UNIVERSAL HYBRID QUANTUM PROCESSORS

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A quantum processor (the programmable gate array) is a quantum network with a fixed structure. A space of states is represented as tensor product of data and program registers. Different unitary operations with the data register correspond to «loaded» programs without any changing or «tuning» of network itself. Due to such property and undesirability of entanglement between program and data registers, universality of quantum processors is subject of rather strong restrictions. Universal «stochastic» quantum gate arrays were developed by different authors. It was also proved, that «deterministic» quantum processors with finite-dimensional space of states may be universal only in approximate sense. In the present paper it is shown that, using hybrid system with continuous and discrete quantum variables, it is possible to suggest a design of strictly universal quantum processors. It is also shown that «deterministic» limit of specific programmable «stochastic» $U(1)$ gates (probability of success becomes unit for infinite program register), discussed by other authors, may be essentially the same kind of hybrid quantum systems used here.

INTRODUCTION

The quantum programmable gate array [1–4] or *quantum processor* [5, 6] is a quantum circuit with fixed structure. Similarly with usual processor here are *data register* $|D\rangle$ and *program register* $|P\rangle$. Different operations $\mathbf{u}$ with data are governed by a state of the program; i. e., it may be described as

$$\mathsf{U} : \big(|P\rangle \otimes |D\rangle\big) \mapsto |P'\rangle \otimes \big(\mathbf{u}_P|D\rangle\big). \tag{1}$$

Each register is a quantum system[2] and may be represented for particular task using qubits [1–4], qudits [5, 6], etc.

It can be simply found [1] that Eq. (1) is compatible with unitary quantum evolution, if different states of *program register* are orthogonal, due to such a requirement number of accessible programs coincides with dimension of Hilbert space, and it produces some challenge for construction of universal quantum processors. Few ways were suggested to avoid such a problem: to use specific «stochastic» design of universal quantum processor [1–3, 6], to construct (nonstochastic) quantum processor with possibility to approximate any gate with given precision [2–5] (it is also traditional approach to universality [7–9], sometimes called «universality in approximate sense» [10]).

Here is discussed an alternative approach for strictly universal quantum processor — use of continuous quantum variables in program register and discrete ones for data, i. e.,

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[2] Usually finite-dimensional.

hybrid quantum computer [12]. In such a case, number of different programs is infinite, and it provides possibility to construct strictly universal hybrid quantum processor for initial («deterministic») design described by Eq. (1). It is enough to provide procedures for one-qubit rotations with three real parameters together with some finite number of two-gates [10, 11].

It is also shown that hybrid quantum gates used in this article can be considered not only as limit of deterministic design [4, 5], but also coincide with «deterministic limit» [3] of a special case of programmable $U(1)$ «stochastic» gates with probability of failure, which tends to zero for infinite program register.

1. CONSTRUCTION OF HYBRID QUANTUM PROCESSORS

In finite-dimensional case, unitary operator U satisfying Eq. (1) can be simply found [4,5]. Let us consider the case with $|P'\rangle = |P\rangle$ in Eq. (1). It was already mentioned that states $|P\rangle$ of program register corresponding to different operators $\mathbf{u}_P$ are orthogonal and, thus, may be chosen as basis. In such a basis, $\mathbf{u}_P$ is simply a set of matrices numbered by integer index P, and operator U (1) can be written as block-diagonal $NM \times NM$ matrix:

$$\mathsf{U} = \begin{pmatrix} \mathbf{u}_1 & & & \\ & \mathbf{u_2} & & 0 \\ & & \ddots & \\ 0 & & & \mathbf{u}_M \end{pmatrix}, \tag{2}$$

with $N \times N$ matrices $\mathbf{u}_P$, if dimensions of program and data registers are M and N, respectively;

$$\mathsf{U} = \sum_{P=1}^{M} |P\rangle\langle P| \otimes \mathbf{u}_P. \tag{3}$$

It is *conditional quantum dynamics* [13]. For quantum computations with qubits $M = 2^m$, $N = 2^n$.

Generalization of hybrid system with program register described by one continuous quantum variable and qubit data register is straightforward. The states of program register may be described as Hilbert space of functions on line $\psi(x)$. In coordinate representation, a basis is

$$|q\rangle = \delta(x - q), \quad \langle q \mid \psi(x)\rangle = \psi(q). \tag{4}$$

To represent some continuous family of gates $\mathbf{u}_{(q)}$ acting on data state, say, phase rotations

$$\boldsymbol{\theta}_{(q)} = \exp{(2\pi i q \boldsymbol{\sigma}_3)}, \tag{5}$$

it is possible to write continuous analog of Eq. (3):

$$\mathsf{U} = \int dq(|q\rangle\langle q| \otimes \mathbf{u}_{(q)}), \tag{6}$$

$$\mathsf{U}\big(\psi(x)|s\rangle\big) = \int \delta(x - q)\psi(q)|\mathbf{u}_{(q)}s\rangle dq = \psi(x)|\mathbf{u}_{(x)}s\rangle, \tag{7}$$

where $\otimes$ is omitted because $|\psi\rangle|s\rangle$ can be considered as product of scalar function $\psi(x)$ on complex vector $|s\rangle$. Finally,

$$\mathsf{U}(|q\rangle|s\rangle) = |q\rangle|\mathbf{u}_{(q)}s\rangle. \tag{8}$$

It is also convenient to use momentum basis, i. e.,

$$|\tilde{p}\rangle = \mathrm{e}^{ipx}, \quad \langle\tilde{p}\psi(x)\ |{=}\rangle \int \mathrm{e}^{-ipx}\psi(x)dx \equiv \tilde{\psi}(p), \tag{9}$$

(where $\tilde{\psi}$ is Fourier transform of ψ) and operator $\tilde{\mathsf{U}}$:

$$\tilde{\mathsf{U}} = \int dp\big(|\tilde{p}\rangle\langle\tilde{p}| \otimes \mathbf{u}_{(p)}\big), \tag{10}$$

$$\tilde{\mathsf{U}}\big(\psi(x)|s\rangle\big) = \int \mathrm{e}^{ipx}\Big(\int \mathrm{e}^{-ipx'}\psi(x')dx'\Big)|\mathbf{u}_{(p)}s\rangle dp = \\ = \iint \mathrm{e}^{ip(x-x')}\psi(x')|\mathbf{u}_{(p)}s\rangle dx'dp = -\iint \mathrm{e}^{-ipq}\psi(x-q)|\mathbf{u}_{(p)}s\rangle dqdp. \tag{11}$$

Here $\tilde{\mathsf{U}}$ is not rewriting U in momentum basis, it is the other operator with property:

$$\tilde{\mathsf{U}}(|\tilde{p}\rangle|s\rangle) = |\tilde{p}\rangle|\mathbf{u}_{(p)}s\rangle. \tag{12}$$

Using such an approach with hybrid program register (few continuous variables for different qubit rotations and discrete ones for two-gates like CNOT), it is possible to suggest a design of universal quantum processor with qubits data register.

Hilbert space of hybrid system with k continuous and $M = 2^m$ discrete quantum variables can be considered as space of $\mathbb{C}^M$-valued functions with k variables

$$F(x_1, \ldots, x_k)\colon \mathbb{R}^k \to \mathbb{C}^M.$$

For construction of universal processor, it is possible to use three continuous variables[1] for each qubit together with discrete variables for control of two-qubit gates.

It should be mentioned that it is a rather simplified model. More rigorous consideration for different physical examples may include different functional spaces, distributions, functions localized on discrete set of points, and symbol «$\int$» or scalar product used in formulas above. In such a case it should be defined with necessary care. Due to such a problem, in many works about quantum computations with continuous variables Heisenberg approach and expressions with operators like coordinate $\mathbf{Q}$ and momentum $\mathbf{P}$ are used [14].

Heisenberg approach may simplify description, but hides some subtleties. For example, in many models variables could hardly be called «continuous», because they may be described as set of natural numbers; i. e., terms «infinite», «nonfinite» perhaps fits better for such quantum variables.

Let us consider an example with qubit controlled by above-described continuous variable $\mathbf{u}_{(q)} \equiv \boldsymbol{\theta}_{(q)}$ Eq. (5). In such a case, it is enough to use operator U in Eq. (8) only at the

[1] Angular parametrization of $SU(2)$.

interval of values $0 < q \leqslant 2\pi$ or even consider Hilbert space of periodic functions $\psi(q)$, such as phases. But in such a case in dual space momenta one has only discrete set of values $p \in \mathbb{Z}$ and both spaces are connected by Fourier transform, it is an example of relation between periodical functions of continuous variable and functions defined on infinite, but discontinuous set $\mathbb{Z}$ of integer numbers.

Here is an important issue: the commutation relations like $i[\mathbf{P}, \mathbf{Q}] = 1$ are not compatible with linear algebra of any finite matrix[1], but may be simply satisfied by infinite-dimensional operator algebras, like Schrödinger representation $\mathbf{Q} = x$, $\mathbf{P} = -id/dx$. But here is yet another problem: integer and real numbers are used for representation of infinite quantum variables, but *cardinality* of the sets are different, $\operatorname{card} \mathbb{N} = \aleph_0$, $\operatorname{card} \mathbb{R} = \aleph$. To avoid discussion, related with the cardinality issues, Russell paradox, etc., some formal cardinality $\aleph_{QP}$ of «quantum infinite variables» is used here; i. e., any model of infinite numbers appropriate for introduction of Heisenberg relations.

It should be mentioned that the term «hybrid» is also used with other meaning [15]. Formally, it is a different thing, but for discussed strategy for hybrid quantum processors, these two topics are closely connected. Let us discuss it briefly. For realistic design of quantum computers, it is useful to have some language for joint description with more convenient classical microdevices, which could be used as some base for development of quantum processors. Generally, such a task is very difficult (if possible at all) and has variety of different approaches.

But there is an especially simple idea that could be applied for the model under consideration. The quantum gates and «wires» may be roughly treated as (pseudo)classical, if only elements of computational basis are accepted in a model as states of system, and also gates may not cause any superposition and directly correspond to set of invertible classical logical gates [9].

Really, such a model is still quantum, but has closer relation with usual classical circuits and, thus, may reduce some difficulties in description of hybrid classical-quantum processor design. It was already discussed in [4,5] that from such point of view program register can be treated as pseudoclassical[2]. It was designed with finite number of state in program register.

Similar procedure without difficulties may be extended for continuous case, but now it corresponds to continuous classical variables; i. e., it is similar either to analogous classical control or to more detailed description of usual microprocessor, when inputs and outputs are not described as abstract zeros and ones, but as real dynamically changed classical continuous signals (fields, currents, laser beams, etc.).

2. COMPARISON WITH LIMIT OF «STOCHASTIC» MODELS

In this paper, design of universal hybrid quantum processor was used that could be considered as some limit of approximately universal «deterministic» quantum processors [4,5], when the size of program register formally becomes unlimited. On the other hand, in [2, 3] a design of programmed «stochastic» $U(1)$ gates is considered with probability of success

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[1] It is simple to show, taking trace of the commutator for $D \times D$ matrices: $i\,\mathrm{Tr}\,[\mathbf{P}, \mathbf{Q}] = i\,\mathrm{Tr}\,(\mathbf{PQ}) - i\,\mathrm{Tr}\,(\mathbf{QP}) = 0 \neq \mathrm{Tr}\,1 = D$.

[2] In fact, there was used even more specific design with *intermediate register (bus)* between program and data.

comming arbitrary close to unit with extension of program register and, thus, such design formally also becomes deterministic for infinite size of program register.

Conceptually, the «probabilistic», «stochastic» design of quantum processors [1–3, 6, 16] is a rather tricky question, but it is not discussed here in details.

For our purposes it is enough to use «stochastic» quantum circuit [2, 3] for application of gate $\boldsymbol{\theta}_{(q)}$ (5) with probability of success $p = 1 - 1/M$ for size $M = 2^m$ of (m-qubits) program register with existance of «deterministic» limit $p = 1$ for $M \to \infty$ [3]. Let us, without plunging into discussion about specific problems of «stochastic» model, consider the limit and show that it is essentially the same programmable phase gates discussed in Sec. 1.

The construction is straightforward. For «encoding transformations» $\boldsymbol{\theta}_\alpha$ to state of m-qubits program register in [2, 3] a family of states is used

$$|\Phi_{\alpha,m}\rangle = \bigotimes_{k=0}^{m-1} |\phi_{2^k\alpha}\rangle, \quad \text{where} \quad |\phi_a\rangle \equiv \frac{1}{\sqrt{2}}(\mathrm{e}^{ia/2}|0\rangle + \mathrm{e}^{-ia/2}|1\rangle). \tag{13}$$

It can be rewritten as[1]

$$|\Phi_{\alpha,m}\rangle = \frac{\mathrm{e}^{i\alpha(M-1)/2}}{\sqrt{M}} \sum_{K=0}^{M-1} \mathrm{e}^{-iK\alpha}|K\rangle \quad (M = 2^m) \tag{14}$$

and for $\alpha = -2\pi p/M$ with integer p states (14) coincide with usual *momentum* basis $|\tilde{p}\rangle = |\Phi_{-2\pi p/M,m}\rangle$ $(p \in \mathbb{Z},\ 0 \leqslant p < M)$ of M-dimensional Hilbert space.

Such elements $|\tilde{p}\rangle$ may be used as M orthogonal basic states of program register in «deterministic» quantum processor [c.f. Eq. (3)],

$$\tilde{\mathsf{U}} = \sum_{p=0}^{M-1} |\tilde{p}\rangle\langle\tilde{p}| \otimes \boldsymbol{\theta}_{(2\pi p/M)}. \tag{15}$$

The «deterministic» approach uses only the computational basis $\tilde{p}$, and it prevents the entanglement between program and data registers. Stochastic $U(1)$ approach [2,3] uses $|\Phi_{\alpha,m}\rangle$ with arbitrary α and for finite-dimensional case such states are not always orthogonal, but here it is possible not to discuss the issues related with interpretation of *quantum measurements* used for «probabilistic» calculations of $\boldsymbol{\theta}_{(\alpha)}$ for entangled case $\alpha \neq 2\pi p/M$, because for infinite-dimensional case *all* states $|\Phi_{\alpha,\infty}\rangle$ are orthogonal.

So, continuous (infinite) limit of «stochastic» $U(1)$ programmable gates suggested in [2,3] is essentially[2] the same as deterministic hybrid gate like Eq. (12) discussed in Sec. 1.

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[1] Here is used «inverted» binary notation for $|K\rangle$, $0 \leqslant K < M$, i. e., $|b_0 b_1 b_2 \cdots\rangle$ corresponds to $K = b_0 + 2b_1 + 2^2 b_2 + \ldots$ Another choice is to save standard binary notation, and to change order of terms in initial tensor product Eq. (13) to opposite one.

[2] There is some difference, if state of program register changes in [2, 3], even if there is no entanglement in continuous limit under consideration.

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