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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

THE $N = 2$ GAUSS–BONNET FROM CONFORMAL SUPERGRAVITY

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In a recent paper [1], we constructed a novel class of higher derivative invariants in 4D $N = 2$ supergravity that included, as a special case, the supersymmetric Gauss–Bonnet invariant. Here we give a brief description of these results and highlight the potential applications.

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The low energy effective actions derived from string theory generically involve higher derivative contributions in the Riemann tensor. Although usually suppressed by powers of the Planck scale relative to the leading Einstein term, these contributions can prove significant, for example, when calculating contributions to the conformal anomaly or to the classical entropy of black holes (see, e.g., [2]). In dimensions $d \geqslant 4$, there are three $\mathcal{O}(R^2)$ expressions even under parity. The simplest is the square of the Ricci scalar, $\mathcal{R}^2$. The other two can be taken to be the square of the Weyl tensor,

$$L_W = (C_{abcd})^2 = (R_{abcd})^2 - \frac{4}{d-2}(\mathcal{R}_{ab})^2 + \frac{2}{(d-1)(d-2)}\mathcal{R}^2, \tag{1}$$

which transforms covariantly under Weyl transformations, $g_{mn} \to \mathrm{e}^{-2\Lambda}\, g_{mn}$, and the Gauss–Bonnet combination, which we define in d dimensions by

$$L_{\rm GB} = 6\, e_{[a}{}^m e_b{}^n e_c{}^p e_{d]}{}^q R_{mn}{}^{ab} R_{pq}{}^{cd} = (R_{abcd})^2 - 4(\mathcal{R}_{ab})^2 + \mathcal{R}^2. \tag{2}$$

Of these three expressions, the Gauss–Bonnet combination is unique in leading to a ghost-free higher-order propagator for the graviton[2].

Our interest is in finding the $N = 2$ supersymmetric extension of such invariants in four dimensions. (Although the 4D Gauss–Bonnet invariant is a total derivative, we are interested in cases where it is multiplied by a scalar function ϕ'.) The $N = 1$ extensions of both L_W and $L_{\rm GB}$ are well known, while the $N = 2$ Gauss–Bonnet has remained elusive until its discovery recently in a new class of higher derivative invariants [1].

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[2]This was an important motivation for the recent work in five dimensions, where the supersymmetric Gauss–Bonnet combination was constructed [3].

Let us briefly discuss the situation in $N = 1$ supergravity, following the conventions of [4]. The so-called old minimal supergravity is described in a superspace with three torsion superfields: two complex chiral superfields $W_{\alpha\beta\gamma}$ and R and one real superfield G_a. Within this framework, the Einstein–Hilbert term is given as the full superspace volume integral

$$S_{\mathrm{EH}} = -3M_P^2 \int d^4x\, d^2\theta\, d^2\bar{\theta}\, E \tag{3}$$

with explicit factors of the Planck mass M_P for dimensional reasons. Each of the $\mathcal{O}(\mathcal{R}^2)$ invariants can be constructed. The square of the antiself-dual part of C_{abcd}, corresponding to the sum of L_W and the Pontryagin term, is given by the chiral superspace integral

$$\int d^4x\, d^2\theta\, \mathcal{E}\, W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} = \frac{1}{4} \int d^4x\, e \left(C_{abcd}C^{abcd} - C_{abcd}\tilde{C}^{abcd} + \ldots\right). \tag{4}$$

(Following [1], we use conventions, where ε_{abcd} is imaginary.) The difference between the Gauss–Bonnet and Weyl combinations is given by the full superspace integral

$$S_{\mathrm{GB}} - S_W = \int d^4x\, d^2\theta\, d^2\theta\, E(8R\bar{R} + 4G^a G_a) = \int d^4x\, e\, \left(\frac{2}{3}\mathcal{R}^2 - 2(\mathcal{R}_{ab})^2 + \ldots\right). \tag{5}$$

The most convenient $N = 2$ superspace is $SU(2)$ superspace (see [5] for a discussion), which explicitly gauges super-diffeomorphisms, Lorentz transformations, and $SU(2)$ R-symmetry[1]. The supergeometry is described by four fundamental dimension-1 torsion superfields $W_{\alpha\beta}$, $Y_{\alpha\beta}$, S^{ij}, and G_a. The first three are complex and the last is real. Let us attempt to construct invariants. The two obvious candidates are the full superspace and the chiral superspace volume integrals. The chiral volume integral, including an explicit factor of M_P^2 for dimensional reasons, turns out to give the Einstein–Hilbert Lagrangian[2]

$$S_{\mathrm{EH}} = -\frac{M_P^2}{4} \int d^4x\, d^4\theta\, \mathcal{E} + \text{c.c.} \tag{6}$$

The full superspace volume integral vanishes.

The other possibility is to construct scalar Lagrangians out of the torsion superfields. Because only $W_{\alpha\beta}$ is chiral, there is only one apparent chiral scalar of the correct dimension to give an $\mathcal{O}(R^2)$ invariant, $W^{\alpha\beta}W_{\alpha\beta}$. The corresponding component action is the $N = 2$ analogue of (4)

$$\int d^4x\, d^4\theta\, \mathcal{E}\, W^{\alpha\beta} W_{\alpha\beta} = -\frac{1}{2} \int d^4x\, e\, (C^{abcd}C_{abcd} - C^{abcd}\tilde{C}_{abcd}) + \ldots \tag{7}$$

In analogy with (5), we may also consider full superspace integrals involving torsion superfields, but these lead to component Lagrangians of dimension six and higher:

$$\frac{1}{M_P^2} \int d^4x\, d^4\theta\, d^4\bar{\theta} E\, (S^{ij}S_{ij} + Y^{\alpha\beta}Y_{\alpha\beta} + \ldots) \sim \frac{1}{M_P^2} \int d^4x\, e\, (\mathcal{R}^3 + \mathcal{R}\Box\mathcal{R} + \ldots).$$

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[1] This superspace is versatile. It can be used to describe both conformal supergravity as well as the minimal multiplet of $N = 2$ supergravity (corresponding to conformal supergravity coupled to a vector multiplet compensator).

[2] There is a subtlety: one of the auxiliary fields yields the inconsistent equation of motion $e = 0$. This is rectified by coupling to a nonlinear multiplet, a tensor multiplet, or a hypermultiplet.

This would seem to indicate that a supersymmetric version of $S_{\rm GB} - S_W$ does not exist.

The resolution to this mystery is that there exists an extremely nontrivial dimension-2 combination of torsion superfields which turns out to be chiral,

$$\mathbb{T}_0 \equiv -\frac{1}{6}\bar{\mathcal{D}}_{ij}\bar{S}^{ij} - \bar{Y}_{\dot\alpha\dot\beta}\bar{Y}^{\dot\alpha\dot\beta} - \bar{S}_{ij}\bar{S}^{ij}, \quad \bar{\mathcal{D}}^{\dot\alpha i}\mathbb{T}_0 = 0, \tag{8}$$

and which gives the desired supersymmetric completion,

$$\int d^4x\, d^4\theta\, \mathcal{E}\, \mathbb{T}_0 + \text{c.c.} = S_{\rm GB} - S_W. \tag{9}$$

The origin of this term can be understood in a manifestly superconformal framework. Let us first reformulate the Gauss–Bonnet invariant in conformal gravity, where the entire conformal algebra is gauged, including dilatations (with generator $\mathbb{D}$) and conformal boosts (with generator K_a). The covariant derivative is given by $\nabla_a = e_a{}^m(\partial_m - (1/2)\omega_m{}^{ab}M_{ab} - b_m\mathbb{D} - f_m{}^aK_a)$, and the curvatures are constrained so that the connections $\omega_m{}^{ab}$ and $f_m{}^a$ are composite. The connection b_m can always be gauged away, leaving $e_m{}^a$ as the only independent field. Any quantity in the usual formulation of Poincaré gravity can be described by conformal gravity coupled to a scalar compensator field ϕ of Weyl weight-1. For example, the scalar kinetic term $\phi\Box_{\rm c}\phi := \phi\nabla^a\nabla_a\phi$ can be rewritten using Poincaré covariant derivatives as $\phi(\Box + (1/6)\mathcal{R})\phi$. In the gauge $\phi = 1$, this gives the Einstein–Hilbert Lagrangian. We can similarly construct the Gauss–Bonnet invariant in conformal gravity. Begin by rewriting (2) as $L_{\rm GB} = C^{abcd}C_{abcd} - 2\mathcal{R}^{ab}\mathcal{R}_{ab} + (2/3)\mathcal{R}^2$. The Weyl tensor is identified with the conformal curvature $R(M)_{abcd}$. The Ricci terms can be found by taking the covariant combination

$$\Box_{\rm c}\Box_{\rm c}\ln\phi = \Box\Box\ln\phi + \mathcal{D}^a\left(\frac{2}{3}\mathcal{R}\mathcal{D}_a\ln\phi - 2\mathcal{R}_{ab}\mathcal{D}^b\ln\phi\right) + \frac{1}{6}\Box\mathcal{R} - \frac{1}{2}\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{1}{6}\mathcal{R}^2. \tag{10}$$

It follows that within conformal gravity

$$L_\chi = R(M)^{ab\,cd}\,R(M)_{ab\,cd} + 4\,\Box_{\rm c}\Box_{\rm c}\ln\phi \tag{11}$$

reduces (in the gauge $\phi = 1$) to the Gauss–Bonnet invariant, up to an additional $\Box\mathcal{R}$ term.

We know how to supersymmetrize the first term in (11), so let us focus on the second. The easiest treatment is within conformal superspace [6], the supersymmetric analogue of conformal gravity, where the covariant derivative $\nabla_A = (\nabla_{\alpha i}, \bar\nabla^{\dot\alpha i}, \nabla_a)$ has connections for the full superconformal algebra. The supersymmetric version of the expression (11) is simply

$$S_\chi = \int d^4x\, d^4\theta\, \mathcal{E}\left(-W^{\alpha\beta}W_{\alpha\beta} + 2\,\bar\nabla^4\ln\bar\Phi\right) + \text{c.c.}, \tag{12}$$

where $\bar\nabla^4\ln\bar\Phi \equiv (1/48)\bar\nabla_{ij}\bar\nabla^{ij}\ln\bar\Phi$ for some weight-1 antichiral multiplet $\bar\Phi$. In analogy to (10), we can rewrite this superfield in $SU(2)$ superspace as

$$\bar\nabla^4\ln\bar\Phi = \Delta\ln\bar\Phi - \frac{1}{2}\mathbb{T}_0, \tag{13}$$

where Δ is the $SU(2)$ superspace chiral projection operator, and $\mathbb{T}_0$ is defined in (8). The term $\Delta\ln\bar\Phi$ is itself always chiral and vanishes in the gauge, where $\bar\Phi$ is constant. The remaining expression $\mathbb{T}_0$ is independent of $\bar{\Phi}$ (compare with the last three terms of (10)) and is a nontrivial chiral invariant of $SU(2)$ superspace.

Knowledge of the fully supersymmetric Gauss–Bonnet combination resolves a puzzle in the calculation of the entropy for BPS black holes. The original calculation of [2] found a match between the Wald entropy and the microstate counting from string theory by considering the higher derivative terms involving the Weyl Lagrangian coupled to vector multiplets X^I of the form

$$\int d^4x\, d^4\theta\, \mathcal{E}\, \mathcal{F}(X, W^{\alpha\beta}W_{\alpha\beta}). \tag{14}$$

It has been argued [7] that the same entropy matching could be found by replacing the supersymmetric L_W with the simple expression (2). Because the result of [2] required details of the supersymmetric completion of L_W, it was a mystery why the pure Gauss–Bonnet should match. The resolution to this puzzle is that the *difference* between the Weyl and Gauss–Bonnet actions is $\int d^4x\, d^4\theta\, \mathcal{E}\, \mathbb{T}_0$, and (one can show) $\mathbb{T}_0$ does not contribute to the Wald entropy for BPS black holes. This implies that while one should extend (14) to the more general expression

$$\int d^4x\, d^4\theta\, \mathcal{E}\, \mathcal{F}(X, W^{\alpha\beta}W_{\alpha\beta}, \mathbb{T}_0), \tag{15}$$

the contribution of $\mathbb{T}_0$ *to the Wald entropy* in such expressions is actually vanishing.

We emphasize that the full contribution of $\mathbb{T}_0$ to higher derivative actions (15) is generally *neither topological nor vanishing*. In fact, one example of this class was found recently in [8] when applying dimensional reduction to the 5D gauge-gravitational Chern–Simons term [9]

$$S_{\text{vww}} = \int d^5x\, e c_I \left(\sigma^I R(M)_{mn\,ab} R(M)^{mn\,ab} - \frac{i}{2} \varepsilon^{mnpqr} V_m{}^I R(M)_{np}{}^{ab} R(M)_{qr\,ab} + \ldots \right) \tag{16}$$

involving 5D vector multiplets σ^I. Using the results of [1], the resulting 4D invariant is

$$S_{\text{vww}} = \mathrm{i} \int d^4x\, d^4\theta\, \mathcal{E} c_I\, \frac{X^I}{X^0} \left(W^{\alpha\beta}W_{\alpha\beta} + \frac{2}{3} \bar{\nabla}^4 \ln \bar{X}^0 \right) + \text{c.c.}, \tag{17}$$

where X^I is the 4D reduction of the 5D vector multiplet σ^I, and X^0 is the Kaluza–Klein vector multiplet. Taking the natural gauge where the KK vector multiplet is a constant, one recovers a linear combination of L_W and L_{GB} weighted by a factor of $\operatorname{Im} X^I$, along with its supersymmetric completion involving other components of X^I. The second term in (17) proves to be critical for finding the correct action and equations of motion corresponding to the reduction of the 5D action (16).

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