those living on the cubic lattice. This being emphasized, it is time to turn to conclusions.

10. CONCLUSIONS AND OUTLOOK

In this section we plan to summarize the results of this paper and put them into a perspective for future work and deeper understanding.

Yet, in order to do this, some general considerations are necessary which might be particularly useful for those readers, we hope to have some of them, who do not belong to the community of scientists familiar with the ABC-flows and working on such topics. One of the two authors of this paper was in such a position when he was encouraged by the other author to get involved in the matters dealt with here. This author believes that the same shock that he experienced while entering this research field will probably affect any reader coming from the community of *theoretical theorists* and that the same fundamental philosophical questions that confronted him at the beginning, remaining so far unanswered, will equally bewilder such a reader. We want to emphasize that the roots of such a situation are in the very different mentality, we venture to say *weltaunschauung*, characterizing the community of *theoretical theorists* (in this category we include classical and quantum field theorists, supergravity and superstring theorists, general relativists and the like) and the community of mathematical physicists dealing with dynamical systems and related topics, ABC-flows being one prominent province in that realm. For the sake of shortness, we refer hereafter to such a community as to that of the *dynamical theorists*.

So let us first of all single out this difference in *weltanschauung*.

Both scientific communities share a high level of *mathematization* and rely on sophisticated mathematical structures to formulate their constructions, often they use the very same ones, yet, there is a drastic difference in their attitude toward Mathematics.

In the community of *theoretical theorists*, Mathematics is scanned in the quest for uniqueness, looking for such a priori conceptual choices that might single out a unique and distinctive mathematical framework able to capture the essence of a Physical Law and reduce its understanding to First Principles. *Theoretical theorists* are reductionists: what is most important is not the mathematical structure *per se*, that is utilized to describe a physical process, or its simplified idealized model, rather the conceptual category that singles out such a mathematical structure, in other words, its Principle.

In the community of *dynamical theorists*, Mathematics is mostly appreciated as a source of diversity, praise being obtained for each new solution of any given equation and a bottle of champagne being opened to welcome the discovery of any new model, for instance, of any new Integrable Dynamical System. In the issues addressed in this paper, the main goal is the opposite, namely, to obtain *the most nonintegrable systems*, yet the attitude does not change and any new mathematical example of such a type is equally appreciated and welcome.

Two other essential differences in the *weltanschauung* of the two communities concern the delimitation of the *metatheories* forming the play-ground of model-builders and the role, use, and implementation of *symmetries*. The differences in both these issues are quite relevant while trying to assess the character, relevance, and perspective of the results we have achieved in this paper. So let us dwell on this point.

In *theoretical theory*, the relevant *metatheories* have been established since long time. Up to the seventies of the last century they have been *Classical Lagrangian Field Theories* and their quantum descendants, namely the corresponding *Quantum Field Theories* obtained from canonical quantization of the former. Principal distinction in the vast container of *Lagrangian Field Theories* is the space-time symmetry: Lorentz invariance for all models of Particle Physics, Galilei invariance for some field-theoretical description of Newtonian or Statistical Mechanical Systems, the latter in many instances being interpreted as Wick rotations of the former. The efforts of constructive quantum field theorists allowed one to establish on a firm basis some general results for these metatheories, the most important among which is the Spin-Statistics theorem. From the middle seventies of the last century, both experiments and theory developments pointed to a further restriction of the *metatheory category* from generic *Relativistic Lagrangian Field Theories* to *Matter Coupled Gauge Theories*. Furthermore, a deep conceptual revolution led to replace the concept of fundamental *point-particles* with that of *fundamental extended objects*, *strings* and *superstrings* being the first focus of attention, enlarged, after *the second string revolution* in 1996, to p*-branes*. In these developments symmetry played a fundamental conceptual role: bosonic space-time symmetry was enlarged to *supersymmetry*, a necessary step in order to mix space-time and internal symmetries as proved by Coleman and Mandula [26].

Hence, on the basis of this definition of the *metatheory play-ground*, whose change is possible, but only at the price of a conscious, detailed and motivated philosophical revision of some Fundamental Principles (a scientific revolution in the Kuhn sense), the model building work of *theoretical theorists* is aimed either at the construction of new instances of Lagrangian Field Theories in diverse dimensions, displaying various types of catalogued symmetries as supersymmetry, conformal symmetry, special bosonic gauge-symmetries, or at the understanding of general properties of the former, or, still further, at the derivation of new exact solutions of either new or old Field Theories. In all these procedures, the role and the use of symmetry is well codified and falls into one of the following cases:

A) Symmetry $\mathrm{G}[\mathcal{L}]$ of the Lagrangian $\mathcal{L}$ and hence of the theory. The classification of possible symmetries of the Lagrangian typically amounts to a classification of theories inside the metatheory. The reduction is almost obtained. We have to invent the Principle that favors one symmetry more than another one.

B) Symmetry $\mathrm{G}[\mathcal{S}]$ of a solution $\mathcal{S}$ of the Lagrangian field equations which is a subgroup of the Lagrangian symmetries: $\mathrm{G}[\mathcal{S}] \subset \mathrm{G}[\mathcal{L}]$. Classification of $\mathrm{G}[\mathcal{S}]$ typically amounts to a classification of solutions $\mathcal{S}$ and this becomes particularly relevant if solutions can be interpreted as vacua of the theory. Being an extra functional-like, the energy functional can be typically advocated to select one vacuum symmetry more than another. Here the primary example, with all its far reaching consequences, is the spontaneous symmetry breaking mechanism.

C) Hidden symmetry which is a special declination of case A) or B), when the symmetry of a Lagrangian $\mathcal{L}$ or a solution $\mathcal{S}$ is significantly enhanced to a larger one with respect to the obvious one that dictates the construction principles of $\mathcal{L}$.

D) Dynamical symmetry: when the irreducible unitary representations of some finite or infinite (super) Lie algebra $\mathbb{G}_D$ constitute the quantum states of a theory (inside the metatheory). In this case, dynamics is completely reduced to algebra. Typical instance of this are the two-dimensional conformal field theories where the (super)-Virasoro algebra plays the role of $\mathbb{G}_D$.

In the community of *dynamical theorists*, the landscape is much less defined and clarified. The metatheory reference frame is just the vast and almost all containing setup of *dynamical systems*, *i.e.*, of first-order differential equations, and the main conceptual categories are just those of Poissonian or generalized

Poissonian structures. Integrability and nonintegrability are the main pursued issues with almost no emphasis on the reduction of choices to the First Principles. A vast and mathematically sophisticated literature deals with the construction of a plethora of models each of which mostly plays the role of a Leibniz monad. The role of symmetries in this vast literature is also episodical and not clearly categorized as done in the above list. Indeed, what is missing is the attempt to link symmetry to a philosophically motivated selection principle.

The typical attitude of *dynamical theorists* is that any good *mathematical result* will sooner or later find its place in physical theory and therefore it is worth pursuing. This is superficially very similar to the common belief of *theoretical theorists* that all sound and elegant *mathematical architectures*, including in this category all symmetries, have to be realized in the fabrics of Natural Law. The conceptual difference, however, is enormous and it is hidden in the semantic difference between *result* and *architecture*. What is missing in the attitude of *dynamical theorists* is the reductionist tension toward a small set of Economic First Principles rich of consequences but also strongly selective in the sense that they encompass vast landscapes yet rule out many possibilities.

Having spelled out these *weltanschauung* differences, let us come to the case of the ABC-flows which, with their generalizations, constitute the main topics of the present paper.

Leaving aside the issue of periodic boundary conditions, already addressed in the introduction, the main source of discomfort for one of the two authors and, possibly, for the *theoretical theorist* reader of this paper is that the starting point of the whole thing, namely Euler Equation (1.1), is not a Lagrangian, it is just an equation. This means that the velocity field $\mathrm{U}(t)$ is not uniquely identified as a Lagrangian field and that any of its possible symmetries do not fall in category B) of the above list. This is not too much surprising for any relativist who remembers Einstein's words about the two sides of his field equations. The *left-hand side*, said Einstein, meaning the Einstein tensor formed from the metric, *is written on pure marble*. The *right-hand side*, meaning the stress energy tensor, *is written on deteriorable wood.* Indeed the stress-energy tensor $T_{\mu\nu}$ of perfect fluids, which contains the velocity field, the energy-density, and the pressure, was, according to Einstein, just a modeling of our ignorance, being subject only to the kinematical and almost empty constraint of conservation $\nabla^{\mu} T_{\mu\nu} = 0$. The mission of theoretical physics was, according to Einstein, to transform the wood into marble by bringing the right-hand side to the left, namely by geometrizing it. This is what is currently done in unified field theory models and, in particular, in models of inflation where the stress energy tensor is derived from a field-theoretical Lagrangian. Now what is the Euler equation if not the nonrelativistic analogue of the stress-energy tensor conservation law? So, we can similarly say that the fundamental equation of hydrodynamics (1.1) is a modeling of our ignorance and a priority mission would be that of deriving its main ingredient, namely the velocity field $\mathrm{U}(t)$ from some field-theoretical Lagrangian. This is an extremely urgent question, specially in view of the role that symmetries have in this business.

The relation between points A) and B) in the consideration of symmetries can be provisionally amended by substituting $\mathrm{G}[\mathcal{L}]$ of the missing Lagrangian $\mathcal{L}$ with the group of symmetries of the Euler equation (1.1). On $\mathbb{R}^3$ this is just the full Euclidean group $\mathbb{E}^3$. In the case of a T^3 torus with a polarized constant metric (2.3) related with some chosen lattice Λ, it becomes the group defined in Eq. (5.4), namely, the semidirect product of the lattice point group with the continuous translation group modulo the lattice Λ. Thus, by setting

$$\mathrm{G}[\mathcal{L}] \equiv \mathfrak{G}_\Lambda, \tag{10.1}$$

we can be in business with point A) of the above list.

This point in the discussion offers to us the opportunity to stress the first and probably the **main of our new results**, namely the concept of a *Universal Classifying Group* $\mathfrak{GU}_\Lambda$. Relying on the suggestion of crystallographers who seek modifications of the lattice point groups, named Space-Groups, by the inclusion of quantized translations of (5.3) that cannot be eliminated by conjugation with elements of $\mathfrak{T}^3$, we have advocated the following three points:

1. Frobenius congruence classes define for all lattice Λ an Abelian subgroup

$$\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \mathbb{Z}_{k_3} \sim \mathfrak{TU}_\Lambda \subset \mathfrak{T}^3 \tag{10.2}$$

of quantized translations.

2. The semidirect product

$$\mathfrak{GU}_\Lambda = \mathfrak{P}_\Lambda \ltimes \mathfrak{TU}_\Lambda \tag{10.3}$$

of the lattice point group $\mathfrak{P}_\Lambda$ with the discretized translation group mentioned in Eq. (10.2) constitutes a large discrete group that contains as proper subgroups all possible space groups of crystallography.

3. The above defined group $\mathfrak{GU}_\Lambda$, named by us the Universal Classifying Group, is that apt to organize all solutions of the Beltrami equation (1.14) into irreducible representations and by this token to classify them.

Here comes the second and the most severe point of discomfort both for one of the authors and for our hypothetical *theoretical theorist* reader but, at the same time, here comes also the opportunity to stress the **second main result of the present paper**.

As emphasized in Introduction, the main idea in the whole scientific landscape around ABC-flows is given by Arnold theorem stating that the only velocity fields able to give rise to chaotic streamlines are those that satisfy Beltrami equation (1.14). The effort to substantiate this result in a more topological and abstract way leads to the conception of *contact structures*, *contact one-forms* and their associated *Reeb (like) fields*. Hence, the construction and classification of the Beltrami vector fields is a main priority in this arena of mathematical hydrodynamics.

Our main contribution, thoroughly developed in this paper, is the recognition that Beltrami equation (1.14) is nothing else but the eigenvalue equation for the first-order Laplace–Beltrami operator, the $\star_g \mathrm{d}$ operator that, on a compact Riemannian manifold, as the T^3 torus happens to be, has a discrete spectrum, the eigenfunctions being harmonics of the Universal Classifying Group $\mathfrak{GU}_\Lambda$, that can be systematically constructed with a rather simple algorithm and fall into irreps of the same. Special solutions $\mathcal{S}$ can be characterized by invariance under subgroups:

$$\mathrm{G}\,[\mathcal{S}] \subset \mathfrak{GU}_\Lambda. \tag{10.4}$$

In this way we are in business also with point B) of the above list of symmetry conceptions, yet the discomfort is related with the eigenvalue λ in Eq. (1.14). Who is going to tell us the value of λ? If we do not have a theory from which the Beltrami equation emerges as a field equation, then we do not have any reason to choose one or another of the possible eigenvalues. We are just in the Leibniz monad world. Every solution of the equation is equally admissible and we can just open as many bottles of champagne as there are eigenvectors and eigenfunctions. These are all the available generalizations of the ABC-flows. A priori it might seem that, since we have infinitely many eigenvalues, there are infinitely many eigenfunctions (with their moduli space) and this implies that we will get fully drunk. Yet the number of irreducible representations of a finite group, like $\mathfrak{GU}_\Lambda$, is finite, which sounds a warning that the collection of sparkling wine bottles should also be finite. Indeed, and this is the **third of our main results**, we have proven in this paper that for the cubic lattice there are actually only 48 different types of eigenfunctions which repeat themselves periodically; 48 is a large number of bottles, but with some attention we can survive!

Coming back to the issue of a theory able to produce the Beltrami equation, we observe the following two possibilities:

Possibility One. Being the first-order equation, the Beltrami equation might be interpreted in the context of a field theory as a sort of instantonic equation all of whose solutions are also solutions of the standard second-order equations, although the reverse is not true. If we adopt such an ideology, we can easily single out a simple candidate for such a field theory in Euclidean three dimensions. Let us identify the one-form $\Omega^{[\mathrm{U}]}$ of Eq. (1.7) with a gauge one-form:

$$\mathbf{A} \equiv \Omega^{[\mathrm{U}]}, \tag{10.5}$$

and let us consider the following action functional:

$$\begin{aligned} &\mathcal{A} = \int \mathcal{L}, \quad \mathcal{L} = \alpha\, \mathbf{F} \wedge \star_g \mathbf{F} + \beta\, \mathbf{F} \wedge \mathbf{A}, \\ &\mathbf{F} = \mathrm{d}\mathbf{A}, \end{aligned} \tag{10.6}$$

which describes a standard $\mathrm{U}(1)$ gauge theory with the addition of a Chern Simons term. The second-order field equation of this theory is

$$\mathrm{d} \star_g \mathbf{F} + \frac{\beta}{\alpha}\mathbf{F} = 0 \quad \Leftrightarrow \quad \mathrm{d}\left(\star_g \mathrm{d}\mathbf{A} + \frac{\beta}{\alpha}\mathbf{A}\right) = 0, \tag{10.7}$$

which is certainly satisfied if

$$\star_g \mathrm{d}\mathbf{A} + \frac{\beta}{2\alpha}\mathbf{A} = 0. \tag{10.8}$$

That above is indeed the Beltrami equation with an eigenvalue $\lambda = -\frac{\beta}{\alpha}$ dictated by the ratio of the two coupling constants in the Lagrangian.

Possibility Two. The Beltrami equation is just the field equation of the field theory. In this case we obtain the candidate Lagrangian by making the same identification as in Eq. (10.5) and then writing

$$\begin{aligned} &\mathcal{A} = \int \mathcal{L}, \quad \mathcal{L} = \beta\,\mathbf{A} \wedge \star_g \mathbf{A} + \alpha\,\mathbf{F} \wedge \mathbf{A}, \\ &\mathbf{F} = \mathrm{d}\mathbf{A}. \end{aligned} \tag{10.9}$$

The above action describes a $\mathrm{U}(1)$ Chern Simons gauge theory with the addition of a mass term for the gauge field. Such a mass term might be induced by a Brout–Englert–Higgs mechanism of spontaneous symmetry breaking. In this case equation (10.8) is just the complete field equation.

The two mentioned possibilities* are quite interesting and challenging in view of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence which relates a supersymmetric Chern–Simons gauge theory on the boundary with a supergravity theory in the bulk of an anti-de Sitter space AdS_4 (for the most general form of supersymmetric Chern–Simons theories in $d = 3$, see [28] and for several examples of such theories derived from $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, see [27, 29–31]). We plan to come back to such an issue in a forthcoming publication [32]. What we want to stress here is that adopting such a point of view, one sticks to a Principle that rules out the plethora of Beltrami flows and the feasting with flows of champagne. The eigenvalue in the Beltrami equation is a ratio of coupling constants appearing in the Lagrangian and one has to consider only those flows that fit to it. Furthermore it is to be hoped that the ratio $\lambda = -\beta/2\alpha$ is fixed from other elements of the

*Approximately half month after the appearance of the present paper on the ArXive, a new independent result was published by Ferrara and Sagnotti (*Massive Born–Infeld and Other Dual Pairs*. arXiv:1502.01650), establishing that the two above-mentioned possibilities are actually off-shell equivalent by means of a duality transformation that maps one Lagrangian into the other. On-shell equivalence is evident from our equation (10.7). See also the old paper by Townsend, Pilch, and van Nieuwenhuizen (*Self-duality in Odd Dimensions.* Phys. Lett. B **136** (1984) 38 [Addendum-ibid. B **137** (1984) 443]).

construction, for instance, from the supergravity solution on $\mathrm{AdS}_4 \times$ something. In this case it would obtain an interpretation in terms of First Principles.

Let us now come to a further reason of discomfort in relation with symmetries. In the presentation of our results we have observed that several interesting Arnold–Beltrami flows are obtained by decomposing the irreducible representations of the Universal Classifying Group into irreps of some of its subgroups $\mathrm{H}_i \subset \mathfrak{GU}_\Lambda$. When an identity representation is available in the branching rules, we obtain the Arnold–Beltrami flow invariant under the group H_i. In the existing literature on ABC-flows, there is a general feeling that flows with symmetries have a distinguished and more important role to play than other noninvariant ones, yet nowhere such a role is spelled out in a clear way and there is no well-established hierarchy of concepts for the interpretation of such symmetries. This is still another manifestation of the problems that the lack of a field theoretical basis for the flows does create. The streamlines are solutions of nonlinear equations and for this reason there is no superposition principle. On the other hand, the equation that defines the flows, namely the Beltrami equation, is linear and, as such, it leads to a superposition principle. If one is able to anchor such an equation to the firm ground of a field theory, then the moduli space of the solutions, namely the parameters that fill the irreducible representations of the Classifying Group $\mathfrak{GU}_\Lambda$, can be explored with standard techniques: as it happens in many similar situations, we should expect that the moduli-space points that correspond to an enhanced symmetry are in some sense singular points and they might dominate some appropriate path-integral.

Thus, we believe that all the conceptual problems we have pointed out can be addressed and solved only in the framework of some reasonable field theory for the velocity field U. This is, in our opinion, the priority one issue of this research field.

Finally, in the vein of the above remarks let us come back to the discussion of Eqs. (1.30) and (1.31) of Introduction. We propose that in higher odd dimensions $d = (2\,p + 1)$, instead that by equation (1.31), a *generalized contact structure* be defined by a p-contact form $\alpha^{(p)}$ satisfying the condition

$$\alpha^{(p)} \wedge \mathrm{d}\alpha^{(p)} \neq 0. \tag{10.10}$$

At every point of the manifold $\mathcal{M}_{2p+1}$, the kernel of the contact form is a subspace of the tangent space $\mathrm{T}_p\mathcal{M}$ of codimension p rather than of codimension one. So, as in the classical three-dimensional case, also in $2p + 1$ dimensions the contact form defines a sub-bundle of the tangent bundle, yet with dimension of the fibre equal to $p + 1$ rather than $2p$. The same ideas about maximal nonintegrability of such a bundle can apply, in the sense that its fibres can be prevented from being the tangent spaces of any embedded hypersurface $\Sigma_{p+1} \subset M_{2p+1}$ of dimension $p + 1$. We stress that the *linear* Beltrami equation for such generalized contact forms can be linked to supergravity field equations, in particular, for $p = 3$ to the field equations of M-theory, when one compactifies $\mathcal{M}_{11} = \mathcal{M}_4 \times \mathcal{M}_7$. This is another issue that we plan to investigate.

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THE APPENDICES: TABLES OF CONJUGACY CLASSES, CHARACTERS, BRANCHING RULES AND ORBIT SPLITTINGS

A. DESCRIPTION OF THE UNIVERSAL CLASSIFYING GROUP G_{1536} AND OF ITS SUBGROUPS

In this appendix we provide the list of elements of the Universal Classifying Group G_{1536} and of all its subgroups that happen to be relevant in the constructions discussed in the main text. The most relevant piece of information provided here is the assembling of the group elements into conjugacy classes. This is done both for the Universal Classifying Group G_{1536} and for each of its subgroups relevant to us. This arrangement is essential for the calculation of characters and for the decomposition of any representation into irreducible ones. The group elements are uniquely identified by their code:

$$\left\{\gamma, \frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}\right\}, \quad \gamma \in \mathrm{O}_{24}, \tag{A.1}$$

where γ is an element of the proper octahedral group labeled according to the notation of Eq. (4.5), while $n_1, n_2, n_3 \in \{0, 1, 2, 3\}$ specify a translation. The action of the group element $\left\{\gamma, \frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}\right\}$ on the three Euclidean coordinates $\{x, y, z\}$ is

$$\left\{\gamma, \frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}\right\} : \{x, y, z\} \to \gamma \cdot \{x, y\, z\} + \left\{\frac{n_1}{4}, \frac{n_2}{2}, \frac{n_3}{4}\right\}. \tag{A.2}$$

The subgroups we explicitly describe in the present section are the following ones:

A) The chain of normal subgroups:

$$\mathrm{G}_{1536} \rhd \mathrm{G}_{768} \rhd \mathrm{G}_{256} \rhd \mathrm{G}_{128} \rhd \mathrm{G}_{64}, \tag{A.3}$$

where $\mathrm{G}_{64} \sim \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ is Abelian and corresponds to the compactified translation group. The above chain leads to the following quotient groups:

$$\frac{\mathrm{G}_{1536}}{\mathrm{G}_{768}} \sim \mathbb{Z}_2; \quad \frac{\mathrm{G}_{768}}{\mathrm{G}_{256}} \sim \mathbb{Z}_3; \quad \frac{\mathrm{G}_{256}}{\mathrm{G}_{128}} \sim \mathbb{Z}_2; \quad \frac{\mathrm{G}_{128}}{\mathrm{G}_{64}} \sim \mathbb{Z}_2. \tag{A.4}$$

B) The subgroup $\mathrm{G}_{192} \subset \mathrm{G}_{1536}$, with respect to which the 6-dimensional point orbit remains irreducible. G_{192} is not a normal subgroup but possesses a chain of normal subgroups that make it solvable and allow for the complete calculation of its irreps and character table:

$$\mathrm{G}_{192} \rhd \mathrm{G}_{96} \rhd \mathrm{G}_{48} \rhd \mathrm{G}_{16}. \tag{A.5}$$

C) The subgroup $\mathrm{GF}_{192} \subset \mathrm{G}_{1536}$, with respect to which the 6-dimensional point orbit splits into a pair $\mathbf{3} \oplus \mathbf{3}$ of irreducible representations. The subgroups G_{192} and GF_{192} are isomorphic: $\mathrm{G}_{192} \sim \mathrm{GF}_{192}$, yet they are not conjugate to each other. Indeed, the branching rules of G_{1536}-irreps with respect to either G_{192} or GF_{192} are sometimes different. Just as G_{192} also GF_{192} is not a normal subgroup.

D) The subgroup $\mathrm{Oh}_{48} \subset \mathrm{G}_{192}$ which is isomorphic to the extended octahedral group $\mathrm{O_h}$ of crystallography.

E) The subgroup $\mathrm{GS}_{24} \subset \mathrm{GF}_{192}$ which is isomorphic but not conjugate to the proper octahedral group O_{24} (*i.e.*, the point group) and appears as the group of hidden symmetries of a parameterless Arnold–Beltrami flow derived both from the 6-point orbit and from the lowest-lying 24-point orbit.

F) The subgroup $\mathrm{GP}_{24} \subset \mathrm{G}_{192}$ which is not isomorphic to the proper octahedral group O_{24}, having a different structure of conjugacy classes, and appears as a group of hidden symmetries of a parameterless Arnold–Beltrami flow derived from the 12-point orbit.

G) The subgroup $\mathrm{GK}_{24} \subset \mathrm{GF}_{192}$ which is isomorphic but not conjugate to the group GP_{24}, and also appears as a group of hidden symmetries of a parameterless Arnold–Beltrami flow derived from the 12-point orbit. Both GP_{24} and GK_{24} are isomorphic to the abstract group $\mathrm{A}_2 \times \mathbb{Z}_2$

H) The subgroup $\mathrm{GS}_{32} \subset \mathrm{G}_{192}$ which appears as a group of hidden symmetries of a parameterless Arnold–Beltrami flow derived from the 8-point orbit.

I) The subgroup $\mathrm{GK}_{32} \subset \mathrm{GF}_{192}$ which is isomorphic but not conjugate to GS_{32} and also appears as a group of hidden symmetries of a parameterless Arnold–Beltrami flow derived from the 8-point orbit.

A.1. The Group G_{1536}. In this section, we list all the elements of the space group G_{1536}, organized into their 37 conjugacy classes.

Conjugacy Class $\mathcal{C}_1\left(\mathrm{G}_{1536}\right)$: # *of elements* = 1

$$\left\{ \begin{array}{cccc} 1_1 & 0 & 0 & 0 \end{array} \right\} \tag{A.6}$$

Conjugacy class $\mathcal{C}_2\left(\mathrm{G}_{1536}\right)$: # *of elements* = 1

$$\left\{ \begin{array}{cccc} 1_1 & 1 & 1 & 1 \end{array} \right\} \tag{A.7}$$

Conjugacy class $\mathcal{C}_3\left(\mathrm{G}_{1536}\right)$: # *of elements* = 3

$$\{1_1, 0, 0, 1\} \quad \{1_1, 0, 1, 0\} \quad \{1_1, 1, 0, 0\} \tag{A.8}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{G}_{1536})$: # *of elements* = 3

$$\{1_1,0,1,1\}\quad\{1_1,1,0,1\}\quad\{1_1,1,1,0\}\tag{A.9}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{G}_{1536})$: # *of elements* = 6

$$\begin{array}{ccc}\left\{1_1,0,0,\frac{1}{2}\right\} & \left\{1_1,0,0,\frac{3}{2}\right\} & \left\{1_1,0,\frac{1}{2},0\right\}\\ \left\{1_1,0,\frac{3}{2},0\right\} & \left\{1_1,\frac{1}{2},0,0\right\} & \left\{1_1,\frac{3}{2},0,0\right\}\end{array}\tag{A.10}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{G}_{1536})$: # *of elements* = 6

$$\begin{array}{ccc}\left\{1_1,\frac{1}{2},1,1\right\} & \left\{1_1,1,\frac{1}{2},1\right\} & \left\{1_1,1,1,\frac{1}{2}\right\}\\ \left\{1_1,1,1,\frac{3}{2}\right\} & \left\{1_1,1,\frac{3}{2},1\right\} & \left\{1_1,\frac{3}{2},1,1\right\}\end{array}\tag{A.11}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{G}_{1536})$: # *of elements* = 8

$$\begin{array}{cccc}\left\{1_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{1_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{1_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{1_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\}\\ \left\{1_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{1_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{1_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{1_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}\end{array}\tag{A.12}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{G}_{1536})$: # *of elements* = 12

$$\begin{array}{cccc}\left\{1_1,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{1_1,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{1_1,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{1_1,0,\frac{3}{2},\frac{3}{2}\right\}\\ \left\{1_1,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{1_1,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{1_1,\frac{1}{2},\frac{1}{2},0\right\} & \left\{1_1,\frac{1}{2},\frac{3}{2},0\right\}\\ \left\{1_1,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{1_1,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{1_1,\frac{3}{2},\frac{1}{2},0\right\} & \left\{1_1,\frac{3}{2},\frac{3}{2},0\right\}\end{array}\tag{A.13}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{G}_{1536})$: # *of elements* = 12

$$\begin{array}{cccc}\left\{1_1,0,\frac{1}{2},1\right\} & \left\{1_1,0,1,\frac{1}{2}\right\} & \left\{1_1,0,1,\frac{3}{2}\right\} & \left\{1_1,0,\frac{3}{2},1\right\}\\ \left\{1_1,\frac{1}{2},0,1\right\} & \left\{1_1,\frac{1}{2},1,0\right\} & \left\{1_1,1,0,\frac{1}{2}\right\} & \left\{1_1,1,0,\frac{3}{2}\right\}\\ \left\{1_1,1,\frac{1}{2},0\right\} & \left\{1_1,1,\frac{3}{2},0\right\} & \left\{1_1,\frac{3}{2},0,1\right\} & \left\{1_1,\frac{3}{2},1,0\right\}\end{array}\tag{A.14}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{G}_{1536})$: # *of elements* = 12

$$\begin{array}{cccc}\left\{1_1,\frac{1}{2},\frac{1}{2},1\right\} & \left\{1_1,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{1_1,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{1_1,\frac{1}{2},\frac{3}{2},1\right\}\\ \left\{1_1,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{1_1,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{1_1,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{1_1,1,\frac{3}{2},\frac{3}{2}\right\}\\ \left\{1_1,\frac{3}{2},\frac{1}{2},1\right\} & \left\{1_1,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{1_1,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{1_1,\frac{3}{2},\frac{3}{2},1\right\}\end{array}\tag{A.15}$$

Conjugacy class $\mathcal{C}_{11}\,(\mathrm{G}_{1536})$: # *of elements* = 12

$$\begin{array}{cccc}\{3_1,0,0,0\} & \{3_1,0,1,0\} & \{3_1,1,0,0\} & \{3_1,1,1,0\}\\ \{3_2,0,0,0\} & \{3_2,0,0,1\} & \{3_2,1,0,0\} & \{3_2,1,0,1\}\\ \{3_3,0,0,0\} & \{3_3,0,0,1\} & \{3_3,0,1,0\} & \{3_3,0,1,1\}\end{array}\tag{A.16}$$

Conjugacy class $\mathcal{C}_{12}(\mathrm{G}_{1536})$: # *of elements* = 12

$$\begin{array}{cccc} \{3_1,0,0,1\} & \{3_1,0,1,1\} & \{3_1,1,0,1\} & \{3_1,1,1,1\} \\ \{3_2,0,1,0\} & \{3_2,0,1,1\} & \{3_2,1,1,0\} & \{3_2,1,1,1\} \\ \{3_3,1,0,0\} & \{3_3,1,0,1\} & \{3_3,1,1,0\} & \{3_3,1,1,1\} \end{array} \tag{A.17}$$

Conjugacy class $\mathcal{C}_{13}(\mathrm{G}_{1536})$: # *of elements* = 12

$$\begin{array}{cccc} \left\{3_1,\frac{1}{2},\frac{1}{2},0\right\} & \left\{3_1,\frac{1}{2},\frac{3}{2},0\right\} & \left\{3_1,\frac{3}{2},\frac{1}{2},0\right\} & \left\{3_1,\frac{3}{2},\frac{3}{2},0\right\} \\ \left\{3_2,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{3_2,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{3_2,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{3_2,\frac{3}{2},0,\frac{3}{2}\right\} \\ \left\{3_3,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_3,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_3,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_3,0,\frac{3}{2},\frac{3}{2}\right\} \end{array} \tag{A.18}$$

Conjugacy class $\mathcal{C}_{14}(\mathrm{G}_{1536})$: # *of elements* = 12

$$\begin{array}{cccc} \left\{3_1,\frac{1}{2},\frac{1}{2},1\right\} & \left\{3_1,\frac{1}{2},\frac{3}{2},1\right\} & \left\{3_1,\frac{3}{2},\frac{1}{2},1\right\} & \left\{3_1,\frac{3}{2},\frac{3}{2},1\right\} \\ \left\{3_2,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{3_2,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{3_2,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{3_2,\frac{3}{2},1,\frac{3}{2}\right\} \\ \left\{3_3,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_3,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_3,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_3,1,\frac{3}{2},\frac{3}{2}\right\} \end{array} \tag{A.19}$$

Conjugacy class $\mathcal{C}_{15}(\mathrm{G}_{1536})$: # *of elements* = 24

$$\begin{array}{cccc} \left\{3_1,0,0,\frac{1}{2}\right\} & \left\{3_1,0,0,\frac{3}{2}\right\} & \left\{3_1,0,1,\frac{1}{2}\right\} & \left\{3_1,0,1,\frac{3}{2}\right\} \\ \left\{3_1,1,0,\frac{1}{2}\right\} & \left\{3_1,1,0,\frac{3}{2}\right\} & \left\{3_1,1,1,\frac{1}{2}\right\} & \left\{3_1,1,1,\frac{3}{2}\right\} \\ \left\{3_2,0,\frac{1}{2},0\right\} & \left\{3_2,0,\frac{1}{2},1\right\} & \left\{3_2,0,\frac{3}{2},0\right\} & \left\{3_2,0,\frac{3}{2},1\right\} \\ \left\{3_2,1,\frac{1}{2},0\right\} & \left\{3_2,1,\frac{1}{2},1\right\} & \left\{3_2,1,\frac{3}{2},0\right\} & \left\{3_2,1,\frac{3}{2},1\right\} \\ \left\{3_3,\frac{1}{2},0,0\right\} & \left\{3_3,\frac{1}{2},0,1\right\} & \left\{3_3,\frac{1}{2},1,0\right\} & \left\{3_3,\frac{1}{2},1,1\right\} \\ \left\{3_3,\frac{3}{2},0,0\right\} & \left\{3_3,\frac{3}{2},0,1\right\} & \left\{3_3,\frac{3}{2},1,0\right\} & \left\{3_3,\frac{3}{2},1,1\right\} \end{array} \tag{A.20}$$

Conjugacy class $\mathcal{C}_{16}(\mathrm{G}_{1536})$: # *of elements* = 24

$$\begin{array}{cccc} \left\{3_1,0,\frac{1}{2},0\right\} & \left\{3_1,0,\frac{3}{2},0\right\} & \left\{3_1,\frac{1}{2},0,0\right\} & \left\{3_1,\frac{1}{2},1,0\right\} \\ \left\{3_1,1,\frac{1}{2},0\right\} & \left\{3_1,1,\frac{3}{2},0\right\} & \left\{3_1,\frac{3}{2},0,0\right\} & \left\{3_1,\frac{3}{2},1,0\right\} \\ \left\{3_2,0,0,\frac{1}{2}\right\} & \left\{3_2,0,0,\frac{3}{2}\right\} & \left\{3_2,\frac{1}{2},0,0\right\} & \left\{3_2,\frac{1}{2},0,1\right\} \\ \left\{3_2,1,0,\frac{1}{2}\right\} & \left\{3_2,1,0,\frac{3}{2}\right\} & \left\{3_2,\frac{3}{2},0,0\right\} & \left\{3_2,\frac{3}{2},0,1\right\} \\ \left\{3_3,0,0,\frac{1}{2}\right\} & \left\{3_3,0,0,\frac{3}{2}\right\} & \left\{3_3,0,\frac{1}{2},0\right\} & \left\{3_3,0,\frac{1}{2},1\right\} \\ \left\{3_3,0,1,\frac{1}{2}\right\} & \left\{3_3,0,1,\frac{3}{2}\right\} & \left\{3_3,0,\frac{3}{2},0\right\} & \left\{3_3,0,\frac{3}{2},1\right\} \end{array} \tag{A.21}$$

Conjugacy class $\mathcal{C}_{17}\,(\mathrm{G}_{1536})$: # *of elements* = 24

$$\begin{array}{cccc}
\left\{3_1,0,\frac{1}{2},1\right\} & \left\{3_1,0,\frac{3}{2},1\right\} & \left\{3_1,\frac{1}{2},0,1\right\} & \left\{3_1,\frac{1}{2},1,1\right\} \\
\left\{3_1,1,\frac{1}{2},1\right\} & \left\{3_1,1,\frac{3}{2},1\right\} & \left\{3_1,\frac{3}{2},0,1\right\} & \left\{3_1,\frac{3}{2},1,1\right\} \\
\left\{3_2,0,1,\frac{1}{2}\right\} & \left\{3_2,0,1,\frac{3}{2}\right\} & \left\{3_2,\frac{1}{2},1,0\right\} & \left\{3_2,\frac{1}{2},1,1\right\} \\
\left\{3_2,1,1,\frac{1}{2}\right\} & \left\{3_2,1,1,\frac{3}{2}\right\} & \left\{3_2,\frac{3}{2},1,0\right\} & \left\{3_2,\frac{3}{2},1,1\right\} \\
\left\{3_3,1,0,\frac{1}{2}\right\} & \left\{3_3,1,0,\frac{3}{2}\right\} & \left\{3_3,1,\frac{1}{2},0\right\} & \left\{3_3,1,\frac{1}{2},1\right\} \\
\left\{3_3,1,1,\frac{1}{2}\right\} & \left\{3_3,1,1,\frac{3}{2}\right\} & \left\{3_3,1,\frac{3}{2},0\right\} & \left\{3_3,1,\frac{3}{2},1\right\}
\end{array} \tag{A.22}$$

Conjugacy class $\mathcal{C}_{18}\,(\mathrm{G}_{1536})$: # *of elements* = 24

$$\begin{array}{cccc}
\left\{3_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_2,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_2,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_3,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_3,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}
\end{array} \tag{A.23}$$

Conjugacy class $\mathcal{C}_{19}\,(\mathrm{G}_{1536})$: # *of elements* = 48

$$\begin{array}{cccc}
\left\{3_1,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_1,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{3_1,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},1,\frac{3}{2}\right\} \\
\left\{3_1,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_1,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{3_1,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},1,\frac{3}{2}\right\} \\
\left\{3_2,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_2,\frac{1}{2},\frac{1}{2},0\right\} & \left\{3_2,\frac{1}{2},\frac{1}{2},1\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},0\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{3_2,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_2,\frac{3}{2},\frac{1}{2},0\right\} & \left\{3_2,\frac{3}{2},\frac{1}{2},1\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},0\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{3_3,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{1}{2},0\right\} & \left\{3_3,\frac{1}{2},\frac{1}{2},1\right\} \\
\left\{3_3,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},0\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{3_3,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{1}{2},0\right\} & \left\{3_3,\frac{3}{2},\frac{1}{2},1\right\} \\
\left\{3_3,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},0\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},1\right\}
\end{array} \tag{A.24}$$

Conjugacy class $\mathcal{C}_{20}\,(\mathrm{G}_{1536})$: # *of elements* = 48

$$\begin{array}{llll}
\{4_1,0,0,0\} & \{4_1,0,\frac{1}{2},\frac{1}{2}\} & \{4_1,0,1,1\} & \{4_1,0,\frac{3}{2},\frac{3}{2}\} \\
\{4_1,1,0,0\} & \{4_1,1,\frac{1}{2},\frac{1}{2}\} & \{4_1,1,1,1\} & \{4_1,1,\frac{3}{2},\frac{3}{2}\} \\
\{4_2,0,0,0\} & \{4_2,0,\frac{1}{2},\frac{3}{2}\} & \{4_2,0,1,1\} & \{4_2,0,\frac{3}{2},\frac{1}{2}\} \\
\{4_2,1,0,0\} & \{4_2,1,\frac{1}{2},\frac{3}{2}\} & \{4_2,1,1,1\} & \{4_2,1,\frac{3}{2},\frac{1}{2}\} \\
\{4_3,0,0,0\} & \{4_3,0,0,1\} & \{4_3,\frac{1}{2},\frac{1}{2},0\} & \{4_3,\frac{1}{2},\frac{1}{2},1\} \\
\{4_3,1,1,0\} & \{4_3,1,1,1\} & \{4_3,\frac{3}{2},\frac{3}{2},0\} & \{4_3,\frac{3}{2},\frac{3}{2},1\} \\
\{4_4,0,0,0\} & \{4_4,0,1,0\} & \{4_4,\frac{1}{2},0,\frac{1}{2}\} & \{4_4,\frac{1}{2},1,\frac{1}{2}\} \\
\{4_4,1,0,1\} & \{4_4,1,1,1\} & \{4_4,\frac{3}{2},0,\frac{3}{2}\} & \{4_4,\frac{3}{2},1,\frac{3}{2}\} \\
\{4_5,0,0,0\} & \{4_5,0,1,0\} & \{4_5,\frac{1}{2},0,\frac{3}{2}\} & \{4_5,\frac{1}{2},1,\frac{3}{2}\} \\
\{4_5,1,0,1\} & \{4_5,1,1,1\} & \{4_5,\frac{3}{2},0,\frac{1}{2}\} & \{4_5,\frac{3}{2},1,\frac{1}{2}\} \\
\{4_6,0,0,0\} & \{4_6,0,0,1\} & \{4_6,\frac{1}{2},\frac{3}{2},0\} & \{4_6,\frac{1}{2},\frac{3}{2},1\} \\
\{4_6,1,1,0\} & \{4_6,1,1,1\} & \{4_6,\frac{3}{2},\frac{1}{2},0\} & \{4_6,\frac{3}{2},\frac{1}{2},1\}
\end{array} \tag{A.25}$$

Conjugacy class $\mathcal{C}_{21}\,(\mathrm{G}_{1536})$: # *of elements* = 48

$$\begin{array}{llll}
\{4_1,0,0,1\} & \{4_1,0,\frac{1}{2},\frac{3}{2}\} & \{4_1,0,1,0\} & \{4_1,0,\frac{3}{2},\frac{1}{2}\} \\
\{4_1,1,0,1\} & \{4_1,1,\frac{1}{2},\frac{3}{2}\} & \{4_1,1,1,0\} & \{4_1,1,\frac{3}{2},\frac{1}{2}\} \\
\{4_2,0,0,1\} & \{4_2,0,\frac{1}{2},\frac{1}{2}\} & \{4_2,0,1,0\} & \{4_2,0,\frac{3}{2},\frac{3}{2}\} \\
\{4_2,1,0,1\} & \{4_2,1,\frac{1}{2},\frac{1}{2}\} & \{4_2,1,1,0\} & \{4_2,1,\frac{3}{2},\frac{3}{2}\} \\
\{4_3,0,1,0\} & \{4_3,0,1,1\} & \{4_3,\frac{1}{2},\frac{3}{2},0\} & \{4_3,\frac{1}{2},\frac{3}{2},1\} \\
\{4_3,1,0,0\} & \{4_3,1,0,1\} & \{4_3,\frac{3}{2},\frac{1}{2},0\} & \{4_3,\frac{3}{2},\frac{1}{2},1\} \\
\{4_4,0,0,1\} & \{4_4,0,1,1\} & \{4_4,\frac{1}{2},0,\frac{3}{2}\} & \{4_4,\frac{1}{2},1,\frac{3}{2}\} \\
\{4_4,1,0,0\} & \{4_4,1,1,0\} & \{4_4,\frac{3}{2},0,\frac{1}{2}\} & \{4_4,\frac{3}{2},1,\frac{1}{2}\} \\
\{4_5,0,0,1\} & \{4_5,0,1,1\} & \{4_5,\frac{1}{2},0,\frac{1}{2}\} & \{4_5,\frac{1}{2},1,\frac{1}{2}\} \\
\{4_5,1,0,0\} & \{4_5,1,1,0\} & \{4_5,\frac{3}{2},0,\frac{3}{2}\} & \{4_5,\frac{3}{2},1,\frac{3}{2}\} \\
\{4_6,0,1,0\} & \{4_6,0,1,1\} & \{4_6,\frac{1}{2},\frac{1}{2},0\} & \{4_6,\frac{1}{2},\frac{1}{2},1\} \\
\{4_6,1,0,0\} & \{4_6,1,0,1\} & \{4_6,\frac{3}{2},\frac{3}{2},0\} & \{4_6,\frac{3}{2},\frac{3}{2},1\}
\end{array} \tag{A.26}$$

Conjugacy class $\mathcal{C}_{22}\,(\mathrm{G}_{1536})$: # *of elements* $= 48$

$$\begin{array}{llll}
\left\{4_1,\frac{1}{2},0,0\right\} & \left\{4_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_1,\frac{1}{2},1,1\right\} & \left\{4_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_1,\frac{3}{2},0,0\right\} & \left\{4_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_1,\frac{3}{2},1,1\right\} & \left\{4_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_2,\frac{1}{2},0,0\right\} & \left\{4_2,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_2,\frac{1}{2},1,1\right\} & \left\{4_2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{4_2,\frac{3}{2},0,0\right\} & \left\{4_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_2,\frac{3}{2},1,1\right\} & \left\{4_2,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{4_3,0,0,\frac{1}{2}\right\} & \left\{4_3,0,0,\frac{3}{2}\right\} & \left\{4_3,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_3,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} \\
\left\{4_3,1,1,\frac{1}{2}\right\} & \left\{4_3,1,1,\frac{3}{2}\right\} & \left\{4_3,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{4_3,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_4,0,\frac{1}{2},0\right\} & \left\{4_4,0,\frac{3}{2},0\right\} & \left\{4_4,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_4,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{4_4,1,\frac{1}{2},1\right\} & \left\{4_4,1,\frac{3}{2},1\right\} & \left\{4_4,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_4,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_5,0,\frac{1}{2},0\right\} & \left\{4_5,0,\frac{3}{2},0\right\} & \left\{4_5,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_5,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_5,1,\frac{1}{2},1\right\} & \left\{4_5,1,\frac{3}{2},1\right\} & \left\{4_5,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_5,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{4_6,0,0,\frac{1}{2}\right\} & \left\{4_6,0,0,\frac{3}{2}\right\} & \left\{4_6,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{4_6,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_6,1,1,\frac{1}{2}\right\} & \left\{4_6,1,1,\frac{3}{2}\right\} & \left\{4_6,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_6,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\}
\end{array} \tag{A.27}$$

Conjugacy class $\mathcal{C}_{23}\,(\mathrm{G}_{1536})$: # *of elements* $= 48$

$$\begin{array}{llll}
\left\{4_1,\frac{1}{2},0,1\right\} & \left\{4_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_1,\frac{1}{2},1,0\right\} & \left\{4_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{4_1,\frac{3}{2},0,1\right\} & \left\{4_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_1,\frac{3}{2},1,0\right\} & \left\{4_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{4_2,\frac{1}{2},0,1\right\} & \left\{4_2,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_2,\frac{1}{2},1,0\right\} & \left\{4_2,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_2,\frac{3}{2},0,1\right\} & \left\{4_2,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_2,\frac{3}{2},1,0\right\} & \left\{4_2,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_3,0,1,\frac{1}{2}\right\} & \left\{4_3,0,1,\frac{3}{2}\right\} & \left\{4_3,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{4_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_3,1,0,\frac{1}{2}\right\} & \left\{4_3,1,0,\frac{3}{2}\right\} & \left\{4_3,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_3,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} \\
\left\{4_4,0,\frac{1}{2},1\right\} & \left\{4_4,0,\frac{3}{2},1\right\} & \left\{4_4,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_4,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_4,1,\frac{1}{2},0\right\} & \left\{4_4,1,\frac{3}{2},0\right\} & \left\{4_4,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_4,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{4_5,0,\frac{1}{2},1\right\} & \left\{4_5,0,\frac{3}{2},1\right\} & \left\{4_5,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_5,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{4_5,1,\frac{1}{2},0\right\} & \left\{4_5,1,\frac{3}{2},0\right\} & \left\{4_5,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_5,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_6,0,1,\frac{1}{2}\right\} & \left\{4_6,0,1,\frac{3}{2}\right\} & \left\{4_6,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_6,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} \\
\left\{4_6,1,0,\frac{1}{2}\right\} & \left\{4_6,1,0,\frac{3}{2}\right\} & \left\{4_6,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{4_6,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}
\end{array} \tag{A.28}$$

Conjugacy class $\mathcal{C}_{24}\,(\mathrm{G}_{1536})$: # *of elements* $= 48$

$$
\begin{array}{llll}
\{5_1,0,0,0\} & \{5_1,0,1,0\} & \{5_1,\frac{1}{2},\frac{1}{2},0\} & \{5_1,\frac{1}{2},\frac{3}{2},0\} \\
\{5_1,1,0,0\} & \{5_1,1,1,0\} & \{5_1,\frac{3}{2},\frac{1}{2},0\} & \{5_1,\frac{3}{2},\frac{3}{2},0\} \\
\{5_2,0,0,0\} & \{5_2,0,0,1\} & \{5_2,\frac{1}{2},0,\frac{1}{2}\} & \{5_2,\frac{1}{2},0,\frac{3}{2}\} \\
\{5_2,1,0,0\} & \{5_2,1,0,1\} & \{5_2,\frac{3}{2},0,\frac{1}{2}\} & \{5_2,\frac{3}{2},0,\frac{3}{2}\} \\
\{5_3,0,0,0\} & \{5_3,0,0,1\} & \{5_3,\frac{1}{2},0,\frac{1}{2}\} & \{5_3,\frac{1}{2},0,\frac{3}{2}\} \\
\{5_3,1,0,0\} & \{5_3,1,0,1\} & \{5_3,\frac{3}{2},0,\frac{1}{2}\} & \{5_3,\frac{3}{2},0,\frac{3}{2}\} \\
\{5_4,0,0,0\} & \{5_4,0,1,0\} & \{5_4,\frac{1}{2},\frac{1}{2},0\} & \{5_4,\frac{1}{2},\frac{3}{2},0\} \\
\{5_4,1,0,0\} & \{5_4,1,1,0\} & \{5_4,\frac{3}{2},\frac{1}{2},0\} & \{5_4,\frac{3}{2},\frac{3}{2},0\} \\
\{5_5,0,0,0\} & \{5_5,0,0,1\} & \{5_5,0,\frac{1}{2},\frac{1}{2}\} & \{5_5,0,\frac{1}{2},\frac{3}{2}\} \\
\{5_5,0,1,0\} & \{5_5,0,1,1\} & \{5_5,0,\frac{3}{2},\frac{1}{2}\} & \{5_5,0,\frac{3}{2},\frac{3}{2}\} \\
\{5_6,0,0,0\} & \{5_6,0,0,1\} & \{5_6,0,\frac{1}{2},\frac{1}{2}\} & \{5_6,0,\frac{1}{2},\frac{3}{2}\} \\
\{5_6,0,1,0\} & \{5_6,0,1,1\} & \{5_6,0,\frac{3}{2},\frac{1}{2}\} & \{5_6,0,\frac{3}{2},\frac{3}{2}\}
\end{array}
\tag{A.29}
$$

Conjugacy class $\mathcal{C}_{25}\,(\mathrm{G}_{1536})$: # *of elements* $= 48$

$$
\begin{array}{llll}
\{5_1,0,0,\frac{1}{2}\} & \{5_1,0,1,\frac{1}{2}\} & \{5_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\} & \{5_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{5_1,1,0,\frac{1}{2}\} & \{5_1,1,1,\frac{1}{2}\} & \{5_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{5_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\} \\
\{5_2,0,\frac{3}{2},0\} & \{5_2,0,\frac{3}{2},1\} & \{5_2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} & \{5_2,\frac{1}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_2,1,\frac{3}{2},0\} & \{5_2,1,\frac{3}{2},1\} & \{5_2,\frac{3}{2},\frac{3}{2},\frac{1}{2}\} & \{5_2,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_3,0,\frac{1}{2},0\} & \{5_3,0,\frac{1}{2},1\} & \{5_3,\frac{1}{2},\frac{1}{2},\frac{1}{2}\} & \{5_3,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} \\
\{5_3,1,\frac{1}{2},0\} & \{5_3,1,\frac{1}{2},1\} & \{5_3,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{5_3,\frac{3}{2},\frac{1}{2},\frac{3}{2}\} \\
\{5_4,0,0,\frac{3}{2}\} & \{5_4,0,1,\frac{3}{2}\} & \{5_4,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{5_4,\frac{1}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_4,1,0,\frac{3}{2}\} & \{5_4,1,1,\frac{3}{2}\} & \{5_4,\frac{3}{2},\frac{1}{2},\frac{3}{2}\} & \{5_4,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_5,\frac{1}{2},0,0\} & \{5_5,\frac{1}{2},0,1\} & \{5_5,\frac{1}{2},\frac{1}{2},\frac{1}{2}\} & \{5_5,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} \\
\{5_5,\frac{1}{2},1,0\} & \{5_5,\frac{1}{2},1,1\} & \{5_5,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} & \{5_5,\frac{1}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_6,\frac{3}{2},0,0\} & \{5_6,\frac{3}{2},0,1\} & \{5_6,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{5_6,\frac{3}{2},\frac{1}{2},\frac{3}{2}\} \\
\{5_6,\frac{3}{2},1,0\} & \{5_6,\frac{3}{2},1,1\} & \{5_6,\frac{3}{2},\frac{3}{2},\frac{1}{2}\} & \{5_6,\frac{3}{2},\frac{3}{2},\frac{3}{2}\}
\end{array}
\tag{A.30}
$$

Conjugacy class $\mathcal{C}_{26}\,(\mathrm{G}_{1536})$: # *of elements* = 48

$$\begin{array}{llll}
\{5_1,0,0,1\} & \{5_1,0,1,1\} & \{5_1,\frac{1}{2},\frac{1}{2},1\} & \{5_1,\frac{1}{2},\frac{3}{2},1\} \\
\{5_1,1,0,1\} & \{5_1,1,1,1\} & \{5_1,\frac{3}{2},\frac{1}{2},1\} & \{5_1,\frac{3}{2},\frac{3}{2},1\} \\
\{5_2,0,1,0\} & \{5_2,0,1,1\} & \{5_2,\frac{1}{2},1,\frac{1}{2}\} & \{5_2,\frac{1}{2},1,\frac{3}{2}\} \\
\{5_2,1,1,0\} & \{5_2,1,1,1\} & \{5_2,\frac{3}{2},1,\frac{1}{2}\} & \{5_2,\frac{3}{2},1,\frac{3}{2}\} \\
\{5_3,0,1,0\} & \{5_3,0,1,1\} & \{5_3,\frac{1}{2},1,\frac{1}{2}\} & \{5_3,\frac{1}{2},1,\frac{3}{2}\} \\
\{5_3,1,1,0\} & \{5_3,1,1,1\} & \{5_3,\frac{3}{2},1,\frac{1}{2}\} & \{5_3,\frac{3}{2},1,\frac{3}{2}\} \\
\{5_4,0,0,1\} & \{5_4,0,1,1\} & \{5_4,\frac{1}{2},\frac{1}{2},1\} & \{5_4,\frac{1}{2},\frac{3}{2},1\} \\
\{5_4,1,0,1\} & \{5_4,1,1,1\} & \{5_4,\frac{3}{2},\frac{1}{2},1\} & \{5_4,\frac{3}{2},\frac{3}{2},1\} \\
\{5_5,1,0,0\} & \{5_5,1,0,1\} & \{5_5,1,\frac{1}{2},\frac{1}{2}\} & \{5_5,1,\frac{1}{2},\frac{3}{2}\} \\
\{5_5,1,1,0\} & \{5_5,1,1,1\} & \{5_5,1,\frac{3}{2},\frac{1}{2}\} & \{5_5,1,\frac{3}{2},\frac{3}{2}\} \\
\{5_6,1,0,0\} & \{5_6,1,0,1\} & \{5_6,1,\frac{1}{2},\frac{1}{2}\} & \{5_6,1,\frac{1}{2},\frac{3}{2}\} \\
\{5_6,1,1,0\} & \{5_6,1,1,1\} & \{5_6,1,\frac{3}{2},\frac{1}{2}\} & \{5_6,1,\frac{3}{2},\frac{3}{2}\}
\end{array} \tag{A.31}$$

Conjugacy class $\mathcal{C}_{27}\,(\mathrm{G}_{1536})$: # *of elements* = 48

$$\begin{array}{llll}
\{5_1,0,0,\frac{3}{2}\} & \{5_1,0,1,\frac{3}{2}\} & \{5_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{5_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_1,1,0,\frac{3}{2}\} & \{5_1,1,1,\frac{3}{2}\} & \{5_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\} & \{5_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_2,0,\frac{1}{2},0\} & \{5_2,0,\frac{1}{2},1\} & \{5_2,\frac{1}{2},\frac{1}{2},\frac{1}{2}\} & \{5_2,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} \\
\{5_2,1,\frac{1}{2},0\} & \{5_2,1,\frac{1}{2},1\} & \{5_2,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{5_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\} \\
\{5_3,0,\frac{3}{2},0\} & \{5_3,0,\frac{3}{2},1\} & \{5_3,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} & \{5_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_3,1,\frac{3}{2},0\} & \{5_3,1,\frac{3}{2},1\} & \{5_3,\frac{3}{2},\frac{3}{2},\frac{1}{2}\} & \{5_3,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_4,0,0,\frac{1}{2}\} & \{5_4,0,1,\frac{1}{2}\} & \{5_4,\frac{1}{2},\frac{1}{2},\frac{1}{2}\} & \{5_4,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{5_4,1,0,\frac{1}{2}\} & \{5_4,1,1,\frac{1}{2}\} & \{5_4,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{5_4,\frac{3}{2},\frac{3}{2},\frac{1}{2}\} \\
\{5_5,\frac{3}{2},0,0\} & \{5_5,\frac{3}{2},0,1\} & \{5_5,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{5_5,\frac{3}{2},\frac{1}{2},\frac{3}{2}\} \\
\{5_5,\frac{3}{2},1,0\} & \{5_5,\frac{3}{2},1,1\} & \{5_5,\frac{3}{2},\frac{3}{2},\frac{1}{2}\} & \{5_5,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{5_6,\frac{1}{2},0,0\} & \{5_6,\frac{1}{2},0,1\} & \{5_6,\frac{1}{2},\frac{1}{2},\frac{1}{2}\} & \{5_6,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} \\
\{5_6,\frac{1}{2},1,0\} & \{5_6,\frac{1}{2},1,1\} & \{5_6,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} & \{5_6,\frac{1}{2},\frac{3}{2},\frac{3}{2}\}
\end{array} \tag{A.32}$$

Conjugacy class $\mathcal{C}_{28}\,(\mathrm{G}_{1536})$: # *of elements* $= 48$

$$\begin{array}{cccc}
\left\{5_1,0,\frac{1}{2},0\right\} & \left\{5_1,0,\frac{3}{2},0\right\} & \left\{5_1,\frac{1}{2},0,0\right\} & \left\{5_1,\frac{1}{2},1,0\right\} \\
\left\{5_1,1,\frac{1}{2},0\right\} & \left\{5_1,1,\frac{3}{2},0\right\} & \left\{5_1,\frac{3}{2},0,0\right\} & \left\{5_1,\frac{3}{2},1,0\right\} \\
\left\{5_2,0,0,\frac{1}{2}\right\} & \left\{5_2,0,0,\frac{3}{2}\right\} & \left\{5_2,\frac{1}{2},0,0\right\} & \left\{5_2,\frac{1}{2},0,1\right\} \\
\left\{5_2,1,0,\frac{1}{2}\right\} & \left\{5_2,1,0,\frac{3}{2}\right\} & \left\{5_2,\frac{3}{2},0,0\right\} & \left\{5_2,\frac{3}{2},0,1\right\} \\
\left\{5_3,0,0,\frac{1}{2}\right\} & \left\{5_3,0,0,\frac{3}{2}\right\} & \left\{5_3,\frac{1}{2},0,0\right\} & \left\{5_3,\frac{1}{2},0,1\right\} \\
\left\{5_3,1,0,\frac{1}{2}\right\} & \left\{5_3,1,0,\frac{3}{2}\right\} & \left\{5_3,\frac{3}{2},0,0\right\} & \left\{5_3,\frac{3}{2},0,1\right\} \\
\left\{5_4,0,\frac{1}{2},0\right\} & \left\{5_4,0,\frac{3}{2},0\right\} & \left\{5_4,\frac{1}{2},0,0\right\} & \left\{5_4,\frac{1}{2},1,0\right\} \\
\left\{5_4,1,\frac{1}{2},0\right\} & \left\{5_4,1,\frac{3}{2},0\right\} & \left\{5_4,\frac{3}{2},0,0\right\} & \left\{5_4,\frac{3}{2},1,0\right\} \\
\left\{5_5,0,0,\frac{1}{2}\right\} & \left\{5_5,0,0,\frac{3}{2}\right\} & \left\{5_5,0,\frac{1}{2},0\right\} & \left\{5_5,0,\frac{1}{2},1\right\} \\
\left\{5_5,0,1,\frac{1}{2}\right\} & \left\{5_5,0,1,\frac{3}{2}\right\} & \left\{5_5,0,\frac{3}{2},0\right\} & \left\{5_5,0,\frac{3}{2},1\right\} \\
\left\{5_6,0,0,\frac{1}{2}\right\} & \left\{5_6,0,0,\frac{3}{2}\right\} & \left\{5_6,0,\frac{1}{2},0\right\} & \left\{5_6,0,\frac{1}{2},1\right\} \\
\left\{5_6,0,1,\frac{1}{2}\right\} & \left\{5_6,0,1,\frac{3}{2}\right\} & \left\{5_6,0,\frac{3}{2},0\right\} & \left\{5_6,0,\frac{3}{2},1\right\}
\end{array} \tag{A.33}$$

Conjugacy class $\mathcal{C}_{29}\,(\mathrm{G}_{1536})$: # *of elements* $= 48$

$$\begin{array}{cccc}
\left\{5_1,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{5_1,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{5_1,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{5_1,\frac{1}{2},1,\frac{1}{2}\right\} \\
\left\{5_1,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{5_1,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{5_1,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{5_1,\frac{3}{2},1,\frac{1}{2}\right\} \\
\left\{5_2,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{5_2,0,\frac{3}{2},\frac{3}{2}\right\} & \left\{5_2,\frac{1}{2},\frac{3}{2},0\right\} & \left\{5_2,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{5_2,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{5_2,1,\frac{3}{2},\frac{3}{2}\right\} & \left\{5_2,\frac{3}{2},\frac{3}{2},0\right\} & \left\{5_2,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{5_3,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{5_3,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{5_3,\frac{1}{2},\frac{1}{2},0\right\} & \left\{5_3,\frac{1}{2},\frac{1}{2},1\right\} \\
\left\{5_3,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{5_3,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{5_3,\frac{3}{2},\frac{1}{2},0\right\} & \left\{5_3,\frac{3}{2},\frac{1}{2},1\right\} \\
\left\{5_4,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{5_4,0,\frac{3}{2},\frac{3}{2}\right\} & \left\{5_4,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{5_4,\frac{1}{2},1,\frac{3}{2}\right\} \\
\left\{5_4,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{5_4,1,\frac{3}{2},\frac{3}{2}\right\} & \left\{5_4,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{5_4,\frac{3}{2},1,\frac{3}{2}\right\} \\
\left\{5_5,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{5_5,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{5_5,\frac{1}{2},\frac{1}{2},0\right\} & \left\{5_5,\frac{1}{2},\frac{1}{2},1\right\} \\
\left\{5_5,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{5_5,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{5_5,\frac{1}{2},\frac{3}{2},0\right\} & \left\{5_5,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{5_6,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{5_6,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{5_6,\frac{3}{2},\frac{1}{2},0\right\} & \left\{5_6,\frac{3}{2},\frac{1}{2},1\right\} \\
\left\{5_6,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{5_6,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{5_6,\frac{3}{2},\frac{3}{2},0\right\} & \left\{5_6,\frac{3}{2},\frac{3}{2},1\right\}
\end{array} \tag{A.34}$$

Conjugacy class $\mathcal{C}_{30}\,(\mathrm{G}_{1536})$: # *of elements* = 48

$$
\begin{array}{llll}
\{5_1,0,\frac{1}{2},1\} & \{5_1,0,\frac{3}{2},1\} & \{5_1,\frac{1}{2},0,1\} & \{5_1,\frac{1}{2},1,1\} \\
\{5_1,1,\frac{1}{2},1\} & \{5_1,1,\frac{3}{2},1\} & \{5_1,\frac{3}{2},0,1\} & \{5_1,\frac{3}{2},1,1\} \\
\{5_2,0,1,\frac{1}{2}\} & \{5_2,0,1,\frac{3}{2}\} & \{5_2,\frac{1}{2},1,0\} & \{5_2,\frac{1}{2},1,1\} \\
\{5_2,1,1,\frac{1}{2}\} & \{5_2,1,1,\frac{3}{2}\} & \{5_2,\frac{3}{2},1,0\} & \{5_2,\frac{3}{2},1,1\} \\
\{5_3,0,1,\frac{1}{2}\} & \{5_3,0,1,\frac{3}{2}\} & \{5_3,\frac{1}{2},1,0\} & \{5_3,\frac{1}{2},1,1\} \\
\{5_3,1,1,\frac{1}{2}\} & \{5_3,1,1,\frac{3}{2}\} & \{5_3,\frac{3}{2},1,0\} & \{5_3,\frac{3}{2},1,1\} \\
\{5_4,0,\frac{1}{2},1\} & \{5_4,0,\frac{3}{2},1\} & \{5_4,\frac{1}{2},0,1\} & \{5_4,\frac{1}{2},1,1\} \\
\{5_4,1,\frac{1}{2},1\} & \{5_4,1,\frac{3}{2},1\} & \{5_4,\frac{3}{2},0,1\} & \{5_4,\frac{3}{2},1,1\} \\
\{5_5,1,0,\frac{1}{2}\} & \{5_5,1,0,\frac{3}{2}\} & \{5_5,1,\frac{1}{2},0\} & \{5_5,1,\frac{1}{2},1\} \\
\{5_5,1,1,\frac{1}{2}\} & \{5_5,1,1,\frac{3}{2}\} & \{5_5,1,\frac{3}{2},0\} & \{5_5,1,\frac{3}{2},1\} \\
\{5_6,1,0,\frac{1}{2}\} & \{5_6,1,0,\frac{3}{2}\} & \{5_6,1,\frac{1}{2},0\} & \{5_6,1,\frac{1}{2},1\} \\
\{5_6,1,1,\frac{1}{2}\} & \{5_6,1,1,\frac{3}{2}\} & \{5_6,1,\frac{3}{2},0\} & \{5_6,1,\frac{3}{2},1\}
\end{array}
\tag{A.35}
$$

Conjugacy class $\mathcal{C}_{31}\,(\mathrm{G}_{1536})$: # *of elements* = 48

$$
\begin{array}{llll}
\{5_1,0,\frac{1}{2},\frac{3}{2}\} & \{5_1,0,\frac{3}{2},\frac{3}{2}\} & \{5_1,\frac{1}{2},0,\frac{3}{2}\} & \{5_1,\frac{1}{2},1,\frac{3}{2}\} \\
\{5_1,1,\frac{1}{2},\frac{3}{2}\} & \{5_1,1,\frac{3}{2},\frac{3}{2}\} & \{5_1,\frac{3}{2},0,\frac{3}{2}\} & \{5_1,\frac{3}{2},1,\frac{3}{2}\} \\
\{5_2,0,\frac{1}{2},\frac{1}{2}\} & \{5_2,0,\frac{1}{2},\frac{3}{2}\} & \{5_2,\frac{1}{2},\frac{1}{2},0\} & \{5_2,\frac{1}{2},\frac{1}{2},1\} \\
\{5_2,1,\frac{1}{2},\frac{1}{2}\} & \{5_2,1,\frac{1}{2},\frac{3}{2}\} & \{5_2,\frac{3}{2},\frac{1}{2},0\} & \{5_2,\frac{3}{2},\frac{1}{2},1\} \\
\{5_3,0,\frac{3}{2},\frac{1}{2}\} & \{5_3,0,\frac{3}{2},\frac{3}{2}\} & \{5_3,\frac{1}{2},\frac{3}{2},0\} & \{5_3,\frac{1}{2},\frac{3}{2},1\} \\
\{5_3,1,\frac{3}{2},\frac{1}{2}\} & \{5_3,1,\frac{3}{2},\frac{3}{2}\} & \{5_3,\frac{3}{2},\frac{3}{2},0\} & \{5_3,\frac{3}{2},\frac{3}{2},1\} \\
\{5_4,0,\frac{1}{2},\frac{1}{2}\} & \{5_4,0,\frac{3}{2},\frac{1}{2}\} & \{5_4,\frac{1}{2},0,\frac{1}{2}\} & \{5_4,\frac{1}{2},1,\frac{1}{2}\} \\
\{5_4,1,\frac{1}{2},\frac{1}{2}\} & \{5_4,1,\frac{3}{2},\frac{1}{2}\} & \{5_4,\frac{3}{2},0,\frac{1}{2}\} & \{5_4,\frac{3}{2},1,\frac{1}{2}\} \\
\{5_5,\frac{3}{2},0,\frac{1}{2}\} & \{5_5,\frac{3}{2},0,\frac{3}{2}\} & \{5_5,\frac{3}{2},\frac{1}{2},0\} & \{5_5,\frac{3}{2},\frac{1}{2},1\} \\
\{5_5,\frac{3}{2},1,\frac{1}{2}\} & \{5_5,\frac{3}{2},1,\frac{3}{2}\} & \{5_5,\frac{3}{2},\frac{3}{2},0\} & \{5_5,\frac{3}{2},\frac{3}{2},1\} \\
\{5_6,\frac{1}{2},0,\frac{1}{2}\} & \{5_6,\frac{1}{2},0,\frac{3}{2}\} & \{5_6,\frac{1}{2},\frac{1}{2},0\} & \{5_6,\frac{1}{2},\frac{1}{2},1\} \\
\{5_6,\frac{1}{2},1,\frac{1}{2}\} & \{5_6,\frac{1}{2},1,\frac{3}{2}\} & \{5_6,\frac{1}{2},\frac{3}{2},0\} & \{5_6,\frac{1}{2},\frac{3}{2},1\}
\end{array}
\tag{A.36}
$$

Conjugacy class $\mathcal{C}_{32}\left(\mathrm{G}_{1536}\right)$: # *of elements* = 96

$$
\begin{array}{cccc}
\left\{4_1,0,0,\frac{1}{2}\right\} & \left\{4_1,0,0,\frac{3}{2}\right\} & \left\{4_1,0,\frac{1}{2},0\right\} & \left\{4_1,0,\frac{1}{2},1\right\} \\
\left\{4_1,0,1,\frac{1}{2}\right\} & \left\{4_1,0,1,\frac{3}{2}\right\} & \left\{4_1,0,\frac{3}{2},0\right\} & \left\{4_1,0,\frac{3}{2},1\right\} \\
\left\{4_1,1,0,\frac{1}{2}\right\} & \left\{4_1,1,0,\frac{3}{2}\right\} & \left\{4_1,1,\frac{1}{2},0\right\} & \left\{4_1,1,\frac{1}{2},1\right\} \\
\left\{4_1,1,1,\frac{1}{2}\right\} & \left\{4_1,1,1,\frac{3}{2}\right\} & \left\{4_1,1,\frac{3}{2},0\right\} & \left\{4_1,1,\frac{3}{2},1\right\} \\
\left\{4_2,0,0,\frac{1}{2}\right\} & \left\{4_2,0,0,\frac{3}{2}\right\} & \left\{4_2,0,\frac{1}{2},0\right\} & \left\{4_2,0,\frac{1}{2},1\right\} \\
\left\{4_2,0,1,\frac{1}{2}\right\} & \left\{4_2,0,1,\frac{3}{2}\right\} & \left\{4_2,0,\frac{3}{2},0\right\} & \left\{4_2,0,\frac{3}{2},1\right\} \\
\left\{4_2,1,0,\frac{1}{2}\right\} & \left\{4_2,1,0,\frac{3}{2}\right\} & \left\{4_2,1,\frac{1}{2},0\right\} & \left\{4_2,1,\frac{1}{2},1\right\} \\
\left\{4_2,1,1,\frac{1}{2}\right\} & \left\{4_2,1,1,\frac{3}{2}\right\} & \left\{4_2,1,\frac{3}{2},0\right\} & \left\{4_2,1,\frac{3}{2},1\right\} \\
\left\{4_3,0,\frac{1}{2},0\right\} & \left\{4_3,0,\frac{1}{2},1\right\} & \left\{4_3,0,\frac{3}{2},0\right\} & \left\{4_3,0,\frac{3}{2},1\right\} \\
\left\{4_3,\frac{1}{2},0,0\right\} & \left\{4_3,\frac{1}{2},0,1\right\} & \left\{4_3,\frac{1}{2},1,0\right\} & \left\{4_3,\frac{1}{2},1,1\right\} \\
\left\{4_3,1,\frac{1}{2},0\right\} & \left\{4_3,1,\frac{1}{2},1\right\} & \left\{4_3,1,\frac{3}{2},0\right\} & \left\{4_3,1,\frac{3}{2},1\right\} \\
\left\{4_3,\frac{3}{2},0,0\right\} & \left\{4_3,\frac{3}{2},0,1\right\} & \left\{4_3,\frac{3}{2},1,0\right\} & \left\{4_3,\frac{3}{2},1,1\right\} \\
\left\{4_4,0,0,\frac{1}{2}\right\} & \left\{4_4,0,0,\frac{3}{2}\right\} & \left\{4_4,0,1,\frac{1}{2}\right\} & \left\{4_4,0,1,\frac{3}{2}\right\} \\
\left\{4_4,\frac{1}{2},0,0\right\} & \left\{4_4,\frac{1}{2},0,1\right\} & \left\{4_4,\frac{1}{2},1,0\right\} & \left\{4_4,\frac{1}{2},1,1\right\} \\
\left\{4_4,1,0,\frac{1}{2}\right\} & \left\{4_4,1,0,\frac{3}{2}\right\} & \left\{4_4,1,1,\frac{1}{2}\right\} & \left\{4_4,1,1,\frac{3}{2}\right\} \\
\left\{4_4,\frac{3}{2},0,0\right\} & \left\{4_4,\frac{3}{2},0,1\right\} & \left\{4_4,\frac{3}{2},1,0\right\} & \left\{4_4,\frac{3}{2},1,1\right\} \\
\left\{4_5,0,0,\frac{1}{2}\right\} & \left\{4_5,0,0,\frac{3}{2}\right\} & \left\{4_5,0,1,\frac{1}{2}\right\} & \left\{4_5,0,1,\frac{3}{2}\right\} \\
\left\{4_5,\frac{1}{2},0,0\right\} & \left\{4_5,\frac{1}{2},0,1\right\} & \left\{4_5,\frac{1}{2},1,0\right\} & \left\{4_5,\frac{1}{2},1,1\right\} \\
\left\{4_5,1,0,\frac{1}{2}\right\} & \left\{4_5,1,0,\frac{3}{2}\right\} & \left\{4_5,1,1,\frac{1}{2}\right\} & \left\{4_5,1,1,\frac{3}{2}\right\} \\
\left\{4_5,\frac{3}{2},0,0\right\} & \left\{4_5,\frac{3}{2},0,1\right\} & \left\{4_5,\frac{3}{2},1,0\right\} & \left\{4_5,\frac{3}{2},1,1\right\} \\
\left\{4_6,0,\frac{1}{2},0\right\} & \left\{4_6,0,\frac{1}{2},1\right\} & \left\{4_6,0,\frac{3}{2},0\right\} & \left\{4_6,0,\frac{3}{2},1\right\} \\
\left\{4_6,\frac{1}{2},0,0\right\} & \left\{4_6,\frac{1}{2},0,1\right\} & \left\{4_6,\frac{1}{2},1,0\right\} & \left\{4_6,\frac{1}{2},1,1\right\} \\
\left\{4_6,1,\frac{1}{2},0\right\} & \left\{4_6,1,\frac{1}{2},1\right\} & \left\{4_6,1,\frac{3}{2},0\right\} & \left\{4_6,1,\frac{3}{2},1\right\} \\
\left\{4_6,\frac{3}{2},0,0\right\} & \left\{4_6,\frac{3}{2},0,1\right\} & \left\{4_6,\frac{3}{2},1,0\right\} & \left\{4_6,\frac{3}{2},1,1\right\}
\end{array}
\tag{A.37}
$$

Conjugacy class $\mathcal{C}_{33}\,(\mathrm{G}_{1536})$: $\#$ *of elements* $= 96$

$$
\begin{array}{cccc}
\left\{4_1,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{4_1,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{4_1,\frac{1}{2},\frac{1}{2},0\right\} & \left\{4_1,\frac{1}{2},\frac{1}{2},1\right\} \\
\left\{4_1,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{4_1,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{4_1,\frac{1}{2},\frac{3}{2},0\right\} & \left\{4_1,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{4_1,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{4_1,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{4_1,\frac{3}{2},\frac{1}{2},0\right\} & \left\{4_1,\frac{3}{2},\frac{1}{2},1\right\} \\
\left\{4_1,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{4_1,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{4_1,\frac{3}{2},\frac{3}{2},0\right\} & \left\{4_1,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{4_2,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{4_2,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{4_2,\frac{1}{2},\frac{1}{2},0\right\} & \left\{4_2,\frac{1}{2},\frac{1}{2},1\right\} \\
\left\{4_2,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{4_2,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{4_2,\frac{1}{2},\frac{3}{2},0\right\} & \left\{4_2,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{4_2,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{4_2,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{4_2,\frac{3}{2},\frac{1}{2},0\right\} & \left\{4_2,\frac{3}{2},\frac{1}{2},1\right\} \\
\left\{4_2,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{4_2,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{4_2,\frac{3}{2},\frac{3}{2},0\right\} & \left\{4_2,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{4_3,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{4_3,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{4_3,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{4_3,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_3,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{4_3,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{4_3,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{4_3,\frac{1}{2},1,\frac{3}{2}\right\} \\
\left\{4_3,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{4_3,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{4_3,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{4_3,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_3,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{4_3,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{4_3,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{4_3,\frac{3}{2},1,\frac{3}{2}\right\} \\
\left\{4_4,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{4_4,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{4_4,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{4_4,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_4,\frac{1}{2},\frac{1}{2},0\right\} & \left\{4_4,\frac{1}{2},\frac{1}{2},1\right\} & \left\{4_4,\frac{1}{2},\frac{3}{2},0\right\} & \left\{4_4,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{4_4,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{4_4,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{4_4,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{4_4,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_4,\frac{3}{2},\frac{1}{2},0\right\} & \left\{4_4,\frac{3}{2},\frac{1}{2},1\right\} & \left\{4_4,\frac{3}{2},\frac{3}{2},0\right\} & \left\{4_4,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{4_5,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{4_5,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{4_5,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{4_5,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_5,\frac{1}{2},\frac{1}{2},0\right\} & \left\{4_5,\frac{1}{2},\frac{1}{2},1\right\} & \left\{4_5,\frac{1}{2},\frac{3}{2},0\right\} & \left\{4_5,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{4_5,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{4_5,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{4_5,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{4_5,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_5,\frac{3}{2},\frac{1}{2},0\right\} & \left\{4_5,\frac{3}{2},\frac{1}{2},1\right\} & \left\{4_5,\frac{3}{2},\frac{3}{2},0\right\} & \left\{4_5,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{4_6,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{4_6,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{4_6,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{4_6,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_6,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{4_6,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{4_6,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{4_6,\frac{1}{2},1,\frac{3}{2}\right\} \\
\left\{4_6,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{4_6,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{4_6,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{4_6,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{4_6,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{4_6,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{4_6,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{4_6,\frac{3}{2},1,\frac{3}{2}\right\}
\end{array}
\tag{A.38}
$$

Conjugacy class $\mathcal{C}_{34}\,(\mathrm{G}_{1536})$: # *of elements* = 128

$$\begin{array}{llll}
\{2_1,0,0,0\} & \{2_1,0,\frac{1}{2},\frac{1}{2}\} & \{2_1,0,1,1\} & \{2_1,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_1,\frac{1}{2},0,\frac{3}{2}\} & \{2_1,\frac{1}{2},\frac{1}{2},0\} & \{2_1,\frac{1}{2},1,\frac{1}{2}\} & \{2_1,\frac{1}{2},\frac{3}{2},1\} \\
\{2_1,1,0,1\} & \{2_1,1,\frac{1}{2},\frac{3}{2}\} & \{2_1,1,1,0\} & \{2_1,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_1,\frac{3}{2},0,\frac{1}{2}\} & \{2_1,\frac{3}{2},\frac{1}{2},1\} & \{2_1,\frac{3}{2},1,\frac{3}{2}\} & \{2_1,\frac{3}{2},\frac{3}{2},0\} \\
\{2_2,0,0,0\} & \{2_2,0,\frac{1}{2},\frac{3}{2}\} & \{2_2,0,1,1\} & \{2_2,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_2,\frac{1}{2},0,\frac{1}{2}\} & \{2_2,\frac{1}{2},\frac{1}{2},0\} & \{2_2,\frac{1}{2},1,\frac{3}{2}\} & \{2_2,\frac{1}{2},\frac{3}{2},1\} \\
\{2_2,1,0,1\} & \{2_2,1,\frac{1}{2},\frac{1}{2}\} & \{2_2,1,1,0\} & \{2_2,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_2,\frac{3}{2},0,\frac{3}{2}\} & \{2_2,\frac{3}{2},\frac{1}{2},1\} & \{2_2,\frac{3}{2},1,\frac{1}{2}\} & \{2_2,\frac{3}{2},\frac{3}{2},0\} \\
\{2_3,0,0,0\} & \{2_3,0,\frac{1}{2},\frac{3}{2}\} & \{2_3,0,1,1\} & \{2_3,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_3,\frac{1}{2},0,\frac{1}{2}\} & \{2_3,\frac{1}{2},\frac{1}{2},0\} & \{2_3,\frac{1}{2},1,\frac{3}{2}\} & \{2_3,\frac{1}{2},\frac{3}{2},1\} \\
\{2_3,1,0,1\} & \{2_3,1,\frac{1}{2},\frac{1}{2}\} & \{2_3,1,1,0\} & \{2_3,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_3,\frac{3}{2},0,\frac{3}{2}\} & \{2_3,\frac{3}{2},\frac{1}{2},1\} & \{2_3,\frac{3}{2},1,\frac{1}{2}\} & \{2_3,\frac{3}{2},\frac{3}{2},0\} \\
\{2_4,0,0,0\} & \{2_4,0,\frac{1}{2},\frac{1}{2}\} & \{2_4,0,1,1\} & \{2_4,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_4,\frac{1}{2},0,\frac{1}{2}\} & \{2_4,\frac{1}{2},\frac{1}{2},1\} & \{2_4,\frac{1}{2},1,\frac{3}{2}\} & \{2_4,\frac{1}{2},\frac{3}{2},0\} \\
\{2_4,1,0,1\} & \{2_4,1,\frac{1}{2},\frac{3}{2}\} & \{2_4,1,1,0\} & \{2_4,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_4,\frac{3}{2},0,\frac{3}{2}\} & \{2_4,\frac{3}{2},\frac{1}{2},0\} & \{2_4,\frac{3}{2},1,\frac{1}{2}\} & \{2_4,\frac{3}{2},\frac{3}{2},1\} \\
\{2_5,0,0,0\} & \{2_5,0,\frac{1}{2},\frac{1}{2}\} & \{2_5,0,1,1\} & \{2_5,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_5,\frac{1}{2},0,\frac{3}{2}\} & \{2_5,\frac{1}{2},\frac{1}{2},0\} & \{2_5,\frac{1}{2},1,\frac{1}{2}\} & \{2_5,\frac{1}{2},\frac{3}{2},1\} \\
\{2_5,1,0,1\} & \{2_5,1,\frac{1}{2},\frac{3}{2}\} & \{2_5,1,1,0\} & \{2_5,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_5,\frac{3}{2},0,\frac{1}{2}\} & \{2_5,\frac{3}{2},\frac{1}{2},1\} & \{2_5,\frac{3}{2},1,\frac{3}{2}\} & \{2_5,\frac{3}{2},\frac{3}{2},0\} \\
\{2_6,0,0,0\} & \{2_6,0,\frac{1}{2},\frac{3}{2}\} & \{2_6,0,1,1\} & \{2_6,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_6,\frac{1}{2},0,\frac{3}{2}\} & \{2_6,\frac{1}{2},\frac{1}{2},1\} & \{2_6,\frac{1}{2},1,\frac{1}{2}\} & \{2_6,\frac{1}{2},\frac{3}{2},0\} \\
\{2_6,1,0,1\} & \{2_6,1,\frac{1}{2},\frac{1}{2}\} & \{2_6,1,1,0\} & \{2_6,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_6,\frac{3}{2},0,\frac{1}{2}\} & \{2_6,\frac{3}{2},\frac{1}{2},0\} & \{2_6,\frac{3}{2},1,\frac{3}{2}\} & \{2_6,\frac{3}{2},\frac{3}{2},1\} \\
\{2_7,0,0,0\} & \{2_7,0,\frac{1}{2},\frac{1}{2}\} & \{2_7,0,1,1\} & \{2_7,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_7,\frac{1}{2},0,\frac{1}{2}\} & \{2_7,\frac{1}{2},\frac{1}{2},1\} & \{2_7,\frac{1}{2},1,\frac{3}{2}\} & \{2_7,\frac{1}{2},\frac{3}{2},0\} \\
\{2_7,1,0,1\} & \{2_7,1,\frac{1}{2},\frac{3}{2}\} & \{2_7,1,1,0\} & \{2_7,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_7,\frac{3}{2},0,\frac{3}{2}\} & \{2_7,\frac{3}{2},\frac{1}{2},0\} & \{2_7,\frac{3}{2},1,\frac{1}{2}\} & \{2_7,\frac{3}{2},\frac{3}{2},1\} \\
\{2_8,0,0,0\} & \{2_8,0,\frac{1}{2},\frac{3}{2}\} & \{2_8,0,1,1\} & \{2_8,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_8,\frac{1}{2},0,\frac{3}{2}\} & \{2_8,\frac{1}{2},\frac{1}{2},1\} & \{2_8,\frac{1}{2},1,\frac{1}{2}\} & \{2_8,\frac{1}{2},\frac{3}{2},0\} \\
\{2_8,1,0,1\} & \{2_8,1,\frac{1}{2},\frac{1}{2}\} & \{2_8,1,1,0\} & \{2_8,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_8,\frac{3}{2},0,\frac{1}{2}\} & \{2_8,\frac{3}{2},\frac{1}{2},0\} & \{2_8,\frac{3}{2},1,\frac{3}{2}\} & \{2_8,\frac{3}{2},\frac{3}{2},1\}
\end{array} \tag{A.39}$$

Conjugacy class $\mathcal{C}_{35}\,(\mathrm{G}_{1536})$: # *of elements* = 128

$$\begin{array}{llll}
\left\{2_1,0,0,\frac{1}{2}\right\} & \left\{2_1,0,\frac{1}{2},1\right\} & \left\{2_1,0,1,\frac{3}{2}\right\} & \left\{2_1,0,\frac{3}{2},0\right\} \\
\left\{2_1,\frac{1}{2},0,0\right\} & \left\{2_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_1,\frac{1}{2},1,1\right\} & \left\{2_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_1,1,0,\frac{3}{2}\right\} & \left\{2_1,1,\frac{1}{2},0\right\} & \left\{2_1,1,1,\frac{1}{2}\right\} & \left\{2_1,1,\frac{3}{2},1\right\} \\
\left\{2_1,\frac{3}{2},0,1\right\} & \left\{2_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_1,\frac{3}{2},1,0\right\} & \left\{2_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_2,0,0,\frac{1}{2}\right\} & \left\{2_2,0,\frac{1}{2},0\right\} & \left\{2_2,0,1,\frac{3}{2}\right\} & \left\{2_2,0,\frac{3}{2},1\right\} \\
\left\{2_2,\frac{1}{2},0,1\right\} & \left\{2_2,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_2,\frac{1}{2},1,0\right\} & \left\{2_2,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_2,1,0,\frac{3}{2}\right\} & \left\{2_2,1,\frac{1}{2},1\right\} & \left\{2_2,1,1,\frac{1}{2}\right\} & \left\{2_2,1,\frac{3}{2},0\right\} \\
\left\{2_2,\frac{3}{2},0,0\right\} & \left\{2_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_2,\frac{3}{2},1,1\right\} & \left\{2_2,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_3,0,0,\frac{3}{2}\right\} & \left\{2_3,0,\frac{1}{2},1\right\} & \left\{2_3,0,1,\frac{1}{2}\right\} & \left\{2_3,0,\frac{3}{2},0\right\} \\
\left\{2_3,\frac{1}{2},0,0\right\} & \left\{2_3,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_3,\frac{1}{2},1,1\right\} & \left\{2_3,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_3,1,0,\frac{1}{2}\right\} & \left\{2_3,1,\frac{1}{2},0\right\} & \left\{2_3,1,1,\frac{3}{2}\right\} & \left\{2_3,1,\frac{3}{2},1\right\} \\
\left\{2_3,\frac{3}{2},0,1\right\} & \left\{2_3,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_3,\frac{3}{2},1,0\right\} & \left\{2_3,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_4,0,0,\frac{1}{2}\right\} & \left\{2_4,0,\frac{1}{2},1\right\} & \left\{2_4,0,1,\frac{3}{2}\right\} & \left\{2_4,0,\frac{3}{2},0\right\} \\
\left\{2_4,\frac{1}{2},0,1\right\} & \left\{2_4,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_4,\frac{1}{2},1,0\right\} & \left\{2_4,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_4,1,0,\frac{3}{2}\right\} & \left\{2_4,1,\frac{1}{2},0\right\} & \left\{2_4,1,1,\frac{1}{2}\right\} & \left\{2_4,1,\frac{3}{2},1\right\} \\
\left\{2_4,\frac{3}{2},0,0\right\} & \left\{2_4,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_4,\frac{3}{2},1,1\right\} & \left\{2_4,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_5,0,0,\frac{3}{2}\right\} & \left\{2_5,0,\frac{1}{2},0\right\} & \left\{2_5,0,1,\frac{1}{2}\right\} & \left\{2_5,0,\frac{3}{2},1\right\} \\
\left\{2_5,\frac{1}{2},0,1\right\} & \left\{2_5,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_5,\frac{1}{2},1,0\right\} & \left\{2_5,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_5,1,0,\frac{1}{2}\right\} & \left\{2_5,1,\frac{1}{2},1\right\} & \left\{2_5,1,1,\frac{3}{2}\right\} & \left\{2_5,1,\frac{3}{2},0\right\} \\
\left\{2_5,\frac{3}{2},0,0\right\} & \left\{2_5,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_5,\frac{3}{2},1,1\right\} & \left\{2_5,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_6,0,0,\frac{1}{2}\right\} & \left\{2_6,0,\frac{1}{2},0\right\} & \left\{2_6,0,1,\frac{3}{2}\right\} & \left\{2_6,0,\frac{3}{2},1\right\} \\
\left\{2_6,\frac{1}{2},0,0\right\} & \left\{2_6,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_6,\frac{1}{2},1,1\right\} & \left\{2_6,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_6,1,0,\frac{3}{2}\right\} & \left\{2_6,1,\frac{1}{2},1\right\} & \left\{2_6,1,1,\frac{1}{2}\right\} & \left\{2_6,1,\frac{3}{2},0\right\} \\
\left\{2_6,\frac{3}{2},0,1\right\} & \left\{2_6,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_6,\frac{3}{2},1,0\right\} & \left\{2_6,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_7,0,0,\frac{3}{2}\right\} & \left\{2_7,0,\frac{1}{2},0\right\} & \left\{2_7,0,1,\frac{1}{2}\right\} & \left\{2_7,0,\frac{3}{2},1\right\} \\
\left\{2_7,\frac{1}{2},0,0\right\} & \left\{2_7,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_7,\frac{1}{2},1,1\right\} & \left\{2_7,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_7,1,0,\frac{1}{2}\right\} & \left\{2_7,1,\frac{1}{2},1\right\} & \left\{2_7,1,1,\frac{3}{2}\right\} & \left\{2_7,1,\frac{3}{2},0\right\} \\
\left\{2_7,\frac{3}{2},0,1\right\} & \left\{2_7,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_7,\frac{3}{2},1,0\right\} & \left\{2_7,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_8,0,0,\frac{3}{2}\right\} & \left\{2_8,0,\frac{1}{2},1\right\} & \left\{2_8,0,1,\frac{1}{2}\right\} & \left\{2_8,0,\frac{3}{2},0\right\} \\
\left\{2_8,\frac{1}{2},0,1\right\} & \left\{2_8,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_8,\frac{1}{2},1,0\right\} & \left\{2_8,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_8,1,0,\frac{1}{2}\right\} & \left\{2_8,1,\frac{1}{2},0\right\} & \left\{2_8,1,1,\frac{3}{2}\right\} & \left\{2_8,1,\frac{3}{2},1\right\} \\
\left\{2_8,\frac{3}{2},0,0\right\} & \left\{2_8,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_8,\frac{3}{2},1,1\right\} & \left\{2_8,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\}
\end{array} \tag{A.40}$$

Conjugacy class $\mathcal{C}_{36}\left(\mathrm{G}_{1536}\right)$: $\#$*-elements* = 128

$$
\begin{array}{llll}
\{2_1,0,0,1\} & \{2_1,0,\frac{1}{2},\frac{3}{2}\} & \{2_1,0,1,0\} & \{2_1,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_1,\frac{1}{2},0,\frac{1}{2}\} & \{2_1,\frac{1}{2},\frac{1}{2},1\} & \{2_1,\frac{1}{2},1,\frac{3}{2}\} & \{2_1,\frac{1}{2},\frac{3}{2},0\} \\
\{2_1,1,0,0\} & \{2_1,1,\frac{1}{2},\frac{1}{2}\} & \{2_1,1,1,1\} & \{2_1,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_1,\frac{3}{2},0,\frac{3}{2}\} & \{2_1,\frac{3}{2},\frac{1}{2},0\} & \{2_1,\frac{3}{2},1,\frac{1}{2}\} & \{2_1,\frac{3}{2},\frac{3}{2},1\} \\
\{2_2,0,0,1\} & \{2_2,0,\frac{1}{2},\frac{1}{2}\} & \{2_2,0,1,0\} & \{2_2,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_2,\frac{1}{2},0,\frac{3}{2}\} & \{2_2,\frac{1}{2},\frac{1}{2},1\} & \{2_2,\frac{1}{2},1,\frac{1}{2}\} & \{2_2,\frac{1}{2},\frac{3}{2},0\} \\
\{2_2,1,0,0\} & \{2_2,1,\frac{1}{2},\frac{3}{2}\} & \{2_2,1,1,1\} & \{2_2,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_2,\frac{3}{2},0,\frac{1}{2}\} & \{2_2,\frac{3}{2},\frac{1}{2},0\} & \{2_2,\frac{3}{2},1,\frac{3}{2}\} & \{2_2,\frac{3}{2},\frac{3}{2},1\} \\
\{2_3,0,0,1\} & \{2_3,0,\frac{1}{2},\frac{1}{2}\} & \{2_3,0,1,0\} & \{2_3,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_3,\frac{1}{2},0,\frac{3}{2}\} & \{2_3,\frac{1}{2},\frac{1}{2},1\} & \{2_3,\frac{1}{2},1,\frac{1}{2}\} & \{2_3,\frac{1}{2},\frac{3}{2},0\} \\
\{2_3,1,0,0\} & \{2_3,1,\frac{1}{2},\frac{3}{2}\} & \{2_3,1,1,1\} & \{2_3,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_3,\frac{3}{2},0,\frac{1}{2}\} & \{2_3,\frac{3}{2},\frac{1}{2},0\} & \{2_3,\frac{3}{2},1,\frac{3}{2}\} & \{2_3,\frac{3}{2},\frac{3}{2},1\} \\
\{2_4,0,0,1\} & \{2_4,0,\frac{1}{2},\frac{3}{2}\} & \{2_4,0,1,0\} & \{2_4,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_4,\frac{1}{2},0,\frac{3}{2}\} & \{2_4,\frac{1}{2},\frac{1}{2},0\} & \{2_4,\frac{1}{2},1,\frac{1}{2}\} & \{2_4,\frac{1}{2},\frac{3}{2},1\} \\
\{2_4,1,0,0\} & \{2_4,1,\frac{1}{2},\frac{1}{2}\} & \{2_4,1,1,1\} & \{2_4,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_4,\frac{3}{2},0,\frac{1}{2}\} & \{2_4,\frac{3}{2},\frac{1}{2},1\} & \{2_4,\frac{3}{2},1,\frac{3}{2}\} & \{2_4,\frac{3}{2},\frac{3}{2},0\} \\
\{2_5,0,0,1\} & \{2_5,0,\frac{1}{2},\frac{3}{2}\} & \{2_5,0,1,0\} & \{2_5,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_5,\frac{1}{2},0,\frac{1}{2}\} & \{2_5,\frac{1}{2},\frac{1}{2},1\} & \{2_5,\frac{1}{2},1,\frac{3}{2}\} & \{2_5,\frac{1}{2},\frac{3}{2},0\} \\
\{2_5,1,0,0\} & \{2_5,1,\frac{1}{2},\frac{1}{2}\} & \{2_5,1,1,1\} & \{2_5,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_5,\frac{3}{2},0,\frac{3}{2}\} & \{2_5,\frac{3}{2},\frac{1}{2},0\} & \{2_5,\frac{3}{2},1,\frac{1}{2}\} & \{2_5,\frac{3}{2},\frac{3}{2},1\} \\
\{2_6,0,0,1\} & \{2_6,0,\frac{1}{2},\frac{1}{2}\} & \{2_6,0,1,0\} & \{2_6,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_6,\frac{1}{2},0,\frac{1}{2}\} & \{2_6,\frac{1}{2},\frac{1}{2},0\} & \{2_6,\frac{1}{2},1,\frac{3}{2}\} & \{2_6,\frac{1}{2},\frac{3}{2},1\} \\
\{2_6,1,0,0\} & \{2_6,1,\frac{1}{2},\frac{3}{2}\} & \{2_6,1,1,1\} & \{2_6,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_6,\frac{3}{2},0,\frac{3}{2}\} & \{2_6,\frac{3}{2},\frac{1}{2},1\} & \{2_6,\frac{3}{2},1,\frac{1}{2}\} & \{2_6,\frac{3}{2},\frac{3}{2},0\} \\
\{2_7,0,0,1\} & \{2_7,0,\frac{1}{2},\frac{3}{2}\} & \{2_7,0,1,0\} & \{2_7,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_7,\frac{1}{2},0,\frac{3}{2}\} & \{2_7,\frac{1}{2},\frac{1}{2},0\} & \{2_7,\frac{1}{2},1,\frac{1}{2}\} & \{2_7,\frac{1}{2},\frac{3}{2},1\} \\
\{2_7,1,0,0\} & \{2_7,1,\frac{1}{2},\frac{1}{2}\} & \{2_7,1,1,1\} & \{2_7,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_7,\frac{3}{2},0,\frac{1}{2}\} & \{2_7,\frac{3}{2},\frac{1}{2},1\} & \{2_7,\frac{3}{2},1,\frac{3}{2}\} & \{2_7,\frac{3}{2},\frac{3}{2},0\} \\
\{2_8,0,0,1\} & \{2_8,0,\frac{1}{2},\frac{1}{2}\} & \{2_8,0,1,0\} & \{2_8,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_8,\frac{1}{2},0,\frac{1}{2}\} & \{2_8,\frac{1}{2},\frac{1}{2},0\} & \{2_8,\frac{1}{2},1,\frac{3}{2}\} & \{2_8,\frac{1}{2},\frac{3}{2},1\} \\
\{2_8,1,0,0\} & \{2_8,1,\frac{1}{2},\frac{3}{2}\} & \{2_8,1,1,1\} & \{2_8,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_8,\frac{3}{2},0,\frac{3}{2}\} & \{2_8,\frac{3}{2},\frac{1}{2},1\} & \{2_8,\frac{3}{2},1,\frac{1}{2}\} & \{2_8,\frac{3}{2},\frac{3}{2},0\}
\end{array}
\tag{A.41}
$$

Conjugacy class $\mathcal{C}_{37}\,(\mathrm{G}_{1536})$: $\#$-*elements* = 128

$$
\begin{array}{llll}
\left\{2_1,0,0,\frac{3}{2}\right\} & \left\{2_1,0,\frac{1}{2},0\right\} & \left\{2_1,0,1,\frac{1}{2}\right\} & \left\{2_1,0,\frac{3}{2},1\right\} \\
\left\{2_1,\frac{1}{2},0,1\right\} & \left\{2_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_1,\frac{1}{2},1,0\right\} & \left\{2_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_1,1,0,\frac{1}{2}\right\} & \left\{2_1,1,\frac{1}{2},1\right\} & \left\{2_1,1,1,\frac{3}{2}\right\} & \left\{2_1,1,\frac{3}{2},0\right\} \\
\left\{2_1,\frac{3}{2},0,0\right\} & \left\{2_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_1,\frac{3}{2},1,1\right\} & \left\{2_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_2,0,0,\frac{3}{2}\right\} & \left\{2_2,0,\frac{1}{2},1\right\} & \left\{2_2,0,1,\frac{1}{2}\right\} & \left\{2_2,0,\frac{3}{2},0\right\} \\
\left\{2_2,\frac{1}{2},0,0\right\} & \left\{2_2,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_2,\frac{1}{2},1,1\right\} & \left\{2_2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_2,1,0,\frac{1}{2}\right\} & \left\{2_2,1,\frac{1}{2},0\right\} & \left\{2_2,1,1,\frac{3}{2}\right\} & \left\{2_2,1,\frac{3}{2},1\right\} \\
\left\{2_2,\frac{3}{2},0,1\right\} & \left\{2_2,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_2,\frac{3}{2},1,0\right\} & \left\{2_2,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_3,0,0,\frac{1}{2}\right\} & \left\{2_3,0,\frac{1}{2},0\right\} & \left\{2_3,0,1,\frac{3}{2}\right\} & \left\{2_3,0,\frac{3}{2},1\right\} \\
\left\{2_3,\frac{1}{2},0,1\right\} & \left\{2_3,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_3,\frac{1}{2},1,0\right\} & \left\{2_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_3,1,0,\frac{3}{2}\right\} & \left\{2_3,1,\frac{1}{2},1\right\} & \left\{2_3,1,1,\frac{1}{2}\right\} & \left\{2_3,1,\frac{3}{2},0\right\} \\
\left\{2_3,\frac{3}{2},0,0\right\} & \left\{2_3,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_3,\frac{3}{2},1,1\right\} & \left\{2_3,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_4,0,0,\frac{3}{2}\right\} & \left\{2_4,0,\frac{1}{2},0\right\} & \left\{2_4,0,1,\frac{1}{2}\right\} & \left\{2_4,0,\frac{3}{2},1\right\} \\
\left\{2_4,\frac{1}{2},0,0\right\} & \left\{2_4,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_4,\frac{1}{2},1,1\right\} & \left\{2_4,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_4,1,0,\frac{1}{2}\right\} & \left\{2_4,1,\frac{1}{2},1\right\} & \left\{2_4,1,1,\frac{3}{2}\right\} & \left\{2_4,1,\frac{3}{2},0\right\} \\
\left\{2_4,\frac{3}{2},0,1\right\} & \left\{2_4,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_4,\frac{3}{2},1,0\right\} & \left\{2_4,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_5,0,0,\frac{1}{2}\right\} & \left\{2_5,0,\frac{1}{2},1\right\} & \left\{2_5,0,1,\frac{3}{2}\right\} & \left\{2_5,0,\frac{3}{2},0\right\} \\
\left\{2_5,\frac{1}{2},0,0\right\} & \left\{2_5,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_5,\frac{1}{2},1,1\right\} & \left\{2_5,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_5,1,0,\frac{3}{2}\right\} & \left\{2_5,1,\frac{1}{2},0\right\} & \left\{2_5,1,1,\frac{1}{2}\right\} & \left\{2_5,1,\frac{3}{2},1\right\} \\
\left\{2_5,\frac{3}{2},0,1\right\} & \left\{2_5,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_5,\frac{3}{2},1,0\right\} & \left\{2_5,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_6,0,0,\frac{3}{2}\right\} & \left\{2_6,0,\frac{1}{2},1\right\} & \left\{2_6,0,1,\frac{1}{2}\right\} & \left\{2_6,0,\frac{3}{2},0\right\} \\
\left\{2_6,\frac{1}{2},0,1\right\} & \left\{2_6,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_6,\frac{1}{2},1,0\right\} & \left\{2_6,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_6,1,0,\frac{1}{2}\right\} & \left\{2_6,1,\frac{1}{2},0\right\} & \left\{2_6,1,1,\frac{3}{2}\right\} & \left\{2_6,1,\frac{3}{2},1\right\} \\
\left\{2_6,\frac{3}{2},0,0\right\} & \left\{2_6,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_6,\frac{3}{2},1,1\right\} & \left\{2_6,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_7,0,0,\frac{1}{2}\right\} & \left\{2_7,0,\frac{1}{2},1\right\} & \left\{2_7,0,1,\frac{3}{2}\right\} & \left\{2_7,0,\frac{3}{2},0\right\} \\
\left\{2_7,\frac{1}{2},0,1\right\} & \left\{2_7,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_7,\frac{1}{2},1,0\right\} & \left\{2_7,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_7,1,0,\frac{3}{2}\right\} & \left\{2_7,1,\frac{1}{2},0\right\} & \left\{2_7,1,1,\frac{1}{2}\right\} & \left\{2_7,1,\frac{3}{2},1\right\} \\
\left\{2_7,\frac{3}{2},0,0\right\} & \left\{2_7,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_7,\frac{3}{2},1,1\right\} & \left\{2_7,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_8,0,0,\frac{1}{2}\right\} & \left\{2_8,0,\frac{1}{2},0\right\} & \left\{2_8,0,1,\frac{3}{2}\right\} & \left\{2_8,0,\frac{3}{2},1\right\} \\
\left\{2_8,\frac{1}{2},0,0\right\} & \left\{2_8,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_8,\frac{1}{2},1,1\right\} & \left\{2_8,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_8,1,0,\frac{3}{2}\right\} & \left\{2_8,1,\frac{1}{2},1\right\} & \left\{2_8,1,1,\frac{1}{2}\right\} & \left\{2_8,1,\frac{3}{2},0\right\} \\
\left\{2_8,\frac{3}{2},0,1\right\} & \left\{2_8,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_8,\frac{3}{2},1,0\right\} & \left\{2_8,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}
\end{array}
\tag{A.42}
$$

A.2. The Group G_{768}**.** In this section, we list all the elements of the space group G_{768}, organized into their 32 conjugacy classes.

Conjugacy class $\mathcal{C}_1\,(\mathrm{G}_{768})$: # *of elements* = 1

$$\{1_1, 0, 0, 0\} \tag{A.43}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{G}_{768})$: # *of elements* = 1

$$\{1_1, 1, 1, 1\} \tag{A.44}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{G}_{768})$: # *of elements* = 3

$$\{1_1, 0, 0, 1\} \quad \{1_1, 0, 1, 0\} \quad \{1_1, 1, 0, 0\} \tag{A.45}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{G}_{768})$: # *of elements* = 3

$$\{1_1, 0, 1, 1\} \quad \{1_1, 1, 0, 1\} \quad \{1_1, 1, 1, 0\} \tag{A.46}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{G}_{768})$: # *of elements* = 4

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2}\right\} \tag{A.47}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{G}_{768})$: # *of elements* = 4

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}\right\} \tag{A.48}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{G}_{768})$: # *of elements* = 6

$$\begin{array}{ccc} \left\{1_1, 0, 0, \tfrac{1}{2}\right\} & \left\{1_1, 0, 0, \tfrac{3}{2}\right\} & \left\{1_1, 0, \tfrac{1}{2}, 0\right\} \\ \left\{1_1, 0, \tfrac{3}{2}, 0\right\} & \left\{1_1, \tfrac{1}{2}, 0, 0\right\} & \left\{1_1, \tfrac{3}{2}, 0, 0\right\} \end{array} \tag{A.49}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{G}_{768})$: # *of elements* = 6

$$\begin{array}{ccc} \left\{1_1, 0, \tfrac{1}{2}, 1\right\} & \left\{1_1, 0, \tfrac{3}{2}, 1\right\} & \left\{1_1, \tfrac{1}{2}, 1, 0\right\} \\ \left\{1_1, 1, 0, \tfrac{1}{2}\right\} & \left\{1_1, 1, 0, \tfrac{3}{2}\right\} & \left\{1_1, \tfrac{3}{2}, 1, 0\right\} \end{array} \tag{A.50}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{G}_{768})$: # *of elements* = 6

$$\begin{array}{ccc} \left\{1_1, 0, 1, \tfrac{1}{2}\right\} & \left\{1_1, 0, 1, \tfrac{3}{2}\right\} & \left\{1_1, \tfrac{1}{2}, 0, 1\right\} \\ \left\{1_1, 1, \tfrac{1}{2}, 0\right\} & \left\{1_1, 1, \tfrac{3}{2}, 0\right\} & \left\{1_1, \tfrac{3}{2}, 0, 1\right\} \end{array} \tag{A.51}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{G}_{768})$: # *of elements* = 6

$$\begin{array}{ccc} \left\{1_1, \tfrac{1}{2}, 1, 1\right\} & \left\{1_1, 1, \tfrac{1}{2}, 1\right\} & \left\{1_1, 1, 1, \tfrac{1}{2}\right\} \\ \left\{1_1, 1, 1, \tfrac{3}{2}\right\} & \left\{1_1, 1, \tfrac{3}{2}, 1\right\} & \left\{1_1, \tfrac{3}{2}, 1, 1\right\} \end{array} \tag{A.52}$$

Conjugacy class $\mathcal{C}_{11}\,(\mathrm{G}_{768})$: # *of elements* = 12

$$\left\{1_1,0,\tfrac{1}{2},\tfrac{1}{2}\right\}\ \left\{1_1,0,\tfrac{1}{2},\tfrac{3}{2}\right\}\ \left\{1_1,0,\tfrac{3}{2},\tfrac{1}{2}\right\}\ \left\{1_1,0,\tfrac{3}{2},\tfrac{3}{2}\right\}\ \left\{1_1,\tfrac{1}{2},0,\tfrac{1}{2}\right\}\ \left\{1_1,\tfrac{1}{2},0,\tfrac{3}{2}\right\}$$
$$\left\{1_1,\tfrac{1}{2},\tfrac{1}{2},0\right\}\ \left\{1_1,\tfrac{1}{2},\tfrac{3}{2},0\right\}\ \left\{1_1,\tfrac{3}{2},0,\tfrac{1}{2}\right\}\ \left\{1_1,\tfrac{3}{2},0,\tfrac{3}{2}\right\}\ \left\{1_1,\tfrac{3}{2},\tfrac{1}{2},0\right\}\ \left\{1_1,\tfrac{3}{2},\tfrac{3}{2},0\right\} \tag{A.53}$$

Conjugacy class $\mathcal{C}_{12}\,(\mathrm{G}_{768})$: # *of elements* = 12

$$\left\{1_1,\tfrac{1}{2},\tfrac{1}{2},1\right\}\ \left\{1_1,\tfrac{1}{2},1,\tfrac{1}{2}\right\}\ \left\{1_1,\tfrac{1}{2},1,\tfrac{3}{2}\right\}\ \left\{1_1,\tfrac{1}{2},\tfrac{3}{2},1\right\}\ \left\{1_1,1,\tfrac{1}{2},\tfrac{1}{2}\right\}\ \left\{1_1,1,\tfrac{1}{2},\tfrac{3}{2}\right\}$$
$$\left\{1_1,1,\tfrac{3}{2},\tfrac{1}{2}\right\}\ \left\{1_1,1,\tfrac{3}{2},\tfrac{3}{2}\right\}\ \left\{1_1,\tfrac{3}{2},\tfrac{1}{2},1\right\}\ \left\{1_1,\tfrac{3}{2},1,\tfrac{1}{2}\right\}\ \left\{1_1,\tfrac{3}{2},1,\tfrac{3}{2}\right\}\ \left\{1_1,\tfrac{3}{2},\tfrac{3}{2},1\right\} \tag{A.54}$$

Conjugacy class $\mathcal{C}_{13}\,(\mathrm{G}_{768})$: # *of elements* = 12

$$\{3_1,0,0,0\}\ \{3_1,0,1,0\}\ \{3_1,1,0,0\}\ \{3_1,1,1,0\}\ \{3_2,0,0,0\}\ \{3_2,0,0,1\}$$
$$\{3_2,1,0,0\}\ \{3_2,1,0,1\}\ \{3_3,0,0,0\}\ \{3_3,0,0,1\}\ \{3_3,0,1,0\}\ \{3_3,0,1,1\} \tag{A.55}$$

Conjugacy class $\mathcal{C}_{14}\,(\mathrm{G}_{768})$: # *of elements* = 12

$$\{3_1,0,0,1\}\ \{3_1,0,1,1\}\ \{3_1,1,0,1\}\ \{3_1,1,1,1\}\ \{3_2,0,1,0\}\ \{3_2,0,1,1\}$$
$$\{3_2,1,1,0\}\ \{3_2,1,1,1\}\ \{3_3,1,0,0\}\ \{3_3,1,0,1\}\ \{3_3,1,1,0\}\ \{3_3,1,1,1\} \tag{A.56}$$

Conjugacy class $\mathcal{C}_{15}\,(\mathrm{G}_{768})$: # *of elements* = 12

$$\left\{3_1,0,\tfrac{1}{2},0\right\}\ \left\{3_1,0,\tfrac{3}{2},0\right\}\ \left\{3_1,1,\tfrac{1}{2},0\right\}\ \left\{3_1,1,\tfrac{3}{2},0\right\}\ \left\{3_2,\tfrac{1}{2},0,0\right\}\ \left\{3_2,\tfrac{1}{2},0,1\right\}$$
$$\left\{3_2,\tfrac{3}{2},0,0\right\}\ \left\{3_2,\tfrac{3}{2},0,1\right\}\ \left\{3_3,0,0,\tfrac{1}{2}\right\}\ \left\{3_3,0,0,\tfrac{3}{2}\right\}\ \left\{3_3,0,1,\tfrac{1}{2}\right\}\ \left\{3_3,0,1,\tfrac{3}{2}\right\} \tag{A.57}$$

Conjugacy class $\mathcal{C}_{16}\,(\mathrm{G}_{768})$: # *of elements* = 12

$$\left\{3_1,0,\tfrac{1}{2},1\right\}\ \left\{3_1,0,\tfrac{3}{2},1\right\}\ \left\{3_1,1,\tfrac{1}{2},1\right\}\ \left\{3_1,1,\tfrac{3}{2},1\right\}\ \left\{3_2,\tfrac{1}{2},1,0\right\}\ \left\{3_2,\tfrac{1}{2},1,1\right\}$$
$$\left\{3_2,\tfrac{3}{2},1,0\right\}\ \left\{3_2,\tfrac{3}{2},1,1\right\}\ \left\{3_3,1,0,\tfrac{1}{2}\right\}\ \left\{3_3,1,0,\tfrac{3}{2}\right\}\ \left\{3_3,1,1,\tfrac{1}{2}\right\}\ \left\{3_3,1,1,\tfrac{3}{2}\right\} \tag{A.58}$$

Conjugacy class $\mathcal{C}_{17}\,(\mathrm{G}_{768})$: # *of elements* = 12

$$\left\{3_1,\tfrac{1}{2},0,0\right\}\ \left\{3_1,\tfrac{1}{2},1,0\right\}\ \left\{3_1,\tfrac{3}{2},0,0\right\}\ \left\{3_1,\tfrac{3}{2},1,0\right\}\ \left\{3_2,0,0,\tfrac{1}{2}\right\}\ \left\{3_2,0,0,\tfrac{3}{2}\right\}$$
$$\left\{3_2,1,0,\tfrac{1}{2}\right\}\ \left\{3_2,1,0,\tfrac{3}{2}\right\}\ \left\{3_3,0,\tfrac{1}{2},0\right\}\ \left\{3_3,0,\tfrac{1}{2},1\right\}\ \left\{3_3,0,\tfrac{3}{2},0\right\}\ \left\{3_3,0,\tfrac{3}{2},1\right\} \tag{A.59}$$

Conjugacy class $\mathcal{C}_{18}\,(\mathrm{G}_{768})$: # *of elements* = 12

$$\left\{3_1,\tfrac{1}{2},0,1\right\}\ \left\{3_1,\tfrac{1}{2},1,1\right\}\ \left\{3_1,\tfrac{3}{2},0,1\right\}\ \left\{3_1,\tfrac{3}{2},1,1\right\}\ \left\{3_2,0,1,\tfrac{1}{2}\right\}\ \left\{3_2,0,1,\tfrac{3}{2}\right\}$$
$$\left\{3_2,1,1,\tfrac{1}{2}\right\}\ \left\{3_2,1,1,\tfrac{3}{2}\right\}\ \left\{3_3,1,\tfrac{1}{2},0\right\}\ \left\{3_3,1,\tfrac{1}{2},1\right\}\ \left\{3_3,1,\tfrac{3}{2},0\right\}\ \left\{3_3,1,\tfrac{3}{2},1\right\} \tag{A.60}$$

Conjugacy class $\mathcal{C}_{19}\left(\mathrm{G}_{768}\right)$: # *of elements* = 12

$$\left\{3_1,\tfrac{1}{2},\tfrac{1}{2},0\right\}\ \left\{3_1,\tfrac{1}{2},\tfrac{3}{2},0\right\}\ \left\{3_1,\tfrac{3}{2},\tfrac{1}{2},0\right\}\ \left\{3_1,\tfrac{3}{2},\tfrac{3}{2},0\right\}\ \left\{3_2,\tfrac{1}{2},0,\tfrac{1}{2}\right\}\ \left\{3_2,\tfrac{1}{2},0,\tfrac{3}{2}\right\}$$
$$\left\{3_2,\tfrac{3}{2},0,\tfrac{1}{2}\right\}\ \left\{3_2,\tfrac{3}{2},0,\tfrac{3}{2}\right\}\ \left\{3_3,0,\tfrac{1}{2},\tfrac{1}{2}\right\}\ \left\{3_3,0,\tfrac{1}{2},\tfrac{3}{2}\right\}\ \left\{3_3,0,\tfrac{3}{2},\tfrac{1}{2}\right\}\ \left\{3_3,0,\tfrac{3}{2},\tfrac{3}{2}\right\} \tag{A.61}$$

Conjugacy class $\mathcal{C}_{20}\left(\mathrm{G}_{768}\right)$: # *of elements* = 12

$$\left\{3_1,\tfrac{1}{2},\tfrac{1}{2},1\right\}\ \left\{3_1,\tfrac{1}{2},\tfrac{3}{2},1\right\}\ \left\{3_1,\tfrac{3}{2},\tfrac{1}{2},1\right\}\ \left\{3_1,\tfrac{3}{2},\tfrac{3}{2},1\right\}\ \left\{3_2,\tfrac{1}{2},1,\tfrac{1}{2}\right\}\ \left\{3_2,\tfrac{1}{2},1,\tfrac{3}{2}\right\}$$
$$\left\{3_2,\tfrac{3}{2},1,\tfrac{1}{2}\right\}\ \left\{3_2,\tfrac{3}{2},1,\tfrac{3}{2}\right\}\ \left\{3_3,1,\tfrac{1}{2},\tfrac{1}{2}\right\}\ \left\{3_3,1,\tfrac{1}{2},\tfrac{3}{2}\right\}\ \left\{3_3,1,\tfrac{3}{2},\tfrac{1}{2}\right\}\ \left\{3_3,1,\tfrac{3}{2},\tfrac{3}{2}\right\} \tag{A.62}$$

Conjugacy class $\mathcal{C}_{21}\left(\mathrm{G}_{768}\right)$: # *of elements* = 24

$$\begin{array}{llll}
\left\{3_1,0,0,\frac{1}{2}\right\} & \left\{3_1,0,0,\frac{3}{2}\right\} & \left\{3_1,0,1,\frac{1}{2}\right\} & \left\{3_1,0,1,\frac{3}{2}\right\} \\
\left\{3_1,1,0,\frac{1}{2}\right\} & \left\{3_1,1,0,\frac{3}{2}\right\} & \left\{3_1,1,1,\frac{1}{2}\right\} & \left\{3_1,1,1,\frac{3}{2}\right\} \\
\left\{3_2,0,\frac{1}{2},0\right\} & \left\{3_2,0,\frac{1}{2},1\right\} & \left\{3_2,0,\frac{3}{2},0\right\} & \left\{3_2,0,\frac{3}{2},1\right\} \\
\left\{3_2,1,\frac{1}{2},0\right\} & \left\{3_2,1,\frac{1}{2},1\right\} & \left\{3_2,1,\frac{3}{2},0\right\} & \left\{3_2,1,\frac{3}{2},1\right\} \\
\left\{3_3,\frac{1}{2},0,0\right\} & \left\{3_3,\frac{1}{2},0,1\right\} & \left\{3_3,\frac{1}{2},1,0\right\} & \left\{3_3,\frac{1}{2},1,1\right\} \\
\left\{3_3,\frac{3}{2},0,0\right\} & \left\{3_3,\frac{3}{2},0,1\right\} & \left\{3_3,\frac{3}{2},1,0\right\} & \left\{3_3,\frac{3}{2},1,1\right\}
\end{array} \tag{A.63}$$

Conjugacy class $\mathcal{C}_{22}\left(\mathrm{G}_{768}\right)$: # *of elements* = 24

$$\begin{array}{llll}
\left\{3_1,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_1,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_2,\frac{1}{2},\frac{1}{2},0\right\} & \left\{3_2,\frac{1}{2},\frac{1}{2},1\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},0\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{3_2,\frac{3}{2},\frac{1}{2},0\right\} & \left\{3_2,\frac{3}{2},\frac{1}{2},1\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},0\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{3_3,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{3_3,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},1,\frac{3}{2}\right\} \\
\left\{3_3,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{3_3,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},1,\frac{3}{2}\right\}
\end{array} \tag{A.64}$$

Conjugacy class $\mathcal{C}_{23}\left(\mathrm{G}_{768}\right)$: # *of elements* = 24

$$\begin{array}{llll}
\left\{3_1,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{3_1,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},1,\frac{3}{2}\right\} \\
\left\{3_1,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{3_1,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},1,\frac{3}{2}\right\} \\
\left\{3_2,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_2,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{3_3,\frac{1}{2},\frac{1}{2},0\right\} & \left\{3_3,\frac{1}{2},\frac{1}{2},1\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},0\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},1\right\} \\
\left\{3_3,\frac{3}{2},\frac{1}{2},0\right\} & \left\{3_3,\frac{3}{2},\frac{1}{2},1\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},0\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},1\right\}
\end{array} \tag{A.65}$$

Conjugacy class $\mathcal{C}_{24}\,(\mathrm{G}_{768})$: # *of elements* = 24

$$\begin{array}{cccc}
\left\{3_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\}\\
\left\{3_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}\\
\left\{3_2,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\}\\
\left\{3_2,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}\\
\left\{3_3,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\}\\
\left\{3_3,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}
\end{array} \tag{A.66}$$

Conjugacy class $\mathcal{C}_{25}\,(\mathrm{G}_{768})$: # *of elements* = 64

$$\begin{array}{cccc}
\{2_1,0,0,0\} & \left\{2_1,0,\frac{1}{2},\frac{1}{2}\right\} & \{2_1,0,1,1\} & \left\{2_1,0,\frac{3}{2},\frac{3}{2}\right\}\\
\left\{2_1,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{2_1,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_1,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{2_1,\frac{1}{2},\frac{3}{2},1\right\}\\
\{2_1,1,0,1\} & \left\{2_1,1,\frac{1}{2},\frac{3}{2}\right\} & \{2_1,1,1,0\} & \left\{2_1,1,\frac{3}{2},\frac{1}{2}\right\}\\
\left\{2_1,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{2_1,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_1,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{2_1,\frac{3}{2},\frac{3}{2},0\right\}\\
\{2_2,0,0,0\} & \left\{2_2,0,\frac{1}{2},\frac{3}{2}\right\} & \{2_2,0,1,1\} & \left\{2_2,0,\frac{3}{2},\frac{1}{2}\right\}\\
\left\{2_2,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{2_2,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_2,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{2_2,\frac{1}{2},\frac{3}{2},1\right\}\\
\{2_2,1,0,1\} & \left\{2_2,1,\frac{1}{2},\frac{1}{2}\right\} & \{2_2,1,1,0\} & \left\{2_2,1,\frac{3}{2},\frac{3}{2}\right\}\\
\left\{2_2,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{2_2,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_2,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{2_2,\frac{3}{2},\frac{3}{2},0\right\}\\
\{2_7,0,0,0\} & \left\{2_7,0,\frac{1}{2},\frac{1}{2}\right\} & \{2_7,0,1,1\} & \left\{2_7,0,\frac{3}{2},\frac{3}{2}\right\}\\
\left\{2_7,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{2_7,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_7,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{2_7,\frac{1}{2},\frac{3}{2},0\right\}\\
\{2_7,1,0,1\} & \left\{2_7,1,\frac{1}{2},\frac{3}{2}\right\} & \{2_7,1,1,0\} & \left\{2_7,1,\frac{3}{2},\frac{1}{2}\right\}\\
\left\{2_7,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{2_7,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_7,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{2_7,\frac{3}{2},\frac{3}{2},1\right\}\\
\{2_8,0,0,0\} & \left\{2_8,0,\frac{1}{2},\frac{3}{2}\right\} & \{2_8,0,1,1\} & \left\{2_8,0,\frac{3}{2},\frac{1}{2}\right\}\\
\left\{2_8,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{2_8,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_8,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{2_8,\frac{1}{2},\frac{3}{2},0\right\}\\
\{2_8,1,0,1\} & \left\{2_8,1,\frac{1}{2},\frac{1}{2}\right\} & \{2_8,1,1,0\} & \left\{2_8,1,\frac{3}{2},\frac{3}{2}\right\}\\
\left\{2_8,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{2_8,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_8,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{2_8,\frac{3}{2},\frac{3}{2},1\right\}
\end{array} \tag{A.67}$$

Conjugacy class $\mathcal{C}_{26}\,(\mathrm{G}_{768})$: # *of elements* = 64

$$\begin{array}{cccc}
\left\{2_1,0,0,\frac{1}{2}\right\} & \left\{2_1,0,\frac{1}{2},1\right\} & \left\{2_1,0,1,\frac{3}{2}\right\} & \left\{2_1,0,\frac{3}{2},0\right\} \\
\left\{2_1,\frac{1}{2},0,0\right\} & \left\{2_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_1,\frac{1}{2},1,1\right\} & \left\{2_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_1,1,0,\frac{3}{2}\right\} & \left\{2_1,1,\frac{1}{2},0\right\} & \left\{2_1,1,1,\frac{1}{2}\right\} & \left\{2_1,1,\frac{3}{2},1\right\} \\
\left\{2_1,\frac{3}{2},0,1\right\} & \left\{2_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_1,\frac{3}{2},1,0\right\} & \left\{2_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_2,0,0,\frac{1}{2}\right\} & \left\{2_2,0,\frac{1}{2},0\right\} & \left\{2_2,0,1,\frac{3}{2}\right\} & \left\{2_2,0,\frac{3}{2},1\right\} \\
\left\{2_2,\frac{1}{2},0,1\right\} & \left\{2_2,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_2,\frac{1}{2},1,0\right\} & \left\{2_2,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_2,1,0,\frac{3}{2}\right\} & \left\{2_2,1,\frac{1}{2},1\right\} & \left\{2_2,1,1,\frac{1}{2}\right\} & \left\{2_2,1,\frac{3}{2},0\right\} \\
\left\{2_2,\frac{3}{2},0,0\right\} & \left\{2_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_2,\frac{3}{2},1,1\right\} & \left\{2_2,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_7,0,0,\frac{3}{2}\right\} & \left\{2_7,0,\frac{1}{2},0\right\} & \left\{2_7,0,1,\frac{1}{2}\right\} & \left\{2_7,0,\frac{3}{2},1\right\} \\
\left\{2_7,\frac{1}{2},0,0\right\} & \left\{2_7,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_7,\frac{1}{2},1,1\right\} & \left\{2_7,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_7,1,0,\frac{1}{2}\right\} & \left\{2_7,1,\frac{1}{2},1\right\} & \left\{2_7,1,1,\frac{3}{2}\right\} & \left\{2_7,1,\frac{3}{2},0\right\} \\
\left\{2_7,\frac{3}{2},0,1\right\} & \left\{2_7,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_7,\frac{3}{2},1,0\right\} & \left\{2_7,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_8,0,0,\frac{3}{2}\right\} & \left\{2_8,0,\frac{1}{2},1\right\} & \left\{2_8,0,1,\frac{1}{2}\right\} & \left\{2_8,0,\frac{3}{2},0\right\} \\
\left\{2_8,\frac{1}{2},0,1\right\} & \left\{2_8,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_8,\frac{1}{2},1,0\right\} & \left\{2_8,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_8,1,0,\frac{1}{2}\right\} & \left\{2_8,1,\frac{1}{2},0\right\} & \left\{2_8,1,1,\frac{3}{2}\right\} & \left\{2_8,1,\frac{3}{2},1\right\} \\
\left\{2_8,\frac{3}{2},0,0\right\} & \left\{2_8,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_8,\frac{3}{2},1,1\right\} & \left\{2_8,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\}
\end{array} \tag{A.68}$$

Conjugacy class $\mathcal{C}_{27}\,(\mathrm{G}_{768})$: # *of elements* = 64

$$\begin{array}{cccc}
\{2_1,0,0,1\} & \left\{2_1,0,\frac{1}{2},\frac{3}{2}\right\} & \{2_1,0,1,0\} & \left\{2_1,0,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_1,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{2_1,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_1,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{2_1,\frac{1}{2},\frac{3}{2},0\right\} \\
\{2_1,1,0,0\} & \left\{2_1,1,\frac{1}{2},\frac{1}{2}\right\} & \{2_1,1,1,1\} & \left\{2_1,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_1,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{2_1,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_1,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{2_1,\frac{3}{2},\frac{3}{2},1\right\} \\
\{2_2,0,0,1\} & \left\{2_2,0,\frac{1}{2},\frac{1}{2}\right\} & \{2_2,0,1,0\} & \left\{2_2,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_2,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{2_2,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_2,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{2_2,\frac{1}{2},\frac{3}{2},0\right\} \\
\{2_2,1,0,0\} & \left\{2_2,1,\frac{1}{2},\frac{3}{2}\right\} & \{2_2,1,1,1\} & \left\{2_2,1,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_2,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{2_2,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_2,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{2_2,\frac{3}{2},\frac{3}{2},1\right\} \\
\{2_7,0,0,1\} & \left\{2_7,0,\frac{1}{2},\frac{3}{2}\right\} & \{2_7,0,1,0\} & \left\{2_7,0,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_7,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{2_7,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_7,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{2_7,\frac{1}{2},\frac{3}{2},1\right\} \\
\{2_7,1,0,0\} & \left\{2_7,1,\frac{1}{2},\frac{1}{2}\right\} & \{2_7,1,1,1\} & \left\{2_7,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_7,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{2_7,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_7,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{2_7,\frac{3}{2},\frac{3}{2},0\right\} \\
\{2_8,0,0,1\} & \left\{2_8,0,\frac{1}{2},\frac{1}{2}\right\} & \{2_8,0,1,0\} & \left\{2_8,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_8,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{2_8,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_8,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{2_8,\frac{1}{2},\frac{3}{2},1\right\} \\
\{2_8,1,0,0\} & \left\{2_8,1,\frac{1}{2},\frac{3}{2}\right\} & \{2_8,1,1,1\} & \left\{2_8,1,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_8,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{2_8,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_8,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{2_8,\frac{3}{2},\frac{3}{2},0\right\}
\end{array} \tag{A.69}$$

Conjugacy class $\mathcal{C}_{28}\,(\mathrm{G}_{768})$: # *of elements* = 64

$$
\begin{array}{llll}
\{2_1,0,0,\frac{3}{2}\} & \{2_1,0,\frac{1}{2},0\} & \{2_1,0,1,\frac{1}{2}\} & \{2_1,0,\frac{3}{2},1\} \\
\{2_1,\frac{1}{2},0,1\} & \{2_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{2_1,\frac{1}{2},1,0\} & \{2_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{2_1,1,0,\frac{1}{2}\} & \{2_1,1,\frac{1}{2},1\} & \{2_1,1,1,\frac{3}{2}\} & \{2_1,1,\frac{3}{2},0\} \\
\{2_1,\frac{3}{2},0,0\} & \{2_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{2_1,\frac{3}{2},1,1\} & \{2_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{2_2,0,0,\frac{3}{2}\} & \{2_2,0,\frac{1}{2},1\} & \{2_2,0,1,\frac{1}{2}\} & \{2_2,0,\frac{3}{2},0\} \\
\{2_2,\frac{1}{2},0,0\} & \{2_2,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{2_2,\frac{1}{2},1,1\} & \{2_2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{2_2,1,0,\frac{1}{2}\} & \{2_2,1,\frac{1}{2},0\} & \{2_2,1,1,\frac{3}{2}\} & \{2_2,1,\frac{3}{2},1\} \\
\{2_2,\frac{3}{2},0,1\} & \{2_2,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{2_2,\frac{3}{2},1,0\} & \{2_2,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{2_7,0,0,\frac{1}{2}\} & \{2_7,0,\frac{1}{2},1\} & \{2_7,0,1,\frac{3}{2}\} & \{2_7,0,\frac{3}{2},0\} \\
\{2_7,\frac{1}{2},0,1\} & \{2_7,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{2_7,\frac{1}{2},1,0\} & \{2_7,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{2_7,1,0,\frac{3}{2}\} & \{2_7,1,\frac{1}{2},0\} & \{2_7,1,1,\frac{1}{2}\} & \{2_7,1,\frac{3}{2},1\} \\
\{2_7,\frac{3}{2},0,0\} & \{2_7,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{2_7,\frac{3}{2},1,1\} & \{2_7,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{2_8,0,0,\frac{1}{2}\} & \{2_8,0,\frac{1}{2},0\} & \{2_8,0,1,\frac{3}{2}\} & \{2_8,0,\frac{3}{2},1\} \\
\{2_8,\frac{1}{2},0,0\} & \{2_8,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{2_8,\frac{1}{2},1,1\} & \{2_8,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{2_8,1,0,\frac{3}{2}\} & \{2_8,1,\frac{1}{2},1\} & \{2_8,1,1,\frac{1}{2}\} & \{2_8,1,\frac{3}{2},0\} \\
\{2_8,\frac{3}{2},0,1\} & \{2_8,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{2_8,\frac{3}{2},1,0\} & \{2_8,\frac{3}{2},\frac{3}{2},\frac{3}{2}\}
\end{array}
\tag{A.70}
$$

Conjugacy class $\mathcal{C}_{29}\,(\mathrm{G}_{768})$: # *of elements* = 64

$$
\begin{array}{llll}
\{2_3,0,0,0\} & \{2_3,0,\frac{1}{2},\frac{3}{2}\} & \{2_3,0,1,1\} & \{2_3,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_3,\frac{1}{2},0,\frac{1}{2}\} & \{2_3,\frac{1}{2},\frac{1}{2},0\} & \{2_3,\frac{1}{2},1,\frac{3}{2}\} & \{2_3,\frac{1}{2},\frac{3}{2},1\} \\
\{2_3,1,0,1\} & \{2_3,1,\frac{1}{2},\frac{1}{2}\} & \{2_3,1,1,0\} & \{2_3,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_3,\frac{3}{2},0,\frac{3}{2}\} & \{2_3,\frac{3}{2},\frac{1}{2},1\} & \{2_3,\frac{3}{2},1,\frac{1}{2}\} & \{2_3,\frac{3}{2},\frac{3}{2},0\} \\
\{2_4,0,0,0\} & \{2_4,0,\frac{1}{2},\frac{1}{2}\} & \{2_4,0,1,1\} & \{2_4,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_4,\frac{1}{2},0,\frac{1}{2}\} & \{2_4,\frac{1}{2},\frac{1}{2},1\} & \{2_4,\frac{1}{2},1,\frac{3}{2}\} & \{2_4,\frac{1}{2},\frac{3}{2},0\} \\
\{2_4,1,0,1\} & \{2_4,1,\frac{1}{2},\frac{3}{2}\} & \{2_4,1,1,0\} & \{2_4,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_4,\frac{3}{2},0,\frac{3}{2}\} & \{2_4,\frac{3}{2},\frac{1}{2},0\} & \{2_4,\frac{3}{2},1,\frac{1}{2}\} & \{2_4,\frac{3}{2},\frac{3}{2},1\} \\
\{2_5,0,0,0\} & \{2_5,0,\frac{1}{2},\frac{1}{2}\} & \{2_5,0,1,1\} & \{2_5,0,\frac{3}{2},\frac{3}{2}\} \\
\{2_5,\frac{1}{2},0,\frac{3}{2}\} & \{2_5,\frac{1}{2},\frac{1}{2},0\} & \{2_5,\frac{1}{2},1,\frac{1}{2}\} & \{2_5,\frac{1}{2},\frac{3}{2},1\} \\
\{2_5,1,0,1\} & \{2_5,1,\frac{1}{2},\frac{3}{2}\} & \{2_5,1,1,0\} & \{2_5,1,\frac{3}{2},\frac{1}{2}\} \\
\{2_5,\frac{3}{2},0,\frac{1}{2}\} & \{2_5,\frac{3}{2},\frac{1}{2},1\} & \{2_5,\frac{3}{2},1,\frac{3}{2}\} & \{2_5,\frac{3}{2},\frac{3}{2},0\} \\
\{2_6,0,0,0\} & \{2_6,0,\frac{1}{2},\frac{3}{2}\} & \{2_6,0,1,1\} & \{2_6,0,\frac{3}{2},\frac{1}{2}\} \\
\{2_6,\frac{1}{2},0,\frac{3}{2}\} & \{2_6,\frac{1}{2},\frac{1}{2},1\} & \{2_6,\frac{1}{2},1,\frac{1}{2}\} & \{2_6,\frac{1}{2},\frac{3}{2},0\} \\
\{2_6,1,0,1\} & \{2_6,1,\frac{1}{2},\frac{1}{2}\} & \{2_6,1,1,0\} & \{2_6,1,\frac{3}{2},\frac{3}{2}\} \\
\{2_6,\frac{3}{2},0,\frac{1}{2}\} & \{2_6,\frac{3}{2},\frac{1}{2},0\} & \{2_6,\frac{3}{2},1,\frac{3}{2}\} & \{2_6,\frac{3}{2},\frac{3}{2},1\}
\end{array}
\tag{A.71}
$$

Conjugacy class $\mathcal{C}_{30}\,(\mathrm{G}_{768})$: # *of elements* = 64

$$\begin{array}{llll}
\left\{2_3,0,0,\frac{1}{2}\right\} & \left\{2_3,0,\frac{1}{2},0\right\} & \left\{2_3,0,1,\frac{3}{2}\right\} & \left\{2_3,0,\frac{3}{2},1\right\} \\
\left\{2_3,\frac{1}{2},0,1\right\} & \left\{2_3,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_3,\frac{1}{2},1,0\right\} & \left\{2_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_3,1,0,\frac{3}{2}\right\} & \left\{2_3,1,\frac{1}{2},1\right\} & \left\{2_3,1,1,\frac{1}{2}\right\} & \left\{2_3,1,\frac{3}{2},0\right\} \\
\left\{2_3,\frac{3}{2},0,0\right\} & \left\{2_3,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_3,\frac{3}{2},1,1\right\} & \left\{2_3,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_4,0,0,\frac{3}{2}\right\} & \left\{2_4,0,\frac{1}{2},0\right\} & \left\{2_4,0,1,\frac{1}{2}\right\} & \left\{2_4,0,\frac{3}{2},1\right\} \\
\left\{2_4,\frac{1}{2},0,0\right\} & \left\{2_4,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_4,\frac{1}{2},1,1\right\} & \left\{2_4,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_4,1,0,\frac{1}{2}\right\} & \left\{2_4,1,\frac{1}{2},1\right\} & \left\{2_4,1,1,\frac{3}{2}\right\} & \left\{2_4,1,\frac{3}{2},0\right\} \\
\left\{2_4,\frac{3}{2},0,1\right\} & \left\{2_4,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_4,\frac{3}{2},1,0\right\} & \left\{2_4,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_5,0,0,\frac{1}{2}\right\} & \left\{2_5,0,\frac{1}{2},1\right\} & \left\{2_5,0,1,\frac{3}{2}\right\} & \left\{2_5,0,\frac{3}{2},0\right\} \\
\left\{2_5,\frac{1}{2},0,0\right\} & \left\{2_5,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_5,\frac{1}{2},1,1\right\} & \left\{2_5,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_5,1,0,\frac{3}{2}\right\} & \left\{2_5,1,\frac{1}{2},0\right\} & \left\{2_5,1,1,\frac{1}{2}\right\} & \left\{2_5,1,\frac{3}{2},1\right\} \\
\left\{2_5,\frac{3}{2},0,1\right\} & \left\{2_5,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_5,\frac{3}{2},1,0\right\} & \left\{2_5,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_6,0,0,\frac{3}{2}\right\} & \left\{2_6,0,\frac{1}{2},1\right\} & \left\{2_6,0,1,\frac{1}{2}\right\} & \left\{2_6,0,\frac{3}{2},0\right\} \\
\left\{2_6,\frac{1}{2},0,1\right\} & \left\{2_6,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{2_6,\frac{1}{2},1,0\right\} & \left\{2_6,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_6,1,0,\frac{1}{2}\right\} & \left\{2_6,1,\frac{1}{2},0\right\} & \left\{2_6,1,1,\frac{3}{2}\right\} & \left\{2_6,1,\frac{3}{2},1\right\} \\
\left\{2_6,\frac{3}{2},0,0\right\} & \left\{2_6,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{2_6,\frac{3}{2},1,1\right\} & \left\{2_6,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\}
\end{array} \tag{A.72}$$

Conjugacy class $\mathcal{C}_{31}\,(\mathrm{G}_{768})$: # *of elements* = 64

$$\begin{array}{llll}
\{2_3,0,0,1\} & \left\{2_3,0,\frac{1}{2},\frac{1}{2}\right\} & \{2_3,0,1,0\} & \left\{2_3,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_3,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{2_3,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_3,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{2_3,\frac{1}{2},\frac{3}{2},0\right\} \\
\{2_3,1,0,0\} & \left\{2_3,1,\frac{1}{2},\frac{3}{2}\right\} & \{2_3,1,1,1\} & \left\{2_3,1,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_3,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{2_3,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_3,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{2_3,\frac{3}{2},\frac{3}{2},1\right\} \\
\{2_4,0,0,1\} & \left\{2_4,0,\frac{1}{2},\frac{3}{2}\right\} & \{2_4,0,1,0\} & \left\{2_4,0,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_4,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{2_4,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_4,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{2_4,\frac{1}{2},\frac{3}{2},1\right\} \\
\{2_4,1,0,0\} & \left\{2_4,1,\frac{1}{2},\frac{1}{2}\right\} & \{2_4,1,1,1\} & \left\{2_4,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_4,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{2_4,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_4,\frac{3}{2},1,\frac{3}{2}\right\} & \left\{2_4,\frac{3}{2},\frac{3}{2},0\right\} \\
\{2_5,0,0,1\} & \left\{2_5,0,\frac{1}{2},\frac{3}{2}\right\} & \{2_5,0,1,0\} & \left\{2_5,0,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_5,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{2_5,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_5,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{2_5,\frac{1}{2},\frac{3}{2},0\right\} \\
\{2_5,1,0,0\} & \left\{2_5,1,\frac{1}{2},\frac{1}{2}\right\} & \{2_5,1,1,1\} & \left\{2_5,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_5,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{2_5,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_5,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{2_5,\frac{3}{2},\frac{3}{2},1\right\} \\
\{2_6,0,0,1\} & \left\{2_6,0,\frac{1}{2},\frac{1}{2}\right\} & \{2_6,0,1,0\} & \left\{2_6,0,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_6,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{2_6,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_6,\frac{1}{2},1,\frac{3}{2}\right\} & \left\{2_6,\frac{1}{2},\frac{3}{2},1\right\} \\
\{2_6,1,0,0\} & \left\{2_6,1,\frac{1}{2},\frac{3}{2}\right\} & \{2_6,1,1,1\} & \left\{2_6,1,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_6,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{2_6,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_6,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{2_6,\frac{3}{2},\frac{3}{2},0\right\}
\end{array} \tag{A.73}$$

Conjugacy class $\mathcal{C}_{32}\,(\mathrm{G}_{768})$: # *of elements* = 64

$$\begin{array}{cccc}
\{2_3,0,0,\frac{3}{2}\} & \{2_3,0,\frac{1}{2},1\} & \{2_3,0,1,\frac{1}{2}\} & \{2_3,0,\frac{3}{2},0\} \\
\{2_3,\frac{1}{2},0,0\} & \{2_3,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{2_3,\frac{1}{2},1,1\} & \{2_3,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{2_3,1,0,\frac{1}{2}\} & \{2_3,1,\frac{1}{2},0\} & \{2_3,1,1,\frac{3}{2}\} & \{2_3,1,\frac{3}{2},1\} \\
\{2_3,\frac{3}{2},0,1\} & \{2_3,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{2_3,\frac{3}{2},1,0\} & \{2_3,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{2_4,0,0,\frac{1}{2}\} & \{2_4,0,\frac{1}{2},1\} & \{2_4,0,1,\frac{3}{2}\} & \{2_4,0,\frac{3}{2},0\} \\
\{2_4,\frac{1}{2},0,1\} & \{2_4,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{2_4,\frac{1}{2},1,0\} & \{2_4,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{2_4,1,0,\frac{3}{2}\} & \{2_4,1,\frac{1}{2},0\} & \{2_4,1,1,\frac{1}{2}\} & \{2_4,1,\frac{3}{2},1\} \\
\{2_4,\frac{3}{2},0,0\} & \{2_4,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{2_4,\frac{3}{2},1,1\} & \{2_4,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{2_5,0,0,\frac{3}{2}\} & \{2_5,0,\frac{1}{2},0\} & \{2_5,0,1,\frac{1}{2}\} & \{2_5,0,\frac{3}{2},1\} \\
\{2_5,\frac{1}{2},0,1\} & \{2_5,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{2_5,\frac{1}{2},1,0\} & \{2_5,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{2_5,1,0,\frac{1}{2}\} & \{2_5,1,\frac{1}{2},1\} & \{2_5,1,1,\frac{3}{2}\} & \{2_5,1,\frac{3}{2},0\} \\
\{2_5,\frac{3}{2},0,0\} & \{2_5,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{2_5,\frac{3}{2},1,1\} & \{2_5,\frac{3}{2},\frac{3}{2},\frac{3}{2}\} \\
\{2_6,0,0,\frac{1}{2}\} & \{2_6,0,\frac{1}{2},0\} & \{2_6,0,1,\frac{3}{2}\} & \{2_6,0,\frac{3}{2},1\} \\
\{2_6,\frac{1}{2},0,0\} & \{2_6,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} & \{2_6,\frac{1}{2},1,1\} & \{2_6,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} \\
\{2_6,1,0,\frac{3}{2}\} & \{2_6,1,\frac{1}{2},1\} & \{2_6,1,1,\frac{1}{2}\} & \{2_6,1,\frac{3}{2},0\} \\
\{2_6,\frac{3}{2},0,1\} & \{2_6,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{2_6,\frac{3}{2},1,0\} & \{2_6,\frac{3}{2},\frac{3}{2},\frac{3}{2}\}
\end{array} \tag{A.74}$$

A.3. The Group G_{256}**.** In this section, we list all the elements of the space group G_{256}, organized into their 64 conjugacy classes.

Conjugacy class $\mathcal{C}_{1}\,(\mathrm{G}_{256})$: # *of elements* = 1

$$\{1_1,0,0,0\} \tag{A.75}$$

Conjugacy class $\mathcal{C}_{2}\,(\mathrm{G}_{256})$: # *of elements* = 1

$$\{1_1,0,0,1\} \tag{A.76}$$

Conjugacy class $\mathcal{C}_{3}\,(\mathrm{G}_{256})$: # *of elements* = 1

$$\{1_1,0,1,0\} \tag{A.77}$$

Conjugacy class $\mathcal{C}_{4}\,(\mathrm{G}_{256})$: # *of elements* = 1

$$\{1_1,0,1,1\} \tag{A.78}$$

Conjugacy class $\mathcal{C}_{5}\,(\mathrm{G}_{256})$: # *of elements* = 1

$$\{1_1,1,0,0\} \tag{A.79}$$

Conjugacy class $\mathcal{C}_{6}\,(\mathrm{G}_{256})$: # *of elements* = 1

$$\{1_1,1,0,1\} \tag{A.80}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{G}_{256})$: # *of elements* = 1

$$\{1_1, 1, 1, 0\} \tag{A.81}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{G}_{256})$: # *of elements* = 1

$$\{1_1, 1, 1, 1\} \tag{A.82}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, 0, 0, \tfrac{1}{2}\right\} \quad \left\{1_1, 0, 0, \tfrac{3}{2}\right\} \tag{A.83}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, 0, \tfrac{1}{2}, 0\right\} \quad \left\{1_1, 0, \tfrac{3}{2}, 0\right\} \tag{A.84}$$

Conjugacy class $\mathcal{C}_{11}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, 0, \tfrac{1}{2}, 1\right\} \quad \left\{1_1, 0, \tfrac{3}{2}, 1\right\} \tag{A.85}$$

Conjugacy class $\mathcal{C}_{12}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, 0, 1, \tfrac{1}{2}\right\} \quad \left\{1_1, 0, 1, \tfrac{3}{2}\right\} \tag{A.86}$$

Conjugacy class $\mathcal{C}_{13}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 0, 0\right\} \quad \left\{1_1, \tfrac{3}{2}, 0, 0\right\} \tag{A.87}$$

Conjugacy class $\mathcal{C}_{14}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 0, 1\right\} \quad \left\{1_1, \tfrac{3}{2}, 0, 1\right\} \tag{A.88}$$

Conjugacy class $\mathcal{C}_{15}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 1, 0\right\} \quad \left\{1_1, \tfrac{3}{2}, 1, 0\right\} \tag{A.89}$$

Conjugacy class $\mathcal{C}_{16}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 1, 1\right\} \quad \left\{1_1, \tfrac{3}{2}, 1, 1\right\} \tag{A.90}$$

Conjugacy class $\mathcal{C}_{17}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, 1, 0, \tfrac{1}{2}\right\} \quad \left\{1_1, 1, 0, \tfrac{3}{2}\right\} \tag{A.91}$$

Conjugacy class $\mathcal{C}_{18}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, 1, \tfrac{1}{2}, 0\right\} \quad \left\{1_1, 1, \tfrac{3}{2}, 0\right\} \tag{A.92}$$

Conjugacy class $\mathcal{C}_{19}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, 1, \tfrac{1}{2}, 1\right\} \quad \left\{1_1, 1, \tfrac{3}{2}, 1\right\} \tag{A.93}$$

Conjugacy class $\mathcal{C}_{20}\,(\mathrm{G}_{256})$: # *of elements* = 2

$$\left\{1_1, 1, 1, \tfrac{1}{2}\right\} \quad \left\{1_1, 1, 1, \tfrac{3}{2}\right\} \tag{A.94}$$

Conjugacy class $\mathcal{C}_{21}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{1_1, 0, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, 0, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, 0, \tfrac{3}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, 0, \tfrac{3}{2}, \tfrac{3}{2}\right\} \tag{A.95}$$

Conjugacy class $\mathcal{C}_{22}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{1_1, \tfrac{1}{2}, 0, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{1}{2}, 0, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, 0, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, 0, \tfrac{3}{2}\right\} \tag{A.96}$$

Conjugacy class $\mathcal{C}_{23}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, 0\right\} \quad \left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, 0\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, 0\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, 0\right\} \tag{A.97}$$

Conjugacy class $\mathcal{C}_{24}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2}\right\} \tag{A.98}$$

Conjugacy class $\mathcal{C}_{25}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, 1\right\} \quad \left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, 1\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, 1\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, 1\right\} \tag{A.99}$$

Conjugacy class $\mathcal{C}_{26}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}\right\} \tag{A.100}$$

Conjugacy class $\mathcal{C}_{27}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{1_1, \tfrac{1}{2}, 1, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{1}{2}, 1, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, 1, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, 1, \tfrac{3}{2}\right\} \tag{A.101}$$

Conjugacy class $\mathcal{C}_{28}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{1_1, 1, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, 1, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, 1, \tfrac{3}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, 1, \tfrac{3}{2}, \tfrac{3}{2}\right\} \tag{A.102}$$

Conjugacy class $\mathcal{C}_{29}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\{3_1, 0, 0, 0\} \quad \{3_1, 0, 1, 0\} \quad \{3_1, 1, 0, 0\} \quad \{3_1, 1, 1, 0\} \tag{A.103}$$

Conjugacy class $\mathcal{C}_{30}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\{3_1, 0, 0, 1\} \quad \{3_1, 0, 1, 1\} \quad \{3_1, 1, 0, 1\} \quad \{3_1, 1, 1, 1\} \tag{A.104}$$

Conjugacy class $\mathcal{C}_{31}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_1,0,\tfrac{1}{2},0\right\}\quad\left\{3_1,0,\tfrac{3}{2},0\right\}\quad\left\{3_1,1,\tfrac{1}{2},0\right\}\quad\left\{3_1,1,\tfrac{3}{2},0\right\} \tag{A.105}$$

Conjugacy class $\mathcal{C}_{32}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_1,0,\tfrac{1}{2},1\right\}\quad\left\{3_1,0,\tfrac{3}{2},1\right\}\quad\left\{3_1,1,\tfrac{1}{2},1\right\}\quad\left\{3_1,1,\tfrac{3}{2},1\right\} \tag{A.106}$$

Conjugacy class $\mathcal{C}_{33}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_1,\tfrac{1}{2},0,0\right\}\quad\left\{3_1,\tfrac{1}{2},1,0\right\}\quad\left\{3_1,\tfrac{3}{2},0,0\right\}\quad\left\{3_1,\tfrac{3}{2},1,0\right\} \tag{A.107}$$

Conjugacy class $\mathcal{C}_{34}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_1,\tfrac{1}{2},0,1\right\}\quad\left\{3_1,\tfrac{1}{2},1,1\right\}\quad\left\{3_1,\tfrac{3}{2},0,1\right\}\quad\left\{3_1,\tfrac{3}{2},1,1\right\} \tag{A.108}$$

Conjugacy class $\mathcal{C}_{35}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_1,\tfrac{1}{2},\tfrac{1}{2},0\right\}\quad\left\{3_1,\tfrac{1}{2},\tfrac{3}{2},0\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{1}{2},0\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{3}{2},0\right\} \tag{A.109}$$

Conjugacy class $\mathcal{C}_{36}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_1,\tfrac{1}{2},\tfrac{1}{2},1\right\}\quad\left\{3_1,\tfrac{1}{2},\tfrac{3}{2},1\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{1}{2},1\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{3}{2},1\right\} \tag{A.110}$$

Conjugacy class $\mathcal{C}_{37}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\{3_2,0,0,0\}\quad\{3_2,0,0,1\}\quad\{3_2,1,0,0\}\quad\{3_2,1,0,1\} \tag{A.111}$$

Conjugacy class $\mathcal{C}_{38}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\{3_2,0,0,0\}\quad\{3_2,0,0,1\}\quad\{3_2,1,0,0\}\quad\{3_2,1,0,1\} \tag{A.112}$$

Conjugacy class $\mathcal{C}_{39}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\{3_2,0,1,0\}\quad\{3_2,0,1,1\}\quad\{3_2,1,1,0\}\quad\{3_2,1,1,1\} \tag{A.113}$$

Conjugacy class $\mathcal{C}_{40}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_2,0,1,\tfrac{1}{2}\right\}\quad\left\{3_2,0,1,\tfrac{3}{2}\right\}\quad\left\{3_2,1,1,\tfrac{1}{2}\right\}\quad\left\{3_2,1,1,\tfrac{3}{2}\right\} \tag{A.114}$$

Conjugacy class $\mathcal{C}_{41}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_2,\tfrac{1}{2},0,0\right\}\quad\left\{3_2,\tfrac{1}{2},0,1\right\}\quad\left\{3_2,\tfrac{3}{2},0,0\right\}\quad\left\{3_2,\tfrac{3}{2},0,1\right\} \tag{A.115}$$

Conjugacy class $\mathcal{C}_{42}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_2,\tfrac{1}{2},0,\tfrac{1}{2}\right\}\quad\left\{3_2,\tfrac{1}{2},0,\tfrac{3}{2}\right\}\quad\left\{3_2,\tfrac{3}{2},0,\tfrac{1}{2}\right\}\quad\left\{3_2,\tfrac{3}{2},0,\tfrac{3}{2}\right\} \tag{A.116}$$

Conjugacy class $\mathcal{C}_{43}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_2,\tfrac{1}{2},1,0\right\}\quad\left\{3_2,\tfrac{1}{2},1,1\right\}\quad\left\{3_2,\tfrac{3}{2},1,0\right\}\quad\left\{3_2,\tfrac{3}{2},1,1\right\}\tag{A.117}$$

Conjugacy class $\mathcal{C}_{44}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_2,\tfrac{1}{2},1,\tfrac{1}{2}\right\}\quad\left\{3_2,\tfrac{1}{2},1,\tfrac{3}{2}\right\}\quad\left\{3_2,\tfrac{3}{2},1,\tfrac{1}{2}\right\}\quad\left\{3_2,\tfrac{3}{2},1,\tfrac{3}{2}\right\}\tag{A.118}$$

Conjugacy class $\mathcal{C}_{45}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\{3_3,0,0,0\}\quad\{3_3,0,0,1\}\quad\{3_3,0,1,0\}\quad\{3_3,0,1,1\}\tag{A.119}$$

Conjugacy class $\mathcal{C}_{46}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_3,0,0,\tfrac{1}{2}\right\}\quad\left\{3_3,0,0,\tfrac{3}{2}\right\}\quad\left\{3_3,0,1,\tfrac{1}{2}\right\}\quad\left\{3_3,0,1,\tfrac{3}{2}\right\}\tag{A.120}$$

Conjugacy class $\mathcal{C}_{47}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_3,0,\tfrac{1}{2},0\right\}\quad\left\{3_3,0,\tfrac{1}{2},1\right\}\quad\left\{3_3,0,\tfrac{3}{2},0\right\}\quad\left\{3_3,0,\tfrac{3}{2},1\right\}\tag{A.121}$$

Conjugacy class $\mathcal{C}_{48}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_3,0,\tfrac{1}{2},\tfrac{1}{2}\right\}\quad\left\{3_3,0,\tfrac{1}{2},\tfrac{3}{2}\right\}\quad\left\{3_3,0,\tfrac{3}{2},\tfrac{1}{2}\right\}\quad\left\{3_3,0,\tfrac{3}{2},\tfrac{3}{2}\right\}\tag{A.122}$$

Conjugacy class $\mathcal{C}_{49}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\{3_3,1,0,0\}\quad\{3_3,1,0,1\}\quad\{3_3,1,1,0\}\quad\{3_3,1,1,1\}\tag{A.123}$$

Conjugacy class $\mathcal{C}_{50}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_3,1,0,\tfrac{1}{2}\right\}\quad\left\{3_3,1,0,\tfrac{3}{2}\right\}\quad\left\{3_3,1,1,\tfrac{1}{2}\right\}\quad\left\{3_3,1,1,\tfrac{3}{2}\right\}\tag{A.124}$$

Conjugacy class $\mathcal{C}_{51}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_3,1,\tfrac{1}{2},0\right\}\quad\left\{3_3,1,\tfrac{1}{2},1\right\}\quad\left\{3_3,1,\tfrac{3}{2},0\right\}\quad\left\{3_3,1,\tfrac{3}{2},1\right\}\tag{A.125}$$

Conjugacy class $\mathcal{C}_{52}\,(\mathrm{G}_{256})$: # *of elements* = 4

$$\left\{3_3,1,\tfrac{1}{2},\tfrac{1}{2}\right\}\quad\left\{3_3,1,\tfrac{1}{2},\tfrac{3}{2}\right\}\quad\left\{3_3,1,\tfrac{3}{2},\tfrac{1}{2}\right\}\quad\left\{3_3,1,\tfrac{3}{2},\tfrac{3}{2}\right\}\tag{A.126}$$

Conjugacy class $\mathcal{C}_{53}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc}\left\{3_1,0,0,\tfrac{1}{2}\right\} & \left\{3_1,0,0,\tfrac{3}{2}\right\} & \left\{3_1,0,1,\tfrac{1}{2}\right\} & \left\{3_1,0,1,\tfrac{3}{2}\right\}\\ \left\{3_1,1,0,\tfrac{1}{2}\right\} & \left\{3_1,1,0,\tfrac{3}{2}\right\} & \left\{3_1,1,1,\tfrac{1}{2}\right\} & \left\{3_1,1,1,\tfrac{3}{2}\right\}\end{array}\tag{A.127}$$

Conjugacy class $\mathcal{C}_{54}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{c}\left\{3_1,0,\frac{1}{2},\frac{1}{2}\right\}\ \left\{3_1,0,\frac{1}{2},\frac{3}{2}\right\}\ \left\{3_1,0,\frac{3}{2},\frac{1}{2}\right\}\ \left\{3_1,0,\frac{3}{2},\frac{3}{2}\right\}\\ \left\{3_1,1,\frac{1}{2},\frac{1}{2}\right\}\ \left\{3_1,1,\frac{1}{2},\frac{3}{2}\right\}\ \left\{3_1,1,\frac{3}{2},\frac{1}{2}\right\}\ \left\{3_1,1,\frac{3}{2},\frac{3}{2}\right\}\end{array} \tag{A.128}$$

Conjugacy class $\mathcal{C}_{55}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{c}\left\{3_1,\frac{1}{2},0,\frac{1}{2}\right\}\ \left\{3_1,\frac{1}{2},0,\frac{3}{2}\right\}\ \left\{3_1,\frac{1}{2},1,\frac{1}{2}\right\}\ \left\{3_1,\frac{1}{2},1,\frac{3}{2}\right\}\\ \left\{3_1,\frac{3}{2},0,\frac{1}{2}\right\}\ \left\{3_1,\frac{3}{2},0,\frac{3}{2}\right\}\ \left\{3_1,\frac{3}{2},1,\frac{1}{2}\right\}\ \left\{3_1,\frac{3}{2},1,\frac{3}{2}\right\}\end{array} \tag{A.129}$$

Conjugacy class $\mathcal{C}_{56}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc}\left\{3_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\}\\ \left\{3_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}\end{array} \tag{A.130}$$

Conjugacy class $\mathcal{C}_{57}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc}\left\{3_2,0,\frac{1}{2},0\right\} & \left\{3_2,0,\frac{1}{2},1\right\} & \left\{3_2,0,\frac{3}{2},0\right\} & \left\{3_2,0,\frac{3}{2},1\right\}\\ \left\{3_2,1,\frac{1}{2},0\right\} & \left\{3_2,1,\frac{1}{2},1\right\} & \left\{3_2,1,\frac{3}{2},0\right\} & \left\{3_2,1,\frac{3}{2},1\right\}\end{array} \tag{A.131}$$

Conjugacy class $\mathcal{C}_{58}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc}\left\{3_2,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,0,\frac{3}{2},\frac{3}{2}\right\}\\ \left\{3_2,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,1,\frac{3}{2},\frac{3}{2}\right\}\end{array} \tag{A.132}$$

Conjugacy class $\mathcal{C}_{59}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc}\left\{3_2,\frac{1}{2},\frac{1}{2},0\right\} & \left\{3_2,\frac{1}{2},\frac{1}{2},1\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},0\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},1\right\}\\ \left\{3_2,\frac{3}{2},\frac{1}{2},0\right\} & \left\{3_2,\frac{3}{2},\frac{1}{2},1\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},0\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},1\right\}\end{array} \tag{A.133}$$

Conjugacy class $\mathcal{C}_{60}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc}\left\{3_2,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\}\\ \left\{3_2,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_2,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\}\end{array} \tag{A.134}$$

Conjugacy class $\mathcal{C}_{61}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc}\left\{3_3,\frac{1}{2},0,0\right\} & \left\{3_3,\frac{1}{2},0,1\right\} & \left\{3_3,\frac{1}{2},1,0\right\} & \left\{3_3,\frac{1}{2},1,1\right\}\\ \left\{3_3,\frac{3}{2},0,0\right\} & \left\{3_3,\frac{3}{2},0,1\right\} & \left\{3_3,\frac{3}{2},1,0\right\} & \left\{3_3,\frac{3}{2},1,1\right\}\end{array} \tag{A.135}$$

Conjugacy class $\mathcal{C}_{62}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc} \left\{3_3,\frac{1}{2},0,\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},0,\frac{3}{2}\right\} & \left\{3_3,\frac{1}{2},1,\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},1,\frac{3}{2}\right\} \\ \left\{3_3,\frac{3}{2},0,\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},0,\frac{3}{2}\right\} & \left\{3_3,\frac{3}{2},1,\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},1,\frac{3}{2}\right\} \end{array} \tag{A.136}$$

Conjugacy class $\mathcal{C}_{63}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc} \left\{3_3,\frac{1}{2},\frac{1}{2},0\right\} & \left\{3_3,\frac{1}{2},\frac{1}{2},1\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},0\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},1\right\} \\ \left\{3_3,\frac{3}{2},\frac{1}{2},0\right\} & \left\{3_3,\frac{3}{2},\frac{1}{2},1\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},0\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},1\right\} \end{array} \tag{A.137}$$

Conjugacy class $\mathcal{C}_{64}\,(\mathrm{G}_{256})$: # *of elements* = 8

$$\begin{array}{cccc} \left\{3_3,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\ \left\{3_3,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{3_3,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \end{array} \tag{A.138}$$

A.4. The Group G_{128}**.** In this section, we list all the elements of the space group G_{128}, organized into their 56 conjugacy classes.

Conjugacy class $\mathcal{C}_{1}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\{1_1,0,0,0\} \tag{A.139}$$

Conjugacy class $\mathcal{C}_{2}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\left\{1_1,0,0,\tfrac{1}{2}\right\} \tag{A.140}$$

Conjugacy class $\mathcal{C}_{3}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\{1_1,0,0,1\} \tag{A.141}$$

Conjugacy class $\mathcal{C}_{4}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\left\{1_1,0,0,\tfrac{3}{2}\right\} \tag{A.142}$$

Conjugacy class $\mathcal{C}_{5}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\{1_1,0,1,0\} \tag{A.143}$$

Conjugacy class $\mathcal{C}_{6}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\left\{1_1,0,1,\tfrac{1}{2}\right\} \tag{A.144}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\{1_1, 0, 1, 1\} \tag{A.145}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\left\{1_1, 0, 1, \tfrac{3}{2}\right\} \tag{A.146}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\{1_1, 1, 0, 0\} \tag{A.147}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\left\{1_1, 1, 0, \tfrac{1}{2}\right\} \tag{A.148}$$

Conjugacy class $\mathcal{C}_{11}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\{1_1, 1, 0, 1\} \tag{A.149}$$

Conjugacy class $\mathcal{C}_{12}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\left\{1_1, 1, 0, \tfrac{3}{2}\right\} \tag{A.150}$$

Conjugacy class $\mathcal{C}_{13}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\{1_1, 1, 1, 0\} \tag{A.151}$$

Conjugacy class $\mathcal{C}_{14}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\left\{1_1, 1, 1, \tfrac{1}{2}\right\} \tag{A.152}$$

Conjugacy class $\mathcal{C}_{15}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\{1_1, 1, 1, 1\} \tag{A.153}$$

Conjugacy class $\mathcal{C}_{16}\,(\mathrm{G}_{128})$: # *of elements* = 1

$$\left\{1_1, 1, 1, \tfrac{3}{2}\right\} \tag{A.154}$$

Conjugacy class $\mathcal{C}_{17}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, 0, \tfrac{1}{2}, 0\right\} \quad \left\{1_1, 0, \tfrac{3}{2}, 0\right\} \tag{A.155}$$

Conjugacy class $\mathcal{C}_{18}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, 0, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, 0, \tfrac{3}{2}, \tfrac{1}{2}\right\} \tag{A.156}$$

Conjugacy class $\mathcal{C}_{19}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, 0, \tfrac{1}{2}, 1\right\} \quad \left\{1_1, 0, \tfrac{3}{2}, 1\right\} \tag{A.157}$$

Conjugacy class $\mathcal{C}_{20}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, 0, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, 0, \tfrac{3}{2}, \tfrac{3}{2}\right\} \tag{A.158}$$

Conjugacy class $\mathcal{C}_{21}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 0, 0\right\} \quad \left\{1_1, \tfrac{3}{2}, 0, 0\right\} \tag{A.159}$$

Conjugacy class $\mathcal{C}_{22}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 0, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, 0, \tfrac{1}{2}\right\} \tag{A.160}$$

Conjugacy class $\mathcal{C}_{23}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 0, 1\right\} \quad \left\{1_1, \tfrac{3}{2}, 0, 1\right\} \tag{A.161}$$

Conjugacy class $\mathcal{C}_{24}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 0, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, 0, \tfrac{3}{2}\right\} \tag{A.162}$$

Conjugacy class $\mathcal{C}_{25}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, 0\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, 0\right\} \tag{A.163}$$

Conjugacy class $\mathcal{C}_{26}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2}\right\} \tag{A.164}$$

Conjugacy class $\mathcal{C}_{27}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, 1\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, 1\right\} \tag{A.165}$$

Conjugacy class $\mathcal{C}_{28}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}\right\} \tag{A.166}$$

Conjugacy class $\mathcal{C}_{29}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 1, 0\right\} \quad \left\{1_1, \tfrac{3}{2}, 1, 0\right\} \tag{A.167}$$

Conjugacy class $\mathcal{C}_{30}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 1, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, 1, \tfrac{1}{2}\right\} \tag{A.168}$$

Conjugacy class $\mathcal{C}_{31}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 1, 1\right\} \quad \left\{1_1, \tfrac{3}{2}, 1, 1\right\} \tag{A.169}$$

Conjugacy class $\mathcal{C}_{32}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, 1, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, 1, \tfrac{3}{2}\right\} \tag{A.170}$$

Conjugacy class $\mathcal{C}_{33}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, 0\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, 0\right\} \tag{A.171}$$

Conjugacy class $\mathcal{C}_{34}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\} \tag{A.172}$$

Conjugacy class $\mathcal{C}_{35}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, 1\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, 1\right\} \tag{A.173}$$

Conjugacy class $\mathcal{C}_{36}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right\} \tag{A.174}$$

Conjugacy class $\mathcal{C}_{37}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, 1, \tfrac{1}{2}, 0\right\} \quad \left\{1_1, 1, \tfrac{3}{2}, 0\right\} \tag{A.175}$$

Conjugacy class $\mathcal{C}_{38}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, 1, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{1_1, 1, \tfrac{3}{2}, \tfrac{1}{2}\right\} \tag{A.176}$$

Conjugacy class $\mathcal{C}_{39}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, 1, \tfrac{1}{2}, 1\right\} \quad \left\{1_1, 1, \tfrac{3}{2}, 1\right\} \tag{A.177}$$

Conjugacy class $\mathcal{C}_{40}\,(\mathrm{G}_{128})$: # *of elements* = 2

$$\left\{1_1, 1, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{1_1, 1, \tfrac{3}{2}, \tfrac{3}{2}\right\} \tag{A.178}$$

Conjugacy class $\mathcal{C}_{41}\,(\mathrm{G}_{128})$: # *of elements* = 4

$$\{3_1, 0, 0, 0\} \quad \{3_1, 0, 1, 0\} \quad \{3_1, 1, 0, 0\} \quad \{3_1, 1, 1, 0\} \tag{A.179}$$

Conjugacy class $\mathcal{C}_{42}\,(\mathrm{G}_{128})$: # *of elements* = 4

$$\left\{3_1, 0, 0, \tfrac{1}{2}\right\} \quad \left\{3_1, 0, 1, \tfrac{1}{2}\right\} \quad \left\{3_1, 1, 0, \tfrac{1}{2}\right\} \quad \left\{3_1, 1, 1, \tfrac{1}{2}\right\} \tag{A.180}$$

Conjugacy class $\mathcal{C}_{43}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\{3_1,0,0,1\}\quad\{3_1,0,1,1\}\quad\{3_1,1,0,1\}\quad\{3_1,1,1,1\}\tag{A.181}$$

Conjugacy class $\mathcal{C}_{44}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,0,0,\tfrac{3}{2}\right\}\quad\left\{3_1,0,1,\tfrac{3}{2}\right\}\quad\left\{3_1,1,0,\tfrac{3}{2}\right\}\quad\left\{3_1,1,1,\tfrac{3}{2}\right\}\tag{A.182}$$

Conjugacy class $\mathcal{C}_{45}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,0,\tfrac{1}{2},0\right\}\quad\left\{3_1,0,\tfrac{3}{2},0\right\}\quad\left\{3_1,1,\tfrac{1}{2},0\right\}\quad\left\{3_1,1,\tfrac{3}{2},0\right\}\tag{A.183}$$

Conjugacy class $\mathcal{C}_{46}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,0,\tfrac{1}{2},\tfrac{1}{2}\right\}\quad\left\{3_1,0,\tfrac{3}{2},\tfrac{1}{2}\right\}\quad\left\{3_1,1,\tfrac{1}{2},\tfrac{1}{2}\right\}\quad\left\{3_1,1,\tfrac{3}{2},\tfrac{1}{2}\right\}\tag{A.184}$$

Conjugacy class $\mathcal{C}_{47}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,0,\tfrac{1}{2},1\right\}\quad\left\{3_1,0,\tfrac{3}{2},1\right\}\quad\left\{3_1,1,\tfrac{1}{2},1\right\}\quad\left\{3_1,1,\tfrac{3}{2},1\right\}\tag{A.185}$$

Conjugacy class $\mathcal{C}_{48}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,0,\tfrac{1}{2},\tfrac{3}{2}\right\}\quad\left\{3_1,0,\tfrac{3}{2},\tfrac{3}{2}\right\}\quad\left\{3_1,1,\tfrac{1}{2},\tfrac{3}{2}\right\}\quad\left\{3_1,1,\tfrac{3}{2},\tfrac{3}{2}\right\}\tag{A.186}$$

Conjugacy class $\mathcal{C}_{49}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,\tfrac{1}{2},0,0\right\}\quad\left\{3_1,\tfrac{1}{2},1,0\right\}\quad\left\{3_1,\tfrac{3}{2},0,0\right\}\quad\left\{3_1,\tfrac{3}{2},1,0\right\}\tag{A.187}$$

Conjugacy class $\mathcal{C}_{50}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,\tfrac{1}{2},0,\tfrac{1}{2}\right\}\quad\left\{3_1,\tfrac{1}{2},1,\tfrac{1}{2}\right\}\quad\left\{3_1,\tfrac{3}{2},0,\tfrac{1}{2}\right\}\quad\left\{3_1,\tfrac{3}{2},1,\tfrac{1}{2}\right\}\tag{A.188}$$

Conjugacy class $\mathcal{C}_{51}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,\tfrac{1}{2},0,1\right\}\quad\left\{3_1,\tfrac{1}{2},1,1\right\}\quad\left\{3_1,\tfrac{3}{2},0,1\right\}\quad\left\{3_1,\tfrac{3}{2},1,1\right\}\tag{A.189}$$

Conjugacy class $\mathcal{C}_{52}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,\tfrac{1}{2},0,\tfrac{3}{2}\right\}\quad\left\{3_1,\tfrac{1}{2},1,\tfrac{3}{2}\right\}\quad\left\{3_1,\tfrac{3}{2},0,\tfrac{3}{2}\right\}\quad\left\{3_1,\tfrac{3}{2},1,\tfrac{3}{2}\right\}\tag{A.190}$$

Conjugacy class $\mathcal{C}_{53}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,\tfrac{1}{2},\tfrac{1}{2},0\right\}\quad\left\{3_1,\tfrac{1}{2},\tfrac{3}{2},0\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{1}{2},0\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{3}{2},0\right\}\tag{A.191}$$

Conjugacy class $\mathcal{C}_{54}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\}\quad\left\{3_1,\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2}\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}\right\}\tag{A.192}$$

Conjugacy class $\mathcal{C}_{55}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,\tfrac{1}{2},\tfrac{1}{2},1\right\}\quad\left\{3_1,\tfrac{1}{2},\tfrac{3}{2},1\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{1}{2},1\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{3}{2},1\right\}\tag{A.193}$$

Conjugacy class $\mathcal{C}_{56}\,(\mathrm{G}_{128})$: # *of elements* = 4
$$\left\{3_1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2}\right\}\quad\left\{3_1,\tfrac{1}{2},\tfrac{3}{2},\tfrac{3}{2}\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{1}{2},\tfrac{3}{2}\right\}\quad\left\{3_1,\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}\right\}\tag{A.194}$$

A.5. The Group G_{64}**.** The group G_{64} is Abelian. Hence there are just as many conjugacy classes as there are elements, every conjugacy class containing just one element. For this reason it suffices to list the 64 elements, which are displayed below:

$$\begin{array}{cccc}
\{1_1,0,0,0\} & \{1_1,0,0,\frac{1}{2}\} & \{1_1,0,0,1\} & \{1_1,0,0,\frac{3}{2}\} \\
\{1_1,0,\frac{1}{2},0\} & \{1_1,0,\frac{1}{2},\frac{1}{2}\} & \{1_1,0,\frac{1}{2},1\} & \{1_1,0,\frac{1}{2},\frac{3}{2}\} \\
\{1_1,0,1,0\} & \{1_1,0,1,\frac{1}{2}\} & \{1_1,0,1,1\} & \{1_1,0,1,\frac{3}{2}\} \\
\{1_1,0,\frac{3}{2},0\} & \{1_1,0,\frac{3}{2},\frac{1}{2}\} & \{1_1,0,\frac{3}{2},1\} & \{1_1,0,\frac{3}{2},\frac{3}{2}\} \\
\{1_1,\frac{1}{2},0,0\} & \{1_1,\frac{1}{2},0,\frac{1}{2}\} & \{1_1,\frac{1}{2},0,1\} & \{1_1,\frac{1}{2},0,\frac{3}{2}\} \\
\{1_1,\frac{1}{2},\frac{1}{2},0\} & \{1_1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\} & \{1_1,\frac{1}{2},\frac{1}{2},1\} & \{1_1,\frac{1}{2},\frac{1}{2},\frac{3}{2}\} \\
\{1_1,\frac{1}{2},1,0\} & \{1_1,\frac{1}{2},1,\frac{1}{2}\} & \{1_1,\frac{1}{2},1,1\} & \{1_1,\frac{1}{2},1,\frac{3}{2}\} \\
\{1_1,\frac{1}{2},\frac{3}{2},0\} & \{1_1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\} & \{1_1,\frac{1}{2},\frac{3}{2},1\} & \{1_1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\} \\
\{1_1,1,0,0\} & \{1_1,1,0,\frac{1}{2}\} & \{1_1,1,0,1\} & \{1_1,1,0,\frac{3}{2}\} \\
\{1_1,1,\frac{1}{2},0\} & \{1_1,1,\frac{1}{2},\frac{1}{2}\} & \{1_1,1,\frac{1}{2},1\} & \{1_1,1,\frac{1}{2},\frac{3}{2}\} \\
\{1_1,1,1,0\} & \{1_1,1,1,\frac{1}{2}\} & \{1_1,1,1,1\} & \{1_1,1,1,\frac{3}{2}\} \\
\{1_1,1,\frac{3}{2},0\} & \{1_1,1,\frac{3}{2},\frac{1}{2}\} & \{1_1,1,\frac{3}{2},1\} & \{1_1,1,\frac{3}{2},\frac{3}{2}\} \\
\{1_1,\frac{3}{2},0,0\} & \{1_1,\frac{3}{2},0,\frac{1}{2}\} & \{1_1,\frac{3}{2},0,1\} & \{1_1,\frac{3}{2},0,\frac{3}{2}\} \\
\{1_1,\frac{3}{2},\frac{1}{2},0\} & \{1_1,\frac{3}{2},\frac{1}{2},\frac{1}{2}\} & \{1_1,\frac{3}{2},\frac{1}{2},1\} & \{1_1,\frac{3}{2},\frac{1}{2},\frac{3}{2}\} \\
\{1_1,\frac{3}{2},1,0\} & \{1_1,\frac{3}{2},1,\frac{1}{2}\} & \{1_1,\frac{3}{2},1,1\} & \{1_1,\frac{3}{2},1,\frac{3}{2}\} \\
\{1_1,\frac{3}{2},\frac{3}{2},0\} & \{1_1,\frac{3}{2},\frac{3}{2},\frac{1}{2}\} & \{1_1,\frac{3}{2},\frac{3}{2},1\} & \{1_1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\}
\end{array} \tag{A.195}$$

Abstractly the group G_{64} is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.

A.6. The Group G_{192}**.** In this section, we list all the elements of the space group G_{192}, organized into their 20 conjugacy classes.

Conjugacy class $\mathcal{C}_1\,(\mathrm{G}_{192})$: # *of elements* = 1

$$\{1_1,0,0,0\} \tag{A.196}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{G}_{192})$: # *of elements* = 1

$$\{1_1,1,1,1\} \tag{A.197}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{G}_{192})$: # *of elements* = 3

$$\{1_1,0,0,1\} \quad \{1_1,0,1,0\} \quad \{1_1,1,0,0\} \tag{A.198}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{G}_{192})$: # *of elements* = 3

$$\{1_1,0,1,1\} \quad \{1_1,1,0,1\} \quad \{1_1,1,1,0\} \tag{A.199}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{G}_{192})$: # *of elements* = 3

$$\{3_1,0,0,0\} \quad \{3_2,0,0,0\} \quad \{3_3,0,0,0\} \tag{A.200}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{G}_{192})$: # *of elements* = 3

$$\{3_1,0,0,1\} \quad \{3_2,0,1,0\} \quad \{3_3,1,0,0\} \tag{A.201}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{G}_{192})$: # *of elements* = 3

$$\{3_1,1,1,0\} \quad \{3_2,1,0,1\} \quad \{3_3,0,1,1\} \tag{A.202}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{G}_{192})$: # *of elements* = 3

$$\{3_1,1,1,1\} \quad \{3_2,1,1,1\} \quad \{3_3,1,1,1\} \tag{A.203}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{G}_{192})$: # *of elements* = 6

$$\begin{array}{ccc} \{3_1,0,1,0\} & \{3_1,1,0,0\} & \{3_2,0,0,1\} \\ \{3_2,1,0,0\} & \{3_3,0,0,1\} & \{3_3,0,1,0\} \end{array} \tag{A.204}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{G}_{192})$: # *of elements* = 6

$$\begin{array}{ccc} \{3_1,0,1,1\} & \{3_1,1,0,1\} & \{3_2,0,1,1\} \\ \{3_2,1,1,0\} & \{3_3,1,0,1\} & \{3_3,1,1,0\} \end{array} \tag{A.205}$$

Conjugacy class $\mathcal{C}_{11}\,(\mathrm{G}_{192})$: # *of elements* = 12

$$\begin{array}{ccc} \{4_1,0,0,0\} & \{4_1,0,1,1\} & \{4_2,0,0,0\} \\ \{4_2,0,1,1\} & \{4_3,0,0,0\} & \{4_3,1,1,0\} \\ \{4_4,0,0,0\} & \{4_4,1,0,1\} & \{4_5,0,0,0\} \\ \{4_5,1,0,1\} & \{4_6,0,0,0\} & \{4_6,1,1,0\} \end{array} \tag{A.206}$$

Conjugacy class $\mathcal{C}_{12}\,(\mathrm{G}_{192})$: # *of elements* = 12

$$\begin{array}{ccc} \{4_1,0,0,1\} & \{4_1,0,1,0\} & \{4_2,0,0,1\} \\ \{4_2,0,1,0\} & \{4_3,0,1,0\} & \{4_3,1,0,0\} \\ \{4_4,0,0,1\} & \{4_4,1,0,0\} & \{4_5,0,0,1\} \\ \{4_5,1,0,0\} & \{4_6,0,1,0\} & \{4_6,1,0,0\} \end{array} \tag{A.207}$$

Conjugacy class $\mathcal{C}_{13}\,(\mathrm{G}_{192})$: # *of elements* = 12

$$\begin{array}{ccc} \{4_1,1,0,0\} & \{4_1,1,1,1\} & \{4_2,1,0,0\} \\ \{4_2,1,1,1\} & \{4_3,0,0,1\} & \{4_3,1,1,1\} \\ \{4_4,0,1,0\} & \{4_4,1,1,1\} & \{4_5,0,1,0\} \\ \{4_5,1,1,1\} & \{4_6,0,0,1\} & \{4_6,1,1,1\} \end{array} \tag{A.208}$$

Conjugacy class $\mathcal{C}_{14}\,(\mathrm{G}_{192})$: # *of elements* = 12

$$\begin{array}{ccc} \{4_1,1,0,1\} & \{4_1,1,1,0\} & \{4_2,1,0,1\} \\ \{4_2,1,1,0\} & \{4_3,0,1,1\} & \{4_3,1,0,1\} \\ \{4_4,0,1,1\} & \{4_4,1,1,0\} & \{4_5,0,1,1\} \\ \{4_5,1,1,0\} & \{4_6,0,1,1\} & \{4_6,1,0,1\} \end{array} \tag{A.209}$$

Conjugacy class $\mathcal{C}_{15}\,(\mathrm{G}_{192})$: # *of elements* = 12

$$\begin{array}{ccc} \{5_1,0,0,0\} & \{5_1,1,1,0\} & \{5_2,0,0,0\} \\ \{5_2,1,0,1\} & \{5_3,0,0,0\} & \{5_3,1,0,1\} \\ \{5_4,0,0,0\} & \{5_4,1,1,0\} & \{5_5,0,0,0\} \\ \{5_5,0,1,1\} & \{5_6,0,0,0\} & \{5_6,0,1,1\} \end{array} \tag{A.210}$$

Conjugacy class $\mathcal{C}_{16}\,(\mathrm{G}_{192})$: # *of elements* = 12

$$\begin{array}{ccc} \{5_1,0,0,1\} & \{5_1,1,1,1\} & \{5_2,0,1,0\} \\ \{5_2,1,1,1\} & \{5_3,0,1,0\} & \{5_3,1,1,1\} \\ \{5_4,0,0,1\} & \{5_4,1,1,1\} & \{5_5,1,0,0\} \\ \{5_5,1,1,1\} & \{5_6,1,0,0\} & \{5_6,1,1,1\} \end{array} \tag{A.211}$$

Conjugacy class $\mathcal{C}_{17}\,(\mathrm{G}_{192})$: # *of elements* = 12

$$\begin{array}{ccc} \{5_1,0,1,0\} & \{5_1,1,0,0\} & \{5_2,0,0,1\} \\ \{5_2,1,0,0\} & \{5_3,0,0,1\} & \{5_3,1,0,0\} \\ \{5_4,0,1,0\} & \{5_4,1,0,0\} & \{5_5,0,0,1\} \\ \{5_5,0,1,0\} & \{5_6,0,0,1\} & \{5_6,0,1,0\} \end{array} \tag{A.212}$$

Conjugacy class $\mathcal{C}_{18}\,(\mathrm{G}_{192})$: # *of elements* = 12

$$\begin{array}{ccc} \{5_1,0,1,1\} & \{5_1,1,0,1\} & \{5_2,0,1,1\} \\ \{5_2,1,1,0\} & \{5_3,0,1,1\} & \{5_3,1,1,0\} \\ \{5_4,0,1,1\} & \{5_4,1,0,1\} & \{5_5,1,0,1\} \\ \{5_5,1,1,0\} & \{5_6,1,0,1\} & \{5_6,1,1,0\} \end{array} \tag{A.213}$$

Conjugacy class $\mathcal{C}_{19}\,(\mathrm{G}_{192})$: # *of elements* = 32

$$\begin{array}{cccc} \{2_1,0,0,0\} & \{2_1,0,1,1\} & \{2_1,1,0,1\} & \{2_1,1,1,0\} \\ \{2_2,0,0,0\} & \{2_2,0,1,1\} & \{2_2,1,0,1\} & \{2_2,1,1,0\} \\ \{2_3,0,0,0\} & \{2_3,0,1,1\} & \{2_3,1,0,1\} & \{2_3,1,1,0\} \\ \{2_4,0,0,0\} & \{2_4,0,1,1\} & \{2_4,1,0,1\} & \{2_4,1,1,0\} \\ \{2_5,0,0,0\} & \{2_5,0,1,1\} & \{2_5,1,0,1\} & \{2_5,1,1,0\} \\ \{2_6,0,0,0\} & \{2_6,0,1,1\} & \{2_6,1,0,1\} & \{2_6,1,1,0\} \\ \{2_7,0,0,0\} & \{2_7,0,1,1\} & \{2_7,1,0,1\} & \{2_7,1,1,0\} \\ \{2_8,0,0,0\} & \{2_8,0,1,1\} & \{2_8,1,0,1\} & \{2_8,1,1,0\} \end{array} \tag{A.214}$$

Conjugacy class $\mathcal{C}_{20}\,(\mathrm{G}_{192})$: # *of elements* = 32

$$\begin{array}{cccc} \{2_1,0,0,1\} & \{2_1,0,1,0\} & \{2_1,1,0,0\} & \{2_1,1,1,1\} \\ \{2_2,0,0,1\} & \{2_2,0,1,0\} & \{2_2,1,0,0\} & \{2_2,1,1,1\} \\ \{2_3,0,0,1\} & \{2_3,0,1,0\} & \{2_3,1,0,0\} & \{2_3,1,1,1\} \\ \{2_4,0,0,1\} & \{2_4,0,1,0\} & \{2_4,1,0,0\} & \{2_4,1,1,1\} \\ \{2_5,0,0,1\} & \{2_5,0,1,0\} & \{2_5,1,0,0\} & \{2_5,1,1,1\} \\ \{2_6,0,0,1\} & \{2_6,0,1,0\} & \{2_6,1,0,0\} & \{2_6,1,1,1\} \\ \{2_7,0,0,1\} & \{2_7,0,1,0\} & \{2_7,1,0,0\} & \{2_7,1,1,1\} \\ \{2_8,0,0,1\} & \{2_8,0,1,0\} & \{2_8,1,0,0\} & \{2_8,1,1,1\} \end{array} \tag{A.215}$$

A.7. The Group GF_{192}. In this section, we list all the elements of the space group GF_{192}, organized into their 20 conjugacy classes.

Conjugacy class $\mathcal{C}_1\,(\mathrm{GF}_{192})$: # *of elements* = 1

$$\{1_1,0,0,0\} \tag{A.216}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{GF}_{192})$: # *of elements* = 1

$$\{1_1, 1, 1, 1\} \tag{A.217}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{GF}_{192})$: # *of elements* = 3

$$\{1_1, 0, 0, 1\} \quad \{1_1, 0, 1, 0\} \quad \{1_1, 1, 0, 0\} \tag{A.218}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{GF}_{192})$: # *of elements* = 3

$$\{1_1, 0, 1, 1\} \quad \{1_1, 1, 0, 1\} \quad \{1_1, 1, 1, 0\} \tag{A.219}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{GF}_{192})$: # *of elements* = 3

$$\{3_1, 0, 0, 1\} \quad \{3_2, 1, 1, 0\} \quad \{3_3, 1, 1, 1\} \tag{A.220}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{GF}_{192})$: # *of elements* = 3

$$\{3_1, 0, 0, 0\} \quad \{3_2, 1, 0, 0\} \quad \{3_3, 0, 1, 1\} \tag{A.221}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{GF}_{192})$: # *of elements* = 3

$$\{3_1, 1, 1, 1\} \quad \{3_2, 0, 1, 1\} \quad \{3_3, 1, 0, 0\} \tag{A.222}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{GF}_{192})$: # *of elements* = 3

$$\{3_1, 1, 1, 0\} \quad \{3_2, 0, 0, 1\} \quad \{3_3, 0, 0, 0\} \tag{A.223}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{GF}_{192})$: # *of elements* = 6

$$\begin{array}{lll} \{3_1, 0, 1, 1\} & \{3_1, 1, 0, 1\} & \{3_2, 0, 1, 0\} \\ \{3_2, 1, 1, 1\} & \{3_3, 1, 0, 1\} & \{3_3, 1, 1, 0\} \end{array} \tag{A.224}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{GF}_{192})$: # *of elements* = 6

$$\begin{array}{lll} \{3_1, 0, 1, 0\} & \{3_1, 1, 0, 0\} & \{3_2, 0, 0, 0\} \\ \{3_2, 1, 0, 1\} & \{3_3, 0, 0, 1\} & \{3_3, 0, 1, 0\} \end{array} \tag{A.225}$$

Conjugacy class $\mathcal{C}_{11}\,(\mathrm{GF}_{192})$: # *of elements* = 12

$$\begin{array}{llll} \left\{4_1, \frac{1}{2}, 0, 0\right\} & \left\{4_1, \frac{1}{2}, 1, 1\right\} & \left\{4_2, \frac{3}{2}, 0, 0\right\} & \left\{4_2, \frac{3}{2}, 1, 1\right\} \\ \left\{4_3, 0, 0, \frac{1}{2}\right\} & \left\{4_3, 1, 1, \frac{1}{2}\right\} & \left\{4_4, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right\} & \left\{4_4, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right\} \\ \left\{4_5, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right\} & \left\{4_5, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right\} & \left\{4_6, 0, 0, \frac{3}{2}\right\} & \left\{4_6, 1, 1, \frac{3}{2}\right\} \end{array} \tag{A.226}$$

Conjugacy class $\mathcal{C}_{12}\,(\mathrm{GF}_{192})$: # *of elements* = 12

$$\begin{array}{llll} \left\{4_1,\frac{1}{2},0,1\right\} & \left\{4_1,\frac{1}{2},1,0\right\} & \left\{4_2,\frac{3}{2},0,1\right\} & \left\{4_2,\frac{3}{2},1,0\right\} \\ \left\{4_3,0,1,\frac{1}{2}\right\} & \left\{4_3,1,0,\frac{1}{2}\right\} & \left\{4_4,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} & \left\{4_4,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\ \left\{4_5,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_5,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_6,0,1,\frac{3}{2}\right\} & \left\{4_6,1,0,\frac{3}{2}\right\} \end{array} \tag{A.227}$$

Conjugacy class $\mathcal{C}_{13}\,(\mathrm{GF}_{192})$: # *of elements* = 12

$$\begin{array}{llll} \left\{4_1,\frac{3}{2},0,0\right\} & \left\{4_1,\frac{3}{2},1,1\right\} & \left\{4_2,\frac{1}{2},0,0\right\} & \left\{4_2,\frac{1}{2},1,1\right\} \\ \left\{4_3,0,0,\frac{3}{2}\right\} & \left\{4_3,1,1,\frac{3}{2}\right\} & \left\{4_4,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_4,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} \\ \left\{4_5,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} & \left\{4_5,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{4_6,0,0,\frac{1}{2}\right\} & \left\{4_6,1,1,\frac{1}{2}\right\} \end{array} \tag{A.228}$$

Conjugacy class $\mathcal{C}_{14}\,(\mathrm{GF}_{192})$: # *of elements* = 12

$$\begin{array}{llll} \left\{4_1,\frac{3}{2},0,1\right\} & \left\{4_1,\frac{3}{2},1,0\right\} & \left\{4_2,\frac{1}{2},0,1\right\} & \left\{4_2,\frac{1}{2},1,0\right\} \\ \left\{4_3,0,1,\frac{3}{2}\right\} & \left\{4_3,1,0,\frac{3}{2}\right\} & \left\{4_4,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{4_4,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} \\ \left\{4_5,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{4_5,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} & \left\{4_6,0,1,\frac{1}{2}\right\} & \left\{4_6,1,0,\frac{1}{2}\right\} \end{array} \tag{A.229}$$

Conjugacy class $\mathcal{C}_{15}\,(\mathrm{GF}_{192})$: # *of elements* = 12

$$\begin{array}{llll} \left\{5_1,0,0,\frac{1}{2}\right\} & \left\{5_1,1,1,\frac{1}{2}\right\} & \left\{5_2,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} & \left\{5_2,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} \\ \left\{5_3,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{5_3,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{5_4,0,0,\frac{3}{2}\right\} & \left\{5_4,1,1,\frac{3}{2}\right\} \\ \left\{5_5,\frac{1}{2},0,1\right\} & \left\{5_5,\frac{1}{2},1,0\right\} & \left\{5_6,\frac{3}{2},0,1\right\} & \left\{5_6,\frac{3}{2},1,0\right\} \end{array} \tag{A.230}$$

Conjugacy class $\mathcal{C}_{16}\,(\mathrm{GF}_{192})$: # *of elements* = 12

$$\begin{array}{llll} \left\{5_1,0,0,\frac{3}{2}\right\} & \left\{5_1,1,1,\frac{3}{2}\right\} & \left\{5_2,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{5_2,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} \\ \left\{5_3,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{5_3,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} & \left\{5_4,0,0,\frac{1}{2}\right\} & \left\{5_4,1,1,\frac{1}{2}\right\} \\ \left\{5_5,\frac{3}{2},0,1\right\} & \left\{5_5,\frac{3}{2},1,0\right\} & \left\{5_6,\frac{1}{2},0,1\right\} & \left\{5_6,\frac{1}{2},1,0\right\} \end{array} \tag{A.231}$$

Conjugacy class $\mathcal{C}_{17}\,(\mathrm{GF}_{192})$: # *of elements* = 12

$$\begin{array}{llll} \left\{5_1,0,1,\frac{1}{2}\right\} & \left\{5_1,1,0,\frac{1}{2}\right\} & \left\{5_2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{5_2,\frac{3}{2},\frac{3}{2},\frac{3}{2}\right\} \\ \left\{5_3,\frac{1}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{5_3,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{5_4,0,1,\frac{3}{2}\right\} & \left\{5_4,1,0,\frac{3}{2}\right\} \\ \left\{5_5,\frac{1}{2},0,0\right\} & \left\{5_5,\frac{1}{2},1,1\right\} & \left\{5_6,\frac{3}{2},0,0\right\} & \left\{5_6,\frac{3}{2},1,1\right\} \end{array} \tag{A.232}$$

Conjugacy class $\mathcal{C}_{18}\,(\mathrm{GF}_{192})$: # *of elements* = 12

$$\begin{array}{llll} \left\{5_1,0,1,\frac{3}{2}\right\} & \left\{5_1,1,0,\frac{3}{2}\right\} & \left\{5_2,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{5_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} \\ \left\{5_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} & \left\{5_3,\frac{3}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{5_4,0,1,\frac{1}{2}\right\} & \left\{5_4,1,0,\frac{1}{2}\right\} \\ \left\{5_5,\frac{3}{2},0,0\right\} & \left\{5_5,\frac{3}{2},1,1\right\} & \left\{5_6,\frac{1}{2},0,0\right\} & \left\{5_6,\frac{1}{2},1,1\right\} \end{array} \tag{A.233}$$

Conjugacy class $\mathcal{C}_{19}\,(\mathrm{GF}_{192})$: # *of elements* = 32

$$\begin{array}{cccc}
\left\{2_1,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_1,\frac{1}{2},\frac{3}{2},1\right\} & \left\{2_1,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_1,\frac{3}{2},\frac{3}{2},0\right\} \\
\left\{2_2,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_2,\frac{1}{2},\frac{3}{2},1\right\} & \left\{2_2,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_2,\frac{3}{2},\frac{3}{2},0\right\} \\
\left\{2_3,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_3,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{2_3,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_3,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_4,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_4,0,\frac{3}{2},\frac{3}{2}\right\} & \left\{2_4,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_4,1,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_5,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_5,0,\frac{3}{2},\frac{3}{2}\right\} & \left\{2_5,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_5,1,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_6,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_6,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{2_6,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_6,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_7,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_7,\frac{1}{2},\frac{3}{2},0\right\} & \left\{2_7,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_7,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{2_8,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_8,\frac{1}{2},\frac{3}{2},0\right\} & \left\{2_8,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_8,\frac{3}{2},\frac{3}{2},1\right\}
\end{array} \tag{A.234}$$

Conjugacy class $\mathcal{C}_{20}\,(\mathrm{GF}_{192})$: # *of elements* = 32

$$\begin{array}{cccc}
\left\{2_1,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_1,\frac{1}{2},\frac{3}{2},0\right\} & \left\{2_1,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_1,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{2_2,\frac{1}{2},\frac{1}{2},1\right\} & \left\{2_2,\frac{1}{2},\frac{3}{2},0\right\} & \left\{2_2,\frac{3}{2},\frac{1}{2},0\right\} & \left\{2_2,\frac{3}{2},\frac{3}{2},1\right\} \\
\left\{2_3,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_3,0,\frac{3}{2},\frac{3}{2}\right\} & \left\{2_3,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_3,1,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_4,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_4,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{2_4,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_4,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_5,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_5,0,\frac{3}{2},\frac{1}{2}\right\} & \left\{2_5,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_5,1,\frac{3}{2},\frac{3}{2}\right\} \\
\left\{2_6,0,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_6,0,\frac{3}{2},\frac{3}{2}\right\} & \left\{2_6,1,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_6,1,\frac{3}{2},\frac{1}{2}\right\} \\
\left\{2_7,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_7,\frac{1}{2},\frac{3}{2},1\right\} & \left\{2_7,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_7,\frac{3}{2},\frac{3}{2},0\right\} \\
\left\{2_8,\frac{1}{2},\frac{1}{2},0\right\} & \left\{2_8,\frac{1}{2},\frac{3}{2},1\right\} & \left\{2_8,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_8,\frac{3}{2},\frac{3}{2},0\right\}
\end{array} \tag{A.235}$$

A.8. The Group Oh_{48}**.** In this section, we list all the elements of the space group Oh_{48}, organized into their 10 conjugacy classes that, in this case, are arranged according to the order which is customary in crystallography for the extended octahedral group.

Conjugacy class $\mathcal{C}_1\,(\mathrm{Oh}_{48})$: # *of elements* = 1

$$\{1_1,0,0,0\} \tag{A.236}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{Oh}_{48})$: # *of elements* = 8

$$\begin{array}{cccc}
\{2_1,0,0,0\} & \{2_2,0,0,0\} & \{2_3,0,0,0\} & \{2_4,0,0,0\} \\
\{2_5,0,0,0\} & \{2_6,0,0,0\} & \{2_7,0,0,0\} & \{2_8,0,0,0\}
\end{array} \tag{A.237}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{Oh}_{48})$: # *of elements* = 3

$$\{3_1,0,0,0\} \quad \{3_2,0,0,0\} \quad \{3_3,0,0,0\} \tag{A.238}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{Oh}_{48})$: # *of elements* = 6

$$\begin{array}{ccc} \{4_1,0,0,0\} & \{4_2,0,0,0\} & \{4_3,0,0,0\} \\ \{4_4,0,0,0\} & \{4_5,0,0,0\} & \{4_6,0,0,0\} \end{array} \tag{A.239}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{Oh}_{48})$: # *of elements* = 6

$$\begin{array}{ccc} \{5_1,0,0,0\} & \{5_2,0,0,0\} & \{5_3,0,0,0\} \\ \{5_4,0,0,0\} & \{5_5,0,0,0\} & \{5_6,0,0,0\} \end{array} \tag{A.240}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{Oh}_{48})$: # *of elements* = 1

$$\{1_1,1,1,1\} \tag{A.241}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{Oh}_{48})$: # *of elements* = 8

$$\begin{array}{cccc} \{2_1,1,1,1\} & \{2_2,1,1,1\} & \{2_3,1,1,1\} & \{2_4,1,1,1\} \\ \{2_5,1,1,1\} & \{2_6,1,1,1\} & \{2_7,1,1,1\} & \{2_8,1,1,1\} \end{array} \tag{A.242}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{Oh}_{48})$: # *of elements* = 3

$$\{3_1,1,1,1\} \quad \{3_2,1,1,1\} \quad \{3_3,1,1,1\} \tag{A.243}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{Oh}_{48})$: # *of elements* = 6

$$\begin{array}{ccc} \{4_1,1,1,1\} & \{4_2,1,1,1\} & \{4_3,1,1,1\} \\ \{4_4,1,1,1\} & \{4_5,1,1,1\} & \{4_6,1,1,1\} \end{array} \tag{A.244}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{Oh}_{48})$: # *of elements* = 6

$$\begin{array}{ccc} \{5_1,1,1,1\} & \{5_2,1,1,1\} & \{5_3,1,1,1\} \\ \{5_4,1,1,1\} & \{5_5,1,1,1\} & \{5_6,1,1,1\} \end{array} \tag{A.245}$$

A.9. The Group GS_{24}**.** In this section, we list all the elements of the space group GS_{24}, organized into their 5 conjugacy classes that, in this case, are arranged according to the order which is customary in crystallography for the proper octahedral group.

Conjugacy class $\mathcal{C}_1\,(\mathrm{GS}_{24})$: # *of elements* = 1

$$\{1_1,0,0,0\} \tag{A.246}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{GS}_{24})$: # *of elements* = 8

$$\begin{array}{cccc} \left\{2_1,\frac{3}{2},\frac{1}{2},1\right\} & \left\{2_2,\frac{1}{2},\frac{3}{2},1\right\} & \left\{2_3,1,\frac{1}{2},\frac{1}{2}\right\} & \left\{2_4,0,\frac{3}{2},\frac{3}{2}\right\} \\ \left\{2_5,1,\frac{3}{2},\frac{1}{2}\right\} & \left\{2_6,0,\frac{1}{2},\frac{3}{2}\right\} & \left\{2_7,\frac{1}{2},\frac{3}{2},0\right\} & \left\{2_8,\frac{3}{2},\frac{1}{2},0\right\} \end{array} \tag{A.247}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{GS}_{24})$: # *of elements* = 3

$$\{3_1,1,1,1\} \quad \{3_2,0,1,1\} \quad \{3_3,1,0,0\} \tag{A.248}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{GS}_{24})$: # *of elements* = 6

$$\begin{array}{ccc} \left\{4_1,\frac{1}{2},1,1\right\} & \left\{4_2,\frac{3}{2},1,1\right\} & \left\{4_3,1,1,\frac{1}{2}\right\} \\ \left\{4_4,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\} & \left\{4_5,\frac{3}{2},\frac{1}{2},\frac{1}{2}\right\} & \left\{4_6,0,0,\frac{3}{2}\right\} \end{array} \tag{A.249}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{GS}_{24})$: # *of elements* = 6

$$\begin{array}{ccc} \left\{5_1,0,1,\frac{3}{2}\right\} & \left\{5_2,\frac{3}{2},\frac{1}{2},\frac{3}{2}\right\} & \left\{5_3,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\} \\ \left\{5_4,1,0,\frac{1}{2}\right\} & \left\{5_5,\frac{3}{2},0,0\right\} & \left\{5_6,\frac{1}{2},0,0\right\} \end{array} \tag{A.250}$$

A.10. The Group GP_{24}**.** In this section, we list all the elements of the space group GP_{24}, organized into their 8 conjugacy classes.

Conjugacy class $\mathcal{C}_1\,(\mathrm{GP}_{24})$: # *of elements* = 1

$$\{1_1,0,0,0\} \tag{A.251}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{GP}_{24})$: # *of elements* = 1

$$\{1_1,1,1,1\} \tag{A.252}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{GP}_{24})$: # *of elements* = 3

$$\{3_1,0,0,1\} \quad \{3_2,0,1,0\} \quad \{3_3,1,0,0\} \tag{A.253}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{GP}_{24})$: # *of elements* = 3

$$\{3_1,1,1,0\} \quad \{3_2,1,0,1\} \quad \{3_3,0,1,1\} \tag{A.254}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{GP}_{24})$: # *of elements* = 4

$$\{2_1,0,0,1\} \quad \{2_2,0,1,0\} \quad \{2_7,1,0,0\} \quad \{2_8,1,1,1\} \tag{A.255}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{GP}_{24})$: # *of elements* = 4

$$\{2_1,1,1,0\} \quad \{2_2,1,0,1\} \quad \{2_7,0,1,1\} \quad \{2_8,0,0,0\} \tag{A.256}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{GP}_{24})$: # *of elements* = 4

$$\{2_3,0,0,1\} \quad \{2_4,0,1,0\} \quad \{2_5,1,0,0\} \quad \{2_6,1,1,1\} \tag{A.257}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{GP}_{24})$: # *of elements* = 4

$$\{2_3,1,1,0\} \quad \{2_4,1,0,1\} \quad \{2_5,0,1,1\} \quad \{2_6,0,0,0\} \tag{A.258}$$

A.11. The Group GK_{24}**.** In this section, we list all the elements of the space group GK_{24}, organized into their 8 conjugacy classes.

Conjugacy class $\mathcal{C}_1\,(\mathrm{GK}_{24})$: # *of elements* = 1

$$\{1_1, 0, 0, 0\} \tag{A.259}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{GK}_{24})$: # *of elements* = 1

$$\{1_1, 1, 1, 1\} \tag{A.260}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{GK}_{24})$: # *of elements* = 3

$$\{3_1, 0, 1, 1\} \quad \{3_2, 0, 1, 0\} \quad \{3_3, 1, 1, 0\} \tag{A.261}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{GK}_{24})$: # *of elements* = 3

$$\{3_1, 1, 0, 0\} \quad \{3_2, 1, 0, 1\} \quad \{3_3, 0, 0, 1\} \tag{A.262}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{GK}_{24})$: # *of elements* = 4

$$\left\{2_1, \tfrac{1}{2}, \tfrac{3}{2}, 0\right\} \quad \left\{2_2, \tfrac{1}{2}, \tfrac{1}{2}, 1\right\} \quad \left\{2_7, \tfrac{1}{2}, \tfrac{3}{2}, 1\right\} \quad \left\{2_8, \tfrac{1}{2}, \tfrac{1}{2}, 0\right\} \tag{A.263}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{GK}_{24})$: # *of elements* = 4

$$\left\{2_1, \tfrac{3}{2}, \tfrac{1}{2}, 1\right\} \quad \left\{2_2, \tfrac{3}{2}, \tfrac{3}{2}, 0\right\} \quad \left\{2_7, \tfrac{3}{2}, \tfrac{1}{2}, 0\right\} \quad \left\{2_8, \tfrac{3}{2}, \tfrac{3}{2}, 1\right\} \tag{A.264}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{GK}_{24})$: # *of elements* = 4

$$\left\{2_3, 0, \tfrac{3}{2}, \tfrac{1}{2}\right\} \quad \left\{2_4, 0, \tfrac{1}{2}, \tfrac{1}{2}\right\} \quad \left\{2_5, 1, \tfrac{3}{2}, \tfrac{1}{2}\right\} \quad \left\{2_6, 1, \tfrac{1}{2}, \tfrac{1}{2}\right\} \tag{A.265}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{GK}_{24})$: # *of elements* = 4

$$\left\{2_3, 1, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{2_4, 1, \tfrac{3}{2}, \tfrac{3}{2}\right\} \quad \left\{2_5, 0, \tfrac{1}{2}, \tfrac{3}{2}\right\} \quad \left\{2_6, 0, \tfrac{3}{2}, \tfrac{3}{2}\right\} \tag{A.266}$$

A.12. The Group GS_{32}**.** In this section, we list all the elements of the space group GS_{32}, organized into their 14 conjugacy classes.

Conjugacy class $\mathcal{C}_1\,(\mathrm{GS}_{32})$: # *of elements* = 1

$$\{1_1, 0, 0, 0\} \tag{A.267}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{GS}_{32})$: # *of elements* = 1

$$\{1_1, 0, 1, 1\} \tag{A.268}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{GS}_{32})$: # *of elements* = 1

$$\{3_3, 0, 0, 0\} \tag{A.269}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 1

$$\{3_3, 0, 1, 1\} \tag{A.270}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 2

$$\{1_1, 1, 0, 1\} \quad \{1_1, 1, 1, 0\} \tag{A.271}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 2

$$\{3_1, 0, 0, 0\} \quad \{3_2, 0, 0, 0\} \tag{A.272}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 2

$$\{3_1, 0, 1, 1\} \quad \{3_2, 0, 1, 1\} \tag{A.273}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 2

$$\{3_1, 1, 0, 1\} \quad \{3_2, 1, 1, 0\} \tag{A.274}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 2

$$\{3_1, 1, 1, 0\} \quad \{3_2, 1, 0, 1\} \tag{A.275}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 2

$$\{3_3, 1, 0, 1\} \quad \{3_3, 1, 1, 0\} \tag{A.276}$$

Conjugacy class $\mathcal{C}_{11}\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 4

$$\begin{array}{cc} \{4_1, 0, 0, 1\} & \{4_1, 0, 1, 0\} \\ \{4_2, 0, 0, 1\} & \{4_2, 0, 1, 0\} \end{array} \tag{A.277}$$

Conjugacy class $\mathcal{C}_{12}\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 4

$$\begin{array}{cc} \{4_1, 1, 0, 0\} & \{4_1, 1, 1, 1\} \\ \{4_2, 1, 0, 0\} & \{4_2, 1, 1, 1\} \end{array} \tag{A.278}$$

Conjugacy class $\mathcal{C}_{13}\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 4

$$\begin{array}{cc} \{5_5, 0, 0, 1\} & \{5_5, 0, 1, 0\} \\ \{5_6, 0, 0, 1\} & \{5_6, 0, 1, 0\} \end{array} \tag{A.279}$$

Conjugacy class $\mathcal{C}_{14}\,(\mathrm{GS}_{32})$: $\#$ *of elements* = 4

$$\begin{array}{cc} \{5_5, 1, 0, 0\} & \{5_5, 1, 1, 1\} \\ \{5_6, 1, 0, 0\} & \{5_6, 1, 1, 1\} \end{array} \tag{A.280}$$

A.13. The Group GK_{32}**.** In this section, we list all the elements of the space group GK_{32}, organized into their 14 conjugacy classes.

Conjugacy class $\mathcal{C}_1\,(\mathrm{GK}_{32})$: # *of elements* = 1

$$\{1_1, 0, 0, 0\} \tag{A.281}$$

Conjugacy class $\mathcal{C}_2\,(\mathrm{GK}_{32})$: # *of elements* = 1

$$\{1_1, 1, 0, 1\} \tag{A.282}$$

Conjugacy class $\mathcal{C}_3\,(\mathrm{GK}_{32})$: # *of elements* = 1

$$\{3_2, 0, 1, 1\} \tag{A.283}$$

Conjugacy class $\mathcal{C}_4\,(\mathrm{GK}_{32})$: # *of elements* = 1

$$\{3_2, 1, 1, 0\} \tag{A.284}$$

Conjugacy class $\mathcal{C}_5\,(\mathrm{GK}_{32})$: # *of elements* = 2

$$\{1_1, 0, 1, 1\} \quad \{1_1, 1, 1, 0\} \tag{A.285}$$

Conjugacy class $\mathcal{C}_6\,(\mathrm{GK}_{32})$: # *of elements* = 2

$$\{3_1, 0, 0, 0\} \quad \{3_3, 0, 1, 1\} \tag{A.286}$$

Conjugacy class $\mathcal{C}_7\,(\mathrm{GK}_{32})$: # *of elements* = 2

$$\{3_1, 0, 1, 1\} \quad \{3_3, 1, 0, 1\} \tag{A.287}$$

Conjugacy class $\mathcal{C}_8\,(\mathrm{GK}_{32})$: # *of elements* = 2

$$\{3_1, 1, 0, 1\} \quad \{3_3, 1, 1, 0\} \tag{A.288}$$

Conjugacy class $\mathcal{C}_9\,(\mathrm{GK}_{32})$: # *of elements* = 2

$$\{3_1, 1, 1, 0\} \quad \{3_3, 0, 0, 0\} \tag{A.289}$$

Conjugacy class $\mathcal{C}_{10}\,(\mathrm{GK}_{32})$: # *of elements* = 2

$$\{3_2, 0, 0, 0\} \quad \{3_2, 1, 0, 1\} \tag{A.290}$$

Conjugacy class $\mathcal{C}_{11}\,(\mathrm{GK}_{32})$: # *of elements* = 4

$$\begin{array}{cc} \left\{4_4, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right\} & \left\{4_4, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right\} \\ \left\{4_5, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right\} & \left\{4_5, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right\} \end{array} \tag{A.291}$$

Conjugacy class $\mathcal{C}_{12}\,(\mathrm{GK}_{32})$: # *of elements* = 4

$$\begin{matrix} \left\{4_4, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right\} & \left\{4_4, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right\} \\ \left\{4_5, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right\} & \left\{4_5, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right\} \end{matrix} \tag{A.292}$$

Conjugacy class $\mathcal{C}_{13}\,(\mathrm{GK}_{32})$: # *of elements* = 4

$$\begin{matrix} \left\{5_2, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right\} & \left\{5_2, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right\} \\ \left\{5_3, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right\} & \left\{5_3, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right\} \end{matrix} \tag{A.293}$$

Conjugacy class $\mathcal{C}_{14}\,(\mathrm{GK}_{32})$: # *of elements* = 4

$$\begin{matrix} \left\{5_2, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right\} & \left\{5_2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right\} \\ \left\{5_3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\} & \left\{5_3, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right\} \end{matrix} \tag{A.294}$$

B. CHARACTER TABLES OF THE CONSIDERED DISCRETE GROUPS

In this section we present the results for the irreducible representations of the various groups listed in Sec. A and we display the character tables of each of them. As explained in the main text, the basis to obtain such results has been the implementation in a series of purposely written MATHEMATICA codes of the algorithm described in Subsubsecs. 5.3.1 and 5.3.2.

B.1. Character Table of the Group G_{1536}**.** The big ambient group G_{1536} has 37 conjugacy classes and therefore 37 irreducible representations that are distributed according to the following pattern:

a) 4 irreps of dimension 1, namely $D_1, \ldots, D_4$,

b) 2 irreps of dimension 2, namely $D_5, \ldots, D_6$,

c) 12 irreps of dimension 3, namely $D_6, \ldots, D_{18}$,

d) 10 irreps of dimension 6, namely $D_7, \ldots, D_{28}$,

e) 3 irreps of dimension 8, namely $D_{29}, \ldots, D_{31}$,

f) 6 irreps of dimension 12, namely $D_{32}, \ldots, D_{37}$.

The corresponding character table that we have calculated with the procedures described in the main text is displayed below. For pure typographical reasons we were forced to split the character table in two parts in order to fit it into the page.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_3	1	1	1	1	−1	−1	−1	1	−1	1	1	1	1	1	−1	−1	−1	−1
D_4	1	1	1	1	−1	−1	−1	1	−1	1	1	1	1	1	−1	−1	−1	−1
D_5	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
D_6	2	2	2	2	−2	−2	−2	2	−2	2	2	2	2	2	−2	−2	−2	−2
D_7	3	3	3	3	3	3	3	3	3	3	−1	−1	−1	−1	−1	−1	−1	−1
D_8	3	3	3	3	3	3	3	3	3	3	−1	−1	−1	−1	−1	−1	−1	−1
D_9	3	3	3	3	1	1	−3	−1	1	−1	3	3	−1	−1	1	1	1	−3
D_{10}	3	3	3	3	1	1	−3	−1	1	−1	3	3	−1	−1	1	1	1	−3
D_{11}	3	3	3	3	1	1	−3	−1	1	−1	−1	−1	3	3	−3	1	1	1
D_{12}	3	3	3	3	1	1	−3	−1	1	−1	−1	−1	3	3	−3	1	1	1
D_{13}	3	3	3	3	−1	−1	3	−1	−1	−1	3	3	−1	−1	−1	−1	−1	3
D_{14}	3	3	3	3	−1	−1	3	−1	−1	−1	3	3	−1	−1	−1	−1	−1	3
D_{15}	3	3	3	3	−1	−1	3	−1	−1	−1	−1	−1	3	3	3	−1	−1	−1
D_{16}	3	3	3	3	−1	−1	3	−1	−1	−1	−1	−1	3	3	3	−1	−1	−1
D_{17}	3	3	3	3	−3	−3	−3	3	−3	3	−1	−1	−1	−1	1	1	1	1
D_{18}	3	3	3	3	−3	−3	−3	3	−3	3	−1	−1	−1	−1	1	1	1	1
D_{19}	6	6	6	6	2	2	−6	−2	2	−2	−2	−2	−2	−2	2	−2	−2	2
D_{20}	6	6	6	6	−2	−2	6	−2	−2	−2	−2	−2	−2	−2	−2	2	2	−2
D_{21}	6	−6	2	−2	4	−4	0	2	0	−2	2	−2	2	−2	0	2	−2	0
D_{22}	6	−6	2	−2	4	−4	0	2	0	−2	2	−2	2	−2	0	2	−2	0
D_{23}	6	−6	2	−2	4	−4	0	2	0	−2	−2	2	−2	2	0	−2	2	0
D_{24}	6	−6	2	−2	4	−4	0	2	0	−2	−2	2	−2	2	0	−2	2	0
D_{25}	6	−6	2	−2	−4	4	0	2	0	−2	2	−2	2	−2	0	−2	2	0
D_{26}	6	−6	2	−2	−4	4	0	2	0	−2	2	−2	2	−2	0	−2	2	0
D_{27}	6	−6	2	−2	−4	4	0	2	0	−2	−2	2	−2	2	0	2	−2	0
D_{28}	6	−6	2	−2	−4	4	0	2	0	−2	−2	2	−2	2	0	2	−2	0
D_{29}	8	−8	−8	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{30}	8	−8	−8	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{31}	8	−8	−8	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{32}	12	12	−4	−4	4	4	0	0	−4	0	0	0	0	0	0	0	0	0
D_{33}	12	12	−4	−4	4	4	0	0	−4	0	0	0	0	0	0	0	0	0
D_{34}	12	12	−4	−4	−4	−4	0	0	4	0	0	0	0	0	0	0	0	0
D_{35}	12	12	−4	−4	−4	−4	0	0	4	0	0	0	0	0	0	0	0	0
D_{36}	12	−12	4	−4	0	0	0	−4	0	4	4	−4	−4	4	0	0	0	0
D_{37}	12	−12	4	−4	0	0	0	−4	0	4	−4	4	4	−4	0	0	0	0

(B.1)

0	C_{19}	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
D_3	1	−1	−1	1	1	−1	1	−1	1	1	−1	1	−1	1	−1	1	−1	1	−1
D_4	1	1	1	−1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	1	−1	1	−1
D_5	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1
D_6	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1	−1	1
D_7	−1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	0	0	0	0
D_8	−1	−1	−1	−1	−1	1	1	1	1	1	1	1	1	−1	−1	0	0	0	0
D_9	−1	1	1	−1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	0	0	0	0
D_{10}	−1	−1	−1	1	1	−1	1	−1	1	−1	1	−1	1	−1	1	0	0	0	0
D_{11}	−1	−1	−1	1	1	1	−1	1	−1	1	−1	1	−1	−1	1	0	0	0	0
D_{12}	−1	1	1	−1	−1	−1	1	−1	1	−1	1	−1	1	1	−1	0	0	0	0
D_{13}	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	0	0	0	0
D_{14}	−1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	0	0	0	0
D_{15}	−1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	1	0	0	0	0
D_{16}	−1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	0	0	0	0
D_{17}	−1	−1	−1	1	1	1	−1	1	−1	−1	1	−1	1	1	−1	0	0	0	0
D_{18}	−1	1	1	−1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	0	0	0	0
D_{19}	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{20}	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{21}	0	0	0	0	0	−2	0	2	0	−2	0	2	0	0	0	0	0	0	0
D_{22}	0	0	0	0	0	2	0	−2	0	2	0	−2	0	0	0	0	0	0	0
D_{23}	0	0	0	0	0	0	−2	0	2	0	−2	0	2	0	0	0	0	0	0
D_{24}	0	0	0	0	0	0	2	0	−2	0	2	0	−2	0	0	0	0	0	0
D_{25}	0	0	0	0	0	−2	0	2	0	2	0	−2	0	0	0	0	0	0	0
D_{26}	0	0	0	0	0	2	0	−2	0	−2	0	2	0	0	0	0	0	0	0
D_{27}	0	0	0	0	0	0	−2	0	2	0	2	0	−2	0	0	0	0	0	0
D_{28}	0	0	0	0	0	0	2	0	−2	0	−2	0	2	0	0	0	0	0	0
D_{29}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	−2	0
D_{30}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	$-\sqrt{3}$	1	$\sqrt{3}$
D_{31}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	$\sqrt{3}$	1	$-\sqrt{3}$
D_{32}	0	−2	2	−2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{33}	0	2	−2	2	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{34}	0	2	−2	−2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{35}	0	−2	2	2	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{36}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{37}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(B.2)

B.2. Character Table of the Group G_{768}**.** The group G_{768} has 32 conjugacy classes and therefore 32 irreducible representations that are distributed according to the following pattern:

a) 6 irreps of dimension 1, namely $D_1, \ldots, D_6$,

b) 10 irreps of dimension 3, namely $D_6, \ldots, D_{16}$,

c) 6 irreps of dimension 4, namely $D_{16}, \ldots, D_{22}$,

d) 8 irreps of dimension 6, namely $D_{23}, \ldots, D_{30}$,

e) 2 irreps of dimension 12, namely D_{31}, D_{32}.

The corresponding character table that we have calculated with the procedures described in the main text is displayed below. For pure typographical reasons we were forced to split the character table in two parts in order to fit it into the page.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_4	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1
D_5	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1
D_6	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1
D_7	3	3	3	3	3	3	3	3	3	3	3	3	-1	-1	-1	-1
D_8	3	3	3	3	-3	-3	1	1	1	1	-1	-1	3	3	1	1
D_9	3	3	3	3	-3	-3	1	1	1	1	-1	-1	-1	-1	1	1
D_{10}	3	3	3	3	-3	-3	1	1	1	1	-1	-1	-1	-1	1	1
D_{11}	3	3	3	3	-3	-3	1	1	1	1	-1	-1	-1	-1	-3	-3
D_{12}	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	3	3	-1	-1
D_{13}	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
D_{14}	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	3	3
D_{15}	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
D_{16}	3	3	3	3	-3	-3	-3	-3	-3	-3	3	3	-1	-1	1	1
D_{17}	4	-4	-4	4	$-4i$	$4i$	0	0	0	0	0	0	0	0	0	0
D_{18}	4	-4	-4	4	$-4i$	$4i$	0	0	0	0	0	0	0	0	0	0
D_{19}	4	-4	-4	4	$-4i$	$4i$	0	0	0	0	0	0	0	0	0	0
D_{20}	4	-4	-4	4	$4i$	$-4i$	0	0	0	0	0	0	0	0	0	0
D_{21}	4	-4	-4	4	$4i$	$-4i$	0	0	0	0	0	0	0	0	0	0
D_{22}	4	-4	-4	4	$4i$	$-4i$	0	0	0	0	0	0	0	0	0	0
D_{23}	6	-6	2	-2	0	0	4	0	0	-4	2	-2	2	-2	2	-2
D_{24}	6	-6	2	-2	0	0	4	0	0	-4	2	-2	-2	2	-2	2
D_{25}	6	-6	2	-2	0	0	0	4	-4	0	-2	2	2	-2	-2	2
D_{26}	6	-6	2	-2	0	0	0	4	-4	0	-2	2	-2	2	2	-2
D_{27}	6	-6	2	-2	0	0	0	-4	4	0	-2	2	2	-2	2	-2
D_{28}	6	-6	2	-2	0	0	0	-4	4	0	-2	2	-2	2	-2	2
D_{29}	6	-6	2	-2	0	0	-4	0	0	4	2	-2	2	-2	-2	2
D_{30}	6	-6	2	-2	0	0	-4	0	0	4	2	-2	-2	2	2	-2
D_{31}	12	12	-4	-4	0	0	4	-4	-4	4	0	0	0	0	0	0
D_{32}	12	12	-4	-4	0	0	-4	4	4	-4	0	0	0	0	0	0

(B.3)

0	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	$-\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$(-1)^{2/3}$	$(-1)^{2/3}$	$(-1)^{2/3}$	$(-1)^{2/3}$
D_3	1	1	1	1	1	1	1	1	$(-1)^{2/3}$	$(-1)^{2/3}$	$(-1)^{2/3}$	$(-1)^{2/3}$	$-\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$-\sqrt[3]{-1}$
D_4	-1	-1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1
D_5	-1	-1	1	1	-1	1	1	-1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$(-1)^{2/3}$	$-(-1)^{2/3}$	$(-1)^{2/3}$	$-(-1)^{2/3}$
D_6	-1	-1	1	1	-1	1	1	-1	$(-1)^{2/3}$	$-(-1)^{2/3}$	$(-1)^{2/3}$	$-(-1)^{2/3}$	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$
D_7	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0
D_8	1	1	-1	-1	1	-1	-1	-3	0	0	0	0	0	0	0	0
D_9	1	1	3	3	-3	-1	-1	1	0	0	0	0	0	0	0	0
D_{10}	-3	-3	-1	-1	1	3	-1	1	0	0	0	0	0	0	0	0
D_{11}	1	1	-1	-1	1	-1	3	1	0	0	0	0	0	0	0	0
D_{12}	-1	-1	-1	-1	-1	-1	-1	3	0	0	0	0	0	0	0	0
D_{13}	3	3	-1	-1	-1	3	-1	-1	0	0	0	0	0	0	0	0
D_{14}	-1	-1	-1	-1	-1	-1	3	-1	0	0	0	0	0	0	0	0
D_{15}	-1	-1	3	3	3	-1	-1	-1	0	0	0	0	0	0	0	0
D_{16}	1	1	-1	-1	1	-1	-1	1	0	0	0	0	0	0	0	0
D_{17}	0	0	0	0	0	0	0	0	1	$-i$	-1	i	1	$-i$	-1	i
D_{18}	0	0	0	0	0	0	0	0	$-\sqrt[3]{-1}$	$(-1)^{5/6}$	$\sqrt[3]{-1}$	$-(-1)^{5/6}$	$(-1)^{2/3}$	$\sqrt[6]{-1}$	$-(-1)^{2/3}$	$-\sqrt[6]{-1}$
D_{19}	0	0	0	0	0	0	0	0	$(-1)^{2/3}$	$\sqrt[6]{-1}$	$-(-1)^{2/3}$	$-\sqrt[6]{-1}$	$-\sqrt[3]{-1}$	$(-1)^{5/6}$	$\sqrt[3]{-1}$	$-(-1)^{5/6}$
D_{20}	0	0	0	0	0	0	0	0	1	i	-1	$-i$	1	i	-1	$-i$
D_{21}	0	0	0	0	0	0	0	0	$-\sqrt[3]{-1}$	$-(-1)^{5/6}$	$\sqrt[3]{-1}$	$(-1)^{5/6}$	$(-1)^{2/3}$	$-\sqrt[6]{-1}$	$-(-1)^{2/3}$	$\sqrt[6]{-1}$
D_{22}	0	0	0	0	0	0	0	0	$(-1)^{2/3}$	$-\sqrt[6]{-1}$	$-(-1)^{2/3}$	$\sqrt[6]{-1}$	$-\sqrt[3]{-1}$	$-(-1)^{5/6}$	$\sqrt[3]{-1}$	$(-1)^{5/6}$
D_{23}	2	-2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0
D_{24}	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0
D_{25}	2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0
D_{26}	-2	2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0
D_{27}	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0
D_{28}	2	-2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0
D_{29}	-2	2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0
D_{30}	2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0
D_{31}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{32}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(B.4)

B.3. Character Table of the Group G_{256}**.** The group G_{256} has 64 conjugacy classes and therefore 64 irreducible representations that are distributed according to the following pattern:

a) 32 irreps of dimension 1, namely $D_1, \ldots, D_{32}$,

b) 24 irreps of dimension 2, namely $D_{33}, \ldots, D_{56}$,

c) 8 irreps of dimension 4, namely $D_{57}, \ldots, D_{64}$.

The corresponding character table that we have calculated with the procedures described in the main text is displayed below. For pure typographical reasons we were forced to split the character table in three parts in order to fit it into the page.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_5	1	1	1	1	1	1	1	1	−1	1	1	−1	1	1	1	1	−1	1	1	−1	−1	−1
D_6	1	1	1	1	1	1	1	1	−1	1	1	−1	1	1	1	1	−1	1	1	−1	−1	−1
D_7	1	1	1	1	1	1	1	1	−1	1	1	−1	1	1	1	1	−1	1	1	−1	−1	−1
D_8	1	1	1	1	1	1	1	1	−1	1	1	−1	1	1	1	1	−1	1	1	−1	−1	−1
D_9	1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	1	−1	1
D_{10}	1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	1	−1	1
D_{11}	1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	1	−1	1
D_{12}	1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	1	−1	1
D_{13}	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	−1
D_{14}	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	−1
D_{15}	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	−1
D_{16}	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	−1
D_{17}	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	−1
D_{18}	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	−1
D_{19}	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	−1
D_{20}	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	−1
D_{21}	1	1	1	1	1	1	1	1	−1	1	1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	1
D_{22}	1	1	1	1	1	1	1	1	−1	1	1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	1
D_{23}	1	1	1	1	1	1	1	1	−1	1	1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	1
D_{24}	1	1	1	1	1	1	1	1	−1	1	1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	1
D_{25}	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	−1	−1	1	−1	−1
D_{26}	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	−1	−1	1	−1	−1
D_{27}	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	−1	−1	1	−1	−1
D_{28}	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	−1	−1	1	−1	−1
D_{29}	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1
D_{30}	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1
D_{31}	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1
D_{32}	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1
D_{33}	2	−2	2	−2	2	−2	2	−2	0	2	−2	0	2	−2	2	−2	0	2	−2	0	0	0
D_{34}	2	−2	2	−2	2	−2	2	−2	0	2	−2	0	2	−2	2	−2	0	2	−2	0	0	0

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}
D_{35}	2	−2	2	−2	2	−2	2	−2	0	−2	2	0	2	−2	2	−2	0	−2	2	0	0	0
D_{36}	2	−2	2	−2	2	−2	2	−2	0	−2	2	0	2	−2	2	−2	0	−2	2	0	0	0
D_{37}	2	−2	2	−2	2	−2	2	−2	0	2	−2	0	−2	2	−2	2	0	2	−2	0	0	0
D_{38}	2	−2	2	−2	2	−2	2	−2	0	2	−2	0	−2	2	−2	2	0	2	−2	0	0	0
D_{39}	2	−2	2	−2	2	−2	2	−2	0	−2	2	0	−2	2	−2	2	0	−2	2	0	0	0
D_{40}	2	−2	2	−2	2	−2	2	−2	0	−2	2	0	−2	2	−2	2	0	−2	2	0	0	0
D_{41}	2	2	−2	−2	2	2	−2	−2	2	0	0	−2	2	2	−2	−2	2	0	0	−2	0	2
D_{42}	2	2	−2	−2	2	2	−2	−2	2	0	0	−2	2	2	−2	−2	2	0	0	−2	0	2
D_{43}	2	2	−2	−2	2	2	−2	−2	−2	0	0	2	2	2	−2	−2	−2	0	0	2	0	−2
D_{44}	2	2	−2	−2	2	2	−2	−2	−2	0	0	2	2	2	−2	−2	−2	0	0	2	0	−2
D_{45}	2	2	2	2	−2	−2	−2	−2	2	2	2	2	0	0	0	0	−2	−2	−2	−2	2	0
D_{46}	2	2	2	2	−2	−2	−2	−2	2	2	2	2	0	0	0	0	−2	−2	−2	−2	2	0
D_{47}	2	2	2	2	−2	−2	−2	−2	−2	2	2	−2	0	0	0	0	2	−2	−2	2	−2	0
D_{48}	2	2	2	2	−2	−2	−2	−2	−2	2	2	−2	0	0	0	0	2	−2	−2	2	−2	0
D_{49}	2	2	2	2	−2	−2	−2	−2	2	−2	−2	2	0	0	0	0	−2	2	2	−2	−2	0
D_{50}	2	2	2	2	−2	−2	−2	−2	2	−2	−2	2	0	0	0	0	−2	2	2	−2	−2	0
D_{51}	2	2	2	2	−2	−2	−2	−2	−2	−2	−2	−2	0	0	0	0	2	2	2	2	2	0
D_{52}	2	2	2	2	−2	−2	−2	−2	−2	−2	−2	−2	0	0	0	0	2	2	2	2	2	0
D_{53}	2	2	−2	−2	2	2	−2	−2	2	0	0	−2	−2	−2	2	2	2	0	0	−2	0	−2
D_{54}	2	2	−2	−2	2	2	−2	−2	2	0	0	−2	−2	−2	2	2	2	0	0	−2	0	−2
D_{55}	2	2	−2	−2	2	2	−2	−2	−2	0	0	2	−2	−2	2	2	−2	0	0	2	0	2
D_{56}	2	2	−2	−2	2	2	−2	−2	−2	0	0	2	−2	−2	2	2	−2	0	0	2	0	2
D_{57}	4	−4	−4	4	4	−4	−4	4	0	0	0	0	4	−4	−4	4	0	0	0	0	0	0
D_{58}	4	−4	4	−4	−4	4	−4	4	0	4	−4	0	0	0	0	0	0	−4	4	0	0	0
D_{59}	4	4	−4	−4	−4	−4	4	4	4	0	0	−4	0	0	0	0	−4	0	0	4	0	0
D_{60}	4	−4	−4	4	−4	4	4	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{61}	4	4	−4	−4	−4	−4	4	4	−4	0	0	4	0	0	0	0	4	0	0	−4	0	0
D_{62}	4	−4	−4	4	−4	4	4	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{63}	4	−4	4	−4	−4	4	−4	4	0	−4	4	0	0	0	0	0	0	4	−4	0	0	0
D_{64}	4	−4	−4	4	4	−4	−4	4	0	0	0	0	−4	4	4	−4	0	0	0	0	0	0

(B.5)

0	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	C_{38}	C_{39}	C_{40}	C_{41}	C_{42}	C_{43}	C_{44}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
D_3	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	1	1
D_4	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
D_5	1	−1	1	−1	−1	−1	1	1	1	1	1	1	1	1	1	−1	1	−1	1	−1	1	−1
D_6	1	−1	1	−1	−1	−1	1	1	1	1	1	1	1	1	−1	1	−1	1	−1	1	−1	1
D_7	1	−1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1	1	−1	1	−1
D_8	1	−1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1	1	−1	1
D_9	−1	−1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	−1	1	1	1	1	1	1	1	1
D_{10}	−1	−1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
D_{11}	−1	−1	−1	−1	1	−1	−1	−1	1	1	−1	−1	1	1	1	1	1	1	1	1	1	1
D_{12}	−1	−1	−1	−1	1	−1	−1	−1	1	1	−1	−1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
D_{13}	−1	1	−1	1	−1	1	1	1	−1	−1	1	1	−1	−1	1	−1	1	−1	1	−1	1	−1

0	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	C_{38}	C_{39}	C_{40}	C_{41}	C_{42}	C_{43}	C_{44}
D_{14}	−1	1	−1	1	−1	1	1	1	−1	−1	1	1	−1	−1	−1	1	−1	1	−1	1	−1	1
D_{15}	−1	1	−1	1	−1	1	−1	−1	1	1	−1	−1	1	1	1	−1	1	−1	1	−1	1	−1
D_{16}	−1	1	−1	1	−1	1	−1	−1	1	1	−1	−1	1	1	−1	1	−1	1	−1	1	−1	1
D_{17}	−1	−1	−1	−1	−1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
D_{18}	−1	−1	−1	−1	−1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
D_{19}	−1	−1	−1	−1	−1	1	−1	−1	−1	−1	1	1	1	1	1	1	1	1	−1	−1	−1	−1
D_{20}	−1	−1	−1	−1	−1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	1	1	1
D_{21}	−1	1	−1	1	1	−1	1	1	1	1	−1	−1	−1	−1	1	−1	1	−1	−1	1	−1	1
D_{22}	−1	1	−1	1	1	−1	1	1	1	1	−1	−1	−1	−1	−1	1	−1	1	1	−1	1	−1
D_{23}	−1	1	−1	1	1	−1	−1	−1	−1	−1	1	1	1	1	1	−1	1	−1	−1	1	−1	1
D_{24}	−1	1	−1	1	1	−1	−1	−1	−1	−1	1	1	1	1	−1	1	−1	1	1	−1	1	−1
D_{25}	1	1	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	−1	−1	−1	−1
D_{26}	1	1	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	−1	−1	1	1	1	1
D_{27}	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	1	1	1	1	−1	−1	−1	−1
D_{28}	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1	1
D_{29}	1	−1	1	−1	1	1	1	1	−1	−1	−1	−1	1	1	1	−1	1	−1	−1	1	−1	1
D_{30}	1	−1	1	−1	1	1	1	1	−1	−1	−1	−1	1	1	−1	1	−1	1	1	−1	1	−1
D_{31}	1	−1	1	−1	1	1	−1	−1	1	1	1	1	−1	−1	1	−1	1	−1	−1	1	−1	1
D_{32}	1	−1	1	−1	1	1	−1	−1	1	1	1	1	−1	−1	−1	1	−1	1	1	−1	1	−1
D_{33}	2	0	−2	0	0	0	2	−2	2	−2	2	−2	2	−2	0	0	0	0	0	0	0	0
D_{34}	2	0	−2	0	0	0	−2	2	−2	2	−2	2	−2	2	0	0	0	0	0	0	0	0
D_{35}	−2	0	2	0	0	0	2	−2	−2	2	2	−2	−2	2	0	0	0	0	0	0	0	0
D_{36}	−2	0	2	0	0	0	−2	2	2	−2	−2	2	2	−2	0	0	0	0	0	0	0	0
D_{37}	−2	0	2	0	0	0	2	−2	2	−2	−2	2	−2	2	0	0	0	0	0	0	0	0
D_{38}	−2	0	2	0	0	0	−2	2	−2	2	2	−2	2	−2	0	0	0	0	0	0	0	0
D_{39}	2	0	−2	0	0	0	2	−2	−2	2	−2	2	2	−2	0	0	0	0	0	0	0	0
D_{40}	2	0	−2	0	0	0	−2	2	2	−2	2	−2	−2	2	0	0	0	0	0	0	0	0
D_{41}	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	−2	−2	2	2	−2	−2	2	2
D_{42}	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	2	2	−2	−2	2	2	−2	−2
D_{43}	0	0	0	0	2	0	0	0	0	0	0	0	0	0	−2	2	2	−2	−2	2	2	−2
D_{44}	0	0	0	0	2	0	0	0	0	0	0	0	0	0	2	−2	−2	2	2	−2	−2	2
D_{45}	0	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{46}	0	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{47}	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{48}	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{49}	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{50}	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{51}	0	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{52}	0	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{53}	0	0	0	0	2	0	0	0	0	0	0	0	0	0	−2	−2	2	2	2	2	−2	−2
D_{54}	0	0	0	0	2	0	0	0	0	0	0	0	0	0	2	2	−2	−2	−2	−2	2	2
D_{55}	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	−2	2	2	−2	2	−2	−2	2
D_{56}	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	2	−2	−2	2	−2	2	2	−2
D_{57}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{58}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{59}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

0	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	C_{38}	C_{39}	C_{40}	C_{41}	C_{42}	C_{43}	C_{44}
D_{60}	0	$-4i$	0	$4i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{61}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{62}	0	$4i$	0	$-4i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{63}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(B.6)

0	C_{45}	C_{46}	C_{47}	C_{48}	C_{49}	C_{50}	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	C_{57}	C_{58}	C_{59}	C_{60}	C_{61}	C_{62}	C_{63}	C_{64}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
D_3	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1
D_4	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
D_5	1	−1	1	−1	−1	−1	1	1	1	1	1	1	1	1	1	−1	1	−1	1	−1
D_6	1	−1	1	−1	−1	−1	1	1	1	1	1	1	1	1	−1	1	−1	1	−1	1
D_7	1	−1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1	1	−1
D_8	1	−1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1	1
D_9	−1	−1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	−1	1	1	1	1	1	1
D_{10}	−1	−1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
D_{11}	−1	−1	−1	−1	1	−1	−1	−1	1	1	−1	−1	1	1	1	1	1	1	1	1
D_{12}	−1	−1	−1	−1	1	−1	−1	−1	1	1	−1	−1	1	1	−1	−1	−1	−1	−1	−1
D_{13}	−1	1	−1	1	−1	1	1	1	−1	−1	1	1	−1	−1	1	−1	1	−1	1	−1
D_{14}	−1	1	−1	1	−1	1	1	1	−1	−1	1	1	−1	−1	−1	1	−1	1	−1	1
D_{15}	−1	1	−1	1	−1	1	−1	−1	1	1	−1	−1	1	1	1	−1	1	−1	1	−1
D_{16}	−1	1	−1	1	−1	1	−1	−1	1	1	−1	−1	1	1	−1	1	−1	1	−1	1
D_{17}	−1	−1	−1	−1	−1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1
D_{18}	−1	−1	−1	−1	−1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1
D_{19}	−1	−1	−1	−1	−1	1	−1	−1	−1	−1	1	1	1	1	1	1	1	1	−1	−1
D_{20}	−1	−1	−1	−1	−1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	1
D_{21}	−1	1	−1	1	1	−1	1	1	1	1	−1	−1	−1	−1	1	−1	1	−1	−1	1
D_{22}	−1	1	−1	1	1	−1	1	1	1	1	−1	−1	−1	−1	−1	1	−1	1	1	−1
D_{23}	−1	1	−1	1	1	−1	−1	−1	−1	−1	1	1	1	1	1	−1	1	−1	−1	1
D_{24}	−1	1	−1	1	1	−1	−1	−1	−1	−1	1	1	1	1	−1	1	−1	1	1	−1
D_{25}	1	1	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	−1	−1
D_{26}	1	1	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	−1	−1	1	1
D_{27}	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	1	1	1	1	−1	−1
D_{28}	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1
D_{29}	1	−1	1	−1	1	1	1	1	−1	−1	−1	−1	1	1	1	−1	1	−1	−1	1
D_{30}	1	−1	1	−1	1	1	1	1	−1	−1	−1	−1	1	1	−1	1	−1	1	1	−1
D_{31}	1	−1	1	−1	1	1	−1	−1	1	1	1	1	−1	−1	1	−1	1	−1	−1	1
D_{32}	1	−1	1	−1	1	1	−1	−1	1	1	1	1	−1	−1	−1	1	−1	1	1	−1
D_{33}	2	0	−2	0	0	0	2	−2	2	−2	2	−2	2	−2	0	0	0	0	0	0
D_{34}	2	0	−2	0	0	0	−2	2	−2	2	−2	2	−2	2	0	0	0	0	0	0
D_{35}	−2	0	2	0	0	0	2	−2	−2	2	2	−2	−2	2	0	0	0	0	0	0

0	C_{45}	C_{46}	C_{47}	C_{48}	C_{49}	C_{50}	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	C_{57}	C_{58}	C_{59}	C_{60}	C_{61}	C_{62}	C_{63}	C_{64}
D_{36}	-2	0	2	0	0	0	-2	2	2	-2	-2	2	2	-2	0	0	0	0	0	0
D_{37}	-2	0	2	0	0	0	2	-2	2	-2	-2	2	-2	2	0	0	0	0	0	0
D_{38}	-2	0	2	0	0	0	-2	2	-2	2	2	-2	2	-2	0	0	0	0	0	0
D_{39}	2	0	-2	0	0	0	2	-2	-2	2	-2	2	2	-2	0	0	0	0	0	0
D_{40}	2	0	-2	0	0	0	-2	2	2	-2	2	-2	-2	2	0	0	0	0	0	0
D_{41}	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	-2	-2	2	2	-2	-2
D_{42}	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	2	2	-2	-2	2	2
D_{43}	0	0	0	0	2	0	0	0	0	0	0	0	0	0	-2	2	2	-2	-2	2
D_{44}	0	0	0	0	2	0	0	0	0	0	0	0	0	0	2	-2	-2	2	2	-2
D_{45}	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{46}	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{47}	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{48}	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{49}	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{50}	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{51}	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{52}	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{53}	0	0	0	0	2	0	0	0	0	0	0	0	0	0	-2	-2	2	2	2	2
D_{54}	0	0	0	0	2	0	0	0	0	0	0	0	0	0	2	2	-2	-2	-2	-2
D_{55}	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	-2	2	2	-2	2	-2
D_{56}	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	2	-2	-2	2	-2	2
D_{57}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{58}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{59}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{60}	0	$-4i$	0	$4i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{61}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{62}	0	$4i$	0	$-4i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{63}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(B.7)

B.4. Character Table of the Group G_{128}**.** The group G_{128} has 56 conjugacy classes and therefore 56 irreducible representations that are distributed according to the following pattern:

a) 32 irreps of dimension 1, namely $D_1, \dots, D_{32}$,

b) 24 irreps of dimension 2, namely $D_{33}, \dots, D_{56}$.

The corresponding character table that we have calculated with the procedures described in the main text is displayed below. For pure typographical reasons we were forced to split the character table in three parts in order to fit it into the page.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_3	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}
D_4	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1
D_5	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
D_6	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
D_7	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1
D_8	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1
D_9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1
D_{10}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1
D_{11}	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	-1	$-i$	1
D_{12}	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	-1	$-i$	1
D_{13}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1
D_{14}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1
D_{15}	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	-1	i	1
D_{16}	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	-1	i	1
D_{17}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_{18}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_{19}	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1
D_{20}	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1
D_{21}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
D_{22}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
D_{23}	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1
D_{24}	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1
D_{25}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1
D_{26}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1
D_{27}	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	-1	$-i$	1
D_{28}	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	-1	$-i$	1
D_{29}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1
D_{30}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1
D_{31}	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	-1	i	1
D_{32}	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	-1	i	1
D_{33}	2	2	2	2	-2	-2	-2	-2	2	2	2	2	-2	-2	-2	-2	0	0	0
D_{34}	2	$2i$	-2	$-2i$	-2	$-2i$	2	$2i$	2	$2i$	-2	$-2i$	-2	$-2i$	2	$2i$	0	0	0
D_{35}	2	-2	2	-2	-2	2	-2	2	2	-2	2	-2	-2	2	-2	2	0	0	0
D_{36}	2	$-2i$	-2	$2i$	-2	$2i$	2	$-2i$	2	$-2i$	-2	$2i$	-2	$2i$	2	$-2i$	0	0	0
D_{37}	2	2	2	2	2	2	2	2	-2	-2	-2	-2	-2	-2	-2	-2	2	2	2
D_{38}	2	$2i$	-2	$-2i$	2	$2i$	-2	$-2i$	-2	$-2i$	2	$2i$	-2	$-2i$	2	$2i$	2	$2i$	-2
D_{39}	2	-2	2	-2	2	-2	2	-2	-2	2	-2	2	-2	2	-2	2	2	-2	2
D_{40}	2	$-2i$	-2	$2i$	2	$-2i$	-2	$2i$	-2	$2i$	2	$-2i$	-2	$2i$	2	$-2i$	2	$-2i$	-2
D_{41}	2	2	2	2	-2	-2	-2	-2	-2	-2	-2	-2	2	2	2	2	0	0	0
D_{42}	2	$2i$	-2	$-2i$	-2	$-2i$	2	$2i$	-2	$-2i$	2	$2i$	2	$2i$	-2	$-2i$	0	0	0
D_{43}	2	-2	2	-2	-2	2	-2	2	-2	2	-2	2	2	-2	2	-2	0	0	0
D_{44}	2	$-2i$	-2	$2i$	-2	$2i$	2	$-2i$	-2	$2i$	2	$-2i$	2	$-2i$	-2	$2i$	0	0	0
D_{45}	2	2	2	2	2	2	2	2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
D_{46}	2	$2i$	-2	$-2i$	2	$2i$	-2	$-2i$	-2	$-2i$	2	$2i$	-2	$-2i$	2	$2i$	-2	$-2i$	2
D_{47}	2	-2	2	-2	2	-2	2	-2	-2	2	-2	2	-2	2	-2	2	-2	2	-2
D_{48}	2	$-2i$	-2	$2i$	2	$-2i$	-2	$2i$	-2	$2i$	2	$-2i$	-2	$2i$	2	$-2i$	-2	$2i$	2
D_{49}	2	2	2	2	-2	-2	-2	-2	-2	-2	-2	-2	2	2	2	2	0	0	0
D_{50}	2	$2i$	-2	$-2i$	-2	$-2i$	2	$2i$	-2	$-2i$	2	$2i$	2	$2i$	-2	$-2i$	0	0	0

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}
D_{51}	2	-2	2	-2	-2	2	-2	2	-2	2	-2	2	2	-2	2	-2	0	0	0
D_{52}	2	$-2i$	-2	$2i$	-2	$2i$	2	$-2i$	-2	$2i$	2	$-2i$	2	$-2i$	-2	$2i$	0	0	0
D_{53}	2	2	2	2	-2	-2	-2	-2	2	2	2	2	-2	-2	-2	-2	0	0	0
D_{54}	2	$2i$	-2	$-2i$	-2	$-2i$	2	$2i$	2	$2i$	-2	$-2i$	-2	$-2i$	2	$2i$	0	0	0
D_{55}	2	-2	2	-2	-2	2	-2	2	2	-2	2	-2	-2	2	-2	2	0	0	0
D_{56}	2	$-2i$	-2	$2i$	-2	$2i$	2	$-2i$	2	$-2i$	-2	$2i$	-2	$2i$	2	$-2i$	0	0	0

(B.8)

0	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	C_{38}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_3	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i
D_4	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i
D_5	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
D_6	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
D_7	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$
D_8	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$
D_9	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1
D_{10}	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1
D_{11}	i	1	i	-1	$-i$	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$	1	i	-1	$-i$
D_{12}	i	1	i	-1	$-i$	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$	1	i	-1	$-i$
D_{13}	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1
D_{14}	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1
D_{15}	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	-1	i
D_{16}	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	-1	i
D_{17}	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1
D_{18}	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1
D_{19}	$-i$	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	1	i
D_{20}	$-i$	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	1	i
D_{21}	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1
D_{22}	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1
D_{23}	i	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	1	$-i$
D_{24}	i	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	1	$-i$
D_{25}	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1
D_{26}	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1
D_{27}	i	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$
D_{28}	i	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$
D_{29}	1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
D_{30}	1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
D_{31}	$-i$	-1	i	1	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	1	$-i$	-1	i	-1	i
D_{32}	$-i$	-1	i	1	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	1	$-i$	-1	i	-1	i
D_{33}	0	2	2	2	2	0	0	0	0	-2	-2	-2	-2	0	0	0	0	0	0
D_{34}	0	2	$2i$	-2	$-2i$	0	0	0	0	-2	$-2i$	2	$2i$	0	0	0	0	0	0
D_{35}	0	2	-2	2	-2	0	0	0	0	-2	2	-2	2	0	0	0	0	0	0
D_{36}	0	2	$-2i$	-2	$2i$	0	0	0	0	-2	$2i$	2	$-2i$	0	0	0	0	0	0

0	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	C_{38}
D_{37}	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	-2
D_{38}	$-2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	$-2i$
D_{39}	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	2
D_{40}	$2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	$2i$
D_{41}	0	0	0	0	0	-2	-2	-2	-2	0	0	0	0	2	2	2	2	0	0
D_{42}	0	0	0	0	0	-2	$-2i$	2	$2i$	0	0	0	0	2	$2i$	-2	$-2i$	0	0
D_{43}	0	0	0	0	0	-2	2	-2	2	0	0	0	0	2	-2	2	-2	0	0
D_{44}	0	0	0	0	0	-2	$2i$	2	$-2i$	0	0	0	0	2	$-2i$	-2	$2i$	0	0
D_{45}	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2
D_{46}	$2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	$2i$
D_{47}	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	-2
D_{48}	$-2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	$-2i$
D_{49}	0	0	0	0	0	2	2	2	2	0	0	0	0	-2	-2	-2	-2	0	0
D_{50}	0	0	0	0	0	2	$2i$	-2	$-2i$	0	0	0	0	-2	$-2i$	2	$2i$	0	0
D_{51}	0	0	0	0	0	2	-2	2	-2	0	0	0	0	-2	2	-2	2	0	0
D_{52}	0	0	0	0	0	2	$-2i$	-2	$2i$	0	0	0	0	-2	$2i$	2	$-2i$	0	0
D_{53}	0	-2	-2	-2	-2	0	0	0	0	2	2	2	2	0	0	0	0	0	0
D_{54}	0	-2	$-2i$	2	$2i$	0	0	0	0	2	$2i$	-2	$-2i$	0	0	0	0	0	0
D_{55}	0	-2	2	-2	2	0	0	0	0	2	-2	2	-2	0	0	0	0	0	0
D_{56}	0	-2	$2i$	2	$-2i$	0	0	0	0	2	$-2i$	-2	$2i$	0	0	0	0	0	0

(B.9)

0	C_{39}	C_{40}	C_{41}	C_{42}	C_{43}	C_{44}	C_{45}	C_{46}	C_{47}	C_{48}	C_{49}	C_{50}	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	0
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
D_2	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0
D_3	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	0
D_4	-1	$-i$	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1	i	0
D_5	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	0
D_6	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	0
D_7	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	0
D_8	-1	i	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	1	$-i$	0
D_9	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	0
D_{10}	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	0
D_{11}	1	i	1	i	-1	$-i$	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$	1	i	0
D_{12}	1	i	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$	1	i	1	i	-1	$-i$	0
D_{13}	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	0
D_{14}	-1	1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	0
D_{15}	1	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	0
D_{16}	1	$-i$	-1	i	1	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	1	$-i$	-1	i	0
D_{17}	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	0
D_{18}	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	0
D_{19}	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	-1	$-i$	1	i	-1	$-i$	1	i	0
D_{20}	-1	$-i$	-1	$-i$	1	i	-1	$-i$	1	i	1	i	-1	$-i$	1	i	-1	$-i$	0
D_{21}	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	0
D_{22}	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	0
D_{23}	-1	i	1	$-i$	-1	i	1	$-i$	-1	i	-1	i	1	$-i$	-1	i	1	$-i$	0
D_{24}	-1	i	-1	i	1	$-i$	-1	i	1	$-i$	1	$-i$	-1	i	1	$-i$	-1	i	0

0	C_{39}	C_{40}	C_{41}	C_{42}	C_{43}	C_{44}	C_{45}	C_{46}	C_{47}	C_{48}	C_{49}	C_{50}	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	0
D_{25}	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	0
D_{26}	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	0
D_{27}	1	i	1	i	-1	$-i$	-1	$-i$	1	i	-1	$-i$	1	i	1	i	-1	$-i$	0
D_{28}	1	i	-1	$-i$	1	i	1	i	-1	$-i$	1	i	-1	$-i$	-1	$-i$	1	i	0
D_{29}	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	0
D_{30}	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	0
D_{31}	1	$-i$	1	$-i$	-1	i	-1	i	1	$-i$	-1	i	1	$-i$	1	$-i$	-1	i	0
D_{32}	1	$-i$	-1	i	1	$-i$	1	$-i$	-1	i	1	$-i$	-1	i	-1	i	1	$-i$	0
D_{33}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{34}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{35}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{36}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{37}	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{38}	2	$2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{39}	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{40}	2	$-2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{41}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{42}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{43}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{44}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{45}	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{46}	-2	$-2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{47}	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{48}	-2	$2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{49}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{50}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{51}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{52}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{53}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{54}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{55}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_{56}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(B.10)

B.5. Character Table of the Group G_{64}**.** The group G_{64} is Abelian and has order 64. Hence it has exactly 64 conjugacy classes and 64 irreducible 1-dimensional representations. The corresponding character table that we have calculated with the procedures described in the main text is displayed below. For pure typographical reasons we were forced to split the character table in three parts in order to fit it into the page. In the formulae below $\omega = \exp\left[\frac{2\pi}{3}\,\mathrm{i}\right]$ is a cubic root of the unity.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
D_2	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	0	0	0
D_3	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	0	0	0
D_4	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	0	0	0
D_5	1	1	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	1	1	1	0	0	0
D_6	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	1	ω	ω^2	0	0	0
D_7	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	1	ω^2	1	0	0	0
D_8	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	1	ω^3	ω^2	0	0	0
D_9	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	0	0	0
D_{10}	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	0	0	0

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}
D_{11}	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	0	0	0
D_{12}	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	0	0	0
D_{13}	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	1	1	1	0	0	0
D_{14}	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	1	ω	ω^2	0	0	0
D_{15}	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	1	ω^2	1	0	0	0
D_{16}	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	1	ω^3	ω^2	0	0	0
D_{17}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	ω	ω	ω	0	0	0
D_{18}	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	ω	ω^2	ω^3	0	0	0
D_{19}	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	ω	ω^3	ω	0	0	0
D_{20}	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	ω	1	ω^3	0	0	0
D_{21}	1	1	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	ω	ω	ω	0	0	0
D_{22}	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	ω	ω^2	ω^3	0	0	0
D_{23}	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	ω	ω^3	ω	0	0	0
D_{24}	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	ω	1	ω^3	0	0	0
D_{25}	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	0	0	0
D_{26}	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	ω	ω^2	ω^3	0	0	0
D_{27}	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1	ω	ω^3	ω	0	0	0
D_{28}	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	ω	1	ω^3	0	0	0
D_{29}	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	ω	ω	ω	0	0	0
D_{30}	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	ω	ω^2	ω^3	0	0	0
D_{31}	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	ω	ω^3	ω	0	0	0
D_{32}	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	ω	1	ω^3	0	0	0
D_{33}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	ω^2	ω^2	ω^2	0	0	0
D_{34}	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	ω^2	ω^3	1	0	0	0
D_{35}	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	ω^2	1	ω^2	0	0	0
D_{36}	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	ω^2	ω	1	0	0	0
D_{37}	1	1	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	0	0	0
D_{38}	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	ω^2	ω^3	1	0	0	0
D_{39}	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	ω^2	1	ω^2	0	0	0
D_{40}	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	ω^2	ω	1	0	0	0
D_{41}	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	0	0	0
D_{42}	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	ω^2	ω^3	1	0	0	0
D_{43}	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1	ω^2	1	ω^2	0	0	0
D_{44}	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	ω^2	ω	1	0	0	0
D_{45}	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	ω^2	ω^2	ω^2	0	0	0
D_{46}	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	ω^2	ω^3	1	0	0	0
D_{47}	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	ω^2	1	ω^2	0	0	0
D_{48}	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	ω^2	ω	1	0	0	0
D_{49}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	ω^3	ω^3	ω^3	0	0	0
D_{50}	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	ω^3	1	ω	0	0	0
D_{51}	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	ω^3	ω	ω^3	0	0	0
D_{52}	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	ω^3	ω^2	ω	0	0	0
D_{53}	1	1	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	0	0	0
D_{54}	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	ω^3	1	ω	0	0	0
D_{55}	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	0	0	0
D_{56}	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	ω^3	ω^2	ω	0	0	0

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}
D_{57}	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	0	0	0
D_{58}	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	ω^3	1	ω	0	0	0
D_{59}	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1	ω^3	ω	ω^3	0	0	0
D_{60}	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	ω^3	ω^2	ω	0	0	0
D_{61}	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	ω^3	ω^3	ω^3	0	0	0
D_{62}	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	ω^3	1	ω	0	0	0
D_{63}	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	ω^3	ω	ω^3	0	0	0
D_{64}	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	ω^3	ω^2	ω	0	0	0

(B.11)

0	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	C_{38}	C_{39}	C_{40}	C_{41}	C_{42}	C_{43}	C_{44}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3
D_3	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2
D_4	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω
D_5	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	1	1	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2
D_6	ω^3	1	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω
D_7	ω	ω^3	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1
D_8	ω^3	ω^2	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3
D_9	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1
D_{10}	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3
D_{11}	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2
D_{12}	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω
D_{13}	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2
D_{14}	ω	ω^2	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω
D_{15}	ω^3	ω	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1
D_{16}	ω	1	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3
D_{17}	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
D_{18}	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω
D_{19}	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1
D_{20}	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3
D_{21}	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	1	1	1	1	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	1	1	1	1
D_{22}	1	ω	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3
D_{23}	ω^2	1	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2
D_{24}	1	ω^3	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω
D_{25}	ω^3	ω^3	ω	ω	ω	ω	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2
D_{26}	ω	ω^2	ω	ω^2	ω^3	1	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω
D_{27}	ω^3	ω	ω	ω^3	ω	ω^3	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1
D_{28}	ω	1	ω	1	ω^3	ω^2	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3
D_{29}	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	1	1	1	1
D_{30}	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3
D_{31}	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2
D_{32}	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω
D_{33}	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	1	1	1	1	1	1	1	1	1	1	1	1
D_{34}	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3

0	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	C_{38}	C_{39}	C_{40}	C_{41}	C_{42}	C_{43}	C_{44}
D_{35}	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2
D_{36}	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω
D_{37}	ω^3	ω^3	1	1	1	1	ω	ω	ω	ω	1	1	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2
D_{38}	ω	ω^2	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω
D_{39}	ω^3	ω	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1
D_{40}	ω	1	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3
D_{41}	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1
D_{42}	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3
D_{43}	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2
D_{44}	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω
D_{45}	ω	ω	1	1	1	1	ω^3	ω^3	ω^3	ω^3	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2
D_{46}	ω^3	1	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω
D_{47}	ω	ω^3	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1
D_{48}	ω^3	ω^2	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3
D_{49}	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
D_{50}	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω
D_{51}	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1
D_{52}	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3
D_{53}	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	1	1	1	1
D_{54}	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3
D_{55}	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2
D_{56}	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω
D_{57}	ω	ω	ω^3	ω^3	ω^3	ω^3	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2
D_{58}	ω^3	1	ω^3	1	ω	ω^2	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω
D_{59}	ω	ω^3	ω^3	ω	ω^3	ω	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1
D_{60}	ω^3	ω^2	ω^3	ω^2	ω	1	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3
D_{61}	ω^2	ω^2	ω	ω	ω	ω	1	1	1	1	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	1	1	1	1
D_{62}	1	ω	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3
D_{63}	ω^2	1	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2
D_{64}	1	ω^3	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω

(B.12)

0	C_{45}	C_{46}	C_{47}	C_{48}	C_{49}	C_{50}	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	C_{57}	C_{58}	C_{59}	C_{60}	C_{61}	C_{62}	C_{63}	C_{64}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3
D_3	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2
D_4	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω
D_5	ω^3	ω^3	ω^3	ω^3	1	1	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3
D_6	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2
D_7	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω
D_8	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1
D_9	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2
D_{10}	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω
D_{11}	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1
D_{12}	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3
D_{13}	ω	ω	ω	ω	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω

0	C_{45}	C_{46}	C_{47}	C_{48}	C_{49}	C_{50}	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	C_{57}	C_{58}	C_{59}	C_{60}	C_{61}	C_{62}	C_{63}	C_{64}
D_{14}	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1
D_{15}	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3
D_{16}	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2
D_{17}	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3
D_{18}	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2
D_{19}	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω
D_{20}	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1
D_{21}	ω	ω	ω	ω	ω^3	ω^3	ω^3	ω^3	1	1	1	1	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2
D_{22}	ω	ω^2	ω^3	1	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω
D_{23}	ω	ω^3	ω	ω^3	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1
D_{24}	ω	1	ω^3	ω^2	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3
D_{25}	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω	ω	ω	ω	ω^3	ω^3	ω^3	ω^3	ω	ω	ω	ω
D_{26}	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω	ω^2	ω^3	1	ω^3	1	ω	ω^2	ω	ω^2	ω^3	1
D_{27}	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω	ω^3	ω	ω^3	ω^3	ω	ω^3	ω	ω	ω^3	ω	ω^3
D_{28}	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω	1	ω^3	ω^2	ω^3	ω^2	ω	1	ω	1	ω^3	ω^2
D_{29}	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	1	1	1	1
D_{30}	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3
D_{31}	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2
D_{32}	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω
D_{33}	1	1	1	1	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
D_{34}	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω
D_{35}	1	ω^2	1	ω^2	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1	ω^2	1
D_{36}	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3
D_{37}	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	1	1	1	1	ω	ω	ω	ω
D_{38}	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1
D_{39}	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3
D_{40}	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2
D_{41}	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	1	1	1	1	ω^2	ω^2	ω^2	ω^2	1	1	1	1
D_{42}	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3	ω^2	ω^3	1	ω	1	ω	ω^2	ω^3
D_{43}	ω^2	1	ω^2	1	ω^2	1	ω^2	1	1	ω^2	1	ω^2	ω^2	1	ω^2	1	1	ω^2	1	ω^2
D_{44}	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω	ω^2	ω	1	ω^3	1	ω^3	ω^2	ω
D_{45}	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	1	1	1	1	ω^3	ω^3	ω^3	ω^3
D_{46}	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2
D_{47}	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω
D_{48}	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1
D_{49}	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
D_{50}	ω^2	ω^3	1	ω	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1
D_{51}	ω^2	1	ω^2	1	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3
D_{52}	ω^2	ω	1	ω^3	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2
D_{53}	ω	ω	ω	ω	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3	1	1	1	1
D_{54}	ω	ω^2	ω^3	1	ω	ω^2	ω^3	1	ω^2	ω^3	1	ω	ω^3	1	ω	ω^2	1	ω	ω^2	ω^3
D_{55}	ω	ω^3	ω	ω^3	ω	ω^3	ω	ω^3	ω^2	1	ω^2	1	ω^3	ω	ω^3	ω	1	ω^2	1	ω^2
D_{56}	ω	1	ω^3	ω^2	ω	1	ω^3	ω^2	ω^2	ω	1	ω^3	ω^3	ω^2	ω	1	1	ω^3	ω^2	ω
D_{57}	1	1	1	1	ω	ω	ω	ω	ω^3	ω^3	ω^3	ω^3	ω	ω	ω	ω	ω^3	ω^3	ω^3	ω^3
D_{58}	1	ω	ω^2	ω^3	ω	ω^2	ω^3	1	ω^3	1	ω	ω^2	ω	ω^2	ω^3	1	ω^3	1	ω	ω^2
D_{59}	1	ω^2	1	ω^2	ω	ω^3	ω	ω^3	ω^3	ω	ω^3	ω	ω	ω^3	ω	ω^3	ω^3	ω	ω^3	ω

0	C_{45}	C_{46}	C_{47}	C_{48}	C_{49}	C_{50}	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	C_{57}	C_{58}	C_{59}	C_{60}	C_{61}	C_{62}	C_{63}	C_{64}
D_{60}	1	ω^3	ω^2	ω	ω	1	ω^3	ω^2	ω^3	ω^2	ω	1	ω	1	ω^3	ω^2	ω^3	ω^2	ω	1
D_{61}	ω^3	ω^3	ω^3	ω^3	ω	ω	ω	ω	1	1	1	1	ω^3	ω^3	ω^3	ω^3	ω^2	ω^2	ω^2	ω^2
D_{62}	ω^3	1	ω	ω^2	ω	ω^2	ω^3	1	1	ω	ω^2	ω^3	ω^3	1	ω	ω^2	ω^2	ω^3	1	ω
D_{63}	ω^3	ω	ω^3	ω	ω	ω^3	ω	ω^3	1	ω^2	1	ω^2	ω^3	ω	ω^3	ω	ω^2	1	ω^2	1
D_{64}	ω^3	ω^2	ω	1	ω	1	ω^3	ω^2	1	ω^3	ω^2	ω	ω^3	ω^2	ω	1	ω^2	ω	1	ω^3

(B.13)

B.6. Character Table of the Group G_{192}**.** The group G_{192} has 20 conjugacy classes and therefore it has 20 irreducible representations that are distributed according to the following pattern:

a) 4 irreps of dimension 1, namely $D_1, \ldots, D_4$,

b) 12 irreps of dimension 3, namely $D_5, \ldots, D_{16}$,

c) 2 irreps of dimension 2, namely D_{17}, D_{18},

d) 2 irreps of dimension 6, namely D_{19}, D_{20}.

The character table is displayed below, where by ϵ we have denoted the cubic root of unity $\epsilon = \exp\left[\frac{2\pi}{3}\,\mathrm{i}\right]$.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1
D_3	1	−1	−1	1	1	−1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1
D_4	1	−1	−1	1	1	−1	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	1	−1
D_5	3	−3	−3	3	−1	1	−1	1	1	−1	−1	1	1	−1	1	−1	−1	1	0	0
D_6	3	−3	−3	3	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1	1	−1	0	0
D_7	3	3	3	3	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1	0	0
D_8	3	3	3	3	−1	−1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	0	0
D_9	3	3	−1	−1	−1	3	3	−1	−1	−1	−1	1	−1	1	1	1	−1	−1	0	0
D_{10}	3	3	−1	−1	−1	3	3	−1	−1	−1	1	−1	1	−1	−1	−1	1	1	0	0
D_{11}	3	−3	1	−1	−1	−3	3	1	1	−1	−1	−1	1	1	1	−1	1	−1	0	0
D_{12}	3	−3	1	−1	−1	−3	3	1	1	−1	1	1	−1	−1	−1	1	−1	1	0	0
D_{13}	3	3	−1	−1	3	−1	−1	3	−1	−1	−1	1	−1	1	−1	−1	1	1	0	0
D_{14}	3	3	−1	−1	3	−1	−1	3	−1	−1	1	−1	1	−1	1	1	−1	−1	0	0
D_{15}	3	−3	1	−1	3	1	−1	−3	1	−1	−1	−1	1	1	−1	1	−1	1	0	0
D_{16}	3	−3	1	−1	3	1	−1	−3	1	−1	1	1	−1	−1	1	−1	1	−1	0	0
D_{17}	2	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0	$\epsilon(\epsilon+1)$	$\epsilon(\epsilon+1)$
D_{18}	2	−2	−2	2	2	−2	2	−2	−2	2	0	0	0	0	0	0	0	0	$\epsilon(\epsilon+1)$	$-\epsilon(\epsilon+1)$
D_{19}	6	6	−2	−2	−2	−2	−2	−2	2	2	0	0	0	0	0	0	0	0	0	0
D_{20}	6	−6	2	−2	−2	2	−2	2	−2	2	0	0	0	0	0	0	0	0	0	0

(B.14)

B.7. Character Table of the Group G_{96}**.** The group G_{96} has 16 conjugacy classes and therefore it has 16 irreducible representations that are distributed according to the following pattern:

a) 6 irreps of dimension 1, namely $D_1, \ldots, D_6$,

b) 10 irreps of dimension 3, namely $D_7, \ldots, D_{16}$.

The character table is displayed below, where by ϵ we have denoted the cubic root of unity $\epsilon = \exp\left[\frac{2\pi}{3}\,\mathrm{i}\right]$.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	1	-1	1	-1
D_3	1	1	1	1	1	1	1	1	1	1	1	1	ϵ	ϵ	ϵ^2	ϵ^2
D_4	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	ϵ	$-\epsilon$	ϵ^2	$-\epsilon^2$
D_5	1	1	1	1	1	1	1	1	1	1	1	1	ϵ^2	ϵ^2	ϵ	ϵ
D_6	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	ϵ^2	$-\epsilon^2$	ϵ	$-\epsilon$
D_7	3	-3	-3	3	-1	1	1	-1	1	-1	-1	1	0	0	0	0
D_8	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0
D_9	3	3	-1	-1	-1	-1	3	-1	-1	3	-1	-1	0	0	0	0
D_{10}	3	-3	1	-1	-1	1	-3	-1	1	3	-1	1	0	0	0	0
D_{11}	3	3	-1	-1	-1	3	-1	-1	-1	-1	3	-1	0	0	0	0
D_{12}	3	-3	1	-1	-1	-3	1	-1	1	-1	3	1	0	0	0	0
D_{13}	3	3	-1	-1	3	-1	-1	-1	-1	-1	-1	3	0	0	0	0
D_{14}	3	-3	1	-1	3	1	1	-1	1	-1	-1	-3	0	0	0	0
D_{15}	3	3	-1	-1	-1	-1	-1	3	3	-1	-1	-1	0	0	0	0
D_{16}	3	-3	1	-1	-1	1	1	3	-3	-1	-1	1	0	0	0	0

(B.15)

B.8. Character Table of the Group G_{48}**.** The group G_{48} has 8 conjugacy classes and therefore it has 8 irreducible representations that are distributed according to the following pattern:

a) 3 irreps of dimension 1, namely $D_1, \ldots, D_3$,

b) 5 irreps of dimension 3, namely $D_4, \ldots, D_8$.

The character table is displayed below, where by ϵ we have denoted the cubic root of unity $\epsilon = \exp\left[\frac{2\pi}{3}\,\mathrm{i}\right]$.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
D_1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	ϵ	ϵ^2
D_3	1	1	1	1	1	1	ϵ^2	ϵ
D_4	3	3	-1	-1	-1	-1	0	0
D_5	3	-1	-1	-1	3	-1	0	0
D_6	3	-1	-1	-1	-1	3	0	0
D_7	3	-1	3	-1	-1	-1	0	0
D_8	3	-1	-1	3	-1	-1	0	0

(B.16)

B.9. Character Table of the Group G_{16}**.** The group G_{16} is Abelian with order 16. Therefore it has 16 conjugacy classes and 16 one-dimensional irreducible representations. The character table is displayed below.

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
D_3	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
D_4	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
D_5	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	1
D_6	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1	1	1	−1	−1
D_7	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
D_8	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1
D_9	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
D_{10}	1	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1
D_{11}	1	−1	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1
D_{12}	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1	1	−1	1	−1
D_{13}	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1	1	−1
D_{14}	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	1	−1	−1	1
D_{15}	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1
D_{16}	1	−1	−1	1	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1

(B.17)

B.10. Character Table of the Group GS_{32}**.** The group GS_{32} has 14 conjugacy classes and therefore it has 14 irreducible representations that are distributed according to the following pattern:

a) 8 irreps of dimension 1, namely $D_1, \ldots, D_8$,
b) 6 irreps of dimension 2, namely $D_9, \ldots, D_{14}$.

The character table is displayed below,

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}
D_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1
D_3	1	1	1	1	1	−1	−1	−1	−1	1	1	1	−1	−1
D_4	1	1	1	1	1	−1	−1	−1	−1	1	−1	−1	1	1
D_5	1	1	1	1	−1	1	1	−1	−1	−1	−1	1	−1	1
D_6	1	1	1	1	−1	1	1	−1	−1	−1	1	−1	1	−1
D_7	1	1	1	1	−1	−1	−1	1	1	−1	−1	1	1	−1
D_8	1	1	1	1	−1	−1	−1	1	1	−1	1	−1	−1	1
D_9	2	2	−2	−2	2	0	0	0	0	−2	0	0	0	0
D_{10}	2	2	−2	−2	−2	0	0	0	0	2	0	0	0	0
D_{11}	2	−2	−2	2	0	0	0	2	−2	0	0	0	0	0
D_{12}	2	−2	2	−2	0	2	−2	0	0	0	0	0	0	0
D_{13}	2	−2	2	−2	0	−2	2	0	0	0	0	0	0	0
D_{14}	2	−2	−2	2	0	0	0	−2	2	0	0	0	0	0

(B.18)

B.11. Character Table of the Group GP_{24}**.** The group GP_{24} has 8 conjugacy classes and therefore it has 8 irreducible representations that are distributed according to the following pattern:

a) 6 irreps of dimension 1, namely $D_1, \ldots, D_6$,

b) 2 irreps of dimension 2, namely D_7, D_8.

The character table is displayed below,

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
D_1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	$(-1)^{2/3}$	$(-1)^{2/3}$	$-\sqrt[3]{-1}$	$-\sqrt[3]{-1}$
D_3	1	1	1	1	$-\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$(-1)^{2/3}$	$(-1)^{2/3}$
D_4	1	-1	-1	1	-1	1	-1	1
D_5	1	-1	-1	1	$-(-1)^{2/3}$	$(-1)^{2/3}$	$\sqrt[3]{-1}$	$-\sqrt[3]{-1}$
D_6	1	-1	-1	1	$\sqrt[3]{-1}$	$-\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$
D_7	3	3	-1	-1	0	0	0	0
D_8	3	-3	1	-1	0	0	0	0

(B.19)

B.12. Character Table of the Group Oh_{48}**.** The group Oh_{48} is isomorphic to the extended octahedral group and has 10 conjugacy classes. Therefore it has 10 irreducible representations that are distributed according to the following pattern:

a) 4 irreps of dimension 1, namely $D_1, \ldots, D_4$,

b) 2 irreps of dimension 2, namely D_5, D_6,

c) 4 irreps of dimension 3, namely $D_7, \ldots D_{10}$.

The character table is displayed below,

0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
D_1	1	1	1	1	1	1	1	1	1	1
D_2	1	1	1	1	1	-1	-1	-1	-1	-1
D_3	1	1	1	-1	-1	1	1	1	-1	-1
D_4	1	1	1	-1	-1	-1	-1	-1	1	1
D_5	2	-1	2	0	0	2	-1	2	0	0
D_6	2	-1	2	0	0	-2	1	-2	0	0
D_7	3	0	-1	-1	1	3	0	-1	-1	1
D_8	3	0	-1	-1	1	-3	0	1	1	-1
D_9	3	0	-1	1	-1	3	0	-1	1	-1
D_{10}	3	0	-1	1	-1	-3	0	1	-1	1

(B.20)

B.13. Character Table of the Group GS_{24}**.** The group GS_{24} is isomorphic to the proper octahedral group O_{24} and has 5 conjugacy classes. Therefore it has 5 irreducible representations that are distributed according to the following pattern:

a) 2 irreps of dimension 1, namely $D_1, \ldots, D_2$,

b) 1 irrep of dimension 2, namely D_3,

c) 2 irreps of dimension 3, namely D_4, D_5.

The character table is displayed below,

0	C_1	C_2	C_3	C_4	C_5
D_1	1	1	1	1	1
D_2	1	1	1	-1	-1
D_3	2	-1	2	0	0
D_4	3	0	-1	-1	1
D_5	3	0	-1	1	-1

(B.21)

C. CLASSIFICATION OF THE MOMENTUM VECTORS AND OF THE CORRESPONDING G_{1536} IRREPS

In this section, we list the results obtained by means of a MATHEMATICA computer code relative to decomposition into irreps of the representations of the group G_{1536} generated by the various octahedral group orbits of momentum vectors. We find that there are five types of momentum vectors on the lattice:

A) Momenta of type $\{a, 0, 0\}$ which generate representations of the universal group G_{1536} of dimensions 6.

B) Momenta of type $\{a, a, a\}$ which generate representations of the universal group G_{1536} of dimensions 8.

C) Momenta of type $\{a, a, 0\}$ which generate representations of the universal group G_{1536} of dimensions 12.

D) Momenta of type $\{a, a, b\}$ which generate representations of the universal group G_{1536} of dimensions 24.

E) Momenta of type $\{a, b, c\}$ which generate representations of the universal group G_{1536} of dimensions 48.

In each of the five groups one still has to reduce the entries to $\mathbb{Z}_4$, namely, to consider their equivalence class $\mathrm{mod}\,4$. Each different choice of the pattern of $\mathbb{Z}_4$ classes appearing in an orbit leads to different decomposition of the representation into irreducible representation of G_{1536}. A simple consideration of the combinatorics leads to the conclusion that there are in total 48 cases to be considered. The very significant result is that all of the 37 irreducible representations of G_{1536} appear at least once in the list of these decompositions. Hence for all the irrepses of this group one can find a corresponding Beltrami field and for some irrepses such a Beltrami field admits a few inequivalent realizations.

In the sections below, we list the decompositions of the representations generated by all of the 48 classes of momentum vectors. The numbers 4μ, 4ν, 4ρ, with $\mu, \nu, \rho = 0, \pm 1, \pm 2, \ldots$, represent the equivalence class of $\mathbb{Z}_4$ integers.

C.1. Classes of Momentum Vectors Yielding Orbits of Length 6: $\{$a,0,0$\}$.

Class of the momentum vector $= \{0, 0, 1 + 4\rho\}$
Dimension of the G_{1536} representation $= 6$
Orbit $= D_{23}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{0, 0, 2 + 4\rho\}$
Dimension of the G_{1536} representation $= 6$
Orbit $= D_{19}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{0, 0, 3 + 4\rho\}$
Dimension of the G_{1536} representation $= 6$
Orbit $= D_{24}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{0, 0, 4 + 4\rho\}$
Dimension of the G_{1536} representation $= 6$
Orbit $= D_7[\mathrm{G}_{1536}, 3] + D_8[\mathrm{G}_{1536}, 3]$

C.2. Classes of Momentum Vectors Yielding Orbits of Length 8: $\{$a,a,a$\}$.

Class of the momentum vector $= \{1 + 4\mu, 1 + 4\mu, 1 + 4\mu\}$
Dimension of the G_{1536} representation $= 8$
Orbit $= D_{30}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{2 + 4\mu, 2 + 4\mu, 2 + 4\mu\}$
Dimension of the G_{1536} representation $= 8$
Orbit $= D_6[\mathrm{G}_{1536}, 2] + D_{17}[\mathrm{G}_{1536}, 3] + D_{18}[\mathrm{G}_{1536}, 3]$
Class of the momentum vector $= \{3 + 4\mu, 3 + 4\mu, 3 + 4\mu\}$
Dimension of the G_{1536} representation $= 8$
Orbit $= D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{4 + 4\mu, 4 + 4\mu, 4 + 4\mu\}$
Dimension of the G_{1536} representation $= 8$
Orbit $= D_5[\mathrm{G}_{1536}, 2] + D_7[\mathrm{G}_{1536}, 3] + D_8[\mathrm{G}_{1536}, 3]$

C.3. Classes of Momentum Vectors Yielding Orbits of Length 12: $\{$a,a,0$\}$.

Class of the momentum vector $= \{0, 1 + 4\nu, 1 + 4\nu\}$
Dimension of the G_{1536} representation $= 12$
Orbit $= D_{32}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{0, 2 + 4\nu, 2 + 4\nu\}$
Dimension of the G_{1536} representation $= 12$
Orbit $= D_{13}[\mathrm{G}_{1536}, 3] + D_{15}[\mathrm{G}_{1536}, 3] + D_{20}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{0, 3 + 4\nu, 3 + 4\nu\}$
Dimension of the G_{1536} representation $= 12$
Orbit $= D_{32}[\mathrm{G}_{1536}, 12]$

Class of the momentum vector $= \{0, 4+4\nu, 4+4\nu\}$
Dimension of the G_{1536} representation $= 12$
Orbit $= D_2[\mathrm{G}_{1536}, 1] + D_5[\mathrm{G}_{1536}, 2] + D_7[\mathrm{G}_{1536}, 3] + 2D_8[\mathrm{G}_{1536}, 3]$

C.4. Classes of Momentum Vectors Yielding Orbits of Length 24: $\{$a,a,b$\}$.

Class of the momentum vector $= \{1+4\mu, 1+4\mu, 2+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{34}[\mathrm{G}_{1536}, 12] + D_{35}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{1+4\mu, 1+4\mu, 3+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{29}[\mathrm{G}_{1536}, 8] + D_{30}[\mathrm{G}_{1536}, 8] + D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{1+4\mu, 1+4\mu, 4+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{32}[\mathrm{G}_{1536}, 12] + D_{33}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{1+4\mu, 1+4\mu, 5+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{29}[\mathrm{G}_{1536}, 8] + D_{30}[\mathrm{G}_{1536}, 8] + D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{1+4\mu, 2+4\mu, 2+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{25}[\mathrm{G}_{1536}, 6] + D_{26}[\mathrm{G}_{1536}, 6] + D_{27}[\mathrm{G}_{1536}, 6] + D_{28}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{2+4\mu, 2+4\mu, 6+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_3[\mathrm{G}_{1536}, 1] + D_4[\mathrm{G}_{1536}, 1] + 2D_6[\mathrm{G}_{1536}, 2] + 3D_{17}[\mathrm{G}_{1536}, 3] + 3D_{18}[\mathrm{G}_{1536}, 3]$
Class of the momentum vector $= \{2+4\mu, 2+4\mu, 3+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{25}[\mathrm{G}_{1536}, 6] + D_{26}[\mathrm{G}_{1536}, 6] + D_{27}[\mathrm{G}_{1536}, 6] + D_{28}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{2+4\mu, 2+4\mu, 4+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{13}[\mathrm{G}_{1536}, 3] + D_{14}[\mathrm{G}_{1536}, 3] + D_{15}[\mathrm{G}_{1536}, 3] + D_{16}[\mathrm{G}_{1536}, 3] + 2D_{20}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{1+4\mu, 3+4\mu, 3+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{29}[\mathrm{G}_{1536}, 8] + D_{30}[\mathrm{G}_{1536}, 8] + D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{2+4\mu, 3+4\mu, 3+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{34}[\mathrm{G}_{1536}, 12] + D_{35}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{3+4\mu, 3+4\mu, 7+4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{29}[\mathrm{G}_{1536}, 8] + D_{30}[\mathrm{G}_{1536}, 8] + D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{1+4\mu, 4+4\mu, 4+4\rho\}$
Dimension of the G_{1536} representation $= 24$

Orbit $= D_{21}[\mathrm{G}_{1536}, 6] + D_{22}[\mathrm{G}_{1536}, 6] + D_{23}[\mathrm{G}_{1536}, 6] + D_{24}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{2 + 4\mu, 4 + 4\mu, 4 + 4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_9[\mathrm{G}_{1536}, 3] + D_{10}[\mathrm{G}_{1536}, 3] + D_{11}[\mathrm{G}_{1536}, 3] + D_{12}[\mathrm{G}_{1536}, 3] + {} + 2D_{19}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{3 + 4\mu, 4 + 4\mu, 4 + 4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{21}[\mathrm{G}_{1536}, 6] + D_{22}[\mathrm{G}_{1536}, 6] + D_{23}[\mathrm{G}_{1536}, 6] + D_{24}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{4 + 4\mu, 4 + 4\mu, 8 + 4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_1[\mathrm{G}_{1536}, 1] + D_2[\mathrm{G}_{1536}, 1] + 2D_5[\mathrm{G}_{1536}, 2] + 3D_7[\mathrm{G}_{1536}, 3] + {} + 3D_8[\mathrm{G}_{1536}, 3]$
Class of the momentum vector $= \{3 + 4\mu, 3 + 4\mu, 4 + 4\rho\}$
Dimension of the G_{1536} representation $= 24$
Orbit $= D_{32}[\mathrm{G}_{1536}, 12] + D_{33}[\mathrm{G}_{1536}, 12]$

C.5. Classes of Momentum Vectors Yielding Point Orbits of Length 24 and G_{1536} Representations of Dimensions 48: $\{a, b, c\}$.

Class of the momentum vector $= \{4 + 4\mu, 8 + 4\nu, 12 + 4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_1[\mathrm{G}_{1536}, 1] + 2D_2[\mathrm{G}_{1536}, 1] + 4D_5[\mathrm{G}_{1536}, 2] + 6D_7[\mathrm{G}_{1536}, 3] + {} + 6D_8[\mathrm{G}_{1536}, 3]$
Class of the momentum vector $= \{1 + 4\mu, 4 + 4\nu, 8 + 4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{21}[\mathrm{G}_{1536}, 6] + 2D_{22}[\mathrm{G}_{1536}, 6] + 2D_{23}[\mathrm{G}_{1536}, 6] + 2D_{24}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{2 + 4\mu, 4 + 4\nu, 8 + 4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_9[\mathrm{G}_{1536}, 3] + 2D_{10}[\mathrm{G}_{1536}, 3] + 2D_{11}[\mathrm{G}_{1536}, 3] + 2D_{12}[\mathrm{G}_{1536}, 3] + {} + 4D_{19}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{3 + 4\mu, 4 + 4\nu, 8 + 4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{21}[\mathrm{G}_{1536}, 6] + 2D_{22}[\mathrm{G}_{1536}, 6] + 2D_{23}[\mathrm{G}_{1536}, 6] + 2D_{24}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{1 + 4\mu, 2 + 4\nu, 4 + 4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{36}[\mathrm{G}_{1536}, 12] + 2D_{37}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{1 + 4\mu, 3 + 4\nu, 4 + 4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{32}[\mathrm{G}_{1536}, 12] + 2D_{33}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{2 + 4\mu, 4 + 4\nu, 6 + 4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{13}[\mathrm{G}_{1536}, 3] + 2D_{14}[\mathrm{G}_{1536}, 3] + 2D_{15}[\mathrm{G}_{1536}, 3] + 2D_{16}[\mathrm{G}_{1536}, 3] + {} + 4D_{20}[\mathrm{G}_{1536}, 6]$

Class of the momentum vector $= \{2+4\mu, 3+4\nu, 4+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{36}[\mathrm{G}_{1536}, 12] + 2D_{37}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{1+4\mu, 5+4\nu, 9+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{29}[\mathrm{G}_{1536}, 8] + 2D_{30}[\mathrm{G}_{1536}, 8] + 2D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{1+4\mu, 2+4\nu, 5+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{34}[\mathrm{G}_{1536}, 12] + 2D_{35}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{1+4\mu, 3+4\nu, 5+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{29}[\mathrm{G}_{1536}, 8] + 2D_{30}[\mathrm{G}_{1536}, 8] + 2D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{1+4\mu, 2+4\nu, 6+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{25}[\mathrm{G}_{1536}, 6] + 2D_{26}[\mathrm{G}_{1536}, 6] + 2D_{27}[\mathrm{G}_{1536}, 6] + 2D_{28}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{1+4\mu, 2+4\nu, 3+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{34}[\mathrm{G}_{1536}, 12] + 2D_{35}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{1+4\mu, 3+4\nu, 7+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{29}[\mathrm{G}_{1536}, 8] + 2D_{30}[\mathrm{G}_{1536}, 8] + 2D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{2+4\mu, 6+4\nu, 10+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{3}[\mathrm{G}_{1536}, 1] + 2D_{4}[\mathrm{G}_{1536}, 1] + 4D_{6}[\mathrm{G}_{1536}, 2] + 6D_{17}[\mathrm{G}_{1536}, 3] +$
$+ 6D_{18}[\mathrm{G}_{1536}, 3]$
Class of the momentum vector $= \{2+4\mu, 3+4\nu, 6+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{25}[\mathrm{G}_{1536}, 6] + 2D_{26}[\mathrm{G}_{1536}, 6] + 2D_{27}[\mathrm{G}_{1536}, 6] + 2D_{28}[\mathrm{G}_{1536}, 6]$
Class of the momentum vector $= \{2+4\mu, 3+4\nu, 7+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{34}[\mathrm{G}_{1536}, 12] + 2D_{35}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{3+4\mu, 7+4\nu, 11+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{29}[\mathrm{G}_{1536}, 8] + 2D_{30}[\mathrm{G}_{1536}, 8] + 2D_{31}[\mathrm{G}_{1536}, 8]$
Class of the momentum vector $= \{1+4\mu, 4+4\nu, 5+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{32}[\mathrm{G}_{1536}, 12] + 2D_{33}[\mathrm{G}_{1536}, 12]$
Class of the momentum vector $= \{3+4\mu, 4+4\nu, 7+4\rho\}$
Dimension of the G_{1536} representation $= 48$
Orbit $= 2D_{32}[\mathrm{G}_{1536}, 12] + 2D_{33}[\mathrm{G}_{1536}, 12]$

D. BRANCHING RULES OF G_{1536} IRREPS

In this section, we present the branching rules of all of the 37 irreducible representations of the Universal Classifying Group G_{1536} with respect to all of its 16 subgroups mentioned in Appendix A, namely:

$$\begin{array}{llll} 1)\ \mathrm{G}_{768}; & 5)\ \mathrm{G}_{192}; & 9)\ \mathrm{GF}_{192}; & 13)\ \mathrm{GS}_{24}; \\ 2)\ \mathrm{G}_{256}; & 6)\ \mathrm{G}_{96}; & 10)\ \mathrm{GF}_{96}; & 14)\ \mathrm{GP}_{24}; \\ 3)\ \mathrm{G}_{128}; & 7)\ \mathrm{G}_{48}; & 11)\ \mathrm{GF}_{48}; & 15)\ \mathrm{O}_{24}; \\ 4)\ \mathrm{G}_{64}; & 8)\ \mathrm{G}_{16}; & 12)\ \mathrm{GS}_{32}; & 16)\ \mathrm{Oh}_{48}. \end{array} \tag{D.1}$$

The information contained in the following tables adjoined with the information provided in the tables of Appendix C allows one to spot all cases of the Beltrami vector fields invariant under some group of symmetries including (or not including translations), namely the appearance of a $D_1(H,1)$ representation for some subgroup $H \subset \mathrm{G}_{1536}$ in the branching rule of some $D_x(\mathrm{G}_{1536}, y)$ irreducible representation produced in one of the momentum vector orbits classified in Appendix C. Looking at those tables one realizes that all 37 irreps of G_{1536} appear at least once. Hence any identity representation appearing in any of the following branching rules corresponds to an existing H-invariant Beltrami vector field.

D.1. Branching Rules of the Irreprs of Dimension 1 and 2.

$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{16},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{48},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{64},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{96},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{128},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{192},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{256},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{768},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GF}_{48},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GF}_{192},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GF}_{96},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GP}_{24},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GS}_{24},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GS}_{32},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{O}_{24},1]$
$D_1\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{Oh}_{48},1]$
$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{16},1]$
$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{48},1]$
$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{64},1]$
$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{96},1]$

$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{128},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{G}_{192},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{256},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{768},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GF}_{48},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{GF}_{192},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{GF}_{96},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GP}_{24},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{GS}_{24},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{GS}_{32},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{O}_{24},1]$$
$$D_2\,[\mathrm{G}_{1536},1] \;=\; D_3[\mathrm{Oh}_{48},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{16},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{48},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_{43}[\mathrm{G}_{64},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{96},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_{29}[\mathrm{G}_{128},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{G}_{192},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_{29}[\mathrm{G}_{256},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_4[\mathrm{G}_{768},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GF}_{48},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GF}_{192},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GF}_{96},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GP}_{24},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GS}_{24},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{GS}_{32},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{O}_{24},1]$$
$$D_3\,[\mathrm{G}_{1536},1] \;=\; D_3[\mathrm{Oh}_{48},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{16},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{48},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_{43}[\mathrm{G}_{64},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{96},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_{29}[\mathrm{G}_{128},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{G}_{192},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_{29}[\mathrm{G}_{256},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_4[\mathrm{G}_{768},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GF}_{48},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{GF}_{192},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{GF}_{96},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GP}_{24},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_2[\mathrm{GS}_{24},1]$$
$$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{GS}_{32},1]$$

$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{O}_{24},1]$
$D_4\,[\mathrm{G}_{1536},1] \;=\; D_1[\mathrm{Oh}_{48},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; 2D_1[\mathrm{G}_{16},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_2[\mathrm{G}_{48},1] + D_3[\mathrm{G}_{48},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; 2D_1[\mathrm{G}_{64},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_3[\mathrm{G}_{96},1] + D_5[\mathrm{G}_{96},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; 2D_1[\mathrm{G}_{128},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_{17}[\mathrm{G}_{192},2]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; 2D_1[\mathrm{G}_{256},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_2[\mathrm{G}_{768},1] + D_3[\mathrm{G}_{768},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_2[\mathrm{GF}_{48},1] + D_3[\mathrm{GF}_{48},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_{17}[\mathrm{GF}_{192},2]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_3[\mathrm{GF}_{96},2]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_2[\mathrm{GP}_{24},1] + D_3[\mathrm{GP}_{24},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_3[\mathrm{GS}_{24},2]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_1[\mathrm{GS}_{32},1] + D_2[\mathrm{GS}_{32},1]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_3[\mathrm{O}_{24},2]$
$D_5\,[\mathrm{G}_{1536},2] \;=\; D_5[\mathrm{Oh}_{48},2]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; 2D_1[\mathrm{G}_{16},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_2[\mathrm{G}_{48},1] + D_3[\mathrm{G}_{48},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; 2D_{43}[\mathrm{G}_{64},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_3[\mathrm{G}_{96},1] + D_5[\mathrm{G}_{96},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; 2D_{29}[\mathrm{G}_{128},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_{17}[\mathrm{G}_{192},2]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; 2D_{29}[\mathrm{G}_{256},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_5[\mathrm{G}_{768},1] + D_6[\mathrm{G}_{768},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_2[\mathrm{GF}_{48},1] + D_3[\mathrm{GF}_{48},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_{17}[\mathrm{GF}_{192},2]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_3[\mathrm{GF}_{96},2]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_2[\mathrm{GP}_{24},1] + D_3[\mathrm{GP}_{24},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_3[\mathrm{GS}_{24},2]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_1[\mathrm{GS}_{32},1] + D_2[\mathrm{GS}_{32},1]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_3[\mathrm{O}_{24},2]$
$D_6\,[\mathrm{G}_{1536},2] \;=\; D_5[\mathrm{Oh}_{48},2]$

D.2. Branching Rules of the Irreps of Dimensions 3.

$D_7\,[\mathrm{G}_{1536},3] \;=\; D_2[\mathrm{G}_{16},1] + D_3[\mathrm{G}_{16},1] + D_4[\mathrm{G}_{16},1]$
$D_7\,[\mathrm{G}_{1536},3] \;=\; D_4[\mathrm{G}_{48},3]$
$D_7\,[\mathrm{G}_{1536},3] \;=\; 3D_1[\mathrm{G}_{64},1]$
$D_7\,[\mathrm{G}_{1536},3] \;=\; D_8[\mathrm{G}_{96},3]$
$D_7\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{G}_{128},1] + 2D_2[\mathrm{G}_{128},1]$
$D_7\,[\mathrm{G}_{1536},3] \;=\; D_8[\mathrm{G}_{192},3]$

$$
\begin{aligned}
D_7\,[\mathrm{G}_{1536},3] &= D_2[\mathrm{G}_{256},1]+D_3[\mathrm{G}_{256},1]+D_4[\mathrm{G}_{256},1]\\
D_7\,[\mathrm{G}_{1536},3] &= D_7[\mathrm{G}_{768},3]\\
D_7\,[\mathrm{G}_{1536},3] &= D_4[\mathrm{GF}_{48},3]\\
D_7\,[\mathrm{G}_{1536},3] &= D_8[\mathrm{GF}_{192},3]\\
D_7\,[\mathrm{G}_{1536},3] &= D_4[\mathrm{GF}_{96},3]\\
D_7\,[\mathrm{G}_{1536},3] &= D_7[\mathrm{GP}_{24},3]\\
D_7\,[\mathrm{G}_{1536},3] &= D_5[\mathrm{GS}_{24},3]\\
D_7\,[\mathrm{G}_{1536},3] &= D_3[\mathrm{GS}_{32},1]+D_9[\mathrm{GS}_{32},2]\\
D_7\,[\mathrm{G}_{1536},3] &= D_5[\mathrm{O}_{24},3]\\
D_7\,[\mathrm{G}_{1536},3] &= D_9[\mathrm{Oh}_{48},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_2[\mathrm{G}_{16},1]+D_3[\mathrm{G}_{16},1]+D_4[\mathrm{G}_{16},1]\\
D_8\,[\mathrm{G}_{1536},3] &= D_4[\mathrm{G}_{48},3]\\
D_8\,[\mathrm{G}_{1536},3] &= 3D_1[\mathrm{G}_{64},1]\\
D_8\,[\mathrm{G}_{1536},3] &= D_8[\mathrm{G}_{96},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_1[\mathrm{G}_{128},1]+2D_2[\mathrm{G}_{128},1]\\
D_8\,[\mathrm{G}_{1536},3] &= D_7[\mathrm{G}_{192},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_2[\mathrm{G}_{256},1]+D_3[\mathrm{G}_{256},1]+D_4[\mathrm{G}_{256},1]\\
D_8\,[\mathrm{G}_{1536},3] &= D_7[\mathrm{G}_{768},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_4[\mathrm{GF}_{48},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_7[\mathrm{GF}_{192},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_5[\mathrm{GF}_{96},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_7[\mathrm{GP}_{24},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_4[\mathrm{GS}_{24},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_4[\mathrm{GS}_{32},1]+D_9[\mathrm{GS}_{32},2]\\
D_8\,[\mathrm{G}_{1536},3] &= D_4[\mathrm{O}_{24},3]\\
D_8\,[\mathrm{G}_{1536},3] &= D_7[\mathrm{Oh}_{48},3]\\
D_9\,[\mathrm{G}_{1536},3] &= 3D_1[\mathrm{G}_{16},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_1[\mathrm{G}_{48},1]+D_2[\mathrm{G}_{48},1]+D_3[\mathrm{G}_{48},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_3[\mathrm{G}_{64},1]+D_9[\mathrm{G}_{64},1]+D_{33}[\mathrm{G}_{64},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_1[\mathrm{G}_{96},1]+D_3[\mathrm{G}_{96},1]+D_5[\mathrm{G}_{96},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_5[\mathrm{G}_{128},1]+D_9[\mathrm{G}_{128},1]+D_{17}[\mathrm{G}_{128},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_1[\mathrm{G}_{192},1]+D_{17}[\mathrm{G}_{192},2]\\
D_9\,[\mathrm{G}_{1536},3] &= D_5[\mathrm{G}_{256},1]+D_9[\mathrm{G}_{256},1]+D_{17}[\mathrm{G}_{256},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_8[\mathrm{G}_{768},3]\\
D_9\,[\mathrm{G}_{1536},3] &= D_1[\mathrm{GF}_{48},1]+D_2[\mathrm{GF}_{48},1]+D_3[\mathrm{GF}_{48},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_2[\mathrm{GF}_{192},1]+D_{17}[\mathrm{GF}_{192},2]\\
D_9\,[\mathrm{G}_{1536},3] &= D_2[\mathrm{GF}_{96},1]+D_3[\mathrm{GF}_{96},2]\\
D_9\,[\mathrm{G}_{1536},3] &= D_1[\mathrm{GP}_{24},1]+D_2[\mathrm{GP}_{24},1]+D_3[\mathrm{GP}_{24},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_2[\mathrm{GS}_{24},1]+D_3[\mathrm{GS}_{24},2]\\
D_9\,[\mathrm{G}_{1536},3] &= 2D_1[\mathrm{GS}_{32},1]+D_2[\mathrm{GS}_{32},1]\\
D_9\,[\mathrm{G}_{1536},3] &= D_1[\mathrm{O}_{24},1]+D_3[\mathrm{O}_{24},2]\\
D_9\,[\mathrm{G}_{1536},3] &= D_1[\mathrm{Oh}_{48},1]+D_5[\mathrm{Oh}_{48},2]
\end{aligned}
$$

$D_{10}\,[\mathrm{G}_{1536},3] \;= 3D_1[\mathrm{G}_{16},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_1[\mathrm{G}_{48},1]+D_2[\mathrm{G}_{48},1]+D_3[\mathrm{G}_{48},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_3[\mathrm{G}_{64},1]+D_9[\mathrm{G}_{64},1]+D_{33}[\mathrm{G}_{64},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_1[\mathrm{G}_{96},1]+D_3[\mathrm{G}_{96},1]+D_5[\mathrm{G}_{96},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_5[\mathrm{G}_{128},1]+D_9[\mathrm{G}_{128},1]+D_{17}[\mathrm{G}_{128},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_2[\mathrm{G}_{192},1]+D_{17}[\mathrm{G}_{192},2]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_5[\mathrm{G}_{256},1]+D_9[\mathrm{G}_{256},1]+D_{17}[\mathrm{G}_{256},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_8[\mathrm{G}_{768},3]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_1[\mathrm{GF}_{48},1]+D_2[\mathrm{GF}_{48},1]+D_3[\mathrm{GF}_{48},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_1[\mathrm{GF}_{192},1]+D_{17}[\mathrm{GF}_{192},2]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_1[\mathrm{GF}_{96},1]+D_3[\mathrm{GF}_{96},2]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_1[\mathrm{GP}_{24},1]+D_2[\mathrm{GP}_{24},1]+D_3[\mathrm{GP}_{24},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_1[\mathrm{GS}_{24},1]+D_3[\mathrm{GS}_{24},2]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_1[\mathrm{GS}_{32},1]+2D_2[\mathrm{GS}_{32},1]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_2[\mathrm{O}_{24},1]+D_3[\mathrm{O}_{24},2]$
$D_{10}\,[\mathrm{G}_{1536},3] \;= D_3[\mathrm{Oh}_{48},1]+D_5[\mathrm{Oh}_{48},2]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_2[\mathrm{G}_{16},1]+D_3[\mathrm{G}_{16},1]+D_4[\mathrm{G}_{16},1]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_4[\mathrm{G}_{48},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_3[\mathrm{G}_{64},1]+D_9[\mathrm{G}_{64},1]+D_{33}[\mathrm{G}_{64},1]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_8[\mathrm{G}_{96},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_5[\mathrm{G}_{128},1]+D_{10}[\mathrm{G}_{128},1]+D_{18}[\mathrm{G}_{128},1]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_7[\mathrm{G}_{192},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_6[\mathrm{G}_{256},1]+D_{11}[\mathrm{G}_{256},1]+D_{20}[\mathrm{G}_{256},1]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_9[\mathrm{G}_{768},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_4[\mathrm{GF}_{48},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_8[\mathrm{GF}_{192},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_4[\mathrm{GF}_{96},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_7[\mathrm{GP}_{24},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_5[\mathrm{GS}_{24},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_4[\mathrm{GS}_{32},1]+D_9[\mathrm{GS}_{32},2]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_4[\mathrm{O}_{24},3]$
$D_{11}\,[\mathrm{G}_{1536},3] \;= D_7[\mathrm{Oh}_{48},3]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_2[\mathrm{G}_{16},1]+D_3[\mathrm{G}_{16},1]+D_4[\mathrm{G}_{16},1]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_4[\mathrm{G}_{48},3]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_3[\mathrm{G}_{64},1]+D_9[\mathrm{G}_{64},1]+D_{33}[\mathrm{G}_{64},1]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_8[\mathrm{G}_{96},3]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_5[\mathrm{G}_{128},1]+D_{10}[\mathrm{G}_{128},1]+D_{18}[\mathrm{G}_{128},1]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_8[\mathrm{G}_{192},3]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_6[\mathrm{G}_{256},1]+D_{11}[\mathrm{G}_{256},1]+D_{20}[\mathrm{G}_{256},1]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_9[\mathrm{G}_{768},3]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_4[\mathrm{GF}_{48},3]$
$D_{12}\,[\mathrm{G}_{1536},3] \;= D_7[\mathrm{GF}_{192},3]$

$$D_{12}\,[\mathrm{G}_{1536},3] \;=\; D_5[\mathrm{GF}_{96},3]$$
$$D_{12}\,[\mathrm{G}_{1536},3] \;=\; D_7[\mathrm{GP}_{24},3]$$
$$D_{12}\,[\mathrm{G}_{1536},3] \;=\; D_4[\mathrm{GS}_{24},3]$$
$$D_{12}\,[\mathrm{G}_{1536},3] \;=\; D_3[\mathrm{GS}_{32},1]+D_9[\mathrm{GS}_{32},2]$$
$$D_{12}\,[\mathrm{G}_{1536},3] \;=\; D_5[\mathrm{O}_{24},3]$$
$$D_{12}\,[\mathrm{G}_{1536},3] \;=\; D_9[\mathrm{Oh}_{48},3]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; 3D_1[\mathrm{G}_{16},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{G}_{48},1]+D_2[\mathrm{G}_{48},1]+D_3[\mathrm{G}_{48},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_{11}[\mathrm{G}_{64},1]+D_{35}[\mathrm{G}_{64},1]+D_{41}[\mathrm{G}_{64},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{G}_{96},1]+D_3[\mathrm{G}_{96},1]+D_5[\mathrm{G}_{96},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_{13}[\mathrm{G}_{128},1]+D_{21}[\mathrm{G}_{128},1]+D_{25}[\mathrm{G}_{128},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_2[\mathrm{G}_{192},1]+D_{17}[\mathrm{G}_{192},2]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_{13}[\mathrm{G}_{256},1]+D_{21}[\mathrm{G}_{256},1]+D_{25}[\mathrm{G}_{256},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_{12}[\mathrm{G}_{768},3]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{GF}_{48},1]+D_2[\mathrm{GF}_{48},1]+D_3[\mathrm{GF}_{48},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_2[\mathrm{GF}_{192},1]+D_{17}[\mathrm{GF}_{192},2]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_2[\mathrm{GF}_{96},1]+D_3[\mathrm{GF}_{96},2]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{GP}_{24},1]+D_2[\mathrm{GP}_{24},1]+D_3[\mathrm{GP}_{24},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_2[\mathrm{GS}_{24},1]+D_3[\mathrm{GS}_{24},2]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{GS}_{32},1]+2D_2[\mathrm{GS}_{32},1]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_2[\mathrm{O}_{24},1]+D_3[\mathrm{O}_{24},2]$$
$$D_{13}\,[\mathrm{G}_{1536},3] \;=\; D_3[\mathrm{Oh}_{48},1]+D_5[\mathrm{Oh}_{48},2]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; 3D_1[\mathrm{G}_{16},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{G}_{48},1]+D_2[\mathrm{G}_{48},1]+D_3[\mathrm{G}_{48},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_{11}[\mathrm{G}_{64},1]+D_{35}[\mathrm{G}_{64},1]+D_{41}[\mathrm{G}_{64},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{G}_{96},1]+D_3[\mathrm{G}_{96},1]+D_5[\mathrm{G}_{96},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_{13}[\mathrm{G}_{128},1]+D_{21}[\mathrm{G}_{128},1]+D_{25}[\mathrm{G}_{128},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{G}_{192},1]+D_{17}[\mathrm{G}_{192},2]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_{13}[\mathrm{G}_{256},1]+D_{21}[\mathrm{G}_{256},1]+D_{25}[\mathrm{G}_{256},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_{12}[\mathrm{G}_{768},3]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{GF}_{48},1]+D_2[\mathrm{GF}_{48},1]+D_3[\mathrm{GF}_{48},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{GF}_{192},1]+D_{17}[\mathrm{GF}_{192},2]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{GF}_{96},1]+D_3[\mathrm{GF}_{96},2]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{GP}_{24},1]+D_2[\mathrm{GP}_{24},1]+D_3[\mathrm{GP}_{24},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{GS}_{24},1]+D_3[\mathrm{GS}_{24},2]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; 2D_1[\mathrm{GS}_{32},1]+D_2[\mathrm{GS}_{32},1]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{O}_{24},1]+D_3[\mathrm{O}_{24},2]$$
$$D_{14}\,[\mathrm{G}_{1536},3] \;=\; D_1[\mathrm{Oh}_{48},1]+D_5[\mathrm{Oh}_{48},2]$$
$$D_{15}\,[\mathrm{G}_{1536},3] \;=\; D_2[\mathrm{G}_{16},1]+D_3[\mathrm{G}_{16},1]+D_4[\mathrm{G}_{16},1]$$
$$D_{15}\,[\mathrm{G}_{1536},3] \;=\; D_4[\mathrm{G}_{48},3]$$
$$D_{15}\,[\mathrm{G}_{1536},3] \;=\; D_{11}[\mathrm{G}_{64},1]+D_{35}[\mathrm{G}_{64},1]+D_{41}[\mathrm{G}_{64},1]$$
$$D_{15}\,[\mathrm{G}_{1536},3] \;=\; D_8[\mathrm{G}_{96},3]$$

$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{14}[\mathrm{G}_{128},1]+D_{22}[\mathrm{G}_{128},1]+D_{25}[\mathrm{G}_{128},1]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{7}[\mathrm{G}_{192},3]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{16}[\mathrm{G}_{256},1]+D_{23}[\mathrm{G}_{256},1]+D_{26}[\mathrm{G}_{256},1]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{15}[\mathrm{G}_{768},3]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{GF}_{48},3]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{7}[\mathrm{GF}_{192},3]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{5}[\mathrm{GF}_{96},3]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{7}[\mathrm{GP}_{24},3]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{GS}_{24},3]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{GS}_{32},1]+D_{9}[\mathrm{GS}_{32},2]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{O}_{24},3]$
$D_{15}\,[\mathrm{G}_{1536},3]\;=D_{7}[\mathrm{Oh}_{48},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{2}[\mathrm{G}_{16},1]+D_{3}[\mathrm{G}_{16},1]+D_{4}[\mathrm{G}_{16},1]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{G}_{48},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{11}[\mathrm{G}_{64},1]+D_{35}[\mathrm{G}_{64},1]+D_{41}[\mathrm{G}_{64},1]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{8}[\mathrm{G}_{96},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{14}[\mathrm{G}_{128},1]+D_{22}[\mathrm{G}_{128},1]+D_{25}[\mathrm{G}_{128},1]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{8}[\mathrm{G}_{192},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{16}[\mathrm{G}_{256},1]+D_{23}[\mathrm{G}_{256},1]+D_{26}[\mathrm{G}_{256},1]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{15}[\mathrm{G}_{768},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{GF}_{48},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{8}[\mathrm{GF}_{192},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{GF}_{96},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{7}[\mathrm{GP}_{24},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{5}[\mathrm{GS}_{24},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{3}[\mathrm{GS}_{32},1]+D_{9}[\mathrm{GS}_{32},2]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{5}[\mathrm{O}_{24},3]$
$D_{16}\,[\mathrm{G}_{1536},3]\;=D_{9}[\mathrm{Oh}_{48},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{2}[\mathrm{G}_{16},1]+D_{3}[\mathrm{G}_{16},1]+D_{4}[\mathrm{G}_{16},1]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{G}_{48},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=3D_{43}[\mathrm{G}_{64},1]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{8}[\mathrm{G}_{96},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{29}[\mathrm{G}_{128},1]+2D_{30}[\mathrm{G}_{128},1]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{7}[\mathrm{G}_{192},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{30}[\mathrm{G}_{256},1]+D_{31}[\mathrm{G}_{256},1]+D_{32}[\mathrm{G}_{256},1]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{16}[\mathrm{G}_{768},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{GF}_{48},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{8}[\mathrm{GF}_{192},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{GF}_{96},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{7}[\mathrm{GP}_{24},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{5}[\mathrm{GS}_{24},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\;=D_{4}[\mathrm{GS}_{32},1]+D_{9}[\mathrm{GS}_{32},2]$

$D_{17}\,[\mathrm{G}_{1536},3]\ = D_4[\mathrm{O}_{24},3]$
$D_{17}\,[\mathrm{G}_{1536},3]\ = D_7[\mathrm{Oh}_{48},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_2[\mathrm{G}_{16},1] + D_3[\mathrm{G}_{16},1] + D_4[\mathrm{G}_{16},1]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_4[\mathrm{G}_{48},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = 3D_{43}[\mathrm{G}_{64},1]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_8[\mathrm{G}_{96},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_{29}[\mathrm{G}_{128},1] + 2D_{30}[\mathrm{G}_{128},1]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_8[\mathrm{G}_{192},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_{30}[\mathrm{G}_{256},1] + D_{31}[\mathrm{G}_{256},1] + D_{32}[\mathrm{G}_{256},1]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_{16}[\mathrm{G}_{768},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_4[\mathrm{GF}_{48},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_7[\mathrm{GF}_{192},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_5[\mathrm{GF}_{96},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_7[\mathrm{GP}_{24},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_4[\mathrm{GS}_{24},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_3[\mathrm{GS}_{32},1] + D_9[\mathrm{GS}_{32},2]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_5[\mathrm{O}_{24},3]$
$D_{18}\,[\mathrm{G}_{1536},3]\ = D_9[\mathrm{Oh}_{48},3]$

D.3. Branching Rules of the Irreps of Dimensions 6.

$D_{19}\,[\mathrm{G}_{1536},6]\ = 2D_2[\mathrm{G}_{16},1] + 2D_3[\mathrm{G}_{16},1] + 2D_4[\mathrm{G}_{16},1]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = 2D_4[\mathrm{G}_{48},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = 2D_3[\mathrm{G}_{64},1] + 2D_9[\mathrm{G}_{64},1] + 2D_{33}[\mathrm{G}_{64},1]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = 2D_8[\mathrm{G}_{96},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = 2D_6[\mathrm{G}_{128},1] + D_9[\mathrm{G}_{128},1] + D_{10}[\mathrm{G}_{128},1] + D_{17}[\mathrm{G}_{128},1] +$
$\qquad + D_{18}[\mathrm{G}_{128},1]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_7[\mathrm{G}_{192},3] + D_8[\mathrm{G}_{192},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_7[\mathrm{G}_{256},1] + D_8[\mathrm{G}_{256},1] + D_{10}[\mathrm{G}_{256},1] + D_{12}[\mathrm{G}_{256},1] +$
$\qquad + D_{18}[\mathrm{G}_{256},1] + D_{19}[\mathrm{G}_{256},1]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_{10}[\mathrm{G}_{768},3] + D_{11}[\mathrm{G}_{768},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = 2D_4[\mathrm{GF}_{48},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_7[\mathrm{GF}_{192},3] + D_8[\mathrm{GF}_{192},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_4[\mathrm{GF}_{96},3] + D_5[\mathrm{GF}_{96},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = 2D_7[\mathrm{GP}_{24},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_4[\mathrm{GS}_{24},3] + D_5[\mathrm{GS}_{24},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_3[\mathrm{GS}_{32},1] + D_4[\mathrm{GS}_{32},1] + 2D_9[\mathrm{GS}_{32},2]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_4[\mathrm{O}_{24},3] + D_5[\mathrm{O}_{24},3]$
$D_{19}\,[\mathrm{G}_{1536},6]\ = D_7[\mathrm{Oh}_{48},3] + D_9[\mathrm{Oh}_{48},3]$
$D_{20}\,[\mathrm{G}_{1536},6]\ = 2D_2[\mathrm{G}_{16},1] + 2D_3[\mathrm{G}_{16},1] + 2D_4[\mathrm{G}_{16},1]$
$D_{20}\,[\mathrm{G}_{1536},6]\ = 2D_4[\mathrm{G}_{48},3]$
$D_{20}\,[\mathrm{G}_{1536},6]\ = 2D_{11}[\mathrm{G}_{64},1] + 2D_{35}[\mathrm{G}_{64},1] + 2D_{41}[\mathrm{G}_{64},1]$
$D_{20}\,[\mathrm{G}_{1536},6]\ = 2D_8[\mathrm{G}_{96},3]$

$$
\begin{aligned}
D_{20}\,[\mathrm{G}_{1536},6] &= D_{13}[\mathrm{G}_{128},1]+D_{14}[\mathrm{G}_{128},1]+D_{21}[\mathrm{G}_{128},1]+\\
&\quad +D_{22}[\mathrm{G}_{128},1]+2D_{26}[\mathrm{G}_{128},1]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{7}[\mathrm{G}_{192},3]+D_{8}[\mathrm{G}_{192},3]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{14}[\mathrm{G}_{256},1]+D_{15}[\mathrm{G}_{256},1]+D_{22}[\mathrm{G}_{256},1]+D_{24}[\mathrm{G}_{256},1]+\\
&\quad +D_{27}[\mathrm{G}_{256},1]+D_{28}[\mathrm{G}_{256},1]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{13}[\mathrm{G}_{768},3]+D_{14}[\mathrm{G}_{768},3]\\
D_{20}\,[\mathrm{G}_{1536},6] &= 2D_{4}[\mathrm{GF}_{48},3]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{7}[\mathrm{GF}_{192},3]+D_{8}[\mathrm{GF}_{192},3]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{4}[\mathrm{GF}_{96},3]+D_{5}[\mathrm{GF}_{96},3]\\
D_{20}\,[\mathrm{G}_{1536},6] &= 2D_{7}[\mathrm{GP}_{24},3]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{4}[\mathrm{GS}_{24},3]+D_{5}[\mathrm{GS}_{24},3]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{3}[\mathrm{GS}_{32},1]+D_{4}[\mathrm{GS}_{32},1]+2D_{9}[\mathrm{GS}_{32},2]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{4}[\mathrm{O}_{24},3]+D_{5}[\mathrm{O}_{24},3]\\
D_{20}\,[\mathrm{G}_{1536},6] &= D_{7}[\mathrm{Oh}_{48},3]+D_{9}[\mathrm{Oh}_{48},3]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{6}[\mathrm{G}_{16},1]+D_{7}[\mathrm{G}_{16},1]+D_{9}[\mathrm{G}_{16},1]+D_{11}[\mathrm{G}_{16},1]+\\
&\quad +D_{15}[\mathrm{G}_{16},1]+D_{16}[\mathrm{G}_{16},1]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{6}[\mathrm{G}_{48},3]+D_{7}[\mathrm{G}_{48},3]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{2}[\mathrm{G}_{64},1]+D_{4}[\mathrm{G}_{64},1]+D_{5}[\mathrm{G}_{64},1]+D_{13}[\mathrm{G}_{64},1]+\\
&\quad +D_{17}[\mathrm{G}_{64},1]+D_{49}[\mathrm{G}_{64},1]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{12}[\mathrm{G}_{96},3]+D_{14}[\mathrm{G}_{96},3]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{3}[\mathrm{G}_{128},1]+D_{7}[\mathrm{G}_{128},1]+D_{33}[\mathrm{G}_{128},2]+D_{37}[\mathrm{G}_{128},2]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{12}[\mathrm{G}_{192},3]+D_{15}[\mathrm{G}_{192},3]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{33}[\mathrm{G}_{256},2]+D_{42}[\mathrm{G}_{256},2]+D_{45}[\mathrm{G}_{256},2]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{23}[\mathrm{G}_{768},6]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{5}[\mathrm{GF}_{48},3]+D_{8}[\mathrm{GF}_{48},3]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{20}[\mathrm{GF}_{192},6]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{10}[\mathrm{GF}_{96},6]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{4}[\mathrm{GP}_{24},1]+D_{5}[\mathrm{GP}_{24},1]+D_{6}[\mathrm{GP}_{24},1]+D_{8}[\mathrm{GP}_{24},3]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{4}[\mathrm{GS}_{24},3]+D_{5}[\mathrm{GS}_{24},3]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{5}[\mathrm{GS}_{32},1]+D_{8}[\mathrm{GS}_{32},1]+D_{12}[\mathrm{GS}_{32},2]+D_{14}[\mathrm{GS}_{32},2]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{2}[\mathrm{O}_{24},1]+D_{3}[\mathrm{O}_{24},2]+D_{5}[\mathrm{O}_{24},3]\\
D_{21}\,[\mathrm{G}_{1536},6] &= D_{4}[\mathrm{Oh}_{48},1]+D_{6}[\mathrm{Oh}_{48},2]+D_{10}[\mathrm{Oh}_{48},3]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{6}[\mathrm{G}_{16},1]+D_{7}[\mathrm{G}_{16},1]+D_{9}[\mathrm{G}_{16},1]+D_{11}[\mathrm{G}_{16},1]+\\
&\quad +D_{15}[\mathrm{G}_{16},1]+D_{16}[\mathrm{G}_{16},1]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{6}[\mathrm{G}_{48},3]+D_{7}[\mathrm{G}_{48},3]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{2}[\mathrm{G}_{64},1]+D_{4}[\mathrm{G}_{64},1]+D_{5}[\mathrm{G}_{64},1]+D_{13}[\mathrm{G}_{64},1]+\\
&\quad +D_{17}[\mathrm{G}_{64},1]+D_{49}[\mathrm{G}_{64},1]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{12}[\mathrm{G}_{96},3]+D_{14}[\mathrm{G}_{96},3]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{3}[\mathrm{G}_{128},1]+D_{7}[\mathrm{G}_{128},1]+D_{33}[\mathrm{G}_{128},2]+D_{37}[\mathrm{G}_{128},2]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{11}[\mathrm{G}_{192},3]+D_{16}[\mathrm{G}_{192},3]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{33}[\mathrm{G}_{256},2]+D_{42}[\mathrm{G}_{256},2]+D_{45}[\mathrm{G}_{256},2]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{23}[\mathrm{G}_{768},6]
\end{aligned}
$$

$$
\begin{aligned}
D_{22}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{GF}_{48},3]+D_8[\mathrm{GF}_{48},3]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{20}[\mathrm{GF}_{192},6]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_{10}[\mathrm{GF}_{96},6]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{GP}_{24},1]+D_5[\mathrm{GP}_{24},1]+D_6[\mathrm{GP}_{24},1]+D_8[\mathrm{GP}_{24},3]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{GS}_{24},3]+D_5[\mathrm{GS}_{24},3]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{GS}_{32},1]+D_7[\mathrm{GS}_{32},1]+D_{12}[\mathrm{GS}_{32},2]+D_{14}[\mathrm{GS}_{32},2]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_1[\mathrm{O}_{24},1]+D_3[\mathrm{O}_{24},2]+D_4[\mathrm{O}_{24},3]\\
D_{22}\,[\mathrm{G}_{1536},6] &= D_2[\mathrm{Oh}_{48},1]+D_6[\mathrm{Oh}_{48},2]+D_8[\mathrm{Oh}_{48},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{G}_{16},1]+D_8[\mathrm{G}_{16},1]+D_{10}[\mathrm{G}_{16},1]+D_{12}[\mathrm{G}_{16},1]+\\
&\quad+D_{13}[\mathrm{G}_{16},1]+D_{14}[\mathrm{G}_{16},1]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{G}_{48},3]+D_8[\mathrm{G}_{48},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_2[\mathrm{G}_{64},1]+D_4[\mathrm{G}_{64},1]+D_5[\mathrm{G}_{64},1]+D_{13}[\mathrm{G}_{64},1]+\\
&\quad+D_{17}[\mathrm{G}_{64},1]+D_{49}[\mathrm{G}_{64},1]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_{10}[\mathrm{G}_{96},3]+D_{16}[\mathrm{G}_{96},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{G}_{128},1]+D_8[\mathrm{G}_{128},1]+D_{33}[\mathrm{G}_{128},2]+D_{37}[\mathrm{G}_{128},2]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_{20}[\mathrm{G}_{192},6]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_{34}[\mathrm{G}_{256},2]+D_{41}[\mathrm{G}_{256},2]+D_{46}[\mathrm{G}_{256},2]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_{24}[\mathrm{G}_{768},6]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{GF}_{48},3]+D_7[\mathrm{GF}_{48},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_{12}[\mathrm{GF}_{192},3]+D_{15}[\mathrm{GF}_{192},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{GF}_{96},3]+D_9[\mathrm{GF}_{96},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= 2D_8[\mathrm{GP}_{24},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_1[\mathrm{GS}_{24},1]+D_3[\mathrm{GS}_{24},2]+D_4[\mathrm{GS}_{24},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_{10}[\mathrm{GS}_{32},2]+D_{11}[\mathrm{GS}_{32},2]+D_{13}[\mathrm{GS}_{32},2]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{O}_{24},3]+D_5[\mathrm{O}_{24},3]\\
D_{23}\,[\mathrm{G}_{1536},6] &= D_8[\mathrm{Oh}_{48},3]+D_{10}[\mathrm{Oh}_{48},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{G}_{16},1]+D_8[\mathrm{G}_{16},1]+D_{10}[\mathrm{G}_{16},1]+D_{12}[\mathrm{G}_{16},1]+\\
&\quad+D_{13}[\mathrm{G}_{16},1]+D_{14}[\mathrm{G}_{16},1]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{G}_{48},3]+D_8[\mathrm{G}_{48},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_2[\mathrm{G}_{64},1]+D_4[\mathrm{G}_{64},1]+D_5[\mathrm{G}_{64},1]+D_{13}[\mathrm{G}_{64},1]+\\
&\quad+D_{17}[\mathrm{G}_{64},1]+D_{49}[\mathrm{G}_{64},1]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_{10}[\mathrm{G}_{96},3]+D_{16}[\mathrm{G}_{96},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{G}_{128},1]+D_8[\mathrm{G}_{128},1]+D_{33}[\mathrm{G}_{128},2]+D_{37}[\mathrm{G}_{128},2]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_{20}[\mathrm{G}_{192},6]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_{34}[\mathrm{G}_{256},2]+D_{41}[\mathrm{G}_{256},2]+D_{46}[\mathrm{G}_{256},2]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_{24}[\mathrm{G}_{768},6]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{GF}_{48},3]+D_7[\mathrm{GF}_{48},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_{11}[\mathrm{GF}_{192},3]+D_{16}[\mathrm{GF}_{192},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_7[\mathrm{GF}_{96},3]+D_8[\mathrm{GF}_{96},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= 2D_8[\mathrm{GP}_{24},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_2[\mathrm{GS}_{24},1]+D_3[\mathrm{GS}_{24},2]+D_5[\mathrm{GS}_{24},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_{10}[\mathrm{GS}_{32},2]+D_{11}[\mathrm{GS}_{32},2]+D_{13}[\mathrm{GS}_{32},2]
\end{aligned}
$$

$$
\begin{aligned}
D_{24}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{O}_{24},3] + D_5[\mathrm{O}_{24},3]\\
D_{24}\,[\mathrm{G}_{1536},6] &= D_8[\mathrm{Oh}_{48},3] + D_{10}[\mathrm{Oh}_{48},3]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{G}_{16},1] + D_7[\mathrm{G}_{16},1] + D_9[\mathrm{G}_{16},1] + D_{11}[\mathrm{G}_{16},1] +\\
&\quad + D_{15}[\mathrm{G}_{16},1] + D_{16}[\mathrm{G}_{16},1]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{G}_{48},3] + D_7[\mathrm{G}_{48},3]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_{27}[\mathrm{G}_{64},1] + D_{39}[\mathrm{G}_{64},1] + D_{42}[\mathrm{G}_{64},1] + D_{44}[\mathrm{G}_{64},1] +\\
&\quad + D_{47}[\mathrm{G}_{64},1] + D_{59}[\mathrm{G}_{64},1]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_{12}[\mathrm{G}_{96},3] + D_{14}[\mathrm{G}_{96},3]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_{27}[\mathrm{G}_{128},1] + D_{31}[\mathrm{G}_{128},1] + D_{47}[\mathrm{G}_{128},2] + D_{55}[\mathrm{G}_{128},2]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_{12}[\mathrm{G}_{192},3] + D_{15}[\mathrm{G}_{192},3]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_{39}[\mathrm{G}_{256},2] + D_{51}[\mathrm{G}_{256},2] + D_{56}[\mathrm{G}_{256},2]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_{29}[\mathrm{G}_{768},6]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{GF}_{48},3] + D_8[\mathrm{GF}_{48},3]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_{20}[\mathrm{GF}_{192},6]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_{10}[\mathrm{GF}_{96},6]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{GP}_{24},1] + D_5[\mathrm{GP}_{24},1] + D_6[\mathrm{GP}_{24},1] + D_8[\mathrm{GP}_{24},3]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{GS}_{24},3] + D_5[\mathrm{GS}_{24},3]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{GS}_{32},1] + D_8[\mathrm{GS}_{32},1] + D_{12}[\mathrm{GS}_{32},2] + D_{14}[\mathrm{GS}_{32},2]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_2[\mathrm{O}_{24},1] + D_3[\mathrm{O}_{24},2] + D_5[\mathrm{O}_{24},3]\\
D_{25}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{Oh}_{48},1] + D_6[\mathrm{Oh}_{48},2] + D_{10}[\mathrm{Oh}_{48},3]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{G}_{16},1] + D_7[\mathrm{G}_{16},1] + D_9[\mathrm{G}_{16},1] + D_{11}[\mathrm{G}_{16},1] +\\
&\quad + D_{15}[\mathrm{G}_{16},1] + D_{16}[\mathrm{G}_{16},1]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{G}_{48},3] + D_7[\mathrm{G}_{48},3]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_{27}[\mathrm{G}_{64},1] + D_{39}[\mathrm{G}_{64},1] + D_{42}[\mathrm{G}_{64},1] + D_{44}[\mathrm{G}_{64},1] +\\
&\quad + D_{47}[\mathrm{G}_{64},1] + D_{59}[\mathrm{G}_{64},1]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_{12}[\mathrm{G}_{96},3] + D_{14}[\mathrm{G}_{96},3]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_{27}[\mathrm{G}_{128},1] + D_{31}[\mathrm{G}_{128},1] + D_{47}[\mathrm{G}_{128},2] +\\
&\quad + D_{55}[\mathrm{G}_{128},2]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_{11}[\mathrm{G}_{192},3] + D_{16}[\mathrm{G}_{192},3]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_{39}[\mathrm{G}_{256},2] + D_{51}[\mathrm{G}_{256},2] + D_{56}[\mathrm{G}_{256},2]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_{29}[\mathrm{G}_{768},6]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{GF}_{48},3] + D_8[\mathrm{GF}_{48},3]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_{20}[\mathrm{GF}_{192},6]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_{10}[\mathrm{GF}_{96},6]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{GP}_{24},1] + D_5[\mathrm{GP}_{24},1] + D_6[\mathrm{GP}_{24},1] + D_8[\mathrm{GP}_{24},3]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_4[\mathrm{GS}_{24},3] + D_5[\mathrm{GS}_{24},3]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_6[\mathrm{GS}_{32},1] + D_7[\mathrm{GS}_{32},1] + D_{12}[\mathrm{GS}_{32},2] + D_{14}[\mathrm{GS}_{32},2]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_1[\mathrm{O}_{24},1] + D_3[\mathrm{O}_{24},2] + D_4[\mathrm{O}_{24},3]\\
D_{26}\,[\mathrm{G}_{1536},6] &= D_2[\mathrm{Oh}_{48},1] + D_6[\mathrm{Oh}_{48},2] + D_8[\mathrm{Oh}_{48},3]\\
D_{27}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{G}_{16},1] + D_8[\mathrm{G}_{16},1] + D_{10}[\mathrm{G}_{16},1] + D_{12}[\mathrm{G}_{16},1] +\\
&\quad + D_{13}[\mathrm{G}_{16},1] + D_{14}[\mathrm{G}_{16},1]\\
D_{27}\,[\mathrm{G}_{1536},6] &= D_5[\mathrm{G}_{48},3] + D_8[\mathrm{G}_{48},3]
\end{aligned}
$$

$$D_{27}[\mathrm{G}_{1536},6] = D_{27}[\mathrm{G}_{64},1]+D_{39}[\mathrm{G}_{64},1]+D_{42}[\mathrm{G}_{64},1]+D_{44}[\mathrm{G}_{64},1]+ + D_{47}[\mathrm{G}_{64},1]+D_{59}[\mathrm{G}_{64},1]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{10}[\mathrm{G}_{96},3]+D_{16}[\mathrm{G}_{96},3]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{28}[\mathrm{G}_{128},1]+D_{32}[\mathrm{G}_{128},1]+D_{47}[\mathrm{G}_{128},2]+D_{55}[\mathrm{G}_{128},2]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{20}[\mathrm{G}_{192},6]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{40}[\mathrm{G}_{256},2]+D_{52}[\mathrm{G}_{256},2]+D_{55}[\mathrm{G}_{256},2]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{30}[\mathrm{G}_{768},6]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{6}[\mathrm{GF}_{48},3]+D_{7}[\mathrm{GF}_{48},3]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{12}[\mathrm{GF}_{192},3]+D_{15}[\mathrm{GF}_{192},3]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{6}[\mathrm{GF}_{96},3]+D_{9}[\mathrm{GF}_{96},3]$$
$$D_{27}[\mathrm{G}_{1536},6] = 2D_{8}[\mathrm{GP}_{24},3]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{1}[\mathrm{GS}_{24},1]+D_{3}[\mathrm{GS}_{24},2]+D_{4}[\mathrm{GS}_{24},3]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{10}[\mathrm{GS}_{32},2]+D_{11}[\mathrm{GS}_{32},2]+D_{13}[\mathrm{GS}_{32},2]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{4}[\mathrm{O}_{24},3]+D_{5}[\mathrm{O}_{24},3]$$
$$D_{27}[\mathrm{G}_{1536},6] = D_{8}[\mathrm{Oh}_{48},3]+D_{10}[\mathrm{Oh}_{48},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{5}[\mathrm{G}_{16},1]+D_{8}[\mathrm{G}_{16},1]+D_{10}[\mathrm{G}_{16},1]+D_{12}[\mathrm{G}_{16},1]+ + D_{13}[\mathrm{G}_{16},1]+D_{14}[\mathrm{G}_{16},1]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{5}[\mathrm{G}_{48},3]+D_{8}[\mathrm{G}_{48},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{27}[\mathrm{G}_{64},1]+D_{39}[\mathrm{G}_{64},1]+D_{42}[\mathrm{G}_{64},1]+D_{44}[\mathrm{G}_{64},1]+ + D_{47}[\mathrm{G}_{64},1]+D_{59}[\mathrm{G}_{64},1]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{10}[\mathrm{G}_{96},3]+D_{16}[\mathrm{G}_{96},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{28}[\mathrm{G}_{128},1]+D_{32}[\mathrm{G}_{128},1]+D_{47}[\mathrm{G}_{128},2]+D_{55}[\mathrm{G}_{128},2]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{20}[\mathrm{G}_{192},6]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{40}[\mathrm{G}_{256},2]+D_{52}[\mathrm{G}_{256},2]+D_{55}[\mathrm{G}_{256},2]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{30}[\mathrm{G}_{768},6]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{6}[\mathrm{GF}_{48},3]+D_{7}[\mathrm{GF}_{48},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{11}[\mathrm{GF}_{192},3]+D_{16}[\mathrm{GF}_{192},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{7}[\mathrm{GF}_{96},3]+D_{8}[\mathrm{GF}_{96},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = 2D_{8}[\mathrm{GP}_{24},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{2}[\mathrm{GS}_{24},1]+D_{3}[\mathrm{GS}_{24},2]+D_{5}[\mathrm{GS}_{24},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{10}[\mathrm{GS}_{32},2]+D_{11}[\mathrm{GS}_{32},2]+D_{13}[\mathrm{GS}_{32},2]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{4}[\mathrm{O}_{24},3]+D_{5}[\mathrm{O}_{24},3]$$
$$D_{28}[\mathrm{G}_{1536},6] = D_{8}[\mathrm{Oh}_{48},3]+D_{10}[\mathrm{Oh}_{48},3]$$

D.4. Branching Rules of the Irreps of Dimension 8.

$$D_{29}[\mathrm{G}_{1536},8] = 2D_{1}[\mathrm{G}_{16},1]+2D_{2}[\mathrm{G}_{16},1]+2D_{3}[\mathrm{G}_{16},1]+2D_{4}[\mathrm{G}_{16},1]$$
$$D_{29}[\mathrm{G}_{1536},8] = 2D_{1}[\mathrm{G}_{48},1]+2D_{4}[\mathrm{G}_{48},3]$$
$$D_{29}[\mathrm{G}_{1536},8] = D_{22}[\mathrm{G}_{64},1]+D_{24}[\mathrm{G}_{64},1]+D_{30}[\mathrm{G}_{64},1]+D_{32}[\mathrm{G}_{64},1]+ + D_{54}[\mathrm{G}_{64},1]+D_{56}[\mathrm{G}_{64},1]+D_{62}[\mathrm{G}_{64},1]+D_{64}[\mathrm{G}_{64},1]$$
$$D_{29}[\mathrm{G}_{1536},8] = 2D_{2}[\mathrm{G}_{96},1]+2D_{7}[\mathrm{G}_{96},3]$$
$$D_{29}[\mathrm{G}_{1536},8] = D_{42}[\mathrm{G}_{128},2]+D_{44}[\mathrm{G}_{128},2]+D_{50}[\mathrm{G}_{128},2]+D_{52}[\mathrm{G}_{128},2]$$
$$D_{29}[\mathrm{G}_{1536},8] = D_{3}[\mathrm{G}_{192},1]+D_{4}[\mathrm{G}_{192},1]+D_{5}[\mathrm{G}_{192},3]+D_{6}[\mathrm{G}_{192},3]$$

$$
\begin{aligned}
D_{29}\,[\mathrm{G}_{1536},8] &= D_{60}[\mathrm{G}_{256},4]+D_{62}[\mathrm{G}_{256},4]\\
D_{29}\,[\mathrm{G}_{1536},8] &= D_{17}[\mathrm{G}_{768},4]+D_{20}[\mathrm{G}_{768},4]\\
D_{29}\,[\mathrm{G}_{1536},8] &= 2D_{1}[\mathrm{GF}_{48},1]+2D_{4}[\mathrm{GF}_{48},3]\\
D_{29}\,[\mathrm{G}_{1536},8] &= D_{3}[\mathrm{GF}_{192},1]+D_{4}[\mathrm{GF}_{192},1]+D_{5}[\mathrm{GF}_{192},3]+D_{6}[\mathrm{GF}_{192},3]\\
D_{29}\,[\mathrm{G}_{1536},8] &= D_{1}[\mathrm{GF}_{96},1]+D_{2}[\mathrm{GF}_{96},1]+D_{4}[\mathrm{GF}_{96},3]+D_{5}[\mathrm{GF}_{96},3]\\
D_{29}\,[\mathrm{G}_{1536},8] &= 2D_{4}[\mathrm{GP}_{24},1]+2D_{8}[\mathrm{GP}_{24},3]\\
D_{29}\,[\mathrm{G}_{1536},8] &= D_{1}[\mathrm{GS}_{24},1]+D_{2}[\mathrm{GS}_{24},1]+D_{4}[\mathrm{GS}_{24},3]+D_{5}[\mathrm{GS}_{24},3]\\
D_{29}\,[\mathrm{G}_{1536},8] &= D_{1}[\mathrm{GS}_{32},1]+D_{2}[\mathrm{GS}_{32},1]+D_{3}[\mathrm{GS}_{32},1]+D_{4}[\mathrm{GS}_{32},1]+\\
&\quad+2D_{9}[\mathrm{GS}_{32},2]\\
D_{29}\,[\mathrm{G}_{1536},8] &= D_{1}[\mathrm{O}_{24},1]+D_{2}[\mathrm{O}_{24},1]+D_{4}[\mathrm{O}_{24},3]+D_{5}[\mathrm{O}_{24},3]\\
D_{29}\,[\mathrm{G}_{1536},8] &= D_{2}[\mathrm{Oh}_{48},1]+D_{4}[\mathrm{Oh}_{48},1]+D_{8}[\mathrm{Oh}_{48},3]+D_{10}[\mathrm{Oh}_{48},3]\\
D_{30}\,[\mathrm{G}_{1536},8] &= 2D_{1}[\mathrm{G}_{16},1]+2D_{2}[\mathrm{G}_{16},1]+2D_{3}[\mathrm{G}_{16},1]+2D_{4}[\mathrm{G}_{16},1]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{2}[\mathrm{G}_{48},1]+D_{3}[\mathrm{G}_{48},1]+2D_{4}[\mathrm{G}_{48},3]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{22}[\mathrm{G}_{64},1]+D_{24}[\mathrm{G}_{64},1]+D_{30}[\mathrm{G}_{64},1]+D_{32}[\mathrm{G}_{64},1]+\\
&\quad+D_{54}[\mathrm{G}_{64},1]+D_{56}[\mathrm{G}_{64},1]+D_{62}[\mathrm{G}_{64},1]+D_{64}[\mathrm{G}_{64},1]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{4}[\mathrm{G}_{96},1]+D_{6}[\mathrm{G}_{96},1]+2D_{7}[\mathrm{G}_{96},3]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{42}[\mathrm{G}_{128},2]+D_{44}[\mathrm{G}_{128},2]+D_{50}[\mathrm{G}_{128},2]+D_{52}[\mathrm{G}_{128},2]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{5}[\mathrm{G}_{192},3]+D_{6}[\mathrm{G}_{192},3]+D_{18}[\mathrm{G}_{192},2]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{60}[\mathrm{G}_{256},4]+D_{62}[\mathrm{G}_{256},4]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{18}[\mathrm{G}_{768},4]+D_{22}[\mathrm{G}_{768},4]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{2}[\mathrm{GF}_{48},1]+D_{3}[\mathrm{GF}_{48},1]+2D_{4}[\mathrm{GF}_{48},3]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{5}[\mathrm{GF}_{192},3]+D_{6}[\mathrm{GF}_{192},3]+D_{18}[\mathrm{GF}_{192},2]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{3}[\mathrm{GF}_{96},2]+D_{4}[\mathrm{GF}_{96},3]+D_{5}[\mathrm{GF}_{96},3]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{5}[\mathrm{GP}_{24},1]+D_{6}[\mathrm{GP}_{24},1]+2D_{8}[\mathrm{GP}_{24},3]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{3}[\mathrm{GS}_{24},2]+D_{4}[\mathrm{GS}_{24},3]+D_{5}[\mathrm{GS}_{24},3]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{1}[\mathrm{GS}_{32},1]+D_{2}[\mathrm{GS}_{32},1]+D_{3}[\mathrm{GS}_{32},1]+D_{4}[\mathrm{GS}_{32},1]+\\
&\quad+2D_{9}[\mathrm{GS}_{32},2]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{3}[\mathrm{O}_{24},2]+D_{4}[\mathrm{O}_{24},3]+D_{5}[\mathrm{O}_{24},3]\\
D_{30}\,[\mathrm{G}_{1536},8] &= D_{6}[\mathrm{Oh}_{48},2]+D_{8}[\mathrm{Oh}_{48},3]+D_{10}[\mathrm{Oh}_{48},3]\\
D_{31}\,[\mathrm{G}_{1536},8] &= 2D_{1}[\mathrm{G}_{16},1]+2D_{2}[\mathrm{G}_{16},1]+2D_{3}[\mathrm{G}_{16},1]+2D_{4}[\mathrm{G}_{16},1]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{2}[\mathrm{G}_{48},1]+D_{3}[\mathrm{G}_{48},1]+2D_{4}[\mathrm{G}_{48},3]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{22}[\mathrm{G}_{64},1]+D_{24}[\mathrm{G}_{64},1]+D_{30}[\mathrm{G}_{64},1]+D_{32}[\mathrm{G}_{64},1]+\\
&\quad+D_{54}[\mathrm{G}_{64},1]+D_{56}[\mathrm{G}_{64},1]+D_{62}[\mathrm{G}_{64},1]+D_{64}[\mathrm{G}_{64},1]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{4}[\mathrm{G}_{96},1]+D_{6}[\mathrm{G}_{96},1]+2D_{7}[\mathrm{G}_{96},3]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{42}[\mathrm{G}_{128},2]+D_{44}[\mathrm{G}_{128},2]+D_{50}[\mathrm{G}_{128},2]+D_{52}[\mathrm{G}_{128},2]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{5}[\mathrm{G}_{192},3]+D_{6}[\mathrm{G}_{192},3]+D_{18}[\mathrm{G}_{192},2]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{60}[\mathrm{G}_{256},4]+D_{62}[\mathrm{G}_{256},4]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{19}[\mathrm{G}_{768},4]+D_{21}[\mathrm{G}_{768},4]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{2}[\mathrm{GF}_{48},1]+D_{3}[\mathrm{GF}_{48},1]+2D_{4}[\mathrm{GF}_{48},3]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{5}[\mathrm{GF}_{192},3]+D_{6}[\mathrm{GF}_{192},3]+D_{18}[\mathrm{GF}_{192},2]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{3}[\mathrm{GF}_{96},2]+D_{4}[\mathrm{GF}_{96},3]+D_{5}[\mathrm{GF}_{96},3]\\
D_{31}\,[\mathrm{G}_{1536},8] &= D_{5}[\mathrm{GP}_{24},1]+D_{6}[\mathrm{GP}_{24},1]+2D_{8}[\mathrm{GP}_{24},3]
\end{aligned}
$$

$D_{31}\,[\mathrm{G}_{1536},8] = D_3[\mathrm{GS}_{24},2] + D_4[\mathrm{GS}_{24},3] + D_5[\mathrm{GS}_{24},3]$

$D_{31}\,[\mathrm{G}_{1536},8] = D_1[\mathrm{GS}_{32},1] + D_2[\mathrm{GS}_{32},1] + D_3[\mathrm{GS}_{32},1] + D_4[\mathrm{GS}_{32},1] + 2D_9[\mathrm{GS}_{32},2]$

$D_{31}\,[\mathrm{G}_{1536},8] = D_3[\mathrm{O}_{24},2] + D_4[\mathrm{O}_{24},3] + D_5[\mathrm{O}_{24},3]$

$D_{31}\,[\mathrm{G}_{1536},8] = D_6[\mathrm{Oh}_{48},2] + D_8[\mathrm{Oh}_{48},3] + D_{10}[\mathrm{Oh}_{48},3]$

D.5. Branching Rules of the Irreps of Dimension 12.

$D_{32}\,[\mathrm{G}_{1536},12] = D_5[\mathrm{G}_{16},1] + D_6[\mathrm{G}_{16},1] + D_7[\mathrm{G}_{16},1] + D_8[\mathrm{G}_{16},1] + D_9[\mathrm{G}_{16},1] + D_{10}[\mathrm{G}_{16},1] + D_{11}[\mathrm{G}_{16},1] + D_{12}[\mathrm{G}_{16},1] + D_{13}[\mathrm{G}_{16},1] + D_{14}[\mathrm{G}_{16},1] + D_{15}[\mathrm{G}_{16},1] + D_{16}[\mathrm{G}_{16},1]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_5[\mathrm{G}_{48},3] + D_6[\mathrm{G}_{48},3] + D_7[\mathrm{G}_{48},3] + D_8[\mathrm{G}_{48},3]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_6[\mathrm{G}_{64},1] + D_8[\mathrm{G}_{64},1] + D_{14}[\mathrm{G}_{64},1] + D_{16}[\mathrm{G}_{64},1] + D_{18}[\mathrm{G}_{64},1] + D_{20}[\mathrm{G}_{64},1] + D_{21}[\mathrm{G}_{64},1] + D_{29}[\mathrm{G}_{64},1] + D_{50}[\mathrm{G}_{64},1] + D_{52}[\mathrm{G}_{64},1] + D_{53}[\mathrm{G}_{64},1] + D_{61}[\mathrm{G}_{64},1]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_9[\mathrm{G}_{96},3] + D_{11}[\mathrm{G}_{96},3] + D_{13}[\mathrm{G}_{96},3] + D_{15}[\mathrm{G}_{96},3]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_{34}[\mathrm{G}_{128},2] + D_{36}[\mathrm{G}_{128},2] + D_{38}[\mathrm{G}_{128},2] + D_{40}[\mathrm{G}_{128},2] + D_{41}[\mathrm{G}_{128},2] + D_{49}[\mathrm{G}_{128},2]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_9[\mathrm{G}_{192},3] + D_{13}[\mathrm{G}_{192},3] + D_{19}[\mathrm{G}_{192},6]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_{57}[\mathrm{G}_{256},4] + D_{58}[\mathrm{G}_{256},4] + D_{59}[\mathrm{G}_{256},4]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_{31}[\mathrm{G}_{768},12]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_5[\mathrm{GF}_{48},3] + D_6[\mathrm{GF}_{48},3] + D_7[\mathrm{GF}_{48},3] + D_8[\mathrm{GF}_{48},3]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_9[\mathrm{GF}_{192},3] + D_{13}[\mathrm{GF}_{192},3] + D_{19}[\mathrm{GF}_{192},6]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_6[\mathrm{GF}_{96},3] + D_8[\mathrm{GF}_{96},3] + D_{10}[\mathrm{GF}_{96},6]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_1[\mathrm{GP}_{24},1] + D_2[\mathrm{GP}_{24},1] + D_3[\mathrm{GP}_{24},1] + 3D_7[\mathrm{GP}_{24},3]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_2[\mathrm{GS}_{24},1] + D_3[\mathrm{GS}_{24},2] + 2D_4[\mathrm{GS}_{24},3] + D_5[\mathrm{GS}_{24},3]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_6[\mathrm{GS}_{32},1] + D_8[\mathrm{GS}_{32},1] + D_{10}[\mathrm{GS}_{32},2] + D_{11}[\mathrm{GS}_{32},2] + D_{12}[\mathrm{GS}_{32},2] + D_{13}[\mathrm{GS}_{32},2] + D_{14}[\mathrm{GS}_{32},2]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_2[\mathrm{O}_{24},1] + D_3[\mathrm{O}_{24},2] + 2D_4[\mathrm{O}_{24},3] + D_5[\mathrm{O}_{24},3]$

$D_{32}\,[\mathrm{G}_{1536},12] = D_3[\mathrm{Oh}_{48},1] + D_5[\mathrm{Oh}_{48},2] + 2D_7[\mathrm{Oh}_{48},3] + D_9[\mathrm{Oh}_{48},3]$

$D_{33}\,[\mathrm{G}_{1536},12] = D_5[\mathrm{G}_{16},1] + D_6[\mathrm{G}_{16},1] + D_7[\mathrm{G}_{16},1] + D_8[\mathrm{G}_{16},1] + D_9[\mathrm{G}_{16},1] + D_{10}[\mathrm{G}_{16},1] + D_{11}[\mathrm{G}_{16},1] + D_{12}[\mathrm{G}_{16},1] + D_{13}[\mathrm{G}_{16},1] + D_{14}[\mathrm{G}_{16},1] + D_{15}[\mathrm{G}_{16},1] + D_{16}[\mathrm{G}_{16},1]$

$D_{33}\,[\mathrm{G}_{1536},12] = D_5[\mathrm{G}_{48},3] + D_6[\mathrm{G}_{48},3] + D_7[\mathrm{G}_{48},3] + D_8[\mathrm{G}_{48},3]$

$D_{33}\,[\mathrm{G}_{1536},12] = D_6[\mathrm{G}_{64},1] + D_8[\mathrm{G}_{64},1] + D_{14}[\mathrm{G}_{64},1] + D_{16}[\mathrm{G}_{64},1] + D_{18}[\mathrm{G}_{64},1] + D_{20}[\mathrm{G}_{64},1] + D_{21}[\mathrm{G}_{64},1] + D_{29}[\mathrm{G}_{64},1] + D_{50}[\mathrm{G}_{64},1] + D_{52}[\mathrm{G}_{64},1] + D_{53}[\mathrm{G}_{64},1] + D_{61}[\mathrm{G}_{64},1]$

$D_{33}\,[\mathrm{G}_{1536},12] = D_9[\mathrm{G}_{96},3] + D_{11}[\mathrm{G}_{96},3] + D_{13}[\mathrm{G}_{96},3] + D_{15}[\mathrm{G}_{96},3]$

$D_{33}\,[\mathrm{G}_{1536},12] = D_{34}[\mathrm{G}_{128},2] + D_{36}[\mathrm{G}_{128},2] + D_{38}[\mathrm{G}_{128},2] + D_{40}[\mathrm{G}_{128},2] + D_{41}[\mathrm{G}_{128},2] + D_{49}[\mathrm{G}_{128},2]$

$D_{33}\,[\mathrm{G}_{1536},12] = D_{10}[\mathrm{G}_{192},3] + D_{14}[\mathrm{G}_{192},3] + D_{19}[\mathrm{G}_{192},6]$

$D_{33}\,[\mathrm{G}_{1536},12] = D_{57}[\mathrm{G}_{256},4] + D_{58}[\mathrm{G}_{256},4] + D_{59}[\mathrm{G}_{256},4]$

$D_{33}\,[\mathrm{G}_{1536},12] = D_{31}[\mathrm{G}_{768},12]$

$$
\begin{aligned}
D_{33}\,[\mathrm{G}_{1536},12] &= D_5[\mathrm{GF}_{48},3]+D_6[\mathrm{GF}_{48},3]+D_7[\mathrm{GF}_{48},3]+D_8[\mathrm{GF}_{48},3]\\
D_{33}\,[\mathrm{G}_{1536},12] &= D_{10}[\mathrm{GF}_{192},3]+D_{14}[\mathrm{GF}_{192},3]+D_{19}[\mathrm{GF}_{192},6]\\
D_{33}\,[\mathrm{G}_{1536},12] &= D_7[\mathrm{GF}_{96},3]+D_9[\mathrm{GF}_{96},3]+D_{10}[\mathrm{GF}_{96},6]\\
D_{33}\,[\mathrm{G}_{1536},12] &= D_1[\mathrm{GP}_{24},1]+D_2[\mathrm{GP}_{24},1]+D_3[\mathrm{GP}_{24},1]+3D_7[\mathrm{GP}_{24},3]\\
D_{33}\,[\mathrm{G}_{1536},12] &= D_1[\mathrm{GS}_{24},1]+D_3[\mathrm{GS}_{24},2]+D_4[\mathrm{GS}_{24},3]+2D_5[\mathrm{GS}_{24},3]\\
D_{33}\,[\mathrm{G}_{1536},12] &= D_5[\mathrm{GS}_{32},1]+D_7[\mathrm{GS}_{32},1]+D_{10}[\mathrm{GS}_{32},2]+D_{11}[\mathrm{GS}_{32},2]+\\
&\quad +D_{12}[\mathrm{GS}_{32},2]+D_{13}[\mathrm{GS}_{32},2]+D_{14}[\mathrm{GS}_{32},2]\\
D_{33}\,[\mathrm{G}_{1536},12] &= D_1[\mathrm{O}_{24},1]+D_3[\mathrm{O}_{24},2]+D_4[\mathrm{O}_{24},3]+2D_5[\mathrm{O}_{24},3]\\
D_{33}\,[\mathrm{G}_{1536},12] &= D_1[\mathrm{Oh}_{48},1]+D_5[\mathrm{Oh}_{48},2]+D_7[\mathrm{Oh}_{48},3]+2D_9[\mathrm{Oh}_{48},3]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_5[\mathrm{G}_{16},1]+D_6[\mathrm{G}_{16},1]+D_7[\mathrm{G}_{16},1]+D_8[\mathrm{G}_{16},1]+\\
&\quad +D_9[\mathrm{G}_{16},1]+D_{10}[\mathrm{G}_{16},1]+D_{11}[\mathrm{G}_{16},1]+D_{12}[\mathrm{G}_{16},1]+\\
&\quad +D_{13}[\mathrm{G}_{16},1]+D_{14}[\mathrm{G}_{16},1]+D_{15}[\mathrm{G}_{16},1]+D_{16}[\mathrm{G}_{16},1]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_5[\mathrm{G}_{48},3]+D_6[\mathrm{G}_{48},3]+D_7[\mathrm{G}_{48},3]+D_8[\mathrm{G}_{48},3]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_{23}[\mathrm{G}_{64},1]+D_{26}[\mathrm{G}_{64},1]+D_{28}[\mathrm{G}_{64},1]+D_{31}[\mathrm{G}_{64},1]+\\
&\quad +D_{38}[\mathrm{G}_{64},1]+D_{40}[\mathrm{G}_{64},1]+D_{46}[\mathrm{G}_{64},1]+D_{48}[\mathrm{G}_{64},1]+\\
&\quad +D_{55}[\mathrm{G}_{64},1]+D_{58}[\mathrm{G}_{64},1]+D_{60}[\mathrm{G}_{64},1]+D_{63}[\mathrm{G}_{64},1]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_9[\mathrm{G}_{96},3]+D_{11}[\mathrm{G}_{96},3]+D_{13}[\mathrm{G}_{96},3]+D_{15}[\mathrm{G}_{96},3]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_{43}[\mathrm{G}_{128},2]+D_{46}[\mathrm{G}_{128},2]+D_{48}[\mathrm{G}_{128},2]+D_{51}[\mathrm{G}_{128},2]+\\
&\quad +D_{54}[\mathrm{G}_{128},2]+D_{56}[\mathrm{G}_{128},2]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_{10}[\mathrm{G}_{192},3]+D_{14}[\mathrm{G}_{192},3]+D_{19}[\mathrm{G}_{192},6]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_{61}[\mathrm{G}_{256},4]+D_{63}[\mathrm{G}_{256},4]+D_{64}[\mathrm{G}_{256},4]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_{32}[\mathrm{G}_{768},12]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_5[\mathrm{GF}_{48},3]+D_6[\mathrm{GF}_{48},3]+D_7[\mathrm{GF}_{48},3]+D_8[\mathrm{GF}_{48},3]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_9[\mathrm{GF}_{192},3]+D_{13}[\mathrm{GF}_{192},3]+D_{19}[\mathrm{GF}_{192},6]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_6[\mathrm{GF}_{96},3]+D_8[\mathrm{GF}_{96},3]+D_{10}[\mathrm{GF}_{96},6]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_1[\mathrm{GP}_{24},1]+D_2[\mathrm{GP}_{24},1]+D_3[\mathrm{GP}_{24},1]+3D_7[\mathrm{GP}_{24},3]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_2[\mathrm{GS}_{24},1]+D_3[\mathrm{GS}_{24},2]+2D_4[\mathrm{GS}_{24},3]+D_5[\mathrm{GS}_{24},3]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_5[\mathrm{GS}_{32},1]+D_7[\mathrm{GS}_{32},1]+D_{10}[\mathrm{GS}_{32},2]+D_{11}[\mathrm{GS}_{32},2]+\\
&\quad +D_{12}[\mathrm{GS}_{32},2]+D_{13}[\mathrm{GS}_{32},2]+D_{14}[\mathrm{GS}_{32},2]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_1[\mathrm{O}_{24},1]+D_3[\mathrm{O}_{24},2]+D_4[\mathrm{O}_{24},3]+2D_5[\mathrm{O}_{24},3]\\
D_{34}\,[\mathrm{G}_{1536},12] &= D_1[\mathrm{Oh}_{48},1]+D_5[\mathrm{Oh}_{48},2]+D_7[\mathrm{Oh}_{48},3]+2D_9[\mathrm{Oh}_{48},3]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_5[\mathrm{G}_{16},1]+D_6[\mathrm{G}_{16},1]+D_7[\mathrm{G}_{16},1]+D_8[\mathrm{G}_{16},1]+\\
&\quad +D_9[\mathrm{G}_{16},1]+D_{10}[\mathrm{G}_{16},1]+D_{11}[\mathrm{G}_{16},1]+D_{12}[\mathrm{G}_{16},1]+\\
&\quad +D_{13}[\mathrm{G}_{16},1]+D_{14}[\mathrm{G}_{16},1]+D_{15}[\mathrm{G}_{16},1]+D_{16}[\mathrm{G}_{16},1]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_5[\mathrm{G}_{48},3]+D_6[\mathrm{G}_{48},3]+D_7[\mathrm{G}_{48},3]+D_8[\mathrm{G}_{48},3]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{23}[\mathrm{G}_{64},1]+D_{26}[\mathrm{G}_{64},1]+D_{28}[\mathrm{G}_{64},1]+D_{31}[\mathrm{G}_{64},1]+\\
&\quad +D_{38}[\mathrm{G}_{64},1]+D_{40}[\mathrm{G}_{64},1]+D_{46}[\mathrm{G}_{64},1]+D_{48}[\mathrm{G}_{64},1]+\\
&\quad +D_{55}[\mathrm{G}_{64},1]+D_{58}[\mathrm{G}_{64},1]+D_{60}[\mathrm{G}_{64},1]+D_{63}[\mathrm{G}_{64},1]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_9[\mathrm{G}_{96},3]+D_{11}[\mathrm{G}_{96},3]+D_{13}[\mathrm{G}_{96},3]+D_{15}[\mathrm{G}_{96},3]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{43}[\mathrm{G}_{128},2]+D_{46}[\mathrm{G}_{128},2]+D_{48}[\mathrm{G}_{128},2]+D_{51}[\mathrm{G}_{128},2]+\\
&\quad +D_{54}[\mathrm{G}_{128},2]+D_{56}[\mathrm{G}_{128},2]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_9[\mathrm{G}_{192},3]+D_{13}[\mathrm{G}_{192},3]+D_{19}[\mathrm{G}_{192},6]
\end{aligned}
$$

$$
\begin{aligned}
D_{35}\,[\mathrm{G}_{1536},12] &= D_{61}[\mathrm{G}_{256},4]+D_{63}[\mathrm{G}_{256},4]+D_{64}[\mathrm{G}_{256},4]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{32}[\mathrm{G}_{768},12]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{5}[\mathrm{GF}_{48},3]+D_{6}[\mathrm{GF}_{48},3]+D_{7}[\mathrm{GF}_{48},3]+D_{8}[\mathrm{GF}_{48},3]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{10}[\mathrm{GF}_{192},3]+D_{14}[\mathrm{GF}_{192},3]+D_{19}[\mathrm{GF}_{192},6]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{7}[\mathrm{GF}_{96},3]+D_{9}[\mathrm{GF}_{96},3]+D_{10}[\mathrm{GF}_{96},6]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{1}[\mathrm{GP}_{24},1]+D_{2}[\mathrm{GP}_{24},1]+D_{3}[\mathrm{GP}_{24},1]+3D_{7}[\mathrm{GP}_{24},3]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{1}[\mathrm{GS}_{24},1]+D_{3}[\mathrm{GS}_{24},2]+D_{4}[\mathrm{GS}_{24},3]+2D_{5}[\mathrm{GS}_{24},3]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{6}[\mathrm{GS}_{32},1]+D_{8}[\mathrm{GS}_{32},1]+D_{10}[\mathrm{GS}_{32},2]+D_{11}[\mathrm{GS}_{32},2]+\\
&\quad +D_{12}[\mathrm{GS}_{32},2]+D_{13}[\mathrm{GS}_{32},2]+D_{14}[\mathrm{GS}_{32},2]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{2}[\mathrm{O}_{24},1]+D_{3}[\mathrm{O}_{24},2]+2D_{4}[\mathrm{O}_{24},3]+D_{5}[\mathrm{O}_{24},3]\\
D_{35}\,[\mathrm{G}_{1536},12] &= D_{3}[\mathrm{Oh}_{48},1]+D_{5}[\mathrm{Oh}_{48},2]+2D_{7}[\mathrm{Oh}_{48},3]+D_{9}[\mathrm{Oh}_{48},3]\\
D_{36}\,[\mathrm{G}_{1536},12] &= 2D_{6}[\mathrm{G}_{16},1]+2D_{7}[\mathrm{G}_{16},1]+2D_{9}[\mathrm{G}_{16},1]+2D_{11}[\mathrm{G}_{16},1]+\\
&\quad +2D_{15}[\mathrm{G}_{16},1]+2D_{16}[\mathrm{G}_{16},1]\\
D_{36}\,[\mathrm{G}_{1536},12] &= 2D_{6}[\mathrm{G}_{48},3]+2D_{7}[\mathrm{G}_{48},3]\\
D_{36}\,[\mathrm{G}_{1536},12] &= D_{7}[\mathrm{G}_{64},1]+D_{10}[\mathrm{G}_{64},1]+D_{12}[\mathrm{G}_{64},1]+D_{15}[\mathrm{G}_{64},1]+\\
&\quad +D_{19}[\mathrm{G}_{64},1]+D_{25}[\mathrm{G}_{64},1]+D_{34}[\mathrm{G}_{64},1]+D_{36}[\mathrm{G}_{64},1]+\\
&\quad +D_{37}[\mathrm{G}_{64},1]+D_{45}[\mathrm{G}_{64},1]+D_{51}[\mathrm{G}_{64},1]+D_{57}[\mathrm{G}_{64},1]\\
D_{36}\,[\mathrm{G}_{1536},12] &= 2D_{12}[\mathrm{G}_{96},3]+2D_{14}[\mathrm{G}_{96},3]\\
D_{36}\,[\mathrm{G}_{1536},12] &= D_{11}[\mathrm{G}_{128},1]+D_{15}[\mathrm{G}_{128},1]+D_{19}[\mathrm{G}_{128},1]+D_{23}[\mathrm{G}_{128},1]+\\
&\quad +D_{35}[\mathrm{G}_{128},2]+D_{39}[\mathrm{G}_{128},2]+D_{45}[\mathrm{G}_{128},2]+D_{53}[\mathrm{G}_{128},2]\\
D_{36}\,[\mathrm{G}_{1536},12] &= D_{11}[\mathrm{G}_{192},3]+D_{12}[\mathrm{G}_{192},3]+D_{15}[\mathrm{G}_{192},3]+D_{16}[\mathrm{G}_{192},3]\\
D_{36}\,[\mathrm{G}_{1536},12] &= D_{35}[\mathrm{G}_{256},2]+D_{37}[\mathrm{G}_{256},2]+D_{44}[\mathrm{G}_{256},2]+D_{47}[\mathrm{G}_{256},2]+
\end{aligned}
$$

$$
\begin{aligned}
&+D_{49}[\mathrm{G}_{256},2]+D_{54}[\mathrm{G}_{256},2]\\
D_{36}\,[\mathrm{G}_{1536},12] &= D_{25}[\mathrm{G}_{768},6]+D_{27}[\mathrm{G}_{768},6]\\
D_{36}\,[\mathrm{G}_{1536},12] &= 2D_{5}[\mathrm{GF}_{48},3]+2D_{8}[\mathrm{GF}_{48},3]\\
D_{36}\,[\mathrm{G}_{1536},12] &= 2D_{20}[\mathrm{GF}_{192},6]\\
D_{36}\,[\mathrm{G}_{1536},12] &= 2D_{10}[\mathrm{GF}_{96},6]\\
D_{36}\,[\mathrm{G}_{1536},12] &= 2D_{4}[\mathrm{GP}_{24},1]+2D_{5}[\mathrm{GP}_{24},1]+2D_{6}[\mathrm{GP}_{24},1]+2D_{8}[\mathrm{GP}_{24},3]\\
D_{36}\,[\mathrm{G}_{1536},12] &= 2D_{4}[\mathrm{GS}_{24},3]+2D_{5}[\mathrm{GS}_{24},3]\\
D_{36}\,[\mathrm{G}_{1536},12] &= D_{5}[\mathrm{GS}_{32},1]+D_{6}[\mathrm{GS}_{32},1]+D_{7}[\mathrm{GS}_{32},1]+D_{8}[\mathrm{GS}_{32},1]+2D_{12}[\mathrm{GS}_{32},2]+\\
&2D_{14}[\mathrm{GS}_{32},2]\\
D_{36}\,[\mathrm{G}_{1536},12] &= D_{1}[\mathrm{O}_{24},1]+D_{2}[\mathrm{O}_{24},1]+2D_{3}[\mathrm{O}_{24},2]+D_{4}[\mathrm{O}_{24},3]+\\
&\quad +D_{5}[\mathrm{O}_{24},3]\\
D_{36}\,[\mathrm{G}_{1536},12] &= D_{2}[\mathrm{Oh}_{48},1]+D_{4}[\mathrm{Oh}_{48},1]+2D_{6}[\mathrm{Oh}_{48},2]+D_{8}[\mathrm{Oh}_{48},3]+\\
&\quad +D_{10}[\mathrm{Oh}_{48},3]\\
D_{37}\,[\mathrm{G}_{1536},12] &= 2D_{5}[\mathrm{G}_{16},1]+2D_{8}[\mathrm{G}_{16},1]+2D_{10}[\mathrm{G}_{16},1]+2D_{12}[\mathrm{G}_{16},1]+\\
&\quad +2D_{13}[\mathrm{G}_{16},1]+2D_{14}[\mathrm{G}_{16},1]\\
D_{37}\,[\mathrm{G}_{1536},12] &= 2D_{5}[\mathrm{G}_{48},3]+2D_{8}[\mathrm{G}_{48},3]\\
D_{37}\,[\mathrm{G}_{1536},12] &= D_{7}[\mathrm{G}_{64},1]+D_{10}[\mathrm{G}_{64},1]+D_{12}[\mathrm{G}_{64},1]+D_{15}[\mathrm{G}_{64},1]+\\
&\quad +D_{19}[\mathrm{G}_{64},1]+D_{25}[\mathrm{G}_{64},1]+D_{34}[\mathrm{G}_{64},1]+D_{36}[\mathrm{G}_{64},1]+\\
&\quad +D_{37}[\mathrm{G}_{64},1]+D_{45}[\mathrm{G}_{64},1]+D_{51}[\mathrm{G}_{64},1]+D_{57}[\mathrm{G}_{64},1]
\end{aligned}
$$

$$D_{37}\,[\mathrm{G}_{1536},12] \;=\; 2D_{10}[\mathrm{G}_{96},3]+2D_{16}[\mathrm{G}_{96},3]$$
$$\begin{aligned}D_{37}\,[\mathrm{G}_{1536},12] \;=\;& D_{12}[\mathrm{G}_{128},1]+D_{16}[\mathrm{G}_{128},1]+D_{20}[\mathrm{G}_{128},1]+D_{24}[\mathrm{G}_{128},1]+\\&+D_{35}[\mathrm{G}_{128},2]+D_{39}[\mathrm{G}_{128},2]+D_{45}[\mathrm{G}_{128},2]+D_{53}[\mathrm{G}_{128},2]\end{aligned}$$

E. OTHER RELEVANT SUBGROUPS

In this appendix, we list the additional chains of subgroups of the Universal Classifying Group G_{1536} that emerge in the analysis of the classical model of ABC-flows. As in the previous cases, we just give for each of them the conjugacy classes of which they are composed. The interesting network of interrelation between these subgroups, which explains the various cases and subcases of ABC-flows is thoroughly discussed in the main text.

E.1. The Group $\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}$.

Conjugacy Class $\mathcal{C}_1\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\{1_1 \quad 0 \quad 0 \quad 0\} \tag{E.1}$$

Conjugacy Class $\mathcal{C}_2\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\{1_1 \quad 0 \quad 1 \quad 0\} \tag{E.2}$$

Conjugacy Class $\mathcal{C}_3\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\{1_1 \quad 1 \quad 0 \quad 1\} \tag{E.3}$$

Conjugacy Class $\mathcal{C}_4\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\{1_1 \quad 1 \quad 1 \quad 1\} \tag{E.4}$$

Conjugacy Class $\mathcal{C}_5\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\{3_2 \quad 0 \quad 0 \quad 1\} \tag{E.5}$$

Conjugacy Class $\mathcal{C}_6\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\{3_2 \quad 0 \quad 1 \quad 1\} \tag{E.6}$$

Conjugacy Class $\mathcal{C}_7\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\{3_2 \quad 1 \quad 0 \quad 0\} \tag{E.7}$$

Conjugacy Class $\mathcal{C}_8\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\{3_2 \quad 1 \quad 1 \quad 0\} \tag{E.8}$$

Conjugacy Class $\mathcal{C}_9\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$
$$\begin{array}{l}\{1_1 \quad 0 \quad 0 \quad 1\}\\ \{1_1 \quad 1 \quad 0 \quad 0\}\end{array} \tag{E.9}$$

$$\textit{Conjugacy Class } \mathcal{C}_{10}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right) \begin{array}{l} \{1_1 \quad 0 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.10}$$

$$\textit{Conjugacy Class } \mathcal{C}_{11}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right) \begin{array}{l} \{1_1 \quad 0 \quad 1 \quad 1\} \\ \{1_1 \quad 1 \quad 1 \quad 0\} \end{array} \tag{E.11}$$

$$\textit{Conjugacy Class } \mathcal{C}_{12}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right) \begin{array}{l} \{1_1 \quad 1 \quad \frac{1}{2} \quad 1\} \\ \{1_1 \quad 1 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.12}$$

$$\textit{Conjugacy Class } \mathcal{C}_{13}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right) \begin{array}{l} \{3_2 \quad 0 \quad 0 \quad 0\} \\ \{3_2 \quad 1 \quad 0 \quad 1\} \end{array} \tag{E.13}$$

$$\textit{Conjugacy Class } \mathcal{C}_{14}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right) \begin{array}{l} \{3_2 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{3_2 \quad 0 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.14}$$

$$\textit{Conjugacy Class } \mathcal{C}_{15}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right) \begin{array}{l} \{3_2 \quad 0 \quad 1 \quad 0\} \\ \{3_2 \quad 1 \quad 1 \quad 1\} \end{array} \tag{E.15}$$

$$\textit{Conjugacy Class } \mathcal{C}_{16}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right) \begin{array}{l} \{3_2 \quad 1 \quad \frac{1}{2} \quad 0\} \\ \{3_2 \quad 1 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.16}$$

$$\textit{Conjugacy Class } \mathcal{C}_{17}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right) \begin{array}{l} \{1_1 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 1\} \\ \{1_1 \quad 1 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 1 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.17}$$

$$\textit{Conjugacy Class } \mathcal{C}_{18}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$$

$$\begin{array}{l} \{3_1 \quad 0 \quad 0 \quad 0\} \\ \{3_1 \quad 0 \quad 1 \quad 0\} \\ \{3_3 \quad 0 \quad 0 \quad 1\} \\ \{3_3 \quad 0 \quad 1 \quad 1\} \end{array} \tag{E.18}$$

Conjugacy Class $\mathcal{C}_{19}\left(\mathrm{G}_{128}^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{llll} \{3_1 & 0 & 0 & 1\} \\ \{3_1 & 0 & 1 & 1\} \\ \{3_3 & 1 & 0 & 1\} \\ \{3_3 & 1 & 1 & 1\} \end{array} \tag{E.19}$$

Conjugacy Class $\mathcal{C}_{20}\left(\mathrm{G}_{128}^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{llll} \{3_1 & 0 & \frac{1}{2} & 0\} \\ \{3_1 & 0 & \frac{3}{2} & 0\} \\ \{3_3 & 0 & \frac{1}{2} & 1\} \\ \{3_3 & 0 & \frac{3}{2} & 1\} \end{array} \tag{E.20}$$

Conjugacy Class $\mathcal{C}_{21}\left(\mathrm{G}_{128}^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{llll} \{3_1 & 0 & \frac{1}{2} & 1\} \\ \{3_1 & 0 & \frac{3}{2} & 1\} \\ \{3_3 & 1 & \frac{1}{2} & 1\} \\ \{3_3 & 1 & \frac{3}{2} & 1\} \end{array} \tag{E.21}$$

Conjugacy Class $\mathcal{C}_{22}\left(\mathrm{G}_{128}^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{llll} \{3_1 & 1 & 0 & 0\} \\ \{3_1 & 1 & 1 & 0\} \\ \{3_3 & 0 & 0 & 0\} \\ \{3_3 & 0 & 1 & 0\} \end{array} \tag{E.22}$$

Conjugacy Class $\mathcal{C}_{23}\left(\mathrm{G}_{128}^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{llll} \{3_1 & 1 & 0 & 1\} \\ \{3_1 & 1 & 1 & 1\} \\ \{3_3 & 1 & 0 & 0\} \\ \{3_3 & 1 & 1 & 0\} \end{array} \tag{E.23}$$

Conjugacy Class $\mathcal{C}_{24}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l}\{3_1 \quad 1 \quad \frac{1}{2} \quad 0\}\\ \{3_1 \quad 1 \quad \frac{3}{2} \quad 0\}\\ \{3_3 \quad 0 \quad \frac{1}{2} \quad 0\}\\ \{3_3 \quad 0 \quad \frac{3}{2} \quad 0\}\end{array} \tag{E.24}$$

Conjugacy Class $\mathcal{C}_{25}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l}\{3_1 \quad 1 \quad \frac{1}{2} \quad 1\}\\ \{3_1 \quad 1 \quad \frac{3}{2} \quad 1\}\\ \{3_3 \quad 1 \quad \frac{1}{2} \quad 0\}\\ \{3_3 \quad 1 \quad \frac{3}{2} \quad 0\}\end{array} \tag{E.25}$$

Conjugacy Class $\mathcal{C}_{26}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l}\{3_2 \quad 0 \quad \frac{1}{2} \quad 0\}\\ \{3_2 \quad 0 \quad \frac{3}{2} \quad 0\}\\ \{3_2 \quad 1 \quad \frac{1}{2} \quad 1\}\\ \{3_2 \quad 1 \quad \frac{3}{2} \quad 1\}\end{array} \tag{E.26}$$

Conjugacy Class $\mathcal{C}_{27}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l}\{5_2 \quad \frac{1}{2} \quad 0 \quad \frac{1}{2}\}\\ \{5_2 \quad \frac{3}{2} \quad 0 \quad \frac{3}{2}\}\\ \{5_3 \quad \frac{1}{2} \quad 0 \quad \frac{3}{2}\}\\ \{5_3 \quad \frac{3}{2} \quad 0 \quad \frac{1}{2}\}\end{array} \tag{E.27}$$

Conjugacy Class $\mathcal{C}_{28}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l}\{5_2 \quad \frac{1}{2} \quad 0 \quad \frac{3}{2}\}\\ \{5_2 \quad \frac{3}{2} \quad 0 \quad \frac{1}{2}\}\\ \{5_3 \quad \frac{1}{2} \quad 0 \quad \frac{1}{2}\}\\ \{5_3 \quad \frac{3}{2} \quad 0 \quad \frac{3}{2}\}\end{array} \tag{E.28}$$

Conjugacy Class $\mathcal{C}_{29}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l}\{5_2 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\}\\ \{5_2 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{3}{2}\}\\ \{5_3 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{3}{2}\}\\ \{5_3 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{1}{2}\}\end{array} \tag{E.29}$$

Conjugacy Class $\mathcal{C}_{30}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{5_2 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2}\} \\ \{5_2 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\} \\ \{5_3 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2}\} \\ \{5_3 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\} \end{array} \tag{E.30}$$

Conjugacy Class $\mathcal{C}_{31}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{5_2 & \frac{1}{2} & 1 & \frac{1}{2}\} \\ \{5_2 & \frac{3}{2} & 1 & \frac{3}{2}\} \\ \{5_3 & \frac{1}{2} & 1 & \frac{3}{2}\} \\ \{5_3 & \frac{3}{2} & 1 & \frac{1}{2}\} \end{array} \tag{E.31}$$

Conjugacy Class $\mathcal{C}_{32}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{5_2 & \frac{1}{2} & 1 & \frac{3}{2}\} \\ \{5_2 & \frac{3}{2} & 1 & \frac{1}{2}\} \\ \{5_3 & \frac{1}{2} & 1 & \frac{1}{2}\} \\ \{5_3 & \frac{3}{2} & 1 & \frac{3}{2}\} \end{array} \tag{E.32}$$

Conjugacy Class $\mathcal{C}_{33}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{5_2 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2}\} \\ \{5_2 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\} \\ \{5_3 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2}\} \\ \{5_3 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\} \end{array} \tag{E.33}$$

Conjugacy Class $\mathcal{C}_{34}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{5_2 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2}\} \\ \{5_2 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2}\} \\ \{5_3 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\} \\ \{5_3 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2}\} \end{array} \tag{E.34}$$

Conjugacy Class $\mathcal{C}_{35}\left(\mathrm{G}_{128}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}
\{4_4 & \frac{1}{2} & 0 & \frac{1}{2}\} \\
\{4_4 & \frac{1}{2} & 1 & \frac{1}{2}\} \\
\{4_4 & \frac{3}{2} & 0 & \frac{3}{2}\} \\
\{4_4 & \frac{3}{2} & 1 & \frac{3}{2}\} \\
\{4_5 & \frac{1}{2} & 0 & \frac{3}{2}\} \\
\{4_5 & \frac{1}{2} & 1 & \frac{3}{2}\} \\
\{4_5 & \frac{3}{2} & 0 & \frac{1}{2}\} \\
\{4_5 & \frac{3}{2} & 1 & \frac{1}{2}\}
\end{array} \tag{E.35}$$

Conjugacy Class $\mathcal{C}_{36}\left(\mathrm{G}_{128}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}
\{4_4 & \frac{1}{2} & 0 & \frac{3}{2}\} \\
\{4_4 & \frac{1}{2} & 1 & \frac{3}{2}\} \\
\{4_4 & \frac{3}{2} & 0 & \frac{1}{2}\} \\
\{4_4 & \frac{3}{2} & 1 & \frac{1}{2}\} \\
\{4_5 & \frac{1}{2} & 0 & \frac{1}{2}\} \\
\{4_5 & \frac{1}{2} & 1 & \frac{1}{2}\} \\
\{4_5 & \frac{3}{2} & 0 & \frac{3}{2}\} \\
\{4_5 & \frac{3}{2} & 1 & \frac{3}{2}\}
\end{array} \tag{E.36}$$

Conjugacy Class $\mathcal{C}_{37}\left(\mathrm{G}_{128}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}
\{4_4 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\} \\
\{4_4 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2}\} \\
\{4_4 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2}\} \\
\{4_4 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\} \\
\{4_5 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2}\} \\
\{4_5 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2}\} \\
\{4_5 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\} \\
\{4_5 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2}\}
\end{array} \tag{E.37}$$

Conjugacy Class $\mathcal{C}_{38}\left(\mathrm{G}_{128}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll} \{4_4 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2}\} \\ \{4_4 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2}\} \\ \{4_4 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\} \\ \{4_4 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2}\} \\ \{4_5 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\} \\ \{4_5 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2}\} \\ \{4_5 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2}\} \\ \{4_5 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\} \end{array} \tag{E.38}$$

E.2. The Group $\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}$.

Conjugacy Class $\mathcal{C}_{1}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 0\} \tag{E.39}$$

Conjugacy Class $\mathcal{C}_{2}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 1\} \tag{E.40}$$

Conjugacy Class $\mathcal{C}_{3}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 0 \quad 1 \quad 0\} \tag{E.41}$$

Conjugacy Class $\mathcal{C}_{4}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 0 \quad 1 \quad 1\} \tag{E.42}$$

Conjugacy Class $\mathcal{C}_{5}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 1 \quad 0 \quad 0\} \tag{E.43}$$

Conjugacy Class $\mathcal{C}_{6}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 1 \quad 0 \quad 1\} \tag{E.44}$$

Conjugacy Class $\mathcal{C}_{7}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 1 \quad 1 \quad 0\} \tag{E.45}$$

Conjugacy Class $\mathcal{C}_{8}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 1 \quad 1 \quad 1\} \tag{E.46}$$

Conjugacy Class $\mathcal{C}_{9}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 0 \quad 0 \quad 0\} \tag{E.47}$$

Conjugacy Class $\mathcal{C}_{10}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 0 \quad 0 \quad 1\} \tag{E.48}$$

Conjugacy Class $\mathcal{C}_{11}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 0 \quad 1 \quad 0\} \tag{E.49}$$

Conjugacy Class $\mathcal{C}_{12}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 0 \quad 1 \quad 1\} \tag{E.50}$$

Conjugacy Class $\mathcal{C}_{13}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 1 \quad 0 \quad 0\} \tag{E.51}$$

Conjugacy Class $\mathcal{C}_{14}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 1 \quad 0 \quad 1\} \tag{E.52}$$

Conjugacy Class $\mathcal{C}_{15}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 1 \quad 1 \quad 0\} \tag{E.53}$$

Conjugacy Class $\mathcal{C}_{16}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 1 \quad 1 \quad 1\} \tag{E.54}$$

Conjugacy Class $\mathcal{C}_{17}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{1_1 \quad 0 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.55}$$

Conjugacy Class $\mathcal{C}_{18}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{1_1 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.56}$$

Conjugacy Class $\mathcal{C}_{19}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{1_1 \quad 1 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 1 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.57}$$

Conjugacy Class $\mathcal{C}_{20}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{1_1 \quad 1 \quad \frac{1}{2} \quad 1\} \\ \{1_1 \quad 1 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.58}$$

Conjugacy Class $\mathcal{C}_{21}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 0 & 0 & 0\} \\ \{3_1 & 0 & 1 & 0\}\end{array} \tag{E.59}$$

Conjugacy Class $\mathcal{C}_{22}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 0 & 0 & 1\} \\ \{3_1 & 0 & 1 & 1\}\end{array} \tag{E.60}$$

Conjugacy Class $\mathcal{C}_{23}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 0 & \frac{1}{2} & 0\} \\ \{3_1 & 0 & \frac{3}{2} & 0\}\end{array} \tag{E.61}$$

Conjugacy Class $\mathcal{C}_{24}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 0 & \frac{1}{2} & 1\} \\ \{3_1 & 0 & \frac{3}{2} & 1\}\end{array} \tag{E.62}$$

Conjugacy Class $\mathcal{C}_{25}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 1 & 0 & 0\} \\ \{3_1 & 1 & 1 & 0\}\end{array} \tag{E.63}$$

Conjugacy Class $\mathcal{C}_{26}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 1 & 0 & 1\} \\ \{3_1 & 1 & 1 & 1\}\end{array} \tag{E.64}$$

Conjugacy Class $\mathcal{C}_{27}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 1 & \frac{1}{2} & 0\} \\ \{3_1 & 1 & \frac{3}{2} & 0\}\end{array} \tag{E.65}$$

Conjugacy Class $\mathcal{C}_{28}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 1 & \frac{1}{2} & 1\} \\ \{3_1 & 1 & \frac{3}{2} & 1\}\end{array} \tag{E.66}$$

Conjugacy Class $\mathcal{C}_{29}\left(\mathrm{G}_{64}^{(\mathrm{A,B},0)}\right)$

$$\begin{array}{llll}\{3_2 & 0 & \frac{1}{2} & 0\} \\ \{3_2 & 0 & \frac{3}{2} & 0\}\end{array} \tag{E.67}$$

Conjugacy Class $\mathcal{C}_{30}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_2 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{3_2 \quad 0 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.68}$$

Conjugacy Class $\mathcal{C}_{31}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_2 \quad 1 \quad \frac{1}{2} \quad 0\} \\ \{3_2 \quad 1 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.69}$$

Conjugacy Class $\mathcal{C}_{32}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_2 \quad 1 \quad \frac{1}{2} \quad 1\} \\ \{3_2 \quad 1 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.70}$$

Conjugacy Class $\mathcal{C}_{33}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_3 \quad 0 \quad 0 \quad 0\} \\ \{3_3 \quad 0 \quad 1 \quad 0\} \end{array} \tag{E.71}$$

Conjugacy Class $\mathcal{C}_{34}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_3 \quad 0 \quad 0 \quad 1\} \\ \{3_3 \quad 0 \quad 1 \quad 1\} \end{array} \tag{E.72}$$

Conjugacy Class $\mathcal{C}_{35}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_3 \quad 0 \quad \frac{1}{2} \quad 0\} \\ \{3_3 \quad 0 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.73}$$

Conjugacy Class $\mathcal{C}_{36}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_3 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{3_3 \quad 0 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.74}$$

Conjugacy Class $\mathcal{C}_{37}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_3 \quad 1 \quad 0 \quad 0\} \\ \{3_3 \quad 1 \quad 1 \quad 0\} \end{array} \tag{E.75}$$

Conjugacy Class $\mathcal{C}_{38}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{l} \{3_3 \quad 1 \quad 0 \quad 1\} \\ \{3_3 \quad 1 \quad 1 \quad 1\} \end{array} \tag{E.76}$$

Conjugacy Class $\mathcal{C}_{39}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_3 & 1 & \frac{1}{2} & 0\} \\ \{3_3 & 1 & \frac{3}{2} & 0\} \end{array} \tag{E.77}$$

Conjugacy Class $\mathcal{C}_{40}\left(\mathrm{G}_{64}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_3 & 1 & \frac{1}{2} & 1\} \\ \{3_3 & 1 & \frac{3}{2} & 1\} \end{array} \tag{E.78}$$

E.3. The Group $\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}$.

Conjugacy Class $\mathcal{C}_{1}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 0\} \tag{E.79}$$

Conjugacy Class $\mathcal{C}_{2}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 1\} \tag{E.80}$$

Conjugacy Class $\mathcal{C}_{3}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 0 \quad 1 \quad 0\} \tag{E.81}$$

Conjugacy Class $\mathcal{C}_{4}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{1_1 \quad 0 \quad 1 \quad 1\} \tag{E.82}$$

Conjugacy Class $\mathcal{C}_{5}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 0 \quad 0 \quad 0\} \tag{E.83}$$

Conjugacy Class $\mathcal{C}_{6}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 0 \quad 0 \quad 1\} \tag{E.84}$$

Conjugacy Class $\mathcal{C}_{7}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 0 \quad 1 \quad 0\} \tag{E.85}$$

Conjugacy Class $\mathcal{C}_{8}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\{3_2 \quad 0 \quad 1 \quad 1\} \tag{E.86}$$

Conjugacy Class $\mathcal{C}_{9}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{1_1 & 0 & \frac{1}{2} & 0\} \\ \{1_1 & 0 & \frac{3}{2} & 0\} \end{array} \tag{E.87}$$

Conjugacy Class $\mathcal{C}_{10}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{1_1 & 0 & \frac{1}{2} & 1\} \\ \{1_1 & 0 & \frac{3}{2} & 1\} \end{array} \tag{E.88}$$

Conjugacy Class $\mathcal{C}_{11}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_1 & 1 & 0 & 0\} \\ \{3_1 & 1 & 1 & 0\} \end{array} \tag{E.89}$$

Conjugacy Class $\mathcal{C}_{12}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_1 & 1 & 0 & 1\} \\ \{3_1 & 1 & 1 & 1\} \end{array} \tag{E.90}$$

Conjugacy Class $\mathcal{C}_{13}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_1 & 1 & \frac{1}{2} & 0\} \\ \{3_1 & 1 & \frac{3}{2} & 0\} \end{array} \tag{E.91}$$

Conjugacy Class $\mathcal{C}_{14}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_1 & 1 & \frac{1}{2} & 1\} \\ \{3_1 & 1 & \frac{3}{2} & 1\} \end{array} \tag{E.92}$$

Conjugacy Class $\mathcal{C}_{15}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_2 & 0 & \frac{1}{2} & 0\} \\ \{3_2 & 0 & \frac{3}{2} & 0\} \end{array} \tag{E.93}$$

Conjugacy Class $\mathcal{C}_{16}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_2 & 0 & \frac{1}{2} & 1\} \\ \{3_2 & 0 & \frac{3}{2} & 1\} \end{array} \tag{E.94}$$

Conjugacy Class $\mathcal{C}_{17}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_3 & 1 & 0 & 0\} \\ \{3_3 & 1 & 1 & 0\} \end{array} \tag{E.95}$$

Conjugacy Class $\mathcal{C}_{18}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{cccc} \{3_3 & 1 & 0 & 1\} \\ \{3_3 & 1 & 1 & 1\} \end{array} \tag{E.96}$$

Conjugacy Class $\mathcal{C}_{19}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{3_3 & 1 & \frac{1}{2} & 0\} \\ \{3_3 & 1 & \frac{3}{2} & 0\}\end{array} \tag{E.97}$$

Conjugacy Class $\mathcal{C}_{20}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{3_3 & 1 & \frac{1}{2} & 1\} \\ \{3_3 & 1 & \frac{3}{2} & 1\}\end{array} \tag{E.98}$$

E.4. The Group $\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}$.

Conjugacy Class $\mathcal{C}_{1}\left(\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{1_1 & 0 & 0 & 0\}\end{array} \tag{E.99}$$

Conjugacy Class $\mathcal{C}_{2}\left(\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{1_1 & 0 & 1 & 0\}\end{array} \tag{E.100}$$

Conjugacy Class $\mathcal{C}_{3}\left(\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{3_2 & 0 & 0 & 1\}\end{array} \tag{E.101}$$

Conjugacy Class $\mathcal{C}_{4}\left(\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{3_2 & 0 & 1 & 1\}\end{array} \tag{E.102}$$

Conjugacy Class $\mathcal{C}_{5}\left(\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{1_1 & 0 & \frac{1}{2} & 0\} \\ \{1_1 & 0 & \frac{3}{2} & 0\}\end{array} \tag{E.103}$$

Conjugacy Class $\mathcal{C}_{6}\left(\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 1 & 0 & 1\} \\ \{3_1 & 1 & 1 & 1\}\end{array} \tag{E.104}$$

Conjugacy Class $\mathcal{C}_{7}\left(\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{3_1 & 1 & \frac{1}{2} & 1\} \\ \{3_1 & 1 & \frac{3}{2} & 1\}\end{array} \tag{E.105}$$

Conjugacy Class $\mathcal{C}_{8}\left(\mathrm{G}_{16}^{(\mathrm{A},\mathrm{B},0)}\right)$

$$\begin{array}{llll}\{3_2 & 0 & \frac{1}{2} & 1\} \\ \{3_2 & 0 & \frac{3}{2} & 1\}\end{array} \tag{E.106}$$

Conjugacy Class $\mathcal{C}_9\left(\mathrm{G}_{16}^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{cccc} \{3_3 & 1 & 0 & 0\} \\ \{3_3 & 1 & 1 & 0\} \end{array} \tag{E.107}$$

Conjugacy Class $\mathcal{C}_{10}\left(\mathrm{G}_{16}^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{cccc} \{3_3 & 1 & \frac{1}{2} & 0\} \\ \{3_3 & 1 & \frac{3}{2} & 0\} \end{array} \tag{E.108}$$

E.5. The Group $\mathrm{G}_8^{(\mathrm{A,B,0})}$.

Conjugacy Class $\mathcal{C}_1\left(\mathrm{G}_8^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{cccc} \{1_1 & 0 & 0 & 0\} \end{array} \tag{E.109}$$

Conjugacy Class $\mathcal{C}_2\left(\mathrm{G}_8^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{cccc} \{1_1 & 0 & 1 & 0\} \end{array} \tag{E.110}$$

Conjugacy Class $\mathcal{C}_3\left(\mathrm{G}_8^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{cccc} \{1_1 & 0 & \frac{1}{2} & 0\} \\ \{1_1 & 0 & \frac{3}{2} & 0\} \end{array} \tag{E.111}$$

Conjugacy Class $\mathcal{C}_4\left(\mathrm{G}_8^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{cccc} \{3_1 & 1 & 0 & 1\} \\ \{3_1 & 1 & 1 & 1\} \end{array} \tag{E.112}$$

Conjugacy Class $\mathcal{C}_5\left(\mathrm{G}_8^{(\mathrm{A,B,0})}\right)$

$$\begin{array}{cccc} \{3_1 & 1 & \frac{1}{2} & 1\} \\ \{3_1 & 1 & \frac{3}{2} & 1\} \end{array} \tag{E.113}$$

E.6. The Group $\mathrm{G}_4^{(\mathrm{A,B,0})}$.

Abelian: every element is a conjugacy class

$$\begin{array}{c} \{1_1, 0, 0, 0\} \\ \left\{1_1, 0, \frac{1}{2}, 0\right\} \\ \{1_1, 0, 1, 0\} \\ \left\{1_1, 0, \frac{3}{2}, 0\right\} \end{array} \tag{E.114}$$

E.7. The Group $\mathrm{G}_{256}^{(\mathrm{A},0,0)}$.

Conjugacy Class $\mathcal{C}_1\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 0\} \tag{E.115}$$

Conjugacy Class $\mathcal{C}_2\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 1\} \tag{E.116}$$

Conjugacy Class $\mathcal{C}_3\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 1 \quad 1 \quad 0\} \tag{E.117}$$

Conjugacy Class $\mathcal{C}_4\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 1 \quad 1 \quad 1\} \tag{E.118}$$

Conjugacy Class $\mathcal{C}_5\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad 0 \quad 1 \quad 0\} \\ \{1_1 \quad 1 \quad 0 \quad 0\} \end{array} \tag{E.119}$$

Conjugacy Class $\mathcal{C}_6\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad 0 \quad 1 \quad 1\} \\ \{1_1 \quad 1 \quad 0 \quad 1\} \end{array} \tag{E.120}$$

Conjugacy Class $\mathcal{C}_7\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad 0 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 0\} \\ \{1_1 \quad \frac{1}{2} \quad 0 \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad 0 \quad 0\} \end{array} \tag{E.121}$$

Conjugacy Class $\mathcal{C}_8\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 1\} \\ \{1_1 \quad \frac{1}{2} \quad 0 \quad 1\} \\ \{1_1 \quad \frac{3}{2} \quad 0 \quad 1\} \end{array} \tag{E.122}$$

Conjugacy Class $\mathcal{C}_9\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad \frac{1}{2} \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad \frac{1}{2} \quad \frac{3}{2} \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.123}$$

Conjugacy Class $\mathcal{C}_{10}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{1_1 & \frac{1}{2} & \frac{1}{2} & 1\} \\ \{1_1 & \frac{1}{2} & \frac{3}{2} & 1\} \\ \{1_1 & \frac{3}{2} & \frac{1}{2} & 1\} \\ \{1_1 & \frac{3}{2} & \frac{3}{2} & 1\} \end{array} \tag{E.124}$$

Conjugacy Class $\mathcal{C}_{11}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{1_1 & \frac{1}{2} & 1 & 0\} \\ \{1_1 & 1 & \frac{1}{2} & 0\} \\ \{1_1 & 1 & \frac{3}{2} & 0\} \\ \{1_1 & \frac{3}{2} & 1 & 0\} \end{array} \tag{E.125}$$

Conjugacy Class $\mathcal{C}_{12}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{1_1 & \frac{1}{2} & 1 & 1\} \\ \{1_1 & 1 & \frac{1}{2} & 1\} \\ \{1_1 & 1 & \frac{3}{2} & 1\} \\ \{1_1 & \frac{3}{2} & 1 & 1\} \end{array} \tag{E.126}$$

Conjugacy Class $\mathcal{C}_{13}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & 0 & 0 & 0\} \\ \{3_1 & 0 & 1 & 0\} \\ \{3_1 & 1 & 0 & 0\} \\ \{3_1 & 1 & 1 & 0\} \end{array} \tag{E.127}$$

Conjugacy Class $\mathcal{C}_{14}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & 0 & 0 & 1\} \\ \{3_1 & 0 & 1 & 1\} \\ \{3_1 & 1 & 0 & 1\} \\ \{3_1 & 1 & 1 & 1\} \end{array} \tag{E.128}$$

Conjugacy Class $\mathcal{C}_{15}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & \frac{1}{2} & \frac{1}{2} & 0\} \\ \{3_1 & \frac{1}{2} & \frac{3}{2} & 0\} \\ \{3_1 & \frac{3}{2} & \frac{1}{2} & 0\} \\ \{3_1 & \frac{3}{2} & \frac{3}{2} & 0\} \end{array} \tag{E.129}$$

Conjugacy Class $\mathcal{C}_{16}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & \frac{1}{2} & \frac{1}{2} & 1\} \\ \{3_1 & \frac{1}{2} & \frac{3}{2} & 1\} \\ \{3_1 & \frac{3}{2} & \frac{1}{2} & 1\} \\ \{3_1 & \frac{3}{2} & \frac{3}{2} & 1\} \end{array} \tag{E.130}$$

Conjugacy Class $\mathcal{C}_{17}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & 0 & 0 & 0\} \\ \{3_2 & 1 & 0 & 0\} \\ \{3_3 & 0 & 0 & 1\} \\ \{3_3 & 0 & 1 & 1\} \end{array} \tag{E.131}$$

Conjugacy Class $\mathcal{C}_{18}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & 0 & 0 & 1\} \\ \{3_2 & 1 & 0 & 1\} \\ \{3_3 & 0 & 0 & 0\} \\ \{3_3 & 0 & 1 & 0\} \end{array} \tag{E.132}$$

Conjugacy Class $\mathcal{C}_{19}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & 0 & 1 & 0\} \\ \{3_2 & 1 & 1 & 0\} \\ \{3_3 & 1 & 0 & 1\} \\ \{3_3 & 1 & 1 & 1\} \end{array} \tag{E.133}$$

Conjugacy Class $\mathcal{C}_{20}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & 0 & 1 & 1\} \\ \{3_2 & 1 & 1 & 1\} \\ \{3_3 & 1 & 0 & 0\} \\ \{3_3 & 1 & 1 & 0\} \end{array} \tag{E.134}$$

Conjugacy Class $\mathcal{C}_{21}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & 0 & 0\} \\ \{3_2 & \frac{3}{2} & 0 & 0\} \\ \{3_3 & 0 & \frac{1}{2} & 1\} \\ \{3_3 & 0 & \frac{3}{2} & 1\} \end{array} \tag{E.135}$$

Conjugacy Class $\mathcal{C}_{22}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & 0 & 1\} \\ \{3_2 & \frac{3}{2} & 0 & 1\} \\ \{3_3 & 0 & \frac{1}{2} & 0\} \\ \{3_3 & 0 & \frac{3}{2} & 0\} \end{array} \tag{E.136}$$

Conjugacy Class $\mathcal{C}_{23}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & 1 & 0\} \\ \{3_2 & \frac{3}{2} & 1 & 0\} \\ \{3_3 & 1 & \frac{1}{2} & 1\} \\ \{3_3 & 1 & \frac{3}{2} & 1\} \end{array} \tag{E.137}$$

Conjugacy Class $\mathcal{C}_{24}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & 1 & 1\} \\ \{3_2 & \frac{3}{2} & 1 & 1\} \\ \{3_3 & 1 & \frac{1}{2} & 0\} \\ \{3_3 & 1 & \frac{3}{2} & 0\} \end{array} \tag{E.138}$$

Conjugacy Class $\mathcal{C}_{25}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & 0 & \frac{1}{2} & 0\} \\ \{3_1 & 0 & \frac{3}{2} & 0\} \\ \{3_1 & \frac{1}{2} & 0 & 0\} \\ \{3_1 & \frac{1}{2} & 1 & 0\} \\ \{3_1 & 1 & \frac{1}{2} & 0\} \\ \{3_1 & 1 & \frac{3}{2} & 0\} \\ \{3_1 & \frac{3}{2} & 0 & 0\} \\ \{3_1 & \frac{3}{2} & 1 & 0\} \end{array} \tag{E.139}$$

Conjugacy Class $\mathcal{C}_{26}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{3_1 \quad 0 \quad \tfrac{1}{2} \quad 1\} \\ \{3_1 \quad 0 \quad \tfrac{3}{2} \quad 1\} \\ \{3_1 \quad \tfrac{1}{2} \quad 0 \quad 1\} \\ \{3_1 \quad \tfrac{1}{2} \quad 1 \quad 1\} \\ \{3_1 \quad 1 \quad \tfrac{1}{2} \quad 1\} \\ \{3_1 \quad 1 \quad \tfrac{3}{2} \quad 1\} \\ \{3_1 \quad \tfrac{3}{2} \quad 0 \quad 1\} \\ \{3_1 \quad \tfrac{3}{2} \quad 1 \quad 1\} \end{array} \tag{E.140}$$

Conjugacy Class $\mathcal{C}_{27}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{3_2 \quad 0 \quad \tfrac{1}{2} \quad 0\} \\ \{3_2 \quad 0 \quad \tfrac{3}{2} \quad 0\} \\ \{3_2 \quad 1 \quad \tfrac{1}{2} \quad 0\} \\ \{3_2 \quad 1 \quad \tfrac{3}{2} \quad 0\} \\ \{3_3 \quad \tfrac{1}{2} \quad 0 \quad 1\} \\ \{3_3 \quad \tfrac{1}{2} \quad 1 \quad 1\} \\ \{3_3 \quad \tfrac{3}{2} \quad 0 \quad 1\} \\ \{3_3 \quad \tfrac{3}{2} \quad 1 \quad 1\} \end{array} \tag{E.141}$$

Conjugacy Class $\mathcal{C}_{28}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{3_2 \quad 0 \quad \tfrac{1}{2} \quad 1\} \\ \{3_2 \quad 0 \quad \tfrac{3}{2} \quad 1\} \\ \{3_2 \quad 1 \quad \tfrac{1}{2} \quad 1\} \\ \{3_2 \quad 1 \quad \tfrac{3}{2} \quad 1\} \\ \{3_3 \quad \tfrac{1}{2} \quad 0 \quad 0\} \\ \{3_3 \quad \tfrac{1}{2} \quad 1 \quad 0\} \\ \{3_3 \quad \tfrac{3}{2} \quad 0 \quad 0\} \\ \{3_3 \quad \tfrac{3}{2} \quad 1 \quad 0\} \end{array} \tag{E.142}$$

Conjugacy Class $\mathcal{C}_{29}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & \frac{1}{2} & 0\} \\ \{3_2 & \frac{1}{2} & \frac{3}{2} & 0\} \\ \{3_2 & \frac{3}{2} & \frac{1}{2} & 0\} \\ \{3_2 & \frac{3}{2} & \frac{3}{2} & 0\} \\ \{3_3 & \frac{1}{2} & \frac{1}{2} & 1\} \\ \{3_3 & \frac{1}{2} & \frac{3}{2} & 1\} \\ \{3_3 & \frac{3}{2} & \frac{1}{2} & 1\} \\ \{3_3 & \frac{3}{2} & \frac{3}{2} & 1\} \end{array} \tag{E.143}$$

Conjugacy Class $\mathcal{C}_{30}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & \frac{1}{2} & 1\} \\ \{3_2 & \frac{1}{2} & \frac{3}{2} & 1\} \\ \{3_2 & \frac{3}{2} & \frac{1}{2} & 1\} \\ \{3_2 & \frac{3}{2} & \frac{3}{2} & 1\} \\ \{3_3 & \frac{1}{2} & \frac{1}{2} & 0\} \\ \{3_3 & \frac{1}{2} & \frac{3}{2} & 0\} \\ \{3_3 & \frac{3}{2} & \frac{1}{2} & 0\} \\ \{3_3 & \frac{3}{2} & \frac{3}{2} & 0\} \end{array} \tag{E.144}$$

Conjugacy Class $\mathcal{C}_{31}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{4_3 & 0 & 0 & \frac{1}{2}\} \\ \{4_3 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\} \\ \{4_3 & 1 & 1 & \frac{1}{2}\} \\ \{4_3 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2}\} \\ \{4_6 & 0 & 0 & \frac{3}{2}\} \\ \{4_6 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2}\} \\ \{4_6 & 1 & 1 & \frac{3}{2}\} \\ \{4_6 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2}\} \end{array} \tag{E.145}$$

Conjugacy Class $\mathcal{C}_{32}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l}\{4_3 \quad 0 \quad 0 \quad \frac{3}{2}\}\\ \{4_3 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{3}{2}\}\\ \{4_3 \quad 1 \quad 1 \quad \frac{3}{2}\}\\ \{4_3 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2}\}\\ \{4_6 \quad 0 \quad 0 \quad \frac{1}{2}\}\\ \{4_6 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{1}{2}\}\\ \{4_6 \quad 1 \quad 1 \quad \frac{1}{2}\}\\ \{4_6 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{1}{2}\}\end{array} \tag{E.146}$$

Conjugacy Class $\mathcal{C}_{33}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l}\{4_3 \quad 0 \quad 1 \quad \frac{1}{2}\}\\ \{4_3 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{1}{2}\}\\ \{4_3 \quad 1 \quad 0 \quad \frac{1}{2}\}\\ \{4_3 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{1}{2}\}\\ \{4_6 \quad 0 \quad 1 \quad \frac{3}{2}\}\\ \{4_6 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{3}{2}\}\\ \{4_6 \quad 1 \quad 0 \quad \frac{3}{2}\}\\ \{4_6 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2}\}\end{array} \tag{E.147}$$

Conjugacy Class $\mathcal{C}_{34}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l}\{4_3 \quad 0 \quad 1 \quad \frac{3}{2}\}\\ \{4_3 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{3}{2}\}\\ \{4_3 \quad 1 \quad 0 \quad \frac{3}{2}\}\\ \{4_3 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{3}{2}\}\\ \{4_6 \quad 0 \quad 1 \quad \frac{1}{2}\}\\ \{4_6 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\}\\ \{4_6 \quad 1 \quad 0 \quad \frac{1}{2}\}\\ \{4_6 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{1}{2}\}\end{array} \tag{E.148}$$

Conjugacy Class $\mathcal{C}_{35}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll}
\{4_3 & 0 & \frac{1}{2} & \frac{1}{2}\} \\
\{4_3 & 0 & \frac{3}{2} & \frac{1}{2}\} \\
\{4_3 & \frac{1}{2} & 0 & \frac{1}{2}\} \\
\{4_3 & \frac{1}{2} & 1 & \frac{1}{2}\} \\
\{4_3 & 1 & \frac{1}{2} & \frac{1}{2}\} \\
\{4_3 & 1 & \frac{3}{2} & \frac{1}{2}\} \\
\{4_3 & \frac{3}{2} & 0 & \frac{1}{2}\} \\
\{4_3 & \frac{3}{2} & 1 & \frac{1}{2}\} \\
\{4_6 & 0 & \frac{1}{2} & \frac{3}{2}\} \\
\{4_6 & 0 & \frac{3}{2} & \frac{3}{2}\} \\
\{4_6 & \frac{1}{2} & 0 & \frac{3}{2}\} \\
\{4_6 & \frac{1}{2} & 1 & \frac{3}{2}\} \\
\{4_6 & 1 & \frac{1}{2} & \frac{3}{2}\} \\
\{4_6 & 1 & \frac{3}{2} & \frac{3}{2}\} \\
\{4_6 & \frac{3}{2} & 0 & \frac{3}{2}\} \\
\{4_6 & \frac{3}{2} & 1 & \frac{3}{2}\}
\end{array} \tag{E.149}$$

Conjugacy Class $\mathcal{C}_{36}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll}
\{4_3 & 0 & \frac{1}{2} & \frac{3}{2}\} \\
\{4_3 & 0 & \frac{3}{2} & \frac{3}{2}\} \\
\{4_3 & \frac{1}{2} & 0 & \frac{3}{2}\} \\
\{4_3 & \frac{1}{2} & 1 & \frac{3}{2}\} \\
\{4_3 & 1 & \frac{1}{2} & \frac{3}{2}\} \\
\{4_3 & 1 & \frac{3}{2} & \frac{3}{2}\} \\
\{4_3 & \frac{3}{2} & 0 & \frac{3}{2}\} \\
\{4_3 & \frac{3}{2} & 1 & \frac{3}{2}\} \\
\{4_6 & 0 & \frac{1}{2} & \frac{1}{2}\} \\
\{4_6 & 0 & \frac{3}{2} & \frac{1}{2}\} \\
\{4_6 & \frac{1}{2} & 0 & \frac{1}{2}\} \\
\{4_6 & \frac{1}{2} & 1 & \frac{1}{2}\} \\
\{4_6 & 1 & \frac{1}{2} & \frac{1}{2}\} \\
\{4_6 & 1 & \frac{3}{2} & \frac{1}{2}\} \\
\{4_6 & \frac{3}{2} & 0 & \frac{1}{2}\} \\
\{4_6 & \frac{3}{2} & 1 & \frac{1}{2}\}
\end{array} \tag{E.150}$$

Conjugacy Class $\mathcal{C}_{37}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{5_1 \quad 0 \quad 0 \quad \frac{1}{2}\} \\ \{5_1 \quad 0 \quad 1 \quad \frac{1}{2}\} \\ \{5_1 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\} \\ \{5_1 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{1}{2}\} \\ \{5_1 \quad 1 \quad 0 \quad \frac{1}{2}\} \\ \{5_1 \quad 1 \quad 1 \quad \frac{1}{2}\} \\ \{5_1 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{1}{2}\} \\ \{5_1 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{1}{2}\} \\ \{5_4 \quad 0 \quad 0 \quad \frac{3}{2}\} \\ \{5_4 \quad 0 \quad 1 \quad \frac{3}{2}\} \\ \{5_4 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{3}{2}\} \\ \{5_4 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{3}{2}\} \\ \{5_4 \quad 1 \quad 0 \quad \frac{3}{2}\} \\ \{5_4 \quad 1 \quad 1 \quad \frac{3}{2}\} \\ \{5_4 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{3}{2}\} \\ \{5_4 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2}\} \end{array} \tag{E.151}$$

Conjugacy Class $\mathcal{C}_{38}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{5_1 \quad 0 \quad 0 \quad \frac{3}{2}\} \\ \{5_1 \quad 0 \quad 1 \quad \frac{3}{2}\} \\ \{5_1 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{3}{2}\} \\ \{5_1 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{3}{2}\} \\ \{5_1 \quad 1 \quad 0 \quad \frac{3}{2}\} \\ \{5_1 \quad 1 \quad 1 \quad \frac{3}{2}\} \\ \{5_1 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{3}{2}\} \\ \{5_1 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2}\} \\ \{5_4 \quad 0 \quad 0 \quad \frac{1}{2}\} \\ \{5_4 \quad 0 \quad 1 \quad \frac{1}{2}\} \\ \{5_4 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\} \\ \{5_4 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{1}{2}\} \\ \{5_4 \quad 1 \quad 0 \quad \frac{1}{2}\} \\ \{5_4 \quad 1 \quad 1 \quad \frac{1}{2}\} \\ \{5_4 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{1}{2}\} \\ \{5_4 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{1}{2}\} \end{array} \tag{E.152}$$

Conjugacy Class $\mathcal{C}_{39}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll}
\{5_1 & 0 & \frac{1}{2} & \frac{1}{2}\} \\
\{5_1 & 0 & \frac{3}{2} & \frac{1}{2}\} \\
\{5_1 & \frac{1}{2} & 0 & \frac{1}{2}\} \\
\{5_1 & \frac{1}{2} & 1 & \frac{1}{2}\} \\
\{5_1 & 1 & \frac{1}{2} & \frac{1}{2}\} \\
\{5_1 & 1 & \frac{3}{2} & \frac{1}{2}\} \\
\{5_1 & \frac{3}{2} & 0 & \frac{1}{2}\} \\
\{5_1 & \frac{3}{2} & 1 & \frac{1}{2}\} \\
\{5_4 & 0 & \frac{1}{2} & \frac{3}{2}\} \\
\{5_4 & 0 & \frac{3}{2} & \frac{3}{2}\} \\
\{5_4 & \frac{1}{2} & 0 & \frac{3}{2}\} \\
\{5_4 & \frac{1}{2} & 1 & \frac{3}{2}\} \\
\{5_4 & 1 & \frac{1}{2} & \frac{3}{2}\} \\
\{5_4 & 1 & \frac{3}{2} & \frac{3}{2}\} \\
\{5_4 & \frac{3}{2} & 0 & \frac{3}{2}\} \\
\{5_4 & \frac{3}{2} & 1 & \frac{3}{2}\}
\end{array} \tag{E.153}$$

Conjugacy Class $\mathcal{C}_{40}\left(\mathrm{G}_{256}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll}
\{5_1 & 0 & \frac{1}{2} & \frac{3}{2}\} \\
\{5_1 & 0 & \frac{3}{2} & \frac{3}{2}\} \\
\{5_1 & \frac{1}{2} & 0 & \frac{3}{2}\} \\
\{5_1 & \frac{1}{2} & 1 & \frac{3}{2}\} \\
\{5_1 & 1 & \frac{1}{2} & \frac{3}{2}\} \\
\{5_1 & 1 & \frac{3}{2} & \frac{3}{2}\} \\
\{5_1 & \frac{3}{2} & 0 & \frac{3}{2}\} \\
\{5_1 & \frac{3}{2} & 1 & \frac{3}{2}\} \\
\{5_4 & 0 & \frac{1}{2} & \frac{1}{2}\} \\
\{5_4 & 0 & \frac{3}{2} & \frac{1}{2}\} \\
\{5_4 & \frac{1}{2} & 0 & \frac{1}{2}\} \\
\{5_4 & \frac{1}{2} & 1 & \frac{1}{2}\} \\
\{5_4 & 1 & \frac{1}{2} & \frac{1}{2}\} \\
\{5_4 & 1 & \frac{3}{2} & \frac{1}{2}\} \\
\{5_4 & \frac{3}{2} & 0 & \frac{1}{2}\} \\
\{5_4 & \frac{3}{2} & 1 & \frac{1}{2}\}
\end{array} \tag{E.154}$$

E.8. The Group $\mathrm{G}_{128}^{(\mathrm{A},0,0)}$.

Conjugacy Class $\mathcal{C}_1\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 0\} \tag{E.155}$$

Conjugacy Class $\mathcal{C}_2\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 1 \quad 1 \quad 0\} \tag{E.156}$$

Conjugacy Class $\mathcal{C}_3\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{1_1 \quad 0 \quad 1 \quad 0\} \\ \{1_1 \quad 1 \quad 0 \quad 0\} \end{array} \tag{E.157}$$

Conjugacy Class $\mathcal{C}_4\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{1_1 \quad 0 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 0\} \\ \{1_1 \quad \frac{1}{2} \quad 0 \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad 0 \quad 0\} \end{array} \tag{E.158}$$

Conjugacy Class $\mathcal{C}_5\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{1_1 \quad \frac{1}{2} \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad \frac{1}{2} \quad \frac{3}{2} \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.159}$$

Conjugacy Class $\mathcal{C}_6\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{1_1 \quad \frac{1}{2} \quad 1 \quad 0\} \\ \{1_1 \quad 1 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 1 \quad \frac{3}{2} \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad 1 \quad 0\} \end{array} \tag{E.160}$$

Conjugacy Class $\mathcal{C}_7\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{3_1 \quad 0 \quad 0 \quad 1\} \\ \{3_1 \quad 0 \quad 1 \quad 1\} \\ \{3_1 \quad 1 \quad 0 \quad 1\} \\ \{3_1 \quad 1 \quad 1 \quad 1\} \end{array} \tag{E.161}$$

Conjugacy Class $\mathcal{C}_8\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & \frac{1}{2} & \frac{1}{2} & 1\} \\ \{3_1 & \frac{1}{2} & \frac{3}{2} & 1\} \\ \{3_1 & \frac{3}{2} & \frac{1}{2} & 1\} \\ \{3_1 & \frac{3}{2} & \frac{3}{2} & 1\} \end{array} \tag{E.162}$$

Conjugacy Class $\mathcal{C}_9\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & 0 & 0 & 1\} \\ \{3_2 & 1 & 0 & 1\} \\ \{3_3 & 0 & 0 & 0\} \\ \{3_3 & 0 & 1 & 0\} \end{array} \tag{E.163}$$

Conjugacy Class $\mathcal{C}_{10}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & 0 & 1 & 1\} \\ \{3_2 & 1 & 1 & 1\} \\ \{3_3 & 1 & 0 & 0\} \\ \{3_3 & 1 & 1 & 0\} \end{array} \tag{E.164}$$

Conjugacy Class $\mathcal{C}_{11}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & 0 & 1\} \\ \{3_2 & \frac{3}{2} & 0 & 1\} \\ \{3_3 & 0 & \frac{1}{2} & 0\} \\ \{3_3 & 0 & \frac{3}{2} & 0\} \end{array} \tag{E.165}$$

Conjugacy Class $\mathcal{C}_{12}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & 1 & 1\} \\ \{3_2 & \frac{3}{2} & 1 & 1\} \\ \{3_3 & 1 & \frac{1}{2} & 0\} \\ \{3_3 & 1 & \frac{3}{2} & 0\} \end{array} \tag{E.166}$$

Conjugacy Class $\mathcal{C}_{13}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & 0 & \frac{1}{2} & 1\} \\ \{3_1 & 0 & \frac{3}{2} & 1\} \\ \{3_1 & \frac{1}{2} & 0 & 1\} \\ \{3_1 & \frac{1}{2} & 1 & 1\} \\ \{3_1 & 1 & \frac{1}{2} & 1\} \\ \{3_1 & 1 & \frac{3}{2} & 1\} \\ \{3_1 & \frac{3}{2} & 0 & 1\} \\ \{3_1 & \frac{3}{2} & 1 & 1\} \end{array} \tag{E.167}$$

Conjugacy Class $\mathcal{C}_{14}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & 0 & \frac{1}{2} & 1\} \\ \{3_2 & 0 & \frac{3}{2} & 1\} \\ \{3_2 & 1 & \frac{1}{2} & 1\} \\ \{3_2 & 1 & \frac{3}{2} & 1\} \\ \{3_3 & \frac{1}{2} & 0 & 0\} \\ \{3_3 & \frac{1}{2} & 1 & 0\} \\ \{3_3 & \frac{3}{2} & 0 & 0\} \\ \{3_3 & \frac{3}{2} & 1 & 0\} \end{array} \tag{E.168}$$

Conjugacy Class $\mathcal{C}_{15}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & \frac{1}{2} & 1\} \\ \{3_2 & \frac{1}{2} & \frac{3}{2} & 1\} \\ \{3_2 & \frac{3}{2} & \frac{1}{2} & 1\} \\ \{3_2 & \frac{3}{2} & \frac{3}{2} & 1\} \\ \{3_3 & \frac{1}{2} & \frac{1}{2} & 0\} \\ \{3_3 & \frac{1}{2} & \frac{3}{2} & 0\} \\ \{3_3 & \frac{3}{2} & \frac{1}{2} & 0\} \\ \{3_3 & \frac{3}{2} & \frac{3}{2} & 0\} \end{array} \tag{E.169}$$

Conjugacy Class $\mathcal{C}_{16}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{4_3 \quad 0 \quad 0 \quad \tfrac{1}{2}\} \\ \{4_3 \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\} \\ \{4_3 \quad 1 \quad 1 \quad \tfrac{1}{2}\} \\ \{4_3 \quad \tfrac{3}{2} \quad \tfrac{3}{2} \quad \tfrac{1}{2}\} \\ \{4_6 \quad 0 \quad 0 \quad \tfrac{3}{2}\} \\ \{4_6 \quad \tfrac{1}{2} \quad \tfrac{3}{2} \quad \tfrac{3}{2}\} \\ \{4_6 \quad 1 \quad 1 \quad \tfrac{3}{2}\} \\ \{4_6 \quad \tfrac{3}{2} \quad \tfrac{1}{2} \quad \tfrac{3}{2}\} \end{array} \tag{E.170}$$

Conjugacy Class $\mathcal{C}_{17}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{4_3 \quad 0 \quad 1 \quad \tfrac{1}{2}\} \\ \{4_3 \quad \tfrac{1}{2} \quad \tfrac{3}{2} \quad \tfrac{1}{2}\} \\ \{4_3 \quad 1 \quad 0 \quad \tfrac{1}{2}\} \\ \{4_3 \quad \tfrac{3}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\} \\ \{4_6 \quad 0 \quad 1 \quad \tfrac{3}{2}\} \\ \{4_6 \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{3}{2}\} \\ \{4_6 \quad 1 \quad 0 \quad \tfrac{3}{2}\} \\ \{4_6 \quad \tfrac{3}{2} \quad \tfrac{3}{2} \quad \tfrac{3}{2}\} \end{array} \tag{E.171}$$

Conjugacy Class $\mathcal{C}_{18}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{4_3 \quad 0 \quad \tfrac{1}{2} \quad \tfrac{1}{2}\} \\ \{4_3 \quad 0 \quad \tfrac{3}{2} \quad \tfrac{1}{2}\} \\ \{4_3 \quad \tfrac{1}{2} \quad 0 \quad \tfrac{1}{2}\} \\ \{4_3 \quad \tfrac{1}{2} \quad 1 \quad \tfrac{1}{2}\} \\ \{4_3 \quad 1 \quad \tfrac{1}{2} \quad \tfrac{1}{2}\} \\ \{4_3 \quad 1 \quad \tfrac{3}{2} \quad \tfrac{1}{2}\} \\ \{4_3 \quad \tfrac{3}{2} \quad 0 \quad \tfrac{1}{2}\} \\ \{4_3 \quad \tfrac{3}{2} \quad 1 \quad \tfrac{1}{2}\} \\ \{4_6 \quad 0 \quad \tfrac{1}{2} \quad \tfrac{3}{2}\} \\ \{4_6 \quad 0 \quad \tfrac{3}{2} \quad \tfrac{3}{2}\} \\ \{4_6 \quad \tfrac{1}{2} \quad 0 \quad \tfrac{3}{2}\} \\ \{4_6 \quad \tfrac{1}{2} \quad 1 \quad \tfrac{3}{2}\} \\ \{4_6 \quad 1 \quad \tfrac{1}{2} \quad \tfrac{3}{2}\} \\ \{4_6 \quad 1 \quad \tfrac{3}{2} \quad \tfrac{3}{2}\} \\ \{4_6 \quad \tfrac{3}{2} \quad 0 \quad \tfrac{3}{2}\} \\ \{4_6 \quad \tfrac{3}{2} \quad 1 \quad \tfrac{3}{2}\} \end{array} \tag{E.172}$$

Conjugacy Class $\mathcal{C}_{19}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll}
\{5_1 & 0 & 0 & \frac{3}{2}\} \\
\{5_1 & 0 & 1 & \frac{3}{2}\} \\
\{5_1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2}\} \\
\{5_1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2}\} \\
\{5_1 & 1 & 0 & \frac{3}{2}\} \\
\{5_1 & 1 & 1 & \frac{3}{2}\} \\
\{5_1 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2}\} \\
\{5_1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\} \\
\{5_4 & 0 & 0 & \frac{1}{2}\} \\
\{5_4 & 0 & 1 & \frac{1}{2}\} \\
\{5_4 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\} \\
\{5_4 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2}\} \\
\{5_4 & 1 & 0 & \frac{1}{2}\} \\
\{5_4 & 1 & 1 & \frac{1}{2}\} \\
\{5_4 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\} \\
\{5_4 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2}\}
\end{array} \tag{E.173}$$

Conjugacy Class $\mathcal{C}_{20}\left(\mathrm{G}_{128}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll}
\{5_1 & 0 & \frac{1}{2} & \frac{3}{2}\} \\
\{5_1 & 0 & \frac{3}{2} & \frac{3}{2}\} \\
\{5_1 & \frac{1}{2} & 0 & \frac{3}{2}\} \\
\{5_1 & \frac{1}{2} & 1 & \frac{3}{2}\} \\
\{5_1 & 1 & \frac{1}{2} & \frac{3}{2}\} \\
\{5_1 & 1 & \frac{3}{2} & \frac{3}{2}\} \\
\{5_1 & \frac{3}{2} & 0 & \frac{3}{2}\} \\
\{5_1 & \frac{3}{2} & 1 & \frac{3}{2}\} \\
\{5_4 & 0 & \frac{1}{2} & \frac{1}{2}\} \\
\{5_4 & 0 & \frac{3}{2} & \frac{1}{2}\} \\
\{5_4 & \frac{1}{2} & 0 & \frac{1}{2}\} \\
\{5_4 & \frac{1}{2} & 1 & \frac{1}{2}\} \\
\{5_4 & 1 & \frac{1}{2} & \frac{1}{2}\} \\
\{5_4 & 1 & \frac{3}{2} & \frac{1}{2}\} \\
\{5_4 & \frac{3}{2} & 0 & \frac{1}{2}\} \\
\{5_4 & \frac{3}{2} & 1 & \frac{1}{2}\}
\end{array} \tag{E.174}$$

E.9. The Group $\mathrm{G}_{64}^{(\mathrm{A},0,0)}$.

Conjugacy Class $\mathcal{C}_1\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 0\} \tag{E.175}$$

Conjugacy Class $\mathcal{C}_2\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 0 \quad 1 \quad 0\} \tag{E.176}$$

Conjugacy Class $\mathcal{C}_3\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 1 \quad 0 \quad 0\} \tag{E.177}$$

Conjugacy Class $\mathcal{C}_4\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 1 \quad 1 \quad 0\} \tag{E.178}$$

Conjugacy Class $\mathcal{C}_5\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad 0 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.179}$$

Conjugacy Class $\mathcal{C}_6\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad \frac{1}{2} \quad 0 \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad 0 \quad 0\} \end{array} \tag{E.180}$$

Conjugacy Class $\mathcal{C}_7\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad \frac{1}{2} \quad 1 \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad 1 \quad 0\} \end{array} \tag{E.181}$$

Conjugacy Class $\mathcal{C}_8\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{1_1 \quad 1 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 1 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.182}$$

Conjugacy Class $\mathcal{C}_9\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{3_2 \quad 0 \quad 0 \quad 1\} \\ \{3_2 \quad 1 \quad 0 \quad 1\} \end{array} \tag{E.183}$$

Conjugacy Class $\mathcal{C}_{10}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l} \{3_2 \quad 0 \quad 1 \quad 1\} \\ \{3_2 \quad 1 \quad 1 \quad 1\} \end{array} \tag{E.184}$$

Conjugacy Class $\mathcal{C}_{11}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & 0 & 1\} \\ \{3_2 & \frac{3}{2} & 0 & 1\} \end{array} \tag{E.185}$$

Conjugacy Class $\mathcal{C}_{12}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_2 & \frac{1}{2} & 1 & 1\} \\ \{3_2 & \frac{3}{2} & 1 & 1\} \end{array} \tag{E.186}$$

Conjugacy Class $\mathcal{C}_{13}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_3 & 0 & 0 & 0\} \\ \{3_3 & 0 & 1 & 0\} \end{array} \tag{E.187}$$

Conjugacy Class $\mathcal{C}_{14}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_3 & 0 & \frac{1}{2} & 0\} \\ \{3_3 & 0 & \frac{3}{2} & 0\} \end{array} \tag{E.188}$$

Conjugacy Class $\mathcal{C}_{15}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_3 & 1 & 0 & 0\} \\ \{3_3 & 1 & 1 & 0\} \end{array} \tag{E.189}$$

Conjugacy Class $\mathcal{C}_{16}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_3 & 1 & \frac{1}{2} & 0\} \\ \{3_3 & 1 & \frac{3}{2} & 0\} \end{array} \tag{E.190}$$

Conjugacy Class $\mathcal{C}_{17}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{1_1 & \frac{1}{2} & \frac{1}{2} & 0\} \\ \{1_1 & \frac{1}{2} & \frac{3}{2} & 0\} \\ \{1_1 & \frac{3}{2} & \frac{1}{2} & 0\} \\ \{1_1 & \frac{3}{2} & \frac{3}{2} & 0\} \end{array} \tag{E.191}$$

Conjugacy Class $\mathcal{C}_{18}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & 0 & 0 & 1\} \\ \{3_1 & 0 & 1 & 1\} \\ \{3_1 & 1 & 0 & 1\} \\ \{3_1 & 1 & 1 & 1\} \end{array} \tag{E.192}$$

Conjugacy Class $\mathcal{C}_{19}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{3_1 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{3_1 \quad 0 \quad \frac{3}{2} \quad 1\} \\ \{3_1 \quad 1 \quad \frac{1}{2} \quad 1\} \\ \{3_1 \quad 1 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.193}$$

Conjugacy Class $\mathcal{C}_{20}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{3_1 \quad \frac{1}{2} \quad 0 \quad 1\} \\ \{3_1 \quad \frac{1}{2} \quad 1 \quad 1\} \\ \{3_1 \quad \frac{3}{2} \quad 0 \quad 1\} \\ \{3_1 \quad \frac{3}{2} \quad 1 \quad 1\} \end{array} \tag{E.194}$$

Conjugacy Class $\mathcal{C}_{21}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{3_1 \quad \frac{1}{2} \quad \frac{1}{2} \quad 1\} \\ \{3_1 \quad \frac{1}{2} \quad \frac{3}{2} \quad 1\} \\ \{3_1 \quad \frac{3}{2} \quad \frac{1}{2} \quad 1\} \\ \{3_1 \quad \frac{3}{2} \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.195}$$

Conjugacy Class $\mathcal{C}_{22}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{3_2 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{3_2 \quad 0 \quad \frac{3}{2} \quad 1\} \\ \{3_2 \quad 1 \quad \frac{1}{2} \quad 1\} \\ \{3_2 \quad 1 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.196}$$

Conjugacy Class $\mathcal{C}_{23}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{3_2 \quad \frac{1}{2} \quad \frac{1}{2} \quad 1\} \\ \{3_2 \quad \frac{1}{2} \quad \frac{3}{2} \quad 1\} \\ \{3_2 \quad \frac{3}{2} \quad \frac{1}{2} \quad 1\} \\ \{3_2 \quad \frac{3}{2} \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.197}$$

Conjugacy Class $\mathcal{C}_{24}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{c} \{3_3 \quad \frac{1}{2} \quad 0 \quad 0\} \\ \{3_3 \quad \frac{1}{2} \quad 1 \quad 0\} \\ \{3_3 \quad \frac{3}{2} \quad 0 \quad 0\} \\ \{3_3 \quad \frac{3}{2} \quad 1 \quad 0\} \end{array} \tag{E.198}$$

Conjugacy Class $\mathcal{C}_{25}\left(\mathrm{G}_{64}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l}\{3_3 \quad \frac{1}{2} \quad \frac{1}{2} \quad 0\} \\ \{3_3 \quad \frac{1}{2} \quad \frac{3}{2} \quad 0\} \\ \{3_3 \quad \frac{3}{2} \quad \frac{1}{2} \quad 0\} \\ \{3_3 \quad \frac{3}{2} \quad \frac{3}{2} \quad 0\}\end{array} \tag{E.199}$$

E.10. The Group $\mathrm{G}_{32}^{(\mathrm{A},0,0)}$.

Conjugacy Class $\mathcal{C}_{1}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 0\} \tag{E.200}$$

Conjugacy Class $\mathcal{C}_{2}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 0 \quad 1 \quad 0\} \tag{E.201}$$

Conjugacy Class $\mathcal{C}_{3}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 1 \quad 0 \quad 0\} \tag{E.202}$$

Conjugacy Class $\mathcal{C}_{4}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\{1_1 \quad 1 \quad 1 \quad 0\} \tag{E.203}$$

Conjugacy Class $\mathcal{C}_{5}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l}\{1_1 \quad 0 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 0\}\end{array} \tag{E.204}$$

Conjugacy Class $\mathcal{C}_{6}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l}\{1_1 \quad \frac{1}{2} \quad 0 \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad 0 \quad 0\}\end{array} \tag{E.205}$$

Conjugacy Class $\mathcal{C}_{7}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l}\{1_1 \quad \frac{1}{2} \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad \frac{3}{2} \quad 0\}\end{array} \tag{E.206}$$

Conjugacy Class $\mathcal{C}_{8}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{l}\{1_1 \quad \frac{1}{2} \quad 1 \quad 0\} \\ \{1_1 \quad \frac{3}{2} \quad 1 \quad 0\}\end{array} \tag{E.207}$$

Conjugacy Class $\mathcal{C}_9\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{1_1 & \frac{1}{2} & \frac{3}{2} & 0\} \\ \{1_1 & \frac{3}{2} & \frac{1}{2} & 0\} \end{array} \tag{E.208}$$

Conjugacy Class $\mathcal{C}_{10}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{1_1 & 1 & \frac{1}{2} & 0\} \\ \{1_1 & 1 & \frac{3}{2} & 0\} \end{array} \tag{E.209}$$

Conjugacy Class $\mathcal{C}_{11}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & 0 & 0 & 1\} \\ \{3_1 & 0 & 1 & 1\} \\ \{3_1 & 1 & 0 & 1\} \\ \{3_1 & 1 & 1 & 1\} \end{array} \tag{E.210}$$

Conjugacy Class $\mathcal{C}_{12}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & 0 & \frac{1}{2} & 1\} \\ \{3_1 & 0 & \frac{3}{2} & 1\} \\ \{3_1 & 1 & \frac{1}{2} & 1\} \\ \{3_1 & 1 & \frac{3}{2} & 1\} \end{array} \tag{E.211}$$

Conjugacy Class $\mathcal{C}_{13}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & \frac{1}{2} & 0 & 1\} \\ \{3_1 & \frac{1}{2} & 1 & 1\} \\ \{3_1 & \frac{3}{2} & 0 & 1\} \\ \{3_1 & \frac{3}{2} & 1 & 1\} \end{array} \tag{E.212}$$

Conjugacy Class $\mathcal{C}_{14}\left(\mathrm{G}_{32}^{(\mathrm{A},0,0)}\right)$

$$\begin{array}{llll} \{3_1 & \frac{1}{2} & \frac{1}{2} & 1\} \\ \{3_1 & \frac{1}{2} & \frac{3}{2} & 1\} \\ \{3_1 & \frac{3}{2} & \frac{1}{2} & 1\} \\ \{3_1 & \frac{3}{2} & \frac{3}{2} & 1\} \end{array} \tag{E.213}$$

E.11. The Group $\mathrm{G}_{16}^{(\mathrm{A},0,0)}$.

Abelian: every element is a conjugacy class

$$
\begin{aligned}
&\{1_1, 0, 0, 0\}\\
&\left\{1_1, 0, \tfrac{1}{2}, 0\right\}\\
&\{1_1, 0, 1, 0\}\\
&\left\{1_1, 0, \tfrac{3}{2}, 0\right\}\\
&\left\{1_1, \tfrac{1}{2}, 0, 0\right\}\\
&\left\{1_1, \tfrac{1}{2}, \tfrac{1}{2}, 0\right\}\\
&\left\{1_1, \tfrac{1}{2}, 1, 0\right\}\\
&\left\{1_1, \tfrac{1}{2}, \tfrac{3}{2}, 0\right\}\\
&\{1_1, 1, 0, 0\}\\
&\left\{1_1, 1, \tfrac{1}{2}, 0\right\}\\
&\{1_1, 1, 1, 0\}\\
&\left\{1_1, 1, \tfrac{3}{2}, 0\right\}\\
&\left\{1_1, \tfrac{3}{2}, 0, 0\right\}\\
&\left\{1_1, \tfrac{3}{2}, \tfrac{1}{2}, 0\right\}\\
&\left\{1_1, \tfrac{3}{2}, 1, 0\right\}\\
&\left\{1_1, \tfrac{3}{2}, \tfrac{3}{2}, 0\right\}
\end{aligned}
\tag{E.214}
$$

E.12. The Group $\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}$.

Conjugacy Class $\mathcal{C}_1\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\{1_1 \quad 0 \quad 0 \quad 0\} \tag{E.215}$$

Conjugacy Class $\mathcal{C}_2\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\{1_1 \quad 0 \quad 1 \quad 0\} \tag{E.216}$$

Conjugacy Class $\mathcal{C}_3\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\{3_2 \quad 0 \quad 0 \quad 1\} \tag{E.217}$$

Conjugacy Class $\mathcal{C}_4\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\{3_2 \quad 0 \quad 1 \quad 1\} \tag{E.218}$$

Conjugacy Class $\mathcal{C}_5\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{c} \{1_1 \quad 0 \quad \frac{1}{2} \quad 0\} \\ \{1_1 \quad 0 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.219}$$

Conjugacy Class $\mathcal{C}_6\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{c} \{3_2 \quad 0 \quad \frac{1}{2} \quad 1\} \\ \{3_2 \quad 0 \quad \frac{3}{2} \quad 1\} \end{array} \tag{E.220}$$

Conjugacy Class $\mathcal{C}_7\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{c} \{5_2 \quad \frac{3}{2} \quad 0 \quad \frac{3}{2}\} \\ \{5_3 \quad \frac{1}{2} \quad 0 \quad \frac{3}{2}\} \end{array} \tag{E.221}$$

Conjugacy Class $\mathcal{C}_8\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{c} \{5_2 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{3}{2}\} \\ \{5_3 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{3}{2}\} \end{array} \tag{E.222}$$

Conjugacy Class $\mathcal{C}_9\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{c} \{5_2 \quad \frac{3}{2} \quad 1 \quad \frac{3}{2}\} \\ \{5_3 \quad \frac{1}{2} \quad 1 \quad \frac{3}{2}\} \end{array} \tag{E.223}$$

Conjugacy Class $\mathcal{C}_{10}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{c} \{5_2 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2}\} \\ \{5_3 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{3}{2}\} \end{array} \tag{E.224}$$

Conjugacy Class $\mathcal{C}_{11}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{c} \{3_1 \quad 1 \quad 0 \quad 1\} \\ \{3_1 \quad 1 \quad 1 \quad 1\} \\ \{3_3 \quad 1 \quad 0 \quad 0\} \\ \{3_3 \quad 1 \quad 1 \quad 0\} \end{array} \tag{E.225}$$

Conjugacy Class $\mathcal{C}_{12}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{c} \{3_1 \quad 1 \quad \frac{1}{2} \quad 1\} \\ \{3_1 \quad 1 \quad \frac{3}{2} \quad 1\} \\ \{3_3 \quad 1 \quad \frac{1}{2} \quad 0\} \\ \{3_3 \quad 1 \quad \frac{3}{2} \quad 0\} \end{array} \tag{E.226}$$

Conjugacy Class $\mathcal{C}_{13}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{l} \{4_4 \quad \frac{1}{2} \quad 0 \quad \frac{1}{2}\} \\ \{4_4 \quad \frac{1}{2} \quad 1 \quad \frac{1}{2}\} \\ \{4_5 \quad \frac{3}{2} \quad 0 \quad \frac{1}{2}\} \\ \{4_5 \quad \frac{3}{2} \quad 1 \quad \frac{1}{2}\} \end{array} \tag{E.227}$$

Conjugacy Class $\mathcal{C}_{14}\left(\mathrm{G}_{32}^{(\mathrm{A},\mathrm{A},0)}\right)$

$$\begin{array}{l} \{4_4 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\} \\ \{4_4 \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{1}{2}\} \\ \{4_5 \quad \frac{3}{2} \quad \frac{1}{2} \quad \frac{1}{2}\} \\ \{4_5 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{1}{2}\} \end{array} \tag{E.228}$$

F. SOME LARGE FORMULAE FOR THE MAIN TEXT

In this Appendix, we write in landscape format some formulae quoted in the main text that are too large for the standard view.

For Subsec. 7.2, we write here the form of the translation generators in the lowest-lying octahedral orbit of length 12:

$$
\mathcal{M}^{(12)}_{\{n_1,0,0\}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos(2\pi n_1) & 0 & 0 & 0 & 0 & 0 & \frac{\sin(2\pi n_1)}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \cos(2\pi n_1) & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\sin(2\pi n_1) & 0 & 0 \\
0 & 0 & 0 & 0 & \cos(2\pi n_1) & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\sin(2\pi n_1) & 0 \\
0 & 0 & 0 & 0 & 0 & \cos(2\pi n_1) & 0 & 0 & 0 & 0 & 0 & \frac{\sin(2\pi n_1)}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2}\sin(2\pi n_1) & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_1) & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sin(2\pi n_1)}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_1) & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\sin(2\pi n_1)}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_1) & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{2}\sin(2\pi n_1) & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_1)
\end{pmatrix},
$$

$$
\mathcal{M}^{(12)}_{\{0,n_2,0\}} = \begin{pmatrix}
\cos(2\pi n_2) & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}\sin(2\pi n_2) & 0 & 0 & 0 & 0 & 0 \\
0 & \cos(2\pi n_2) & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}\sin(2\pi n_2) & 0 & 0 & 0 & 0 \\
0 & 0 & \cos(2\pi n_2) & 0 & 0 & 0 & 0 & 0 & -\frac{\sin(2\pi n_2)}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cos(2\pi n_2) & 0 & 0 & 0 & 0 & 0 & \frac{\sin(2\pi n_2)}{\sqrt{2}} \\
\frac{\sin(2\pi n_2)}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_2) & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sin(2\pi n_2)}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_2) & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2}\sin(2\pi n_2) & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{2}\sin(2\pi n_2) & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_2)
\end{pmatrix},
$$

$$\mathcal{M}^{(12)}_{\{0,0,n_3\}} = \begin{pmatrix}
\cos(2\pi n_3) & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\sin(2\pi n_3) & 0 & 0 & 0 & 0 & 0 \\
0 & \cos(2\pi n_3) & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}\sin(2\pi n_3) & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos(2\pi n_3) & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}\sin(2\pi n_3) & 0 & 0 \\
0 & 0 & 0 & 0 & \cos(2\pi n_3) & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\sin(2\pi n_3) & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\sin(2\pi n_3)}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_3) & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sin(2\pi n_3)}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_3) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sin(2\pi n_3)}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_3) & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\sin(2\pi n_3)}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \cos(2\pi n_3) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{F.1}$$

For Subsec. 7.3, we write here the form of the translation generators in the lowest-lying octahedral orbit of length 8:

$$\mathcal{M}^{(8)}_{\{n_1,0,0\}} =$$

$$\begin{pmatrix}
\cos\left(\frac{\pi n_1}{2}\right)+\frac{\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \frac{2\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 \\
0 & \cos\left(\frac{\pi n_1}{2}\right)-\frac{\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \frac{2\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 \\
0 & 0 & \cos\left(\frac{\pi n_1}{2}\right)-\frac{\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \frac{2\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 \\
0 & 0 & 0 & \cos\left(\frac{\pi n_1}{2}\right)+\frac{\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \frac{2\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} \\
-\frac{2\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_1}{2}\right)-\frac{\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 \\
0 & -\frac{2\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_1}{2}\right)+\frac{\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 \\
0 & 0 & -\frac{2\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_1}{2}\right)+\frac{\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 \\
0 & 0 & 0 & -\frac{2\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_1}{2}\right)-\frac{\sin\left(\frac{\pi n_1}{2}\right)}{\sqrt{3}}
\end{pmatrix},$$

$$\mathcal{M}^{(8)}_{\{0,n_2,0\}} = \left(\begin{array}{cccccccc}
\cos\left(\frac{\pi n_2}{2}\right)-\frac{\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 \\
0 & \cos\left(\frac{\pi n_2}{2}\right)+\frac{\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 \\
0 & 0 & \cos\left(\frac{\pi n_2}{2}\right)-\frac{\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \frac{2\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 \\
0 & 0 & 0 & \cos\left(\frac{\pi n_2}{2}\right)+\frac{\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \frac{2\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} \\
\frac{2\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_2}{2}\right)+\frac{\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 \\
0 & \frac{2\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_2}{2}\right)-\frac{\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 \\
0 & 0 & -\frac{2\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_2}{2}\right)+\frac{\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 \\
0 & 0 & 0 & -\frac{2\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_2}{2}\right)-\frac{\sin\left(\frac{\pi n_2}{2}\right)}{\sqrt{3}}
\end{array}\right),$$

$$\mathcal{M}^{(8)}_{\{0,0,n_3\}} = \left(\begin{array}{cccccccc}
\cos\left(\frac{\pi n_3}{2}\right)-\frac{\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 \\
0 & \cos\left(\frac{\pi n_3}{2}\right)-\frac{\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \frac{2\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 \\
0 & 0 & \cos\left(\frac{\pi n_3}{2}\right)+\frac{\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 \\
0 & 0 & 0 & \cos\left(\frac{\pi n_3}{2}\right)+\frac{\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \frac{2\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} \\
\frac{2\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_3}{2}\right)+\frac{\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 \\
0 & -\frac{2\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_3}{2}\right)+\frac{\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 \\
0 & 0 & \frac{2\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_3}{2}\right)-\frac{\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 \\
0 & 0 & 0 & -\frac{2\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}} & 0 & 0 & 0 & \cos\left(\frac{\pi n_3}{2}\right)-\frac{\sin\left(\frac{\pi n_3}{2}\right)}{\sqrt{3}}
\end{array}\right). \tag{F.2}$$

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