JUCYS–MURPHY ELEMENTS FOR BIRMAN–MURAKAMI–WENZL ALGEBRAS

A. P. Isaev
Joint Institute for Nuclear Research, Dubna

O. V. Ogievetsky
Center of Theoretical Physics, Marseille, France

The Birman–Murakami–Wenzl algebra, considered as the quotient of the braid group algebra, possesses the commutative set of Jucys–Murphy elements. We show that the set of Jucys–Murphy elements is maximal commutative for the generic Birman–Murakami–Wenzl algebra and reconstruct the representation theory of the tower of Birman–Murakami–Wenzl algebras.

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INTRODUCTION

Let \( G \) be either the orthogonal group \( SO(N) \) or, for \( N \) even, the symplectic group \( Sp(N) \). Let \( g \) be its Lie algebra and \( U_q(g) \) the corresponding quantum universal enveloping algebra. We denote the space of the defining irreducible representation (irrep) of \( G \) or \( U_q(g) \) by \( V \).

In 1937 R. Brauer [1] introduced a 1-parametric family of algebras \( Br_n(x) \) to describe the centralizer of the action of \( G \) on the tensor powers \( V \otimes^n \). More precisely, fix the value of the parameter \( x, x = N \). The algebra \( Br_n(N) \) has the representation \( \tau: Br_n(N) \to \text{End}(V \otimes^n) \); the image of \( Br_n(N) \) in this representation coincides with the commutant of the action of \( G \) on \( V \otimes^n \). The generators of the algebra \( Br_n(N) \) are expressed in terms of the permutation \( P \) and the operator \( K \) related to the \( G \)-invariant pairing \( K \):

\[
P(v_i \otimes v_j) = v_j \otimes v_i, \quad K(v_i \otimes v_j) = K_{ij} K^{kl} v_k \otimes v_l.
\]

Here \( K^{kl} \) is inverse to \( K_{ij} \), \( K^{kl} K_{lj} = \delta^k_i \). The Brauer algebras play the same role in the representation theory of \( SO(N) \) and \( Sp(N) \) groups as the symmetric groups in the theory of representations of linear groups. The Brauer–Schur–Weyl duality establishes the correspondence between the finite-dimensional irreps of \( SO(N) \), \( Sp(N) \) and the irreps of \( Br_n(N) \).

For quantum deformations \( U_q(g) \), the Brauer algebras \( Br_n(N) \) get \( q \)-deformed as well; instead of \( Br_n(x) \) one now has a 2-parametric family of algebras \( BMW_n(q, \nu) \). These algebras were introduced independently by J. Murakami [2] and by J. Birman and H. Wenzl [3].
The centralizers $\text{End}_{U_q(g)}(V^\otimes n)$ are realized by specific representations $\tau$ of the Birman–Murakami–Wenzl algebras. The value $\nu_{U_q(g)}$ of the parameter $\nu$ depends on $q$ and $g$. In the representation $\tau: \text{BMW}_{n}(q, \nu_{U_q(g)}) \to \text{End}(V^\otimes n)$ the generators of BMW algebras are built with the help of the Yang–Baxter $R$-operator, the $q$-analogue of the permutation $P$ (see [4] for the functorial construction of $R$-matrices of BMW type from the $R$-matrices of $GL$ type). In contrast to the classical case, the $q$-analogue of $K$ is a certain combination of Yang–Baxter $R$-operators, see, e.g., [5,6].

For generic values of the parameters $q$ and $\nu_{U_q(g)}$, the BMW algebra has the same representations as the Brauer algebra. Different aspects of the representation theory were extensively studied in the literature (see, e.g., [7–11] and references therein).

Here we generalize to the BMW algebras the approach of Vershik and Okounkov [12] developed for the representation theory of symmetric groups and adopted to the Hecke algebra case in [13]. The details of the proofs will be published in our forthcoming publication [14].

1. BIRMAN–MURAKAMI–WENZL (BMW) ALGEBRAS

1.1. Braid Group and Its Quotients. The braid group $B_{M+1}$ is generated by elements $\sigma_i$, $i = 1, \ldots, M$, subject to relations:

Braid : \[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \]

Locality : \[ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1. \]

The braid group $B_{M+1}$ is infinite. We shall discuss certain finite-dimensional quotients of $\mathbb{C}B_{M+1}$.

1. The Hecke algebra $H_{M+1}(q)$ is defined by relations

\[ (\sigma_i - q)(\sigma_i + q^{-1}) = 0, \]

where $q$ is a parameter; $\dim(H_M(q)) = M!$.

2. The Birman–Murakami–Wenzl algebra $\text{BMW}_{M+1}(q, \nu)$ is defined by relations

\[ \left\{ \begin{array}{l}
(\sigma_i - q)(\sigma_i + q^{-1})(\sigma_i - \nu) = 0, \\
\kappa_i \sigma_i^{\pm 1} \kappa_i = \nu^{\pm 1} \kappa_i,
\end{array} \right. \]

where

\[ \kappa_i := \frac{(q - \sigma_i)(\sigma_i + q^{-1})}{\nu(q - q^{-1})} \quad (i = 1, \ldots, M), \]

$q$ and $\nu$ are parameters; $\dim(\text{BMW}_M(q, \nu)) = (2M - 1)!!$.

There is a beautiful graphical presentation of the braid group and its finite-dimensional quotients. The generators $\sigma_i \in B_{M+1}$ are depicted by

\[
\sigma_i = \begin{array}{cccccccc}
1 & 2 & \ldots & i & i+1 & i+2 & \ldots & M & M+1
\end{array}
\]
For the locality relation (2) we have $(i + 1 < j < M)$

\[
\sigma_i \sigma_j = \cdots = \sigma_i \sigma_j = \sigma_i \sigma_j.
\]

The braid relation is

\[
\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i = \sigma_i \sigma_{i+1} \sigma_i.
\]

It is sometimes convenient to depict the element (5) by

\[
\kappa_1 = \begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\vdots & & \vdots & & \vdots & & \vdots \\
1 & & i-1 & & i+1 & & i+2 \\
\vdots & & \vdots & & \vdots & & \vdots \\
M & & M+1 & & & & \\
\end{array}
\]

Below we shall omit the reference to the parameters in the notation $H_M(q)$ and $BMW_M(q, \nu)$ and write simply $H_M$ and $BMW_M$.

1.2. Affine BMW Algebras $\alpha BMW_{M+1}$. Affine Birman–Murakami–Wenzl algebras $\alpha BMW_{M+1}$ are extensions of the algebras $BMW_M$. The algebras $\alpha BMW_{M+1}$ are generated by the elements $\{\sigma_1, \ldots, \sigma_M\}$ with relations (1), (2), (4) and the affine element $y_1$ which satisfies

\[
\begin{align*}
\sigma_1 y_1 \sigma_1 y_1 &= y_1 \sigma_1 y_1 \sigma_1, & [\sigma_k, y_1] &= 0 \quad \text{for} \quad k > 1, \\
\kappa_1 y_1 \sigma_1 y_1 \sigma_1 &= c \kappa_1 = \sigma_1 y_1 \sigma_1 y_1 \kappa_1, \\
\kappa_1 y_1^n \kappa_1 &= \hat{z}^{(n)} \kappa_1, & n &= 1, 2, 3, \ldots
\end{align*}
\]

where $c$, $\hat{z}^{(n)}$ are central elements. Initially, for the Brauer algebras, the affine version was introduced by M. Nazarov [15].

Consider the set of affine elements $y_{k+1} = \sigma_k y_k \sigma_k$, $k = 1, 2, \ldots, M$.

The elements $y_k$ $(k = 1, 2, \ldots, M + 1)$ generate a commutative subalgebra $Y_{M+1}$ in $\alpha BMW_{M+1}$.

1.3. Central Elements in $\alpha BMW$ Algebra. We need some information about the center of $\alpha BMW$.

**Proposition 1.** The elements

\[
\hat{Z} = y_1 \cdot y_2 \cdot \cdots y_M, \quad \hat{z}^{(n)}_M = \sum_{k=1}^{M} (y_k^n - c^n y_k^{-n}), \quad n \in \mathbb{N},
\]

are central in the $\alpha BMW_M$ algebra.
Remark. The set of «power sums» \( \hat{z}^{(n)} = \sum_k (y_k^n - c^n y_k^{-n}) \) has the generating function

\[
Z(t) = \sum_{n=1}^{\infty} \hat{z}^{(n)} t^{n-1} = \frac{d}{dt} \log \left( \prod_{k=1}^{\infty} \frac{y_k - ct}{1 - y_k t} \right).
\]

Consider an ascending chain of subalgebras

\[
\alpha BMW_0 \subset \alpha BMW_1 \subset \alpha BMW_2 \subset \ldots \subset \alpha BMW_M \subset \alpha BMW_{M+1},
\]

where \( \alpha BMW_0, \alpha BMW_1 \) and \( \alpha BMW_j \) (\( j > 1 \)) are generated by \( \{ c, \hat{z}^{(n)} \}, \{ c, \hat{z}^{(n)}, y_1 \} \) and \( \{ c, \hat{z}^{(n)}, y_1, \sigma_1, \sigma_2, \ldots, \sigma_{j-1} \} \), respectively. For the corresponding commutative subalgebras we have \( Y_1 \subset Y_2 \subset \cdots \subset Y_M \subset Y_{M+1} \).

**Proposition 2.** Let \( \hat{Z}^{(n)}_k \) be the central elements in the algebra \( \alpha BMW_k, \alpha BMW_k \subset \alpha BMW_{k+2} \), defined by the generating function

\[
\sum_{n=0}^{\infty} \hat{Z}^{(n)}_k t^n = -\frac{\nu}{(q - q^{-1})} + \frac{1}{(1 - ct^2)} + \left( \sum_{n=0}^{\infty} t^n \hat{z}^{(n)} + \frac{\nu}{(q - q^{-1})} - \frac{1}{(1 - ct^2)} \right) \times \prod_{r=1}^{k} \frac{(1 - y_r t^2)(q^2 - cy_r^{-1} t^2)(q^{-2} - cy_r^{-1} t^2)}{(1 - cy_r^{-1} t^2)(q^2 - y_r t)(q^{-2} - y_r t)}. \tag{7}
\]

The following relations hold:

\[
\kappa_{k+1} y_k^n \kappa_{k+1} = \hat{Z}^{(n)}_k \kappa_{k+1} \in \alpha BMW_{k+2} \quad (\hat{Z}^{(n)}_0 \equiv \hat{z}^{(n)}).
\]

Remark. The evaluation map \( \alpha BMW_M \to BMW_M \) is defined by

\[
y_1 \mapsto 1 \Rightarrow c \mapsto \nu^2, \quad \hat{z}^{(n)} \mapsto 1 + \frac{\nu^{-1} - \nu}{q - q^{-1}}. \tag{8}
\]

Under this map the function (7) transforms into the generating function presented in [9].

**1.4. Intertwining Operators in \( \alpha BMW \) Algebra.** Introduce the intertwining elements \( U_{k+1} \in \alpha BMW_{M+1} \) (\( k = 1, \ldots, M \)):

\[
U_{k+1} = [\sigma_k, y_k - cy_k^{-1}]. \tag{9}
\]

**Proposition 3.** The elements \( U_k \) satisfy

\[
U_{k+1} y_k = y_{k+1} U_{k+1}, \quad U_{k+1} y_{k+1} = y_k U_{k+1}, \quad U_{k+1} y_i = y_i U_{k+1} \quad \text{for} \quad i \neq k, k+1,
\]

\[
U_{k+1} [\sigma_k, y_k] = (q y_k - q^{-1} y_{k+1})(q y_{k+1} - q^{-1} y_k) \left( 1 - \frac{c}{y_k y_{k+1}} \right), \tag{10}
\]

\[
U_{k+1} U_k U_{k+1} = U_k U_{k+1} U_k, \quad \kappa_k U_{k+1} = U_{k+1} \kappa_k = 0.
\]

The elements \( U_k \) provide an important information about the spectrum of the affine elements \( \{ y_j \} \).
Lemma 1. The spectrum of the elements $y_j \in \alpha BMW_{M+1}$ satisfies

$$\text{Spec} (y_j) \subset \{ q^{2\mathbb{Z}}, \text{Spec} (y_1), c q^{2\mathbb{Z}}, \text{Spec} (y_1^{-1}) \}, \quad (11)$$

where $\mathbb{Z}$ is the set of integer numbers.

Proof. Induction in $j$. Equation (11) obviously holds for $y_1$. Assume that

$$\text{Spec} (y_{j-1}) \subset \{ q^{2\mathbb{Z}}, \text{Spec} (y_1), c q^{2\mathbb{Z}}, \text{Spec} (y_1^{-1}) \}, \quad j > 1.$$

Let $f$ be the characteristic polynomial of $y_{j-1}$, $f(y_{j-1}) = 0$. Then

$$0 = U_j f(y_{j-1})[\sigma_{j-1}, y_{j-1}] = f(y_j) U_j[\sigma_{j-1}, y_{j-1}] =
= f(y_j) (q^2 y_{j-1} - y_j) (y_j - q^{-2} y_{j-1}) (y_j - c y_j^{-1}) y_j^{-1}.$$

Here we used (10). Thus, $\text{Spec} (y_j) \subset \text{Spec} (y_{j-1}) \cup q^{\pm 2} \cdot \text{Spec} (y_{j-1}) \cup c \cdot \text{Spec} (y_{j-1}^{-1})$. ■

We denote the image of $w \in \alpha BMW_M$ under the evaluation map (8) by $\tilde{w}$, e.g., $y_j \mapsto \tilde{y}_j$.

The Jucys–Murphy (JM) elements $\tilde{y}_j$ ($j = 2, \ldots, M$) are the images of $y_j$:

$$\tilde{y}_j = \sigma_{j-1} \ldots \sigma_2 \sigma_1^2 \sigma_2 \ldots \sigma_{j-1} \in BMW_M.$$

Lemma 1 provides the information about the spectrum of JM elements $\tilde{y}_j$’s.

Corollary. Since $\tilde{y}_1 = 1$ and $\tilde{c} = \nu^2$, it follows from (11) that

$$\text{Spec} (\tilde{y}_j) \subset \{ q^{2\mathbb{Z}}, \nu^2 q^{2\mathbb{Z}} \}.$$

2. REPRESENTATIONS OF AFFINE ALGEBRA $\alpha BMW_2$

2.1. $\alpha BMW_2$ Algebra and Its Modules $V_\rho$. The elements $\{ y_i, y_{i+1}, \sigma_i, \kappa_i \} \in \alpha BMW_M$ (for fixed $i < M$) satisfy

$$(q - q^{-1})\kappa_i = \sigma_i^{-1} - \sigma_i + (q - q^{-1}), \quad (13)$$

$$y_{i+1} = \sigma_i y_i \sigma_i, \quad y_{i+1} y_i = y_i y_{i+1}, \quad \kappa_i y_i \kappa_i = \tilde{Z}_{i+1}^{(n)} \kappa_i, \quad (14)$$

$$y_i y_{i+1} \kappa_i = c \kappa_i = \kappa_i y_{i+1} y_i. \quad (15)$$

The elements $c$ and $\tilde{Z}_{i+1}^{(n)}$ commute with $\{ y_i, y_{i+1}, \sigma_i, \kappa_i \}$. The elements $\{ y_i, y_{i+1}, \sigma_i, \kappa_i \} \in \alpha BMW_M$ generate a subalgebra isomorphic to $\alpha BMW_2$.

Below we investigate representations $\rho$ of $\alpha BMW_2$ for which the generators $\rho (y_i)$ and $\rho (y_{i+1})$ are diagonalizable and $\rho (c) = \nu^2 \cdot \text{Id}$. Let $\psi$ be a common eigenvector of $\rho (y_i)$ and $\rho (y_{i+1})$ with some eigenvalues $a$ and $b$:

$$\rho (y_i) \psi = a \psi, \quad \rho (y_{i+1}) \psi = b \psi.$$  

The element $\tilde{z} = y_i y_{i+1}$ is central in $\alpha BMW_2$. There are two possibilities:

1. $\rho (\kappa_i) \neq 0 \implies \rho (y_i y_{i+1}) = \nu^2 \cdot \text{Id} \implies ab = \nu^2.

\begin{equation}
\tag{16}
2. \rho (\kappa_i) = 0, \text{ the product } ab \text{ is not fixed.}
\end{equation}
To save space we shall often omit the symbol \( \rho \) and denote, slightly abusively, the operator \( \rho(x) \) for \( x \in \alpha BMW \) by the same letter \( x \); this should not lead to a confusion.

Applying the operators from \( \alpha BMW_2 \) to the vector \( \psi \), we produce, in general infinite-dimensional, \( \alpha BMW_2 \)-module \( V_\infty \) spanned by

\[
\begin{align*}
  e_2 &= \psi, \\
  e_1 &= \kappa_1 \psi, \\
  e_3 &= \sigma_1 \psi, \\
  e_4 &= y_i \kappa_1 \psi, \\
  e_5 &= \sigma_1 y_i \kappa_1 \psi, \\
  e_6 &= y_i^2 \kappa_1 \psi, \\
  &\quad \ldots \\
  e_{2k+2} &= y_i^k \kappa_1 \psi, \\
  e_{2k+3} &= \sigma_1 y_i^k \kappa_1 \psi \ (k \geq 1), \\
  &\quad \ldots
\end{align*}
\]

Using relations (13)–(15) for \( \alpha BMW \), one can write down the left action of elements \( \{ y_i, y_{i+1}, \sigma_1, \kappa_1 \} \) on \( V_\infty \). Our aim is to understand when the sequence \( e_j \) can terminate, giving therefore rise to a finite-dimensional module \( V_D \) (of dimension \( D \)) of \( \alpha BMW_2 \), and investigate the (ir)reducibility of \( V_D \).

We distinguish 3 cases for the module \( V_D \):

(i) \( \kappa_1 V_D = 0 \) (i.e. \( \kappa_1 \psi = 0 \) \( \forall \) \( e \in V_D \)) and in particular \( \kappa_1 \psi = 0 \). Therefore, \( e_j = 0 \) for all \( j \neq 2,3 \) and \( V_\infty \) reduces to a 2-dim module with the basis \( \{ e_2, e_3 \} \). In view of (16), the product \( a b \) is not fixed and the irreps coincide with the irreps of the affine Hecke algebra \( \alpha H_2 \) considered in [13].

(ii) \( \kappa_1 V_D \neq 0 \) (i.e., \( \exists \ e \in V_D : \kappa_1 e \neq 0 \)). The module \( V_D \) is extracted from \( V_\infty \) by constraints

\[
e_{2k+4} = \sum_{m=1}^{2k+3} \alpha_m e_m \ (k \geq -1), \quad ab = \nu^2,
\]

with some parameters \( \alpha_m \). The independent basis vectors are \( (e_1, e_2, \ldots, e_{2k+3}) \). The module \( V_D \) has odd dimension.

(iii) \( \kappa_1 V_D \neq 0 \) and additional constraints are

\[
e_{2k+3} \ = \ \sum_{m=1}^{2k+2} \alpha_m e_m \ (k \geq 0), \quad ab = \nu^2.
\]

The independent basis vectors are \( (e_1, e_2, \ldots, e_{2k+2}) \). The module \( V_D \) has even dimension.

Below we consider a version \( \alpha' BMW_2 \) of the affine BMW algebra. The additional requirement for this algebra concerns the spectrum of \( y_i, y_{i+1} \in \alpha' BMW_2 \):

\[
\text{Spec} (y_j) \subset \{ q^{2n}, \nu^2 q^{2n} \}.
\]

The evaluation map (8) descends to the algebra \( \alpha' BMW \) (cf. Corollary after Lemma 1). In particular, for the cases (ii) and (iii) we have

\[
ab = \nu^2 q^{2z}, \quad b = q^{-2z} \quad \text{or} \quad a = q^{2z}, \quad b = \nu^2 q^{-2z}
\]

for some \( z \in \mathbb{Z} \).

**2.2. The Case \( \kappa_1 V_D = 0 \): Hecke Algebra Case [13].** Representations of \( \alpha BMW_2 \) with \( \kappa_1 V_D = 0 \) reduce to representations of the affine Hecke algebra \( \alpha H_2 \). In the basis \( (e_2, e_3) = (\psi, \sigma_1 \psi) \) the matrices of the generators are

\[
\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & q^{-1} \end{pmatrix}, \quad y_i = \begin{pmatrix} 0 & -(q^{-1} - q)b \\ 1 & b \end{pmatrix}, \quad y_{i+1} = \begin{pmatrix} b & (q^{-1} - q)b \\ 0 & a \end{pmatrix},
\]

(19)
where $a \neq b$ (otherwise $y_i, y_{i+1}$ are not diagonalizable). By Lemma 1, we have for $y_i, y_{i+1} \in \alpha'BMW_2$ the eigenvalues $a, b \in \{q^{2\mathbb{Z}}, \nu^2 q^{2\mathbb{Z}}\}$. The 2-dimensional representation (19) contains a 1-dimensional subrepresentation iff $a = q^\pm b$. Graphically these 1- and 2-dimensional irreps of $\alpha'BMW_2$ are visualized in Figs. 1 and 2.

Different paths going from the upper vertex to the lower vertex correspond to different eigenvectors of $y_i, y_{i+1}$. The indices on the edges are eigenvalues of $y_i, y_{i+1}$.

**2.3. $\kappa_iV_D \neq 0$: Odd-Dimensional Representations for $\alpha'BMW_2$.** Using the condition (17) for the reduction $V_\infty$ to $V_{2m+1}$, one can describe odd-dimensional representations of $\alpha'BMW_2$, determine matrices for the action of $y_i, y_{i+1}$ on $V_{2m+1}$ and calculate

$$
\det(y_i) = \prod_{r=1}^{2m+1} y_i^{(r)} = \nu^{2m}, \quad \det(y_{i+1}) = \prod_{r=1}^{2m+1} y_{i+1}^{(r)} = \nu^{2m+2}.
$$

Here for eigenvalues $y_i^{(r)}, y_{i+1}^{(r)} (r = 1, 2, \ldots, 2m + 1)$ of $y_i$ and $y_{i+1}$ we have constraints

$$
y_i^{(r)} y_{i+1}^{(r)} = \nu^2, \quad r = 1, \ldots, 2m + 1.
$$

and (see Eq. (12))

$$
y_i^{(r)} \in \{q^{2\mathbb{Z}}, \nu^2 q^{2\mathbb{Z}}\}, \quad r = 1, \ldots, 2m + 1.
$$

These odd-dimensional irreps are visualized as graphs presented in Fig. 3, where $z_r \in \mathbb{Z}$ and $\sum_{r=1}^{2m+1} z_r = 0$ as it follows from (20). Different paths going from the top vertex to the bottom vertex correspond to different common eigenvectors of $y_i, y_{i+1}$. Indices on upper and lower edges of these paths are the eigenvalues of $y_i$ and $y_{i+1}$, respectively.
Remark. In view of the braid relations $\sigma_i \sigma_{i\pm 1} = \sigma_{i\pm 1} \sigma_i$ and possible eigenvalues of $\sigma$’s for 1-dimensional representations (described in Subsecs. 2.2 and 2.3), we conclude that the following chains of 1-dimensional representations are forbidden:

$$\begin{array}{c}
& a \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
a & 1 & q^{i/2} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
aq^{i/2} & \nu^2 & 1 \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
a & \nu^2q^{i/2} & \nu^2 \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
& * \\
\end{array}$$

where $a = q^{2z}$ or $a = \nu^2q^{2z}$ ($z \in \mathbb{Z}$).

2.4. $\kappa_{iV_D} \neq 0$: Even-Dimensional Representations of $\alpha'BMW_2$. With the help of the conditions (18) we reduce $V_\infty$ to $V_{2m}$, then explicitly construct $(2m) \times (2m)$ matrices for the operators $y_i, y_i + 1$ and calculate their determinants

$$\det (y_i) = \prod_{r=1}^{2m} y_i^{(r)} = \epsilon q^r \nu^{2m-1}, \quad \det (y_{i+1}) = \prod_{r=1}^{2m} y_{i+1}^{(r)} = -\epsilon q^r \nu^{2m+1}, \quad (21)$$

where $y_i^{(r)}, y_{i+1}^{(r)}$ are eigenvalues of $y_i, y_{i+1}$ (we have two possibilities: $\epsilon = \pm 1$). We see from (21) that all $(2m)$ eigenvalues of $y_i, y_{i+1}$ cannot belong to the spectrum (12). More precisely, there is at least one eigenvalue $y_i^{(r)}$ of $y_i$ (and the eigenvalue $y_{i+1}^{(r)}$ of $y_{i+1}$) such that

$$y_i^{(r)}, y_{i+1}^{(r)} \notin \{q^{2z}, \nu^2q^{2z}\}.$$ 

Thus, even-dimensional irreps of $\alpha'BMW_2$ subject to the conditions (18) are not admissible for $\alpha'BMW_2$.

3. REPRESENTATIONS OF BMW ALGEBRAS

3.1. Spec $(\tilde{y}_1, \ldots, \tilde{y}_n)$ and Rules for Strings of Eigenvalues. Now we reconstruct the representation theory of BMW algebras using an approach which generalizes the approach of Okounkov–Vershik [12] for symmetric groups.

The JM elements $\{\tilde{y}_1, \ldots, \tilde{y}_n\}$ generate a commutative subalgebra in $BMW_n$. The basis in the space of an irrep of $BMW_n$ can be chosen to be the common eigenbasis of all $\tilde{y}_i$. Each common eigenvector $v$ of $\tilde{y}_i$,

$$\tilde{y}_i v = a_i v, \quad i = 1, \ldots, n,$$

defines a string $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Denote by Spec $(\tilde{y}_1, \ldots, \tilde{y}_n)$ the set of such strings.

We summarize our results about representations of $\alpha'BMW_2$ and the spectrum of the JM elements $\tilde{y}_i$ in the following Proposition.

**Proposition 4.** Consider the string

$$\alpha = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \in \text{Spec} (\tilde{y}_1, \ldots, \tilde{y}_i, \tilde{y}_{i+1}, \ldots, \tilde{y}_n).$$
Let $v_\alpha$ be the corresponding eigenvector of $\tilde{y}_i$: $\tilde{y}_i v_\alpha = a_i v_\alpha$. Then

(1) $a_i \in \{q^{2z}, \nu^2 q^{2z}\}$;
(2) $a_i \neq a_{i+1}, i = 1, \ldots, n-1$;
(3a) $a_i a_{i+1} \neq \nu^2, a_{i+1} = q^{\pm 2} a_i \Rightarrow \sigma_i \cdot v_\alpha = \pm q^{\pm 1} v_\alpha, \ \kappa_i \cdot v_\alpha = 0$;
(3b) $a_i a_{i+1} \neq \nu^2, a_{i+1} \neq q^{\pm 2} a_i \Rightarrow$

$\alpha' = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \in \text{Spec}(\tilde{y}_1, \ldots, \tilde{y}_i, \tilde{y}_{i+1}, \ldots, \tilde{y}_n), \ \kappa_i \cdot v_\alpha = 0, \ \kappa_i \cdot v_{\alpha'} = 0$;

(4) $a_i a_{i+1} = \nu^2 \Rightarrow \exists$ odd number of strings $\alpha^{(k)} (k = 1, 2, \ldots, 2m + 1):$

$\alpha^{(k)} = (a_1, \ldots, a_{i-1}, a_i^{(k)}, a_{i+1}^{(k)}, a_{i+2}, \ldots, a_n) \in \text{Spec}(\tilde{y}_1, \ldots, \tilde{y}_n) \text{ } \forall k$,

$\alpha \in \{\alpha^{(k)}\}, \ a_i^{(k)} a_{i+1} = \nu^2, \ \prod_{k=1}^{2m+1} a_i^{(k)} = \nu^{2m}, \ \prod_{k=1}^{2m+1} a_{i+1}^{(k)} = \nu^{2m+2}$.

The necessary and sufficient conditions for a string to belong to the common spectrum of $\tilde{y}_i$ are formulated in the following way.

**Proposition 5.** The string $\alpha = (a_1, a_2, \ldots, a_n)$, where $a_i \in \{q^{2z}, \nu^2 q^{2z}\}$, belongs to the set $\text{Spec}(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)$ iff $\alpha$ satisfies the following conditions ($z \in \mathbb{Z}$):

(1) $a_1 = 1$;
(2) $a_1 = \nu^2 q^{-2z} \Rightarrow q^{2z} \in \{a_1, \ldots, a_{i-1}\}$;
(3) $a_i = q^{2z} \Rightarrow \{a_i q^2, a_i q^{-2}\} \cap \{a_1, \ldots, a_{i-1}\} \neq \emptyset, \ z \neq 0$;

(4a) $a_i = a_j = q^{2z} (i < j) \Rightarrow \left\{\begin{array}{l}
either\ \{q^{2z} (i+1), q^{2z} (i-1)\} \subset \{a_{i+1}, \ldots, a_{j-1}\},  \\
or \nu^2 q^{-2z} \in \{a_{i+1}, \ldots, a_{j-1}\} ;
\end{array}\right.$

(4b) $a_i = a_j = \nu^2 q^{2z} (i < j) \Rightarrow \left\{\begin{array}{l}
either\ \{\nu^2 q^{2z} (i+1), \nu^2 q^{2z} (i-1)\} \subset \{a_{i+1}, \ldots, a_{j-1}\},  \\
or q^{-2z} \in \{a_{i+1}, \ldots, a_{j-1}\} ;
\end{array}\right.$

(5a) $a_i = \nu^2 q^{-2z}, \ a_j = q^{2z} (i < j) \Rightarrow q^{2z} \text{ or } \nu^2 q^{-2z} \in \{a_{i+1}, \ldots, a_{j-1}\}$;

(5b) $a_i = q^{2z}, \ a_j = \nu^2 q^{-2z} (i < j) \Rightarrow \nu^2 q^{-2z} \text{ or } q^{2z} \in \{a_{i+1}, \ldots, a_{j-1}\}$.

where in (5a) and (5b) we set $z' = z \pm 1$.

### 3.2. Young Graph for BMW Algebras.

We illustrate the above considerations on the example of the colored (in the sense of [13]) Young graph for the algebra $BMW_5$ (see Fig. 4). This graph contains the whole information about the irreps of $BMW_5$ and the branching rules $BMW_5 \downarrow BMW_4$.

A vertex $\{\lambda; 5\}$ on the lowest level of this graph is labeled by some Young diagram $\lambda$; this vertex corresponds to the irrep $W_{(\lambda; 5)}$ of $BMW_5$ (the notation $\{\lambda; 5\}$ is designed to encode the diagram $\lambda$ and the level on which this diagram is located; the levels are counted starting from 0). Paths going down from the top vertex $\emptyset$ to the lowest level (that is, paths of length 5) correspond to common eigenvectors of the JM elements $\tilde{y}_1, \ldots, \tilde{y}_5$. Paths ending at $\{\lambda; 5\}$ label the basis in $W_{(\lambda; 5)}$. In particular, the number of different paths going down from the top $\emptyset$ to $\{\lambda; 5\}$ is equal to the dimension of the irrep $W_{(\lambda; 5)}$. 


Note that the colored Young graph in Fig. 4 contains subgraphs presented in Figs. 1–3. For example, in Fig. 4 one recognizes rhombic subgraphs (the vertices on the subgraphs are obtained from one another by a rotation) of the type presented in Fig. 2.
Let \((s, t)\) be coordinates of a node in the Young diagram \(\lambda\). To the node \((s, t)\) of the diagram \(\lambda\) we associate a number \(q^{2(s-t)}\) which is called «content»:

\[
\begin{array}{cccc}
1 & q^2 & q^4 & q^6 \\
q^{-2} & 1 & q^2 \\
q^{-4} & & & \\
\end{array}
\]

Then according to the colored Young graph in Fig. 4, at each step down along the path one can add or remove one node (therefore this graph is called the «oscillating» Young graph) and the eigenvalue of the corresponding JM element is determined by the content of the node:

\[
y_9 = \nu^2 q^2
\]

The eigenvalue corresponding to the addition or removal of the \((s, t)\) node is \(q^{2(s-t)}\) or \(\nu^2 q^{-2(s-t)}\), respectively.

Let \(X(n)\) be the set of paths of length \(n\) starting from the top vertex \(\emptyset\) and going down in the Young graph of oscillating Young diagrams. Now we formulate the following Proposition.

**Proposition 6.** There is a bijection between the set \(\text{Spec}(\tilde{y}_1, \ldots, \tilde{y}_n)\) and the set \(X(n)\).

**3.3. Primitive Idempotents.** The colored Young graph (as in Fig. 4) gives also the rule of construction of a complete set of orthogonal primitive idempotents for the BMW algebra. The completeness of the set of orthogonal primitive idempotents is equivalent to the maximality of the commutative set of JM elements. Let \(\{\lambda; n\}\) be a vertex in the Young graph with

\[
\lambda = \begin{array}{cccc}
& n_1 \lambda_{(1)} & n_3 \lambda_{(3)} & n_k \lambda_{(k)} \\
n_2 \lambda_{(2)} & & & \\
\ldots & & & \\
n_3 \lambda_{(3)} & & & \\
\end{array}
\]

\((n_i, \lambda_{(i)})\) are coordinates of the nodes which are in the corners of \(\lambda = [\lambda_{(1)}^{n_1}, \lambda_{(2)}^{n_2-n_1}, \ldots, \lambda_{(k)}^{n_k-n_{k-1}}]\).
Consider any path \( T_{(\lambda,n)} \) going down from the top \( \emptyset \) to this vertex. Let \( E_{T_{(\lambda,n)}} \in BMW_n \) be the primitive idempotent corresponding to \( T_{(\lambda,n)} \). Using the branching rule implied by the Young graph for \( BMW_{n+1} \), we know all possible eigenvalues of the element \( \tilde{y}_{n+1} \) and, therefore, obtain the identity

\[
E_{T_{(\lambda,n)}} \prod_{r=1}^{k+1} \left( \tilde{y}_{n+1} - q^{2(\lambda(r)-n_r-1)} \right) \prod_{r=1}^{k} \left( \tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda(r))} \right) = 0,
\]

where \( \lambda_{(k+1)} = n_0 = 0 \). So, for a new diagram \( \lambda' \) obtained by adding to \( \lambda \) a new node with coordinates \((n_j+1, \lambda(j)+1)\), the corresponding primitive idempotent (after an appropriate normalization) reads

\[
E_{T_{(\lambda',n+1)}} = E_{T_{(\lambda,n)}} \prod_{r=1}^{k+1} \left( \tilde{y}_{n+1} - q^{2(\lambda(r)-n_r-1)} \right) \prod_{r=1}^{k} \left( \tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda(r))} \right).
\]

For a new diagram \( \lambda'' \) which is obtained from \( \lambda \) by removing a node with coordinates \((n_j, \lambda(j))\), we construct the primitive idempotent

\[
E_{T_{(\lambda'',n+1)}} = E_{T_{(\lambda,n)}} \prod_{r=1}^{k+1} \left( \tilde{y}_{n+1} - q^{2(\lambda(r)-n_r-1)} \right) \prod_{r=1}^{k} \left( \tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda(r))} \right).
\]

Using these formulas and the «initial data» \( E_{T_{(\emptyset,0)}} = 1 \), one can deduce step by step explicit expressions for the primitive orthogonal idempotents related to the paths in the BMW Young graph.

4. OUTLOOK

In this paper we reconstructed the representation theory of the tower of the BMW algebras, using the properties of the commutative subalgebras, generated by the Jucys–Murphy elements, in the BMW algebras. This representation theory is of use in the representation theory of the quantum groups \( U_q(osp(N|K)) \) due to the Brauer–Schur–Weyl duality, but also finds applications in physical models. Recently [16] we have formulated integrable chain models with nontrivial boundary conditions in terms of the affine Hecke algebras \( \alpha H_n \) and the affine BMW algebras \( \alpha BMW_n \). The Hamiltonians for these models are special elements of the algebra \( \alpha BMW_n \). For example, for the \( \alpha BMW_n \) algebra we have deduced [16] the Hamiltonians

\[
\mathcal{H} = \sum_{m=1}^{n-1} \left( \sigma_m + \frac{(q-q^{-1})\nu}{\nu + a}\kappa_m \right) + \frac{(q-q^{-1})\xi}{y_1 - \xi}, \tag{22}
\]

where \( \xi^2 = -ac/\nu \) and the parameter \( a \) can take one of two values \( a = \pm q^{\pm1} \). Now different local representations \( \rho \) of the algebra \( \alpha BMW_n \) give different integrable spin chain models with Hamiltonians \( \rho(\mathcal{H}) \) which in particular possess \( U_q(osp(N|K)) \) symmetries for some \( N \) and \( K \). So, representations \( \rho \) of the algebra \( \alpha BMW_n \) are related to the spin chain models of
osp type with \( n \) sites and nontrivial boundary conditions. BMW chains (chains based on the
BMW algebras) describe in a unified way spin chains with \( U_q(osp(N|K)) \) symmetries.

The Hamiltonians for Hecke chain models are obtained from Hamiltonians for BMW chain
models by taking the quotient \( \kappa_j = 0 \). These models were considered in [17, 18]. The Hecke
chains (chain models based on the Hecke algebras) describe in a unified way spin chains
with \( U_q(sl(N|K)) \) symmetries. In [17, 18] we investigated the integrable open chain models
formulated in terms of generators of the Hecke algebra (nonaffine case, \( y_1 = 1 \)). For the open
Hecke chains of finite size, the spectrum of the Hamiltonians with free boundary conditions
is determined [17] for special (corner-type) irreducible representations of the Hecke algebra.
In [18] we investigated the functional equations for the transfer-matrix-type elements of the
Hecke algebra that appeared in the theory of Hecke chains.

We postpone to future publications a construction of the algebra which extends the BMW
algebra by the free algebra with generators labeled by the oscillating Young tableaux (as is
done for the Hecke algebras in [19]).

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