FLIPPING QUBITS

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On a classical level the information can be represented by bits, each of which can be either 0 or 1. Quantum information, on the other hand, consists of qubits which can be represented as two-level quantum systems with one level labeled $|0\rangle$ and the other $|1\rangle$. Unlike bits, qubits cannot only be in one of the two levels, but in any superposition of them as well. This superposition principle makes quantum information fundamentally different from its classical counterpart. One of the most striking difference between the classical and quantum information is as follows: it is not a problem to flip a classical bit, i. e., to change the value of a bit, a 0 to a 1 and vice versa. This is accomplished by a NOT gate. Flipping a qubit, however, is another matter: there exists the fundamental bound which prohibits to flip a qubit prepared in an *arbitrary* state $|\Psi\rangle = \alpha|0\rangle + \beta|0\rangle$ and to obtain the state $|\Psi^{\perp}\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$ which is *orthogonal* to it, i. e., $\langle\Psi^{\perp}|\Psi\rangle = 0$. We experimentally realize the best possible approximation of the qubit flipping that achieves bounds imposed by complete positivity of quantum mechanics.

INTRODUCTION

Let us assume the Poincaré sphere, which represents a state space of a qubit. The points corresponding to $|\Psi\rangle$ and $|\Psi^{\perp}\rangle$ are antipodes of each other. The desired spin-flip operation is therefore the *inversion of the Poincaré sphere* (see Fig. 1).

It is well known that this inversion preserves angles (which is related to the scalar product $|\langle\Phi,\Psi\rangle|$ of rays). Therefore, by the arguments of the Wigner theorem, the ideal spin-flip operation must be implemented either by a unitary or by an anti-unitary operation. Unitary operations correspond to proper rotations of the Poincaré sphere, whereas anti-unitary operations correspond to orthogonal transformations with determinant -1. The spin-flip is an anti-unitary operation; i. e., it is not completely positive.

Due to the fact that the tensor product of an antilinear and a linear operator is not correctly defined, the spin-flip operation cannot be applied to a qubit, while the rest of the world is governed by unitary evolution[1]. On the other hand, if we consider a spin-flip operation, we have in mind a Universal NOT gate flipping an input qubit to its orthogonal state. The gate itself is an operation applied to the qubit, that is just a subsystem of a «whole universe». Therefore, a completely positive operation must be represented. It is well known that any

[1] In fact, exactly this property makes the spin-flip operation so important in all criteria of inseparability for two-qubit systems [1, 2].

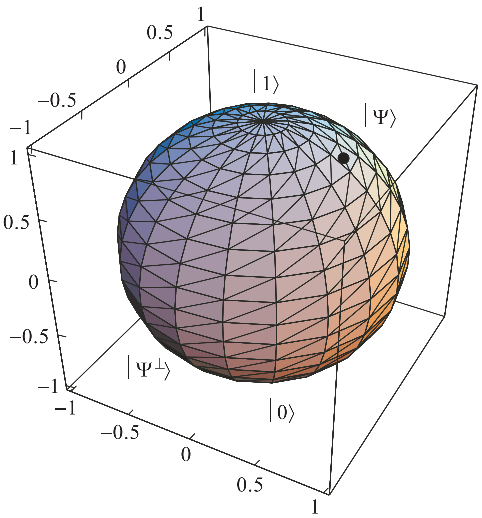

Fig. 1. The state space of a qubit is a Poincaré sphere. Pure states are represented by points on the sphere, while statistical mixtures are points inside the sphere. The Universal NOT operation corresponds to the inversion of the sphere, since the states $|\Psi\rangle$ and $|\Psi^{\perp}\rangle$ are antipodes

completely positive operation on a qubit can be realized by a unitary operation performed on the qubit and the ancillary system. Following this arguments, we see that the ideal *Universal* NOT gate which would flip a qubit in an *arbitrary* state does not exist.

Obviously, if the state of the qubit is known, then we can always perform a flip operation. In this situation the classical and quantum operations share many similar features, since the knowledge of the state is a classical information, which can be manipulated according to the rules of classical information processing (e. g., known states can be copied, flipped, etc.). But the universality of the operation is lost. That is, the gate which would flip the state $|0\rangle \rightarrow |1\rangle$ is not able to perform a flip $|(0\rangle + |1\rangle)/\sqrt{2} \rightarrow (|0\rangle - |1\rangle)/\sqrt{2}$.

Since it is not possible to realize a perfect Universal NOT gate [3] which would flip an arbitrary qubit state, it is of interest to study what is the best approximation to the perfect Universal NOT gate. Here, one can consider two possible scenarios. The first one is based on the measurement of input qubit(s) — using the results of an optimal measurement, one can manufacture an orthogonal qubit, or any desired number of them. Obviously, the fidelity of the NOT operation in this case is equal to the fidelity of estimation of the state of the input qubit(s). The second scenario would be to approximate an anti-unitary transformation on a Hilbert space of the input qubit(s) by a unitary transformation on a larger Hilbert space which describes the input qubit(s) and ancillas.

It has been shown recently that the best achievable fidelity of both flipping scenarios is the same [4–6]. That is, the fidelity of the optimal Universal NOT gate is equal to the fidelity of the best state-estimation performed on input qubits [7–9] (one might say that in order to flip a qubit we have to transform it into a bit). In what follows we briefly describe the unitary transformation realizing the quantum scenario for the spin-flip operation; that is, we present the optimal Universal NOT gate. Then we describe our experiment (see also Ref. [10]) in which we «flip» qubits encoded in polarization states of photons.

1. THEORETICAL DESCRIPTION OF SPIN FLIPPING

Let $\mathcal{H} = \mathbf{C}^2$ denote the two-dimensional Hilbert space of a single qubit. Then the input state of N systems prepared in the pure state $|\Psi\rangle$ is the N-fold tensor product $|\Psi\rangle^{\otimes N} \in \mathcal{H}^{\otimes N}$. The corresponding density matrix is $\sigma \equiv \rho^{\otimes N}$, where $\rho = |\Psi\rangle\langle\Psi|$ is the one-particle density matrix. An important observation is that the vectors $|\Psi\rangle^{\otimes N}$ are invariant under permutations of all N sites; i. e., they belong to the symmetric, or «Bose»-subspace $\mathcal{H}_+^{\otimes N} \subset \mathcal{H}^{\otimes N}$. Thus, as long as we consider only pure input states, we can assume all the input states of the device under consideration to be density operators on $\mathcal{H}_+^{\otimes N}$. We will denote by $\mathcal{S}(\mathcal{H})$ the density operators over a Hilbert space $\mathcal{H}$. Then the U-NOT gate must be a completely positive trace preserving map $T\colon \mathcal{S}\left(\mathcal{H}_+^{\otimes N}\right) \to \mathcal{S}(\mathcal{H})$. Our aim is to design T in such a way that for any pure one-particle state $\rho \in \mathcal{S}(\mathcal{H})$ the output $T(\rho^{\otimes N})$ is as close as possible to the orthogonal qubit state $\rho^\perp = \mathbf{1} - \rho$. In other words, we are trying to make the fidelity $\mathcal{F} := \mathrm{Tr}\,[\rho^\perp T(\rho^{\otimes N})] = 1 - \Delta$ of the optimal complement with the result of the transformation T as close as possible to unity for an arbitrary input state. This corresponds to the problem of finding the minimal value of the error measure $\Delta(T)$ defined as

$$\Delta(T) = \max_{\rho,\mathrm{pure}} \mathrm{Tr}\left[\rho T(\rho^{\otimes N})\right]. \tag{1}$$

Note that this functional Δ is completely unbiased with respect to the choice of input state. More formally, it is invariant with respect to unitary rotations (basis changes) in $\mathcal{H}$. When T is any admissible map, and U is a unitary on $\mathcal{H}$, the map $T_U(\sigma) = U^* T(U^{\otimes N} \sigma U^{*\otimes N})U$ is also admissible and satisfies $\Delta(T_U) = \Delta(T)$. We will show later on that one may look for optimal gates T, minimizing $\Delta(T)$, among the *universal* ones, i. e., the gates satisfying $T_U = T$ for all U. For such U-NOT gates, the maximization can be omitted from the definition (1), because the fidelity $\mathrm{Tr}\left[\rho T(\rho^{\otimes N})\right]$ is independent of ρ.

1.1. Measurement-Based Scenario. An estimation device by definition takes an input state $\sigma \in \mathcal{S}(\mathcal{H}_+^{\otimes N})$ and produces, on every single experiment, an «estimated pure state» $\rho \in \mathcal{S}(\mathcal{H})$. As in any quantum measurement this will not always be the same ρ, even with the same input state ρ, but a random quantity. The estimation device is, therefore, described completely by the probability distribution of pure states it produces for every given input. Still simpler, we will characterize it by the corresponding probability density with respect to the unique normalized measure on the pure states (denoted «$d\Phi$» in integrals), which is also invariant under unitary rotations. For an input state $\sigma \in \mathcal{S}(\mathcal{H}_+^{\otimes N})$, the value of this probability density at the pure state $|\Phi\rangle$ is

$$p(\Phi, \sigma) = (N+1)\langle \Phi^{\otimes N}, \sigma \Phi^{\otimes N}\rangle. \tag{2}$$

To check the normalization, note that $\int d\Phi p(\Phi, \sigma) = \mathrm{Tr}\,[X\sigma]$ for a suitable operator X, because the integral depends linearly on σ. By unitary invariance of the measure «$d\Phi$» this operator commutes with all unitaries of the form $U^{\otimes N}$, and since these operators restricted to $\mathcal{H}_+^{\otimes N}$ form an irreducible representation of the unitary group of $\mathcal{H}$ (for $d = 2$, it is just the spin $N/2$ irreducible representation of $SU(2)$), the operator X is a multiple of the identity. To determine the factor, one inserts $\sigma = \mathbf{1}$, and uses the normalization of «$d\Phi$» to verify that $X = \mathbf{1}$.

Note that the density (2) is proportional to $|\langle \Phi, \Psi\rangle|^{2N}$, when $\sigma = |\Psi^{\otimes N}\rangle\langle\Psi^{\otimes N}|$ is the typical input to such a device: N systems prepared in the same pure state $|\Psi\rangle$. In that case, the probability density is clearly peaked sharply at states $|\Phi\rangle$ which are equal to $|\Psi\rangle$ up to a phase.

Suppose now that we combine the state estimation with the preparation of a new state, which is some function of the estimated state. The overall result will then be the integral of the state valued function with respect to the probability distribution just determined. In the case at hand, the desired function is $f(\Phi) = (\mathbf{1} - |\Phi\rangle\langle\Phi|)$. So the result of the whole measurement-based («classical») scheme is

$$\rho^{(\mathrm{est})} = T(\sigma) = \int d\Phi p(\Phi, \sigma)\,(\mathbf{1} - |\Phi\rangle\langle\Phi|). \tag{3}$$

The fidelity required for the computation of Δ from Eq. (1) is then equal to (see also [8, 9])

$$\Delta = (N+1) \int d\Phi \; |\langle\Phi, \Psi\rangle|^{2N} (1 - |\langle\Phi, \Psi\rangle|^2) = \frac{1}{N+2}, \tag{4}$$

where we have used the fact that the two integrals have exactly the same form (differing only in the choice of N), and that the first integral is just the normalization integral. Since this expression does not depend on ρ, we can drop the maximization in the definition (1) of Δ and find $\Delta(T) = 1/(N+2)$, from which we find that the fidelity of creation of a complement to the original state ρ is

$$\mathcal{F} = \frac{N+1}{N+2}. \tag{5}$$

Finally, we note that the result of the operation (3) can be expressed in the form

$$\rho^{(\mathrm{out})} = s_{_N} \rho^{\perp} + \frac{1 - s_{_N}}{2}\mathbf{1}, \tag{6}$$

with the «scaling» parameter $s_{_N} = N/(N+2)$. From here it is seen that in the limit $N \to \infty$, perfect estimation of the input state can be performed, and, consequently, the perfect complement can be generated. For finite N, the mean fidelity is always smaller than unity. The advantage of the measurement-based scenario is that once the input qubit(s) is measured and its state is estimated, an arbitrary number M of identical (approximately) complemented qubits can be produced with the same fidelity, simply by replacing the output function $f(\Phi) = (\mathbf{1} - |\Phi\rangle\langle\Phi|)$ by $f_M(\Phi) = (\mathbf{1} - |\Phi\rangle\langle\Phi|)^{\otimes M}$.

1.2. Quantum Scenario. Let us now present a transformation which produces complements whose fidelity is the same as those produced by the measurement-based method. Assume we have N input qubits in an unknown state $|\Psi\rangle$, and we are looking for a transformation which generates M qubits at the output in a state as close as possible to the orthogonal state $|\Psi^{\perp}\rangle$. The universality of the proposed transformation has to guarantee that all input states are complemented with the same fidelity. If we want to generate M approximately complemented qubits at the output, the U-NOT gate has to be represented by $2M$ qubits (irrespective of the number, N, of input qubits), M of which serve as ancilla, and M of which become the output complements. We will indicate these subsystems by subscripts «a» = input, «b» = ancilla, and «c» = (prospective) output. The U-NOT gate transformation, U_{NM}, acts on the tensor product of all three systems. The gate is always prepared in some state $|X\rangle_{bc}$, independently of the input state $|\Psi\rangle$. The transformation is determined by the following explicit expression, valid for every unit vector $|\Psi\rangle \in \mathcal{H}$:

$$U_{NM}|N\Psi\rangle_a \otimes |X\rangle_{bc} = \sum_{j=0}^{M} \gamma_j^{(N,M)} |X_j(\Psi)\rangle_{ab} \otimes |\{(M-j)\Psi^{\perp}; j\Psi\}\rangle_c \tag{7}$$

with

$$\gamma_j^{(N,M)} = (-1)^j \binom{N+M-j}{N}^{1/2} \binom{N+M+1}{M}^{-1/2}, \tag{8}$$

where $|N\Psi\rangle_a = |\Psi\rangle^{\otimes N}$ is the input state consisting of N qubits in the same state $|\Psi\rangle$. In the right-hand side of Eq. (7), $|\{(M-j)\Psi^\perp; j\Psi\}\rangle_c$ denotes symmetric and normalized states with $(M-j)$ qubits in the complemented (orthogonal) state $|\Psi^\perp\rangle$ and j qubits in the original state $|\Psi\rangle$. Similarly, the vectors $|X_j(\Psi)\rangle_{ab}$ consist of $N+M$ qubits and are given explicitly by

$$|X_j(\Psi)\rangle_{ab} = |\{(N+M-j)\Psi; j\Psi^\perp\}\rangle_{ab}. \tag{9}$$

Note that with this choice of the coefficients $\gamma_j^{(M,N)}$, the scalar product of the right-hand side with a similar vector, with Ψ replaced by Φ, becomes $\langle\Psi,\Phi\rangle^N$. This is consistent with the unitarity of the operator U_{NM}.

Each of the M qubits at the output of the U-NOT gate is described by the density operator (6) with $s_{_N} = N/(N+2)$, *irrespective* of the number of complements produced. The fidelity of the U-NOT gate depends only on the number of inputs. This means that this U-NOT gate can be thought of as producing an approximate complement and then cloning it, with the quality of the cloning independent of the number of clones produced. The universality of the transformation is directly seen from the «scaled» form of the output operator (6).

We stress that the fidelity of the U-NOT gate (7) is exactly the same as in the measurement-based scenario. Moreover, it also behaves as a classical (measurement-based) gate in a sense that it can generate an arbitrary number of complements with the same fidelity. We have also checked that these cloned complements are pairwise separable.

The $N+M$ qubits at the output of the gate, which do not represent the complements, are individually in the state described by the density operator

$$\rho_j^{(\mathrm{out})} = s\rho + \frac{1-s}{2}1, \quad j = 1, \ldots, N+M, \tag{10}$$

with the scaling factor $s = \dfrac{N}{N+2} + \dfrac{2N}{(N+M)(N+2)}$; i. e., these qubits are the *clones* of the original state with a fidelity of cloning larger than the fidelity of estimation. This fidelity depends on the number, M, of clones produced out of the N originals, and in the limit $M \to \infty$ the fidelity of cloning becomes equal to the fidelity of estimation. These qubits represent the output of the *optimal* $N \to N+M$ cloner introduced by Gisin and Massar [11–13]. This means that the U-NOT gate, as presented by the transformation in Eq. (7), serves also as a universal cloning machine.

1.3. Optimality of U-NOT Gate. At this point, the question arises whether the transformation (7) represents the *optimal* U-NOT gate via quantum scenario. If this is so, then it would mean that the measurement-based and the quantum scenarios realize the U-NOT gate with the same fidelity. In what follows we present a proof due to Werner in Refs. [4, 5].

Theorem. *Let $\mathcal{H}$ be a Hilbert space of dimension $d = 2$. Then among all completely positive trace preserving maps $T : \mathcal{S}\left(\mathcal{H}_+^{\otimes N}\right) \to \mathcal{S}(\mathcal{H})$, the measurement-based U-NOT scenario (3) attains the smallest possible value of the error measure defined by Eq. (1), namely, $\Delta(T) = 1/(N+2)$.*

We have already shown (see Eq. (4)) that for the measurement-based strategy the error Δ attains the value $1/(N+2)$. The more difficult part, however, is to show that no other scheme (i. e., quantum scenario) can do better. Here, we will largely follow the arguments in [14].

Note first that the functional Δ is invariant with respect to unitary rotations (basis changes) in $\mathcal{H}$. When T is any admissible map, and U is a unitary on $\mathcal{H}$, the map $T_U(\sigma) = U^* T(U^{\otimes N} \sigma U^{*\otimes N}) U$ is also admissible and satisfies $\Delta(T_U) = \Delta(T)$. Moreover, the functional Δ is defined as the maximum of a collection of linear functions in T, and is therefore convex. Putting these observations together, we get

$$\Delta(T) \leq \int dU \Delta(T_U) = \Delta(T), \tag{11}$$

where $T = \int dU T_U$ is the average of the rotated operators T_U with respect to the Haar measure on the unitary group. Thus, T is at least as good as T and has the additional «covariance property» $T_U = T$. Without loss we can therefore assume from now on that $T_U = T$ for all U.

An advantage of this assumption is that a very explicit general form for such covariant operations is known by a variant of the Stinespring Dilation Theorem (see [14] for a version adapted to our needs).

The form of T is further simplified in our case by the fact that both representations involved are irreducible: the defining representation of $SU(2)$ on $\mathcal{H}$, and the representation by the operators $U^{\otimes N}$ restricted to the symmetric subspace $\mathcal{H}_+^{\otimes N}$. Then T can be represented as a discrete convex combination $T = \sum_j \lambda_j T_j$, with $\lambda_j \geq 0, \sum_j \lambda_j = 1$, and T_j admissible and covariant maps in their own right, but of an even simpler form. Covariance of T already implies that the maximum can be omitted from the definition (1) of Δ, because the fidelity no longer depends on the pure state chosen. In a convex combination of covariant operators we therefore get

$$\Delta(T) = \sum_j \lambda_j \Delta(T_j). \tag{12}$$

Minimizing this expression is obviously equivalent to minimizing with respect to the discrete parameter j.

We write the general form of the extremal instruments T_j in terms of expectation values of the output state for an observable X on $\mathcal{H}$:

$$\mathrm{Tr}\,\big(T(\sigma)X\big) = \mathrm{Tr}\,\big[\sigma V^*(X \otimes \mathbf{1})V\big], \tag{13}$$

where $V : \mathcal{H}_+^{\otimes N} \to \mathcal{H} \otimes \mathbf{C}^{2j+1}$ is an isometry intertwining of the respective representations of $SU(2)$, namely, the restriction of the operators $U^{\otimes N}$ to $\mathcal{H}_+^{\otimes N}$ (which has spin $N/2$), on the one hand, and the tensor product of the defining representation (spin-$1/2$) with the irreducible spin-j representation, on the other. By the triangle inequality for Clebsch–Gordan reduction, this implies $j = (N/2) \pm (1/2)$, so only two terms appear in the decomposition of T. It remains to compute $\Delta(T_j)$ for these two values.

The basic idea is to use the intertwining property of the isometry V for the generators S_α, J_α, and $L_\alpha, \alpha = 1, 2, 3$ of the $SU(2)$-representations on $\mathcal{H}$, $\mathbf{C}^{2j+1}$ and $\mathcal{H}_+^{\otimes N}$, respectively.

We will show that

$$V^*(S_\alpha \otimes \mathbf{1}_j)V = \mu_j L_\alpha, \tag{14}$$

where μ_j is some constant depending on the choice of j. That such a constant exists is clear from the fact that the left-hand side of this equation is a vector operator (with components labeled by $\alpha = 1, 2, 3$), and the only vector operators in an irreducible representation of $SU(2)$ are multiples of angular momentum (in this case L_α). The constant μ_j can be expressed in terms of a $6j$ symbol, but can also be calculated in an elementary way using the intertwining property, $VL_\alpha = (S_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes J_\alpha)V$, and the fact that the angular momentum squares $\mathbf{J}^2 = \sum_\alpha J_\alpha^2 = j(j+1)$, $\mathbf{S}^2 = 3/4$, and $\mathbf{L}^2 = N/2(N/2+1)$ are multiples of the identity in the irreducible representations involved, and can be treated as scalars:

$$\mu_j \mathbf{L}^2 = \sum_\alpha V^*(S_\alpha \otimes \mathbf{1}_j)VL_\alpha = \mathbf{S}^2 + \sum_\alpha V^*(S_\alpha \otimes J_\alpha)V. \tag{15}$$

The sum in the right-hand side can be obtained as the mixed term of a square, namely, as

$$\frac{1}{2}\Big(\sum_\alpha V^*(S_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes J_\alpha)^2 V - \mathbf{S}^2 - \mathbf{J}^2\Big) = (\mathbf{L}^2 - \mathbf{S}^2 - \mathbf{J}^2). \tag{16}$$

Combining these equations, we find

$$\mu_j = \frac{1}{2} + \frac{\mathbf{S}^2 - \mathbf{J}^2}{2\mathbf{L}^2} \begin{cases} \dfrac{1}{N} & \text{for} \quad j = \dfrac{N}{2} + \dfrac{1}{2} \\ \dfrac{-1}{N+2} & \text{for} \quad j = \dfrac{N}{2} - \dfrac{1}{2} \end{cases}. \tag{17}$$

We combine equations (13) and (14) to get the error quantity Δ from equation (1), with the pure one-particle density matrix $\rho = 1/2\,\mathbf{1} + S_3$:

$$\Delta(T) = \mathrm{Tr}\,(V^*(\rho \otimes \mathbf{1})V\rho^{\otimes N}) = \frac{1}{2}(1 + N\mu_j). \tag{18}$$

With equation (17) we find

$$\Delta(T) = \begin{cases} 1 & \text{for} \quad j = \dfrac{N}{2} + \dfrac{1}{2} \\ \dfrac{1}{N+2} & \text{for} \quad j = \dfrac{N}{2} - \dfrac{1}{2} \end{cases}. \tag{19}$$

The first value is the largest possible fidelity for getting the state ρ from a set of N copies of ρ. The fidelity 1 is expected for this trivial task, because taking any one of the copies will do perfectly. On the other hand, the second value is the minimal fidelity, which we were looking for. This clearly coincides with the value (4), so the Theorem is proved.

The Theorem as it stands concerns the task of producing just one particle in the U-NOT state of the input. From the results of the previous section we see that it is valid also in the case of many outputs. We see that the maximum fidelity is achieved by the classical process via estimation: in equation (3) we just have to replace the output state $(\mathbf{1} - |\Phi\rangle\langle\Phi|)$ by the desired tensor power. Hence, once again the optimum is achieved by the scheme based on classical estimation. Incidentally, this shows that the multiple outputs from such a device are completely unentangled, although they may be correlated.

We can conclude this section by saying that in the quantum world governed by unitary operations anti-unitary operations can be performed with the fidelity which is bounded by the amount of classical information potentially available about states of quantum systems.

2. EXPERIMENT

In our experiment we will consider a flipping of a single qubit. In this case the flipping transformation reads

$$U_{11}|\Psi\rangle_a \otimes |X\rangle_{bc} = \sqrt{\frac{2}{3}}|\Psi\Psi\rangle_{ab}|\Psi^{\perp}\rangle_c - \sqrt{\frac{1}{3}}|\{\Psi, \Psi^{\perp}\}\rangle_{ab}|\Psi\rangle_c. \tag{20}$$

To be specific, Eq. (20) describes a process when the original qubit is encoded in the system a, while the flipped qubit is in the system c. The density operator describing the state of the system c at the output is

$$\sigma_c^{(\mathrm{out})} = \frac{1}{3}|\Psi^{\perp}\rangle\langle\Psi^{\perp}| + \frac{1}{3}\mathbf{1}. \tag{21}$$

The fidelity of the spin flipping is $\mathcal{F} = 2/3$.

A natural way to encode a qubit into a physical system is to utilize polarization states of a single photon. In this case, the Universal NOT gate can be realized via the stimulated emission. The key idea of our experiment is based on the proposal that universal quantum machines [15] such as quantum cloner can be realized with the help of stimulated emission in parametric down conversion [16,17]. In particular, let us consider a qubit to be encoded in a polarization state of a photon. This photon is injected as the input state into an optical parametric amplifier (OPA) physically consisting of a nonlinear (NL) BBO (β-barium-borate) crystal cut for Type II phase matching and excited by a pulsed mode-locked ultraviolet laser UV, having pulse duration $\tau \approx 140f$ s and wavelength (wl) $\lambda_p = 397.5$ nm, associated to pulse duration [18]. The relevant modes of the NL three-wave interaction were the spatial modes with wave-vector (wv) k_1 and k_2 each supporting the two horizontal (H) and vertical (V) linear polarizations $(\mathbf{\Pi})$ of the interacting photons, e. g., $\mathbf{\Pi}_{1H}$ is the horizontal polarization unit vector associated with k_1. The OPA was frequency degenerate; i. e., the interacting photons had the same wl's $\lambda = 795$ nm. The action of OPA under suitable conditions can be described by a simplified Hamiltonian

$$\hat{H}_{\mathrm{int}} = \kappa(\hat{a}^{\dagger}_{\Psi}\hat{b}^{\dagger}_{\Psi^{\perp}} - \hat{a}^{\dagger}_{\Psi^{\perp}}\hat{b}^{\dagger}_{\Psi}) + \mathrm{h.\,c.} \tag{22}$$

A property of the device, of key importance in the context of the present work, is its amplifying behaviour with respect to the polarization $\mathbf{\Pi}$ of the interacting photons. It has been shown by theory [16,19] and in a recent experiment on «universal quantum cloning» [17] that the amplification efficiency of this type of OPA under injection by *any* externally injected quantum field, e. g., consisting of a single photon or of a classical «coherent» field, can be made *independent* of the polarization state of the field. In other words, the OPA «gain» is independent of any (*unknown)* polarization state of the injected field. This precisely represents the necessary *universality (U)* property of the U-NOT gate. For this reason in Eq. (22) we have denoted the creation $\hat{a}^\dagger_\Psi$ ($\hat{b}^\dagger_\Psi$) and annihilation $\hat{a}_\Psi$ ($\hat{b}_\Psi$) operators of a photon in mode k_1 (k_2) with subscripts Ψ (or $\Psi^\perp$) indication of the invariance of the process with respect to polarization states of the input photon.

Let us consider the input photon in the mode k_1 to have a polarization Ψ. We will describe this polarization state as $\hat{a}^\dagger_\Psi|0,0\rangle_{k_1} = |1,0\rangle_{k_1}$, where we have used notation introduced by Simon et al. [16]; i. e., the state $|m,n\rangle_{k_1}$ represents a state with m photons of the mode k_1 having the polarization Ψ, while n photons have the polarization $\Psi^\perp$. Initially, there are no excitations in the mode k_2. The initial polarization state of these two modes reads $|1,0\rangle_{k_1} \otimes |0,0\rangle_{k_2}$, and it evolves according the Hamiltonian (22):

$$\exp(-i\hat{H}_{\rm int}t)|1,0\rangle_{k_1} \otimes |0,0\rangle_{k_2} \simeq |1,0\rangle_{k_1} \otimes |0,0\rangle_{k_2} - \\ - i\kappa t\left(\sqrt{2}|2,0\rangle_{k_1} \otimes |0,1\rangle_{k_2} - |1,1\rangle_{k_1} \otimes |1,0\rangle_{k_2}\right). \quad (23)$$

This approximation for the state vector describing the two modes at times $t > 0$ is sufficient since the values κt are usually very small (see below). The zero-order term corresponds to the process when the input photon in the mode k_1 does not interact in the nonlinear medium, while the second term describes the first-order process in the OPA. This second term is formally equal (up to a normalization factor) to the right-hand side of Eq. (20). Here, the state $|2,0\rangle_{k_1}$ describing two photons of the mode k_1 in the polarization state Ψ corresponds to the state $|\Psi\Psi\rangle$. This state-vector describes the cloning of the original photon [16,17]. The vector $|0,1\rangle_{k-2}$ describes the state of the mode k_2 with a single photon with the polarization $\Psi^\perp$. That is, this state vector represents the flipped version of the input.

To see that the stimulated emission is indeed responsible for creation of the flipped qubit, let us compare the state (23) with the output of the OPA when the vacuum is injected into the nonlinear crystal. In this case, to the same order of approximation as above, we obtain

$$\exp(-i\hat{H}_{\rm int}t)|0,0\rangle_{k_1} \otimes |0,0\rangle_{k_2} \simeq |0,0\rangle_{k_1} \otimes |0,0\rangle_{k_2} - \\ - i\kappa t(|1,0\rangle_{k_1} \otimes |0,1\rangle_{k_2} - |0,1\rangle_{k_1} \otimes |1,0\rangle_{k_2}). \quad (24)$$

We see that the flipped qubit described by the state vector $|0,1\rangle_{k_2}$ in the right-hand sides of Eqs. (23) and (24) does appear with different amplitudes corresponding to the ratio of probabilities to be equal to $1:2$. This ratio has been measured in our experiment.

2.1. Universality. On the «microscopic» quantum level the justification of this U-property of the OPA amplifier resides in the $SU(2)$ rotational invariance of the NL interaction Hamiltonian when the spatial orientation of the OPA NL Type II crystal makes it available for the generation of two-photon entangled «singlet» states by Spontaneous Parametric Down Conversion (SPDC), i. e., by injection of the «vacuum field» [16, 19]. However we should note that in the present context the *universality* property, i. e., the $\mathbf{\Pi}$-*insensitivity* of the parametric amplification «gain» g, is a «macroscopic» classical feature of the OPA device. As a consequence, it can be tested equally well either by injection of «classical», e. g., coherent (Glauber) fields or of a «quantum» states of radiation, e. g., a single-photon Fock state. We have carried out successfully both tests, leading to identical results. We only report here the ones corresponding to the injection by attenuated «coherent» laser field (see Fig. 2).

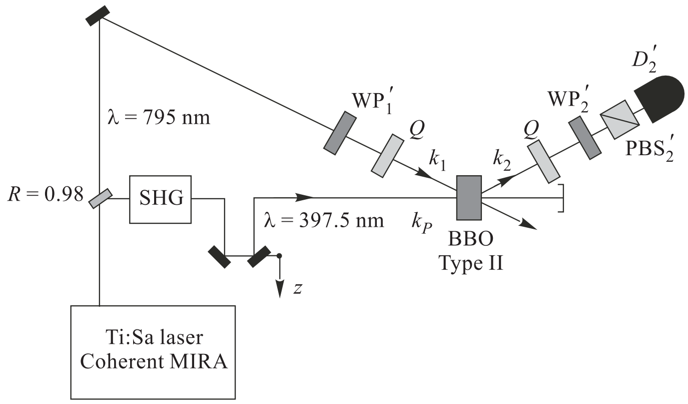

Fig. 2. Schematic description of the experimental verification of the universality of the flip operation

A coherent state of attenuated laser field with wl $\lambda = 795$ nm is used. The source is Ti:Sa Coherent MIRA pulsed laser providing by Second Harmonic Generation (SHG) the OPA «pump» field associated with the spatial mode with wv k_p and wl λ_p. A small portion of the laser radiation at wl λ was directed along the OPA injection mode k_1. The parametric amplification, with calculated «gain» $g = 0.31$, was detected at the OPA output mode k_1

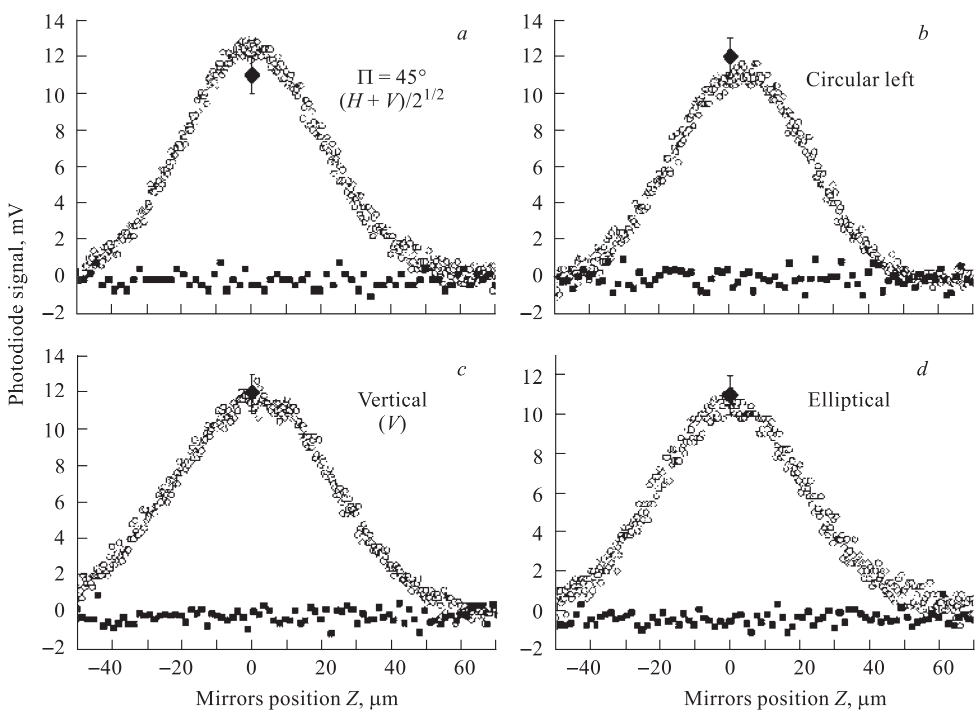

Fig. 3. Experimental verification of the universality of OPA

by D'_1, a Si linear photodiode SGD100. The time superposition in the NL crystal of the «pump» and of the «injection» pulses was assured by micrometric displacements (Z) of a two-mirror optical «trombone». Various **Π**-states of the injected pulse were prepared by the set ($\mathrm{WP}'_1 + Q$) consisting of a Wave-plate (either $\lambda/2$ or $\lambda/4$) and of a 4.3-mm X-cut Quartz plate. These states were then analyzed after amplification and before detection on mode k_2 by an analogous optical set ($\mathrm{WP}'_2 + Q + \mathbf{\Pi}$-*analyzer*), the last device being provided by the *Polarizing Beam Splitter* PBS'_2. In the two experiments reported in the present work, all the 4.5-mm thick X-cut quartz plates (Q) provided the compensation of the unwanted beam walk-off effects due to the birefringence of the NL crystal.

The *universality* condition is demonstrated by the plots of Fig. 3, showing the amplification pulses detected by D'_2 on the OPA output mode, k_2. Each plot corresponds to a definite **Π**-state, $|\Psi\rangle = [\cos(\vartheta/2)|H\rangle + \exp(i\phi)\sin(\vartheta/2)|V\rangle]$, either *linear* — **Π**, i. e., $\vartheta = 0,\ \pi/2,\ \pi$; $\phi = 0$, or *circular* — **Π**, i. e., $\vartheta = \pi/2$; $\phi = \pm\pi/2$, or *elliptical* — **Π**, in the very general case: $\vartheta = 5\pi/18$; $\phi = -\pi/2$. We may check that the corresponding amplification curves, each corresponding to a standard injection pulse_1 with an average photon number $N \approx 5\cdot 10^3$, are almost identical.

For more generality, the *universality* condition as well as the insensitivity of this condition to the value of N is also demonstrated by the *single* experimental data reported, with different scales, at the top of each amplification plot and corresponding to injection pulses with $N \approx 5\cdot 10^2$. Single-photon tests of the same conditions were also carried out with a different experimental setup, as said.

2.2. Optimality. Let us move to the main subject of the present work, i. e., the quantum U-NOT gate. In virtue of the tested *universality* of the OPA amplification, it is of course sufficient to consider here the OPA injection by a *single-photon* in *just one* **Π**-state, for instance in the *vertical* **Π**-state. Accordingly, Fig. 4 shows a layout of the single-photon, $N = 1$, quantum-injection experiment with input state $|\Psi\rangle = |V\rangle$.

Consider the k_p pump mode, i. e., the «towards R» excitation. An SPDC process created single photon-pairs with wl $\lambda = 795$ nm in entangled *singlet* **Π**-*states*, i. e., rotationally invariant, as said. One photon of each pair, emitted over k_1 was reflected by a spherical mirror M onto the NL crystal where it provided the $N = 1$ *quantum injection* into the OPA amplifier excited by the UV «pump» beam associated to the back reflected mode $-k_p$. We consider flipping of a single photon in a state $|\Psi\rangle = |V\rangle$. In this experiment, owing to a spherical mirror M_p with 100 % reflectivity and micrometrically adjustable position Z, the UV pump beam excited the same NL OPA crystal amplifier in both directions k_p and $-k_p$, i. e., correspondingly oriented towards the right (R) and left (L) sides of the figure. Because of the low intensity of the UV beam, the two-photon injection probability $N = 2$

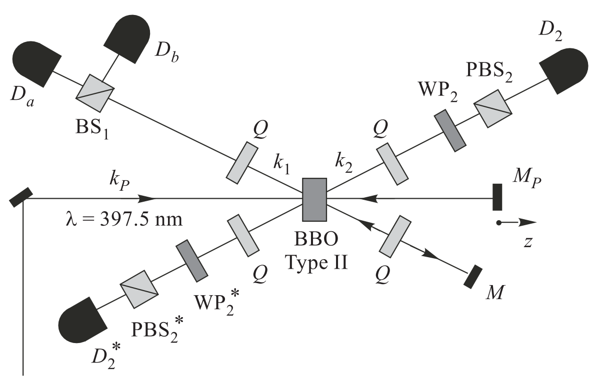

Fig. 4. Experimental realization of the quantum U-NOT gate

has been evaluated to be $\approx 3.5 \cdot 10^{-4}$ smaller that for the $N = 1$ condition. The twin photon emitted over k_2 was $\mathbf{\Pi}$-*selected* by the devices (WP$_2$+ PBS$_2$) and then detected by D_2, thus providing the «trigger» of the overall *conditional experiment*. All detectors in the present experiment were equal active SPCM-AQR14 with quantum efficiency: QE $\approx 55\,\%$. Because of the EPR nonlocality implied by the singlet state, the $\mathbf{\Pi}$-*selection* on channel k_2 provided the realization on k_1 of the state $|\Psi\rangle = |V\rangle$ of the injected photon. As for the previous experiment, all the X-cut quartz plates Q provided the compensation of the unwanted beam walk-off effects due to the birefringence of the NL crystal. Consider the «towards L» amplification, i. e., the amplification process excited by the mode $-k_p$, and do account, in particular, for the OPA output mode k_2. The $\mathbf{\Pi}$-state of the field on that mode was analyzed by the device combination (WP$_2^*$+ PBS$_2^*$) and measured by the detector D_2^*. The detectors D_a, D_b were coupled to the field associated with the mode k_1. The experiment was carried out by detecting the rate of the four-coincidences involving all detectors $[D_2^* D_2 D_a D_b]$.

From the analysis presented by [16, 17], it follows that the state of the field emitted by the OPA indeed realizes the U-NOT gate operation, i. e., the «optimal» realization of the «anticloning» of the authentic qubit originally encoded in the mode k_1. The flipped qubit at the output is in the mode k_2. As has been shown earlier, the state created by the U-NOT gate is not pure. There is a minimal amount of noise induced by the process of flipping which is inevitable in order to preserve complete-positiveness of the Universal NOT gate. This mixed state is described by the density operator (21). The polarization state of the output photon in the mode k_2 in our experiment is indeed described by this density operator.

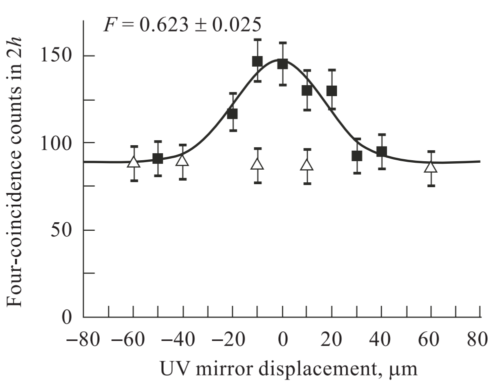

Fig. 5. Experimental verification of the optimality of the U-NOT gate

The plot of Fig. 5 reports our experimental four-coincidence data as function of the time superposition of the UV pump and of the injected single-photon pulses. That superposition was expressed as function of the micrometric displacement Z of the back-reflecting mirror M_p. The height of the central peak expresses the rate measured with the $\mathbf{\Pi}$-*analyzer* of mode k_2 set to measure the «correct» horizontal (H) polarization, i. e., the one orthogonal to the (V) polarization of the $\mathbf{\Pi}$-*state*, $|\Psi\rangle = |V\rangle$ of the injected, input single photon, $N = 1$. On turning by $90\,^\circ$ the $\mathbf{\Pi}$-*analyzer*, the amount of the «noise» contribution is represented by a «flat» curve. In our system, the «noise» was provided by the OPA amplification of the unavoidable «vacuum» state associated with the mode k_1.

Our main result consists of the determination of the ratio R^* between the height of the central peak and the one of the flat «noise» contribution. To understand this ratio, we firstly note that the most efficient stimulation process in the OPA is achieved when a perfect match (overlap) between the input photon and the photon produced by the source is achieved. This situation corresponds to the value of the mirror position Z equal to zero (see Eq. (23)). As soon as the mirror is displaced from the position, the two photons do not overlap properly, and the stimulation is less efficient. Correspondingly, the spin-flip operation is more noisy. In the limit of large displacements Z the spin flipping is totally random due to the fact that the process corresponds to injecting the vacuum into the crystal (see Eq. (24)). The theoretical ratio between the corresponding probabilities is 2. In our experiment, we have found the ratio to be $R^* = (1.66 \pm 0.20)$. This corresponds to a measured value of the fidelity of the U-NOT apparatus: $\mathcal{F}^* = (0.623 \pm 0.025)$ to be compared with the theoretical value: $F = 2/3 = 0.666$. Note that the height of the central peak does not decrease towards zero for large Z's. This effect is due to the finite time-resolution of our four-coincidence electronic apparatus, which is in the nanosecond range. It would totally disappear if the resolution could be pushed into the subpicosecond range, i. e., of the order of the time duration of the OPA pump and injection pulses. By taking a little time to think, it can be easily found that the spurious out of resonance plateau of the central peak should indeed reproduce the size of the «noise» condition measured on the mode k_2. As we can see, this is indeed verified by the experiment.

CONCLUSIONS

In summary, we have realized and investigated experimentally the first quantum machine which performs the best-possible approximation to an anti-unitary quantum operation. In particular, we have realized the universal spin flipping of a qubit (U-NOT gate). The universality of the performed operation is of the paramount importance since in general the fidelity of the flipping should not depend on the state of the input qubit. We have achieved almost optimal fidelity of the gate which is determined by the complete positivity of quantum mechanics.

Acknowledgements. We thank Mark Hillery and Reinhard Werner for many discussions and collaboration on theoretical aspect of the U-NOT gate. This work was supported in part by the European Union projects ATESIST, EQUIP and QUBITS. V. B. would like to thank the SFI for the E. T. S. Walton award.

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