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КОМПЬЮТЕРНЫЕ ТЕХНОЛОГИИ В ФИЗИКЕ

MODELING AND VISUALIZATION OF QUANTUM BIFURCATIONS IN PHASE SPACE

T. V. Gevorgyan [a,1], *G. Yu. Kryuchkyan* [a,b,2]

[a] Institute for Physical Research of the National Academy of Sciences, Ashtarak, Armenia

[b] Yerevan State University, Yerevan

We investigate the quantum-mechanical counterpart of a classical instability in a phase space by the numerical method of quantum trajectories with moving basis. As an application, the model of coupled two oscillators driven by a monochromatic force in the presence of dissipation (intracavity second-harmonic generation) is analyzed. The system of interest is characterized by two bifurcations leading to ranges of instability: the Hopf bifurcation, which connects a steady-state dynamics of the oscillatory modes to a self-pulsing temporal dynamics, and the period-doubling bifurcation. The both two regimes are analyzed in the framework of the semiclassical phase trajectories and the Wigner functions of the oscillatory modes in phase space.

Исследуется проблема неустойчивой динамики с точки зрения квантовой теории в фазовом пространстве в рамках численного метода квантовых траекторий с движущим базисом. В качестве приложения анализируется модель двух связанных диссипативных осцилляторов под действием монохроматической силы (процесс внутрирезонаторной генерации второй гармоники). Эта система характеризуется двумя бифуркациями, определяющими области неустойчивости: бифуркацией Хопфа, которая связывает устойчивую стационарную динамику осцилляторных мод с самопульсирующей временной динамикой, и бифуркацией удвоения периода. Оба эти режима анализируются в рамках полуклассических фазовых траекторий и с помощью функций Вигнера осцилляторных мод.

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INTRODUCTION

What is the quantum-mechanical counterpart of a classical instability and how are bifurcations formed in quantum dynamics? These are important but difficult questions relevant to many phenomena of fundamental interest. These questions arise when we analyze quantum dynamics of nonlinear open system having instability or chaotic behavior in the classical limit. Consideration of these questions, including the role of dissipation in the forming of bifurcation in phase space for the simplest nonlinear systems of quantum optics, is the subject of this paper.

[1]E-mail: t_gevorgyan@ysu.am

[2]E-mail: kryuchkyan@ysu.am

We concern here with an outgrowth of unstable dynamics of interacting cavity modes, which exhibits instabilities. As one of the basis nonlinear phenomenon we consider the process of frequency doubling in $\chi^{(2)}$-nonlinear medium placed in a cavity which supports two resonant modes — the fundamental and second-harmonic modes. In this model the fundamental mode at frequency ω_1 is driven by an external driving field, while the second-harmonic mode at frequency ω_2 is excited through the process of frequency doubling $\omega_1 + \omega_1 = \omega_2$. The system is dissipative because the modes are lost through the partially reflecting mirrors of the cavity. The phenomenon is in many ways an ideal choice of system for the study of fundamental aspects of instability and provides the interesting possibility of a quantum description of instabilities and chaos, when the nonlinearities arise from the elementary quantum processes. The system of interest is characterized by the Hopf bifurcation which connects steady-state regime of mode generation to a temporal periodic regime at the critical value $E_{\rm cr}$ of the pump field amplitude [1]. Beyond the critical value $E_{\rm cr}$, the intensities of the fundamental and second-harmonic modes demonstrate self-pulsing temporal behavior which depends on initial conditions. Relatively far from the Hopf bifurcation, the system exhibits a period-doubling instability.

In a statistical description of open quantum systems, one is led to the master equation, which gives the dynamics of the reduced operator of the system, obtained by tracing over the bath variables. It is known that the master equation for the intracavity frequency doubling shows steady state after long enough time. This implies that all quantum-mechanical expectations reach stable constant values in contrast to the classical limit, where, as we noted, dynamics of the fundamental and second-harmonic modes does not necessarily have a steady state and displays self-pulsing.

There have been many studies of intracavity SHG in various quantum context. We restrict our consideration only to the Wigner functions, that is a distribution in phase space, of the fundamental and second-harmonic modes. Some results in this line have been obtained [2] on the basis of quantum-jump simulation method in deep quantum regime and in the vicinity of Hopf bifurcation. Expanding these results to the above bifurcations range proves to be difficult because in the Fock basis the dimension of the effective Hilbert space is equal to the product of the photon numbers in both modes and becomes very large.

In this paper, we present a new contribution to this problem of «quantum instability» analyzing SHG by the method of quantum state diffusion with a moving basis (MQSD). This method has been proposed in [3] and its advantages for computations were demonstrated in [4]. Using this approach, we expand our results [2] to the above Hopf-bifurcation range on the one hand, and consider also more interesting range of stronger driving field, where the system in classical limit exhibits critical phenomenon of period doubling, on the other.

1. QUANTUM MANIFESTATION OF THE BIFURCATIONS ON THE WIGNER FUNCTIONS

At first, we shortly describe the application of MQSD method for the numerical simulation of Wigner function using the standard definition based on the density matrix. We have for each of the modes

$$W_i(\alpha) = \frac{1}{\pi^2} \int d^2\gamma \, \mathrm{Tr}\,(\rho_i D(\gamma)) \exp\,(\gamma^* \alpha - \gamma \alpha^*), \tag{1}$$

where the reduced density operators for each of the modes are constructed by tracing over the other mode $\rho_{1(2)} = \mathrm{Tr}_{1(2)}(\rho)$ through two-mode density matrix and $|\psi_\xi\rangle$ is a stochastic vector of an individual trajectory

$$\rho = \sum_\xi |\psi_\xi\rangle\langle\psi_\xi|. \tag{2}$$

The MQSD method is very effective for numerical simulation of quantum trajectories. However, the problem is that in this approach the two-mode state vector $|\psi_\xi(t)\rangle$ is expressed in the individual basis depending on the realization. It creates additional definite difficulties for calculation of the density matrix at each time of interest in formula (2), which contains the averaging on the ensemble of all possible states. From the practical point of view, it is useful to operate with state vector $|\psi_\xi(t)\rangle$ reduced to basis which is the same for all realizations of the stochastic process $\xi(t)$. Thus, we express the density operators as

$$\rho_i(t) = \sum_{nm} \rho_{nm}^{(i)}(t)|\sigma, n\rangle\langle\sigma, m| \tag{3}$$

in the basis of excited Fock states with an arbitrary coherent amplitude σ. It gives for the Wigner function (1)

$$W_i(\alpha + \sigma) = \sum_{nm} \rho_{nm}^{(i)} W_{nm}(r, \theta), \tag{4}$$

where W_{nm} are the Fourier coefficients of the Wigner function [5].

Below we shall give the results of numerical analysis of the model in the moving basis using expansion of the state vector $|\psi_\xi(t)\rangle$ in the basis of excited coherent states of two modes as

$$|\psi_\xi(t)\rangle = \sum a_{nm}^\xi(\alpha_\xi, \beta_\xi)|\alpha_\xi, n\rangle_1|\beta_\xi, m\rangle_2, \tag{5}$$

where

$$|\alpha, n\rangle_1 = D_1(\alpha)|n\rangle_1, \quad |\beta, m\rangle_2 = D_2(\beta)|n\rangle_2 \tag{6}$$

are the excited coherent states centered on the complex amplitude $\alpha = \langle a_1\rangle$, $\beta = \langle a_2\rangle$. Here $|n\rangle_1$ and $|m\rangle_2$ are Fock's number states of the fundamental and second-harmonic modes, and D_1 and D_2 are the coherent states displacement operators

$$D_i(\alpha) = \exp{(\alpha a_i^+ + \alpha^* a_i)}. \tag{7}$$

Below the quantum-mechanical origin of the bifurcations is analyzed with the use of Wigner functions, which are calculated on the basis of MQSD for second-harmonic generation.

The interaction Hamiltonian is equal to

$$H_{\mathrm{int}} = \imath\hbar(E a_1^+ - E^* a_1) + \imath\hbar(a_1^{+2} a_2 - a_1^2 a_2^+), \tag{8}$$

where a_1, a_2 are the operators of the modes ω_1 and ω_2, respectively; k is the coupling coefficient between the modes, which is proportional to $\chi^{(2)}$; E is the complex amplitude of the pump field.

In Figs. 1 and 2 we demonstrate both the semiclassical trajectories and the Wigner functions in phase space for the mode ω_2 and for two regimes of the system: (1) in the vicinity of

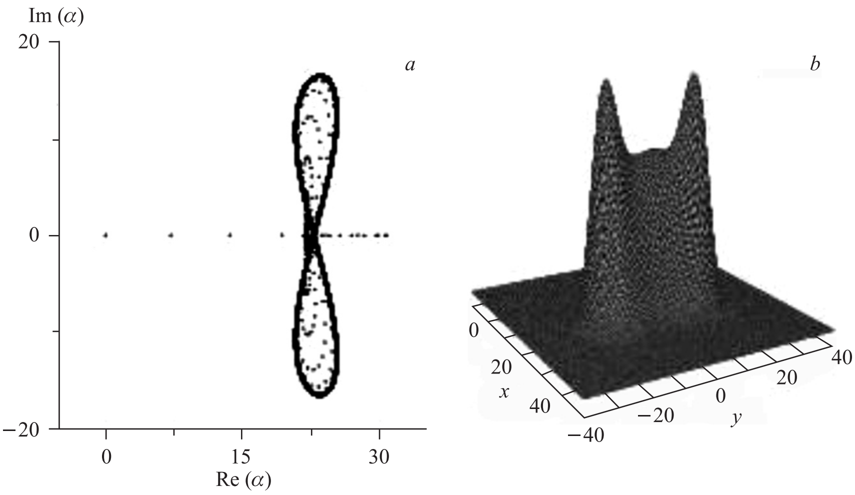

Fig. 1. Semiclassical trajectory (*a*) and the Wigner function (*b*) in the vicinity of Hopf bifurcation for the mode ω_2 and for the parameters $\gamma_1 = \gamma_2 = \gamma,\ \chi/\gamma = 0.1, E/E_{\rm cr} = 1.2, E_{\rm cr}/\gamma = 60$

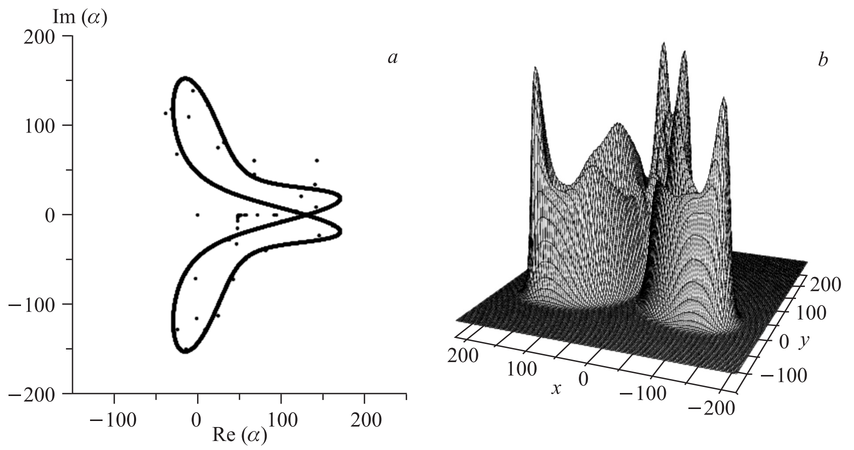

Fig. 2. Semiclassical trajectory (*a*) and the Wigner function (*b*) in the vicinity of the critical point of period doubling for the mode ω_2 and for the parameters $\gamma_1 = \gamma_2 = \gamma,\ \chi/\gamma = 0.1,\ E/E_{\rm cr} = 10,$ $E_{\rm cr} = 60$

the Hopf bifurcation $E \simeq E_{\rm cr}$, $E_{\rm cr} = (2\gamma_1 + \gamma_2)(2\gamma_2(\gamma_1 + \gamma_2)/k^2)$, γ_1, γ_2 are the damping rates of the modes; (2) in the vicinity of the critical point of the period doubling E_d. As we see, the contour plots of both Wigner functions in phase plane generally coincide with the corresponding semiclassical trajectories. It is well known that Hopf bifurcation is determined by the instability of phase variables. Indeed, Fig. 1, *b* describes this situation; the second-harmonic mode initially prepared in vacuum states with one-hump Gaussian Wigner function acquires two-hump structure due to spontaneous symmetry breaking in the bifurcation range. It is natural to connect the occurrence of these sides with unstable dynamics of the phases of modes in semiclassical limit of intracavity second-harmonic generation. Relatively far from the Hopf bifurcation that is realized for more intensive driving forces, the system exhibits the critical phenomenon of period doubling. The numerical calculations based on MQSD are presented in Fig. 2, *b*. As we see, in the vicinity of the period-doubling critical point the Wigner function displays four-hump structures that correspond to four values of phases in semiclassical dynamics of system in the unstable range of period doubling.

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