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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

BLG AND M5

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We discuss the interpretation of the three-dimensional $\mathcal{N} = 8$ superconformal Chern–Simons-matter theory with the gauge group of volume preserving diffeomorphisms as a model describing a six-dimensional self-dual gauge field coupled to scalars and spinors and its possible relation to the M5-brane.

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1. MOTIVATION

Since 2008 there has been a rather intensive activity in the study of three-dimensional superconformal theories describing the interaction of Chern–Simons gauge fields with matter supermultiplets. This activity was inspired by the papers by Bagger and Lambert [1–3], Gustavsson [4] and by Aharony et al. (ABJM) [5] which made a breakthrough in the construction of $d = 3$ conformal gauge theories with $\mathcal{N} = 8$ and $\mathcal{N} = 6$ supersymmetries. By now the dust around this activity seems to settle down and one may pause and calmly overview the developments of this subject.

One of the main motivations behind this activity comes from the AdS_4/CFT3 correspondence, which involves M-theory and, in particular, multiple M2-branes. The hope is that in this way one can make progress in understanding the M-brane theory as a possible microscopic formulation of M-theory, as well as to get a deeper insight into the structure of type IIA string theory compactified to AdS_4 backgrounds.

One of the first persons who addressed the problem of understanding the effective worldvolume theory of multiple M2-branes from this perspective was John Schwarz. In the paper [6] of 2004 he formulated main properties of the theory of multiple M2-branes and tried to construct it. The main features of the theory of N M2-branes conjectured by J. Schwarz are:

• This should be a $3d$ gauge theory with $\mathcal{N} = 8$ linearly realized supersymmetries and the superconformal symmetry $OSp(8|4)$. The argument is based on the fact that $D = 11$ supergravity has the maximally supersymmetric solution with the geometry of $AdS_4 \times S^7$ supported by a nonzero flux of the 4-form gauge field strength. The isometry group of this solution is $OSp(8|4)$. In a dual picture this solution arises as a large-N (or near horizon) limit of a stack of N parallel M2-branes in (orbifolded) flat $D = 11$ space-time, that by Maldacena's AdS/CFT correspondence conjecture is described by a maximally supersymmetric superconformal theory.

• The R-symmetry of the $\mathcal{N} = 8$ supersymmetric theory should be $SO(8)$ since a single M2-brane has 8 scalar modes in a vector representation of $SO(8)$ which correspond to 8 directions in $D = 11$ transversal to the M2-brane worldvolume.

• The gauge group of the theory should include $U(N)$, with N corresponding to the number of M2-branes or, in the dual picture, to N units of magnetic flux through the 7-sphere. This argument has not found a direct evidence in $D = 11$ M-theory yet, but comes from the observation that M-theory is a strong coupling limit of type IIA string theory and, correspondingly, the M2-brane theory is a strong coupling limit of a low-energy effective worldvolume theory of N coincident D2-branes. The latter is known to be a maximally supersymmetric $U(N)$ Yang–Mills theory in three dimensions. This theory is not conformal since the $3d$ YM coupling is dimensionful, but it is believed to have a strong-coupling (or, equivalently, infrared) limit in which the theory becomes conformal and describes the collection of M2-branes in $D = 11$. Further analysis has shown that the gauge group of N coincident M2-branes is, actually, $U(N) \times U(N)$.

• The physical content of the theory of N coincident M2-branes should comprise 8 scalars in fundamental representation of the gauge group as well as their superpartners, 8 Majorana spinor fields. The gauge field, if present, cannot be dynamical, because of the conformal invariance and supersymmetry. Hence it should be described by the conformally invariant Chern–Simons action which does not bring physical degrees of freedom.

• Finally, the M2-brane theory should be invariant under parity transformations, since its D2-brane counterpart is parity invariant. As was pointed out by J. Schwarz, this requirement (at least naively) is in contradiction with the assumption that the gauge field should be of the Chern–Simons type, since the latter violates parity.

Independently of the problems regarding the formulation of the M2-brane theory, the construction of maximally supersymmetric and conformally invariant $d = 3$ theories describing Chern–Simons gauge fields and their interaction with matter supermultiplets are interesting problems *per se*, and they were approached in several papers [6–14]. The net result was the construction of Chern–Simons models with up to $\mathcal{N} = 3$ supersymmetries. The breakthrough in the construction of $\mathcal{N} = 8$ and $\mathcal{N} = 6$ Chern–Simons-matter models has been made only relatively recently in [1–4] and [5]. Less supersymmetric $\mathcal{N} = 4$ theories have been constructed in [15, 16]. The BLG and ABJM constructions (and their subsequent analysis) solved, in particular, the problem of parity conservation and identified the structure of the gauge symmetry of the models which are intended to describe multiple M2-branes in certain $D = 11$ backgrounds.

An interesting outcome of the BLG construction is that it has brought to the attention of a wide community of theoretical and mathematical physicists a new gauge symmetry structure based on the so called 3-algebra that generalizes the notion of the Lie algebra. The 3-algebras, and, in general, n-algebras were introduced by V. Filippov [17]. They are intimately related to the Nambu bracket [18] whose algebraic structure is that of an «∞-algebra».

Another aspect of the BLG model which has been studied rather intensively is its possible interpretation as an effective description of the dynamics of a single M-theory five-brane, rather than of multiple membranes. This relation has been studied from different perspectives. In this contribution we discuss a proposal put forward in [19–21] and further developed in [22]. It is based on the version of the BLG model in which the gauge symmetry is promoted to an infinite-dimensional group of so-called volume-preserving diffeomorphisms. We shall show that indeed, somewhat surprisingly, the BLG model, which is a three-dimensional theory of scalar and spinor fields interacting with gauge fields associated with the group of volume-preserving diffeomorphisms, can be reinterpreted as an effective six-dimensional theory whose physical content is the same as that of an M5-brane propagating in a certain eleven-dimensional superbackground. A related construction based on a mass-deformed BLG model was considered in [23].

2. $\mathcal{N} = 8$ SUPERCONFORMAL CHERN–SIMONS-MATTER THEORY

Let us start with a brief review of main properties of the Bagger–Lambert–Gustavsson model. It is a $3d$ theory which is conformally invariant and $\mathcal{N} = 8$ supersymmetric. Therefore, the superconformal symmetry of the BLG model is the supergroup $OSp(8|4)$. The model contains eight bosonic fields $X^I(x^a)$ ($I = 1, \ldots, 8$; $a = 0, 1, 2$) and 16 fermionic ($3d$ Majorana spinor) fields $\Psi(x)$ taking values in an n-dimensional (fundamental) representation of a gauge algebra g. They interact with a Chern–Simons gauge field $A_a(x)$ ($a = 0, 1, 2$) valued in the adjoint representation of g. Therefore, the matter fields carry the index $A = 1, \ldots, n$ of the fundamental representation, $X^I_A(x)$, $\Psi_A(x)$, and the gauge field carries a couple of indices A, B of the adjoint representation, namely $A_a^{AB}(x)$.

The spinor (16-component) index of $\Psi(x)$ is implicit. As soon as the construction should be related to the description of M2-branes in eleven dimensions, it is convenient to regard $\Psi_a(x)$ as 32-component spinors subject to the κ-symmetry constraint which singles out 16 independent components

$$\Gamma^0\Gamma^1\Gamma^2\Psi = -\Psi, \tag{2.1}$$

where Γ^a ($a = 0, 1, 2$) are 32×32-component gamma matrices along the worldvolume directions of the $3d$ theory. Together with eight Γ^I (transverse to the worldvolume) they form a complete set of $D = 11$ gamma matrices.

The model has eight g-valued scalar degrees of freedom and eight g-valued fermionic ones forming an $\mathcal{N} = 8$ supermultiplet with the linearized $\mathcal{N} = 8$ supersymmetry transformations having the form

$$\delta X^I = i\bar{\epsilon}\Gamma^I\Psi, \quad \delta\Psi = \mathcal{D}_a X^I \Gamma^a \Gamma_I \epsilon + O(X^3, \epsilon), \tag{2.2}$$

where ϵ is the 16-component supersymmetry parameter, Γ^a and Γ^I are 11-dimensional gamma matrices, and $\mathcal{D}_a = \partial_a + A_a$ is the gauge-covariant derivative. The Chern–Simons gauge fields do not carry any physical degrees of freedom. The physical content of the BLG model and its symmetries are similar to those of a certain number of coincident membranes propagating in 11-dimensional superspace. This matching of the physical spectra and the symmetries is the basis of the conjecture that the BLG model provides us with an effective worldvolume description of coincident membranes in 11-dimensional M-theory.

Further analysis of the BLG model actually showed that the requirement of the $\mathcal{N} = 8$ supersymmetry of its action drastically constrains the choice of the gauge groups. Actually, there are essentially only two options [1]:

i) The gauge group of local symmetries is $SO(4) \sim SU(2) \times SU(2)$, or $U(2) \times U(2)$. So the gauge-symmetry index A takes four values. In this case the model describes two coincident membranes [27];

[1] The possible choice of the gauge symmetry becomes much wider if one renounces positive definiteness of the 3-algebra quadratic form [24–26].

ii) The gauge group is a group of volume-preserving diffeomorphisms in an internal three-dimensional space. The gauge-symmetry index A gets replaced by the dependence of the fields of the model on three continuous parameters $y^{\dot a}$ $(\dot a = 1, 2, 3)$. In this case the model describes infinite number of coincident membranes which, according to the assumption of [19–21], can blow up to a single 5-brane.

We shall be interested in the second possibility. So, our fields now depend on three space-time coordinates x^a and three internal coordinates $y^{\dot a}$

$$X^I(x, y), \quad \Psi(x, y), \quad A_{a\dot b}(x, y). \tag{2.3}$$

Note that the gauge field carries both the $3d$ space-time and the internal space index.

In the end we would like to interpret these x^a and $y^{\dot a}$ as coordinates of a six-dimensional space-time $x^\mu = (x^a, y^{\dot a})$ associated with the worldvolume of an M5-brane.

Under the volume-preserving diffeomorphisms of the coordinates $y^{\dot a}$

$$\delta y^{\dot a} = -g\xi^{\dot a}(x, y) \tag{2.4}$$

the fields $X^I(x, y)$ and $\Psi(x, y)$ transform as follows:

$$\delta\Phi = g\xi^{\dot c}(x, y)\partial_{\dot c}\Phi, \tag{2.5}$$

where Φ stands for X^I or Ψ, g is a coupling constant in the theory and

$$\xi^{\dot a} \equiv \varepsilon^{\dot a\dot b\dot c}\partial_{\dot b}\Lambda_{\dot c}(x, y) \tag{2.6}$$

are local gauge parameters such that $\partial_{\dot a}\,\xi^{\dot a} = \partial_{\dot a}\varepsilon^{\dot a\dot b\dot c}\partial_{\dot b}\,\Lambda_{\dot c} \equiv 0$, which is the volume-preserving condition. From Eq. (2.5) it follows that X^I and Ψ transform as scalars.

A defining property of the volume-preserving diffeomorphisms is that if Φ_i $(i = 1, 2, 3)$ are scalar fields with respect to the volume-preserving diffeomorphisms, their Nambu bracket

$$\{\Phi_1, \Phi_2, \Phi_3\} \equiv \varepsilon^{\dot a\dot b\dot c}\partial_{\dot a}\Phi_1\partial_{\dot b}\Phi_2\partial_{\dot c}\Phi_3 \tag{2.7}$$

is also a scalar field. This property is used for constructing diffeomorphism invariant interacting terms for the matter fields in the action of the model.

The gauge field $A_{a\dot b}(x, y)$ transforms under the volume-preserving diffeomorphisms and under additional gauge transformations with the parameter $\Lambda_a(x, y)$ as follows:

$$\delta A_{a\dot b} = \partial_a\Lambda_{\dot b} - \partial_{\dot b}\Lambda_a + g\xi^{\dot c}\partial_{\dot c}A_{a\dot b} + g(\partial_{\dot b}\xi^{\dot c})\,A_{a\dot c}. \tag{2.8}$$

To construct gauge-invariant kinetic terms for the matter fields in the action and to describe their gauge coupling, one introduces the covariant derivative of a scalar field Φ along the $3d$ space-time directions x^a

$$\mathcal{D}_a\Phi = \partial_a\Phi - g\{A_{a\dot b}, x^{\dot b}, \Phi\} = (\partial_a - gB_a{}^{\dot a}\partial_{\dot a})\Phi, \tag{2.9}$$

where

$$B_a{}^{\dot a} \equiv \varepsilon^{\dot b\dot c\dot a}\partial_{\dot b}\,A_{a\dot c}. \tag{2.10}$$

The definition of the covariant derivative $\mathcal{D}_a$ can be extended to any tensor field T [28]

$$\mathcal{D}_a T = (\partial_a - g\,\mathcal{L}_{B_a})\,T, \tag{2.11}$$

where $\mathcal{L}_{B_a}$ is the Lie derivative along the vector field $(B_a)^{\dot{a}}$. It follows from (2.10) that ${B_a}^{\dot{a}}$ is a divergenceless field in the additional three directions $y^{\dot{a}}$

$$\partial_{\dot{a}} {B_a}^{\dot{a}} = 0. \tag{2.12}$$

The ingredients introduced above plus an explicit form of the supersymmetry transformations of the matter fields (2.2) and the gauge field $A_{a\dot{b}}$

$$\delta X^I = i\bar{\epsilon}\Gamma^I\Psi, \quad \delta\Psi = \mathcal{D}_a X^I \Gamma^a \Gamma_I \epsilon - \frac{g^2}{6}\{X^I, X^J, X^K\}\Gamma_{IJK}\epsilon, \tag{2.13}$$

$$\delta A_{a\dot{b}} = -i(\bar{\epsilon}\Gamma_a\Gamma_I\Psi)\partial_{\dot{b}} X^I \tag{2.14}$$

allow one to construct the following $\mathcal{N} = 8$ supersymmetric BLG-like action [19–21, 28]:

$$S = -\int d^3x \int d^3y \left(\frac{1}{2}(\mathcal{D}_a X^I)^2 + \frac{g^4}{12}\{X^I,\, X^J,\, X^K\}^2 + \right.$$
$$\left. + \frac{1}{2}\,\varepsilon^{abc}\, {B_a}^{\dot{a}}\, \partial_b\, A_{c\dot{a}} + \frac{g}{6}\,\varepsilon^{abc}\,\varepsilon_{\dot{a}\dot{b}\dot{c}}\, {B_a}^{\dot{a}}\, {B_b}^{\dot{b}}\, {B_c}^{\dot{c}} + L_F \right), \tag{2.15}$$

where L_F stands for fermionic terms.

We see that the model described by the action (2.15) can be treated as six-dimensional with the six-dimensional space-time being parametrized by the coordinates $x^\mu = (x^a, y^{\dot{b}})$. The manifest space-time symmetry of this construction is $SO(1,2)\times SO(3)$. It is a subgroup of the $6d$ Lorentz group $SO(1,5)$ which is explicitly broken down to $SO(1,2)\times SO(3)$. In [19–21] it was conjectured that the above action describes a $6d$ worldvolume dynamics of a 5-brane of M-theory in an eleven-dimensional supergravity background with a nonzero constant gauge field C_3 along three spacial directions $y^{\dot{a}}$ of M5, i.e., $C_{\dot{a}\dot{b}\dot{c}} = 1/g\,\varepsilon_{\dot{a}\dot{b}\dot{c}}$, which breaks the $SO(1,5)$ Lorentz invariance down to $SO(1,2)\times SO(3)$. The mass deformed version of this model considered in [23] has been interpreted as an M5-brane wrapping an S^3 sphere rather than interacting with the constant background gauge field. To verify these conjectures, one should first of all compare the field content of the given model with that of the M5-brane. So let us leave the BLG model for a moment and review basic properties of the M5-brane.

3. M5-BRANE FROM THE BLG THEORY

Together with the M2-brane, the M5-brane is a fundamental extended object in eleven-dimensional supergravity (or M-theory). It preserves 1/2 supersymmetry of the eleven-dimensional superbackground, which from the M5-brane worldvolume perspective means that the six-dimensional M5-brane effective theory possesses 16-supersymmetries. The worldvolume field content of the M5-brane is given by five scalar fields $X^i(x^\mu)$ ($\mu = 0, 1, \ldots, 5$ and $i = 1, \ldots, 5$), which describe transverse fluctuations of the M5-brane in eleven-dimensional space-time, 16-component fermionic field $\Psi(x^\mu)$, which on the mass shell has 8 physical degrees of freedom, and an antisymmetric (so-called chiral) gauge field $A_{\mu\nu}(x)$ whose field strength $F_{\mu\nu\lambda} = 3\partial_{[\mu}\, A_{\nu\lambda]}$ is self-dual on the $6d$ worldvolume. In the linear approximation the self-duality condition is just the conventional six-dimensional Poincaré duality relation

$$F_{\mu\nu\lambda} = \frac{1}{3!}\, \varepsilon_{\mu\nu\lambda\alpha\beta\gamma}\, F^{\alpha\beta\gamma} \equiv \tilde{F}_{\mu\nu\lambda}, \tag{3.16}$$

while in the full nonlinear theory the self-duality condition becomes a nonpolynomial relation between the field strength F_3 and its Hodge-dual $\tilde{F}_3$. This relation has different (though equivalent) forms in the on-shell superembedding description of the M5-brane [29–31] and in its Lagrangian formulation [32–37].

The gauge symmetry

$$\delta A_{\mu\nu} = \partial_\mu\, \Lambda_\nu - \partial_\nu\, \Lambda_\mu \tag{3.17}$$

and the self-duality of the field strength imply that the gauge field $A_{\mu\nu}$ carries three physical degrees of freedom. Hence, the M5-brane has $8 = 3 + 5$ bosonic and 8 fermionic physical degrees of freedom, i.e., the same number as the BLG model with the gauge group of volume-preserving diffeomorphisms. The difference is that the latter has 8 physical scalar modes and the auxiliary gauge field $A_{a\dot{b}}$, while on the M5-brane worldvolume the eight bosonic physical modes are redistributed between five scalar fields X^i and three physical modes of the gauge field $A_{\mu\nu}$. Therefore, to be able to relate the BLG model to the M5-brane (and assuming that their fermionic sectors match), one should redefine the fields of the bosonic sector of the former to match the bosonic field contents of the latter. Such a redefinition was found in [19].

It is natural to identify the gauge field $A_{a\dot{b}}$ of the BLG model with the corresponding component of the M5-brane chiral gauge field $A_{\mu\nu} = (A_{ab},\, A_{a\dot{b}},\, A_{\dot{a}\dot{b}})$ subject to the $3+3$ splitting of the $SO(1,5)$ vector index μ. It remains to find an analogue of the components A_{ab} and $A_{\dot{a}\dot{b}}$ among the scalar fields of the BLG model. As we shall see, the gauge field A_{ab} does not directly appear in the BLG construction. In this formulation it is a pure gauge degree of freedom which is set to zero. The field $A_{\dot{a}\dot{b}}$ and the five scalar fields X^i are identified as follows [19]. We split the eight BLG scalar fields $X^I(x,y)$ into three fields $X^{\dot{a}}$ and five fields $X^i(x,y)$ and assume that the latter are associated with the five transversal scalar modes of the M5-brane, while the former three are along the M5-brane worldvolume directions associated with $y^{\dot{a}}$. Obviously, such a splitting breaks the $SO(8)$ R-symmetry of the BLG model down to $SO(3) \times SO(5)$ with $SO(5)$ being the R-symmetry of the M5-brane theory.

Since $X^{\dot{a}}(x,y)$ are along the M5-brane worldvolume, the static (vacuum) M5-brane configuration should correspond to «vacuum» values of $X^{\dot{a}}(x,y)$, which are proportional to $y^{\dot{a}}$, and fluctuations around the static solution are described by functions of the form

$$X^{\dot{a}} = \frac{1}{g}\, y^{\dot{a}} + A^{\dot{a}}(x,y) \equiv \frac{1}{g}\, y^{\dot{a}} + \frac{1}{2}\, \varepsilon^{\dot{a}\dot{b}\dot{c}}\, A_{\dot{b}\dot{c}}, \tag{3.18}$$

where in the right-hand side we have dualized the vector field $A^{\dot{a}}$ into the antisymmetric tensor $A_{\dot{b}\dot{c}}$ which is naturally assumed to be associated with the corresponding component of the M5-brane chiral gauge field. The transformation properties of $A_{\dot{b}\dot{c}}$ under the volume-preserving diffeomorphisms are such that $X^{\dot{a}}$ transforms as a scalar (2.5) taking into account the variation (2.4) of the coordinate $y^{\dot{a}}$, namely

$$\delta A_{\dot{a}\dot{b}} = \partial_{\dot{a}}\Lambda_{\dot{b}} - \partial_{\dot{b}}\Lambda_{\dot{a}} + g\xi^{\dot{c}}\partial_{\dot{c}}A_{\dot{a}\dot{b}}. \tag{3.19}$$

At this point we should note that the association of the BLG model fields $A_{a\dot{b}}$ and $A_{\dot{a}\dot{b}}$ with corresponding components of the M5-brane gauge field is not direct. This can be seen by comparing the gauge transformations (3.17) of the M5 gauge field with the gauge transformations of $A_{a\dot{b}}$ and $A_{\dot{a}\dot{b}}$, Eqs. (2.8) and (3.19). They only coincide when $g = 0$. This means that the direct correspondence of the gauge fields of the two models is only possible in the free-field limit, while in the nonlinear case the relation with the M5-brane is much more subtle and requires further study and understanding (see, e.g., [19–23, 38, 39] for the discussion of this issue).

To construct a gauge theory in six-dimensional space-time, one needs the gauge-covariant derivatives $\mathcal{D}_\mu = (\mathcal{D}_a, \mathcal{D}_{\dot{a}})$. The derivative $\mathcal{D}_a$ is already defined in (2.9). Following [19,21], we define the derivative $\mathcal{D}_{\dot{a}}$ as

$$\mathcal{D}_{\dot{a}}\Phi = \frac{g^2}{2}\,\varepsilon_{\dot{a}\dot{b}\dot{c}}\{\Phi, X^{\dot{b}}, X^{\dot{c}}\}, \tag{3.20}$$

where $X^{\dot{a}}$ is defined in (3.18). It is important to realize that the covariant derivative defined in such a way (3.20) transforms any scalar field Φ to a scalar field. Moreover, one can check explicitly that the r.h.s. of (3.20) starts from the plain derivative $\partial_{\dot{a}}$ plus terms depending on the gauge field $A_{\dot{a}}$. The derivative (3.20) can be rewritten in the following way:

$$\mathcal{D}_{\dot{a}}\,\Phi = \det M\, M^{-1}{}_{\dot{a}}{}^{\dot{b}}\partial_{\dot{b}}\Phi, \tag{3.21}$$

where

$$M_{\dot{a}}{}^{\dot{b}} = g\partial_{\dot{a}}X^{\dot{b}} = \delta^{\dot{b}}_{\dot{a}} + g\partial_{\dot{a}}A^{\dot{b}}. \tag{3.22}$$

This matrix $M_{\dot{a}}{}^{\dot{b}}$ plays an important role because it is used to convert a vector-like field $\partial_{\dot{a}}\Phi$ into a scalar-like field $\mathcal{D}_{\dot{a}}\Phi$.

Now we are ready to define the components of the six-dimensional field strength of $A_{\mu\nu}$,

$$\mathcal{H}_{\mu\nu\rho} = (\mathcal{H}_{abc}, \mathcal{H}_{ab\dot{c}}, \mathcal{H}_{a\dot{b}\dot{c}}, \mathcal{H}_{\dot{a}\dot{b}\dot{c}}). \tag{3.23}$$

In fact, only the components $\mathcal{H}_{a\dot{b}\dot{c}}$ and $\mathcal{H}_{\dot{a}\dot{b}\dot{c}}$ can be defined at this point because we have no field A_{ab} in the action (2.15). These components appear in the commutators of the gauge-covariant derivatives [19, 21],

$$[\mathcal{D}_{\dot{a}},\, \mathcal{D}_{\dot{b}}]\Phi = -g^2\{\mathcal{H}_{\dot{a}\dot{b}\dot{f}},\, X^{\dot{f}}, \Phi\}, \tag{3.24}$$

$$[\mathcal{D}_{a},\, \mathcal{D}_{\dot{b}}]\Phi = -g^2\{\mathcal{H}_{a\dot{b}\dot{f}},\, X^{\dot{f}}, \Phi\}. \tag{3.25}$$

They have the following explicit expressions in terms of the gauge fields:

$$\begin{aligned}\mathcal{H}_{\dot{1}\dot{2}\dot{3}} &= \partial_{\dot{a}}A^{\dot{a}} + \frac{g}{2}(\partial_{\dot{a}}A^{\dot{a}}\partial_{\dot{b}}A^{\dot{b}} - \partial_{\dot{b}}A^{\dot{a}}\partial_{\dot{a}}A^{\dot{b}}) + \frac{g^2}{6}\varepsilon_{\dot{a}\dot{b}\dot{c}}\varepsilon^{\dot{d}\dot{f}\dot{e}}\partial_{\dot{d}}A^{\dot{a}}\partial_{\dot{f}}A^{\dot{b}}\partial_{\dot{e}}A^{\dot{c}} \equiv \\ &\equiv \frac{1}{g}(\det M - 1),\end{aligned} \tag{3.26}$$

$$\mathcal{H}_{a\dot{b}\dot{c}} = \partial_a A_{\dot{b}\dot{c}} - \partial_{\dot{b}}A_{a\dot{c}} + \partial_{\dot{c}}A_{a\dot{b}} - g\varepsilon^{\dot{d}\dot{e}\dot{f}}\partial_{\dot{d}}A_{a\dot{e}}\partial_{\dot{f}}A_{\dot{b}\dot{c}} \equiv \varepsilon_{\dot{a}\dot{b}\dot{c}}\mathcal{D}_a X^{\dot{a}}. \tag{3.27}$$

It is easy to see that in the limit $g \to 0$ the components (3.26) and (3.27) of the field strength coincide with (3.31) and (3.32) below.

It is quite straightforward to check that the action (2.15) can be rewritten as

$$S = -\int d^6x \left(\frac{1}{2}(\mathcal{D}_a X^{\dot{b}})^2 + \frac{g^4}{2}\{X^{\dot{1}}, X^{\dot{2}}, X^{\dot{3}}\}^2 + \right.$$
$$\left. + \frac{1}{2g^2} + \frac{1}{2}\varepsilon^{abc} B_a{}^{\dot{a}} \partial_b A_{c\dot{a}} + g \det B_a{}^{\dot{a}} \right) =$$
$$= -\int d^6x \left(\frac{1}{4}\mathcal{H}_{a\dot{b}\dot{c}}\mathcal{H}^{a\dot{b}\dot{c}} + \frac{1}{12}\mathcal{H}_{\dot{a}\dot{b}\dot{c}}\mathcal{H}^{\dot{a}\dot{b}\dot{c}} + \frac{1}{2}\varepsilon^{abc} B_a{}^{\dot{a}} \partial_b A_{c\dot{a}} + g \det B_a{}^{\dot{a}} \right). \quad (3.28)$$

The action (3.28) deserves several comments.

• As was conjectured in [19–21] and proved in [22], the action (3.28) describes a chiral gauge field whose nonlinear field strength $\mathcal{H}_{\mu\nu\rho}$ obeys the usual self-duality condition

$$\mathcal{H}_{\mu\nu\lambda} = \frac{1}{3!}\varepsilon_{\mu\nu\lambda\alpha\beta\gamma}\mathcal{H}^{\alpha\beta\gamma} \equiv \tilde{\mathcal{H}}_{\mu\nu\lambda}. \quad (3.29)$$

• The action (3.28) depends only on the components $\mathcal{H}_{a\dot{b}\dot{c}}$ and $\mathcal{H}_{\dot{a}\dot{b}\dot{c}}$, while the components $\mathcal{H}_{abc}$ and $\mathcal{H}_{ab\dot{c}}$ remained undefined. Further we will show, following [22], that these components can be uniquely defined in such a way that the complete field strength $\mathcal{H}_{\mu\nu\rho}$ does satisfy the self-duality condition (3.29) and the Bianchi identity.

• The action (3.28) has a noncovariant form because of the explicit presence of the gauge potential $A_{a\dot{a}}$ rather than the field strengths only. The use of the components $\mathcal{H}_{abc}$ and $\mathcal{H}_{ab\dot{c}}$ allows one to rewrite the action in a gauge-covariant form [22].

3.1. Six-Dimensional Self-duality in the Linear Case. Let us now concentrate on the free field limit of the action (3.28). We will show that the self-duality condition (3.16) arises as the solution of the gauge field equations derived from the action (3.28). This exercise is quite simple, but it clearly explains the general procedure which remains roughly the same in the nonlinear case as well.

In the limit $g \to 0$ the action (3.28) reduces to

$$S = -\frac{1}{4}F_{a\dot{b}\dot{c}}(F - \tilde{f})^{a\dot{b}\dot{c}} - \frac{1}{12}F_{\dot{a}\dot{b}\dot{c}}F^{\dot{a}\dot{b}\dot{c}}, \quad (3.30)$$

where

$$F_{a\dot{b}\dot{c}} = \partial_a A_{\dot{b}\dot{c}} - \partial_{\dot{b}} A_{a\dot{c}} + \partial_{\dot{c}} A_{a\dot{b}}, \quad (3.31)$$

$$F_{\dot{a}\dot{b}\dot{c}} = \partial_{\dot{a}} A_{\dot{b}\dot{c}} - \partial_{\dot{b}} A_{\dot{a}\dot{c}} + \partial_{\dot{c}} A_{\dot{a}\dot{b}}, \quad (3.32)$$

$$\tilde{f}_{a\dot{b}\dot{c}} = \frac{1}{2}\varepsilon_{abc}\varepsilon_{\dot{b}\dot{c}\dot{a}} f^{bc\dot{a}} \quad (3.33)$$

and

$$f_{ab\dot{c}} = \partial_a A_{b\dot{c}} - \partial_b A_{a\dot{c}}. \quad (3.34)$$

Here ε_{abc} and $\varepsilon_{\dot{a}\dot{b}\dot{c}}$ are the antisymmetric unit tensors invariant under $SO(1,2)$ and $SO(3)$, respectively.

Note that the tensor (3.34), as well as the Lagrangian (3.30), does not contain the components A_{ab} of the gauge potential. Because of this the Lagrangian (3.30) is invariant under the gauge transformations

$$\delta A_{a\dot{b}} = \partial_a \Lambda_{\dot{b}} - \partial_{\dot{b}} \Lambda_a, \quad \delta A_{\dot{a}\dot{b}} = \partial_{\dot{a}} \Lambda_{\dot{b}} - \partial_{\dot{b}} \Lambda_{\dot{a}} \quad (3.35)$$

only modulo a total derivative. We can restore the complete gauge invariance of the Lagrangian, following [22], by extending the tensor (3.34) to the fully fledged gauge-invariant field strength

$$F_{ab\dot{c}} = \partial_a A_{b\dot{c}} - \partial_b A_{a\dot{c}} + \partial_{\dot{c}} A_{ab}, \tag{3.36}$$

introducing the field strength

$$F_{abc} = \partial_a A_{bc} - \partial_b A_{ac} + \partial_c A_{ab} \tag{3.37}$$

and adding to the Lagrangian certain terms depending on A_{ab} in such a way that they enter the Lagrangian as total derivatives and hence do not modify corresponding equations of motion. With these terms the action takes the form

$$S = -\frac{1}{4}\int d^6x \left[F_{a\dot{b}\dot{c}}(F^{a\dot{b}\dot{c}} - \tilde{F}^{a\dot{b}\dot{c}}) + \frac{1}{3}F_{\dot{a}\dot{b}\dot{c}}(F^{\dot{a}\dot{b}\dot{c}} - \tilde{F}^{\dot{a}\dot{b}\dot{c}}) \right]. \tag{3.38}$$

Since the component A_{ab} enters this action under a total derivative, in addition to the conventional gauge symmetry (3.17), the action (3.38) is also invariant under the following local transformations:

$$\delta A_{ab} = \Phi_{ab}(x^\mu). \tag{3.39}$$

The equations of motion obtained from (3.38) have the following form:

$$\frac{\delta S}{\delta A^{a\dot{b}}} = 0 \ \Rightarrow\ \partial_{\dot{c}}\,(F^{a\dot{b}\dot{c}} - \tilde{F}^{a\dot{b}\dot{c}}) = 0, \tag{3.40}$$

$$\frac{\delta S}{\delta A^{\dot{a}\dot{b}}} = 0 \ \Rightarrow\ \partial_a\, F^{a\dot{b}\dot{c}} + \partial_{\dot{a}}\, F^{\dot{a}\dot{b}\dot{c}} = 0. \tag{3.41}$$

It can be shown [19, 22] that, with the use of the gauge symmetry (3.39), these second-order equations reduce to the first-order self-duality condition (3.16). Thus, the action (3.38) indeed describes the free chiral two-form gauge field in six dimensions.

Note that, though the action (3.38) does not possess manifest six-dimensional Lorentz symmetry $SO(1,5)$ but only $SO(1,2)\times SO(3)$, on the mass shell the full $6d$ Lorentz invariance gets restored, since the self-duality condition (3.16) is $SO(1,5)$ invariant. This prompts to look for a hidden $6d$ Lorentz symmetry of the action. And indeed there is such a symmetry [22]. Since the action is manifestly invariant under the $SO(1,2)\times SO(3)$ subgroup of $SO(1,5)$, it is enough to check its invariance under the variation of the gauge fields with the Lorentz parameters $\lambda_a{}^{\dot{b}} = -\lambda^{\dot{b}}{}_a$ corresponding to the coset generators $SO(1,5)/SO(1,2)\times SO(3)$. In the gauge $A_{ab}=0$ the action turns out to be invariant under the following transformations:

$$\begin{aligned} \delta A^{a\dot{a}} &= \lambda^a_{\dot{b}} A^{\dot{b}\dot{a}} + \lambda^b_{\dot{c}}(x_b\partial^{\dot{c}} - x^{\dot{c}}\partial_b)A^{a\dot{a}} + \lambda_{c\dot{d}}\, x^{\dot{d}}(F^{ca\dot{a}} - \tilde{F}^{ca\dot{a}}), \\ \delta A^{\dot{a}\dot{b}} &= -\lambda^{\dot{a}}_a A^{a\dot{b}} + \lambda^{\dot{b}}_b A^{b\dot{a}} + \lambda^b_{\dot{c}}(x_b\partial^{\dot{c}} - x^{\dot{c}}\partial_b)A^{\dot{a}\dot{b}}, \end{aligned} \tag{3.42}$$

which are the $SO(1,5)/SO(1,2)\times SO(3)$ Lorentz transformations modified by the last term in $\delta A^{a\dot{a}}$ proportional to the (anti-)self-dual component of the field strength. Hence, on the mass shell (when the self-duality condition is satisfied) the transformations take the conventional form.

Note that the Lagrangian formulation of the dynamics of the six-dimensional self-dual field based on the action (3.38) is different from the formulation considered previously in [40–43]. In the previous formulation the chiral field action was constructed by breaking manifest $SO(1,5)$ Lorentz symmetry down to $SO(5)$ or $SO(1,4)$, i.e., by splitting the $SO(1,5)$ vector indices into $1+5$, and not into $3+3$ as in the case considered above. In [43] the complete $SO(1,5)$ covariance of the action was restored by introducing a single auxiliary scalar field. It turns out that also in this new alternative formulation the $SO(1,5)$ covariance can be restored, but for this one now needs a triplet of auxiliary scalar fields (see [22] for details).

The actions of the two formulations differ by terms quadratic in (anti-)self-dual components of the gauge field strength. Thus, on the mass shell the two formulations are equivalent. It would be of interest to understand whether the difference of the two chiral-field actions off the mass shell may lead to different results upon quantization. For instance, the two formulations may complement each other when the chiral field is considered in topologically nontrivial backgrounds.

3.2. Nonlinear Self-duality. Let us turn back to the action (3.28) and show that also in the presence of the nonlinear terms it describes a chiral gauge field in six-dimensional Minkowski space. The equations of motion which follow from this action are

$$\mathcal{D}_a \tilde{\mathcal{H}}^{ab\dot{c}} + \mathcal{D}_{\dot{a}} \mathcal{H}^{\dot{a}b\dot{c}} = 0, \tag{3.43}$$

$$\mathcal{D}_a \mathcal{H}^{a\dot{b}\dot{c}} + \mathcal{D}_{\dot{a}} \mathcal{H}^{\dot{a}\dot{b}\dot{c}} = 0. \tag{3.44}$$

Our goal is to show that the general solution to these equations reduces to the self-duality of the nonlinear field strength $\mathcal{H}_3$. We start with Eq. (3.43) and multiply it by $M_{\dot{c}}^{-1\dot{d}}$ to get

$$M_{\dot{c}}^{-1\dot{d}}\, \mathcal{D}_a \tilde{\mathcal{H}}^{ab\dot{c}} + M_{\dot{c}}^{-1\dot{d}}\, \mathcal{D}_{\dot{a}}\, \mathcal{H}^{\dot{a}b\dot{c}} = 0. \tag{3.45}$$

In view of the definition (3.27) of the field strength $\mathcal{H}^{\dot{a}b\dot{c}} = -\mathcal{H}^{b\dot{a}\dot{c}}$ and the identity (3.21), the second term of this equation can be written as a total partial derivative

$$\begin{aligned} M_{\dot{c}}^{-1\dot{d}} \mathcal{D}_{\dot{a}} \mathcal{H}^{\dot{a}b\dot{c}} &= \det M \varepsilon^{\dot{c}\dot{a}\dot{f}} M_{\dot{c}}^{-1\dot{d}} M_{\dot{a}}^{-1\dot{b}} \partial_{\dot{b}} \mathcal{D}^b X_{\dot{f}} = \\ &= \varepsilon^{\dot{d}\dot{b}\dot{c}} \partial_{\dot{b}} (M_{\dot{c}}{}^{\dot{f}} \mathcal{D}^b X_{\dot{f}}) = \frac{1}{2} \varepsilon^{\dot{d}\dot{b}\dot{c}} \partial_{\dot{b}} (M_{\dot{c}}{}^{\dot{f}} \varepsilon_{\dot{f}\dot{a}\dot{k}} \mathcal{H}^{b\dot{a}\dot{k}}). \end{aligned} \tag{3.46}$$

The first term of (3.45) can also be presented as a total partial derivative

$$M_{\dot{c}}^{-1\dot{d}}\, \mathcal{D}_a \tilde{\mathcal{H}}^{ab\dot{c}} = \varepsilon^{bac} M_{\dot{c}}^{-1\dot{d}} \mathcal{D}_a \mathcal{D}_c X^{\dot{c}} = -\varepsilon^{bac} \varepsilon^{\dot{a}\dot{b}\dot{d}} \partial_{\dot{a}} (\partial_a\, A_{c\dot{b}} + \frac{g}{2} \varepsilon_{\dot{b}\dot{c}\dot{f}} B_a^{\dot{c}} B_c{}^{\dot{f}}), \tag{3.47}$$

where $B_a^{\dot{c}}$ is defined in (2.10).

Substituting (3.46) and (3.47) into Eq. (3.45), we get the Bianchi-like equation which, upon taking off the total derivative (in topologically trivial spaces), produces the duality relation

$$\mathcal{H}^{b\dot{a}\dot{c}} = \frac{1}{2}\, \varepsilon^{bcd} \varepsilon^{\dot{a}\dot{c}\dot{b}}\, \mathcal{H}_{cd\dot{b}} \equiv \tilde{\mathcal{H}}^{b\dot{a}\dot{c}}, \tag{3.48}$$

where $\mathcal{H}_{cd\dot{b}}$ is

$$\begin{aligned} \mathcal{H}_{ab\dot{c}} &= M_{\dot{c}}^{-1\dot{b}} (F_{ab\dot{b}} + g \varepsilon^{\dot{a}\dot{e}\dot{k}} \varepsilon^{\dot{d}\dot{f}\dot{g}} \varepsilon_{\dot{k}\dot{g}\dot{b}} \partial_{\dot{a}} A_{a\dot{e}} \partial_{\dot{d}} A_{b\dot{f}}) = \\ &= M_{\dot{c}}^{-1\dot{d}} (F_{ab\dot{d}} + g \varepsilon_{\dot{d}\dot{a}\dot{b}} B_a{}^{\dot{a}} B_b{}^{\dot{b}}). \end{aligned} \tag{3.49}$$

At this step the components A_{ab} of the gauge potential have appeared in $F_{ab\dot b}$ as a result of the integration of Eq. (3.45). Substituting the above duality relation back into Eq. (3.43), we get the Bianchi identity

$$\mathcal{D}_a\tilde{\mathcal{H}}^{ab\dot c} + \mathcal{D}_{\dot a}\tilde{\mathcal{H}}^{\dot a b\dot c} = 0. \tag{3.50}$$

We can now proceed and solve the second field equation (3.44). Multiplying it by $M_{\dot a}{}^{\dot d}\,\varepsilon_{\dot d\dot b\dot c}$, we get

$$M_{\dot a}{}^{\dot d}\varepsilon_{\dot d\dot b\dot c}\mathcal{D}_a\mathcal{H}^{a\dot b\dot c} + 2M_{\dot a}{}^{\dot d}\mathcal{D}_{\dot d}\mathcal{H}_{\dot 1\dot 2\dot 3} = 0. \tag{3.51}$$

Using the definition (3.26) of $\mathcal{H}_{\dot a\dot b\dot c}$ and the identity (3.21), one finds that the second term of this equation is a total derivative

$$2M_{\dot a}{}^{\dot d}\mathcal{D}_{\dot d}\,\mathcal{H}_{\dot 1\dot 2\dot 3} = \frac{1}{g}\partial_{\dot a}((\det M)^2 - 1), \tag{3.52}$$

where in the r.h.s. we have introduced the unit constant to ensure that the integral of (3.52) does not diverge when $g \;\to\; 0$ and $\det M \;\to\; 1$. Similarly, one can check that the first term in (3.51) is also a total derivative modulo the duality relation (3.48). Upon some algebra we finally get

$$\begin{aligned} M_{\dot a}{}^{\dot d}\,\varepsilon_{\dot d\dot b\dot c}\,\mathcal{D}_a\mathcal{H}^{a\dot b\dot c} &= \partial_{\dot a}\,(\varepsilon^{abc}\,\partial_a\,A_{bc} - \frac{g}{2}\mathcal{H}_{a\dot b\dot c}\,\mathcal{H}^{a\dot b\dot c} - g\varepsilon^{abc}B_a{}^{\dot b}F_{bc\dot b} - 4g^2\,\det B_a{}^{\dot b}) + \\ &\quad + 2\mathcal{D}_a\left(M_{\dot a}{}^{\dot d}\left(\mathcal{D}^a X_{\dot d} - \frac{1}{2}\,\varepsilon^{abc}\mathcal{H}_{bc\dot d}\right)\right). \end{aligned} \tag{3.53}$$

Notice that the first term is a total derivative and the last term is proportional to the duality relation (3.48). Therefore, when the duality relation (3.48) is satisfied, Eq. (3.51) can be integrated to produce the last missing field strength $\mathcal{H}_{abc}$,

$$\begin{aligned} \frac{1}{6}\,\varepsilon^{abc}\,\mathcal{H}_{abc} &= \frac{1}{1+\det M}\times \\ &\quad\times\left(\frac{1}{3}\varepsilon^{abc}F_{abc} - \frac{g}{2}\mathcal{H}_{a\dot b\dot c}\mathcal{H}^{a\dot b\dot c} - g\varepsilon^{abc}B_a{}^{\dot b}F_{bc\dot b} - 4g^2\det B_a{}^{\dot b}\right) = \\ &= \frac{1}{2+(g/6)\varepsilon^{\dot a\dot b\dot c}\mathcal{H}_{\dot a\dot b\dot c}}\left(\frac{1}{3}\varepsilon^{abc}F_{abc} - \frac{g}{2}\mathcal{H}_{a\dot b\dot c}\mathcal{H}^{a\dot b\dot c} - g\varepsilon^{abc}B_a{}^{\dot b}F_{bc\dot b} - 4g^2\det B_a{}^{\dot b}\right), \end{aligned} \tag{3.54}$$

where $F_{ab\dot b}$ and F_{abc} are the linear parts of the field strength. In view of this definition, both the duality relation

$$\mathcal{H}_{\dot a\dot b\dot c} = -\frac{1}{6}\,\varepsilon_{\dot a\dot b\dot c}\,\varepsilon^{abc}\,\mathcal{H}_{abc} \tag{3.55}$$

and the Bianchi identity

$$\mathcal{D}_a\,\tilde{\mathcal{H}}^{a\dot b\dot c} + \mathcal{D}_{\dot a}\,\tilde{\mathcal{H}}^{\dot a\dot b\dot c} = 0 \tag{3.56}$$

are satisfied. As a result, we have shown that the field strength $\mathcal{H}_{\mu\nu\rho}$ with the $SO(1,2)\times SO(3)$ components defined in (3.26), (3.27), (3.49) and (3.54) satisfies the self-duality condition (3.29).

3.3. Gauge-Covariant Action. The knowledge of the explicit form (3.54) of $\mathcal{H}_{abc}$ allows us to rewrite the action (3.28) (modulo total derivatives) in the equivalent but gauge-covariant form

$$S = -\int d^6x \left(\frac{1}{8}\mathcal{H}_{a\dot{b}\dot{c}}\mathcal{H}^{a\dot{b}\dot{c}} + \frac{1}{12}\mathcal{H}_{\dot{a}\dot{b}\dot{c}}\mathcal{H}^{\dot{a}\dot{b}\dot{c}} - \right.$$
$$\left. - \frac{1}{144}\varepsilon^{abc}\mathcal{H}_{abc}\mathcal{H}_{\dot{a}\dot{b}\dot{c}}\varepsilon^{\dot{a}\dot{b}\dot{c}} - \frac{1}{12g}\varepsilon^{abc}\mathcal{H}_{abc}\right). \quad (3.57)$$

Note that, as one can check directly, the potential A_{ab} enters the action (3.57) only under a total derivative and hence can be dropped out modulo boundary terms. This means that the action (3.57) is invariant under the local symmetry (3.39).

Note also that the last term in (3.57) is of Chern–Simons type and it can be interpreted as a coupling of the 5-brane to the constant background field C_3 which has the nonzero components $C_{\dot{a}\dot{b}\dot{c}} = (1/g)\varepsilon_{\dot{a}\dot{b}\dot{c}}$ along the $y^{\dot{a}}$ directions of the 5-brane. It can thus be rewritten in the Chern–Simons form similar to that of the M5-brane action

$$\int d^6x \frac{1}{12g}\varepsilon^{abc}\mathcal{H}_{abc} = \frac{1}{2}\int \mathcal{H}_3 \wedge C_3,$$

where the field strength $\mathcal{H}_3$ and C_3 are regarded as $D = 6$ three-forms. The presence of the constant background field C_3 obviously breaks the $D = 6$ Lorentz invariance.

The action (3.57) is invariant under the following gauge transformations:

$$\begin{aligned}
&\delta_\Lambda A_{\dot{a}\dot{b}} = \partial_{\dot{a}}\Lambda_{\dot{b}} - \partial_{\dot{b}}\Lambda_{\dot{a}} + g\,\xi^{\dot{c}}\,\partial_{\dot{c}}A_{\dot{a}\dot{b}},\\
&\delta_\Lambda A_{a\dot{b}} = \partial_a\Lambda_{\dot{b}} - \partial_{\dot{b}}\Lambda_a + g\,\xi^{\dot{c}}\,\partial_{\dot{c}}A_{a\dot{b}} + g(\partial_{\dot{b}}\,\xi^{\dot{c}})\,A_{a\dot{c}},\\
&\delta_\Lambda A_{ab} = \partial_a\Lambda_b - \partial_b\Lambda_a + g\,\xi^{\dot{b}}\,\partial_{\dot{b}}A_{ab} + g\,(A_{a\dot{c}}\,\partial_b\xi^{\dot{c}} - A_{b\dot{c}}\,\partial_a\xi^{\dot{c}}).
\end{aligned} \quad (3.58)$$

These transformations are nothing but a noncommutative deformation of usual (Abelian) ones (3.35). It is well known that similar noncommutative Abelian gauge symmetry arises in noncommutative electrodynamics associated with an effective action for the bosonic string in a constant antisymmetric B-field [44]. In [44] it was shown that one can make a nonlocal redefinition of gauge fields and parameters to bring the noncommutative gauge transformations to the standard form of the Abelian transformations. Such a field redefinition is usually referred to as the Seiberg–Witten map. A similar Seiberg–Witten map for the model under consideration (3.57), which relates the gauge transformations (3.58) and (3.35), was considered in [21] in the assumption that it is associated with a single 5-brane in a strong constant C_3-field background. The relation of this model to the complete nonlinear formulations of the M5-brane [29–31, 35–37] remains to be clarified yet.

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REFERENCES

1. *Bagger J., Lambert N.* Modeling Multiple M2's // Phys. Rev. D. 2007. V. 75. P. 045020; hep-th/0611108.
2. *Bagger J., Lambert N.* Gauge Symmetry and Supersymmetry of Multiple M2-branes // Phys. Rev. D. 2008. V. 77. P. 065008; hep-th/0711.0955.
3. *Bagger J., Lambert N.* Comments on Multiple M2-branes // JHEP. 2008. V. 02. P. 105; hep-th/0712.3738.
4. *Gustavsson A.* Algebraic Structures on Parallel M2-branes // Nucl. Phys. B. 2009. V. 811. P. 66–76; hep-th/0709.1260.
5. *Aharony O. et al.* $N = 6$ Superconformal Chern–Simons-Matter Theories, M2-branes and Their Gravity Duals // JHEP. 2008. V. 10. P. 091; hep-th0806.1218.
6. *Schwarz J. H.* Superconformal Chern–Simons Theories // JHEP. 2004. V. 11. P. 078; hep-th/0411077.
7. *Zupnik B. M., Pak D. G.* Superfield Formulation of the Simplest Three-Dimensional Gauge Theories and Conformal Supergravities // Theor. Math. Phys. 1988. V. 77. P. 1070–1076.
8. *Lee C.-K., Lee K.-M., Weinberg E. J.* Supersymmetry and Self-dual Chern–Simons Systems // Phys. Lett. B. 1990. V. 243. P. 105–108.
9. *Ivanov E. A.* Chern–Simons Matter Systems with Manifest $N = 2$ Supersymmetry // Phys. Lett. B. 1991. V. 268. P. 203–208.
10. *Gates J., James S., Nishino H.* Remarks on the $N = 2$ Supersymmetric Chern-Simons Theories // Phys. Lett. B. 1992. V. 281. P. 72–80.
11. *Nishino H., Gates J., James S.* Chern–Simons Theories with Supersymmetries in Three Dimensions // Intern. J. Mod. Phys. A. 1993. V. 8. P .3371–3422.
12. *Brooks R., Gates J., James S.* Extended Supersymmetry and SuperBF Gauge Theories // Nucl. Phys. B. 1994. V. 432. P. 205–224; hep-th/9407147.
13. *Kao H.-C.* Self-dual Yang–Mills Chern–Simons Higgs Systems with an $N = 3$ Extended Supersymmetry // Phys. Rev. D. 1994. V. 50. P. 2881–2892.
14. *Kao H.-C., Lee K.-M., Lee T.* The Chern–Simons Coefficient in Supersymmetric Yang–Mills Chern–Simons Theories // Phys. Lett. B. 1996. V. 373. P. 94–99; hep-th/9506170.
15. *Gaiotto D., Witten E.* Janus Configurations, Chern–Simons Couplings, and the Theta Angle in $N = 4$ Super Yang–Mills Theory. hep-th/0804.2907.
16. *Hosomichi K. et al.* $N = 4$ Superconformal Chern–Simons Theories with Hyper and Twisted Hyper Multiplets // JHEP. 2008. V. 07. P. 091; hep-th/0805.3662.
17. *Filippov V.* n-Lie Algebras // Sibirsk. Mat. Zh. 1985. V. 26. P. 126–140; 191.
18. *Nambu Y.* Generalized Hamiltonian Dynamics // Phys. Rev. D. 1973. V. 7. P. 2405–2414.
19. *Ho P.-M., Matsuo Y.* M5 from M2 // JHEP. 2008. V. 06. P. 105; hep-th/0804.3629.
20. *Ho P.-M., Imamura Y., Matsuo Y.* M2 to D2 Revisited // JHEP. 2008. V. 07. P. 003; hep-th/0805.1202.
21. *Ho P.-M. et al.* M5-brane in Three-Form Flux and Multiple M2-branes // JHEP. 2008. V. 08. P. 014; hep-th/0805.2898.
22. *Pasti P. et al.* BLG-Motivated Lagrangian Formulation for the Chiral Two-Form Gauge Field in $D = 6$ and M5-branes. hep-th/0907.4596.
23. *Gustavsson A.* M5 Brane from Mass Deformed BLG Theory. hep-th/0909.2518.
24. *Gomis J., Milanesi G., Russo J. G.* Bagger–Lambert Theory for General Lie Algebras // JHEP. 2008. V. 06. P. 075; hep-th/0805.1012.

25. *Benvenuti S. et al.* $N = 8$ Superconformal Gauge Theories and M2 Branes // JHEP. 2009. V. 01. P. 078; hep-th/0805.1087.
26. *Gomis J. et al.* Supersymmetric Yang–Mills Theory from Lorentzian Three Algebras // JHEP. 2008. V. 08. P. 094; hep-th/0806.0738.
27. *Van Raamsdonk M.* Comments on the Bagger–Lambert Theory and Multiple M2-branes // JHEP. 2008. V. 05. P. 105; hep-th/0803.3803.
28. *Bandos I. A., Townsend P. K.* SDiff Gauge Theory and the M2 Condensate // JHEP. 009. V. 02. P. 013; hep-th/0808.1583.
29. *Howe P. S., Sezgin E.* $D = 11$, $p = 5$ // Phys. Lett. B. 1997. V. 394. P. 62–66; hep-th/9611008.
30. *Howe P. S., Sezgin E., West P. C.* Covariant Field Equations of the M-theory Five-Brane // Phys. Lett. B. 1997. V. 399. P. 49–59; hep-th/9702008.
31. *Howe P. S., Sezgin E., West P. C.* The Six-Dimensional Self-dual Tensor // Phys. Lett. B. 1997. V. 400. P. 255–259; hep-th/9702111.
32. *Perry M., Schwarz J. H.* Interacting Chiral Gauge Fields in Six-Dimensions and Born–Infeld Theory // Nucl. Phys. B. 1997. V. 489. P. 47–64; hep-th/9611065.
33. *Schwarz J. H.* Coupling a Self-dual Tensor to Gravity in Six Dimensions // Phys. Lett. B. 1997. V. 395. P. 191–195; hep-th/9701008.
34. *Pasti P., Sorokin D. P., Tonin M.* Covariant Action for a $D = 11$ Five-Brane with the Chiral Field // Phys. Lett. B. 1997. V. 398. P. 41–46; hep-th/9701037.
35. *Bandos I. A. et al.* Covariant Action for the Super-Five-Brane of M-theory // Phys. Rev. Lett. 1997. V. 78. P. 4332–4334; hep-th/9701149.
36. *Aganagic M. et al.* World-Volume Action of the M-theory Five-Brane // Nucl. Phys. B. 1997. V. 496. P. 191–214; hep-th/9701166.
37. *Bandos I. A. et al.* On the Equivalence of Different Formulations of the M Theory Five-Brane // Phys. Lett. B. 1997. V. 408. P. 135–141; hep-th/9703127.
38. *Bandos I. A., Townsend P. K.* Light-Cone M5 and Multiple M2-branes // Class. Quant. Grav. 2008. V. 25. P. 245003; hep-th/0806.4777.
39. *Furuuchi K., Takimi T.* String Solitons in the M5-brane Worldvolume Action with Nambu–Poisson Structure and Seiberg–Witten Map // JHEP. 2009. V. 08. P.050.
40. *Henneaux M., Teitelboim C.* Dynamics of Chiral (Self-dual) P-forms // Phys. Lett. B. 1988. V. 206. P. 650.
41. *Schwarz J. H., Sen A.* Duality Symmetric Actions // Nucl. Phys. B. 1994. V. 411. P. 35–63; hep-th/9304154.
42. *Bengtsson I., Kleppe A.* On Chiral P-forms // Intern. J. Mod. Phys. A. 1997. V. 12. P. 3397–3412; hep-th/9609102.
43. *Pasti P., Sorokin D. P., Tonin M.* On Lorentz Invariant Actions for Chiral P-forms // Phys. Rev. D. 1997. V. 55. P. 6292–6298; hep-th/9611100.
44. *Seiberg N., Witten E.* String Theory and Noncommutative Geometry // JHEP. 1999. V. 09. P. 032; hep-th/9908142.