Письма в ЭЧАЯ. 2014. Т. 11, № 7(191). С. 1529–1532

REALIZATIONS OF LIE ALGEBRAS

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Two main approaches to the construction of realizations are discussed. The practical calculation algorithm, based on the method of I. Shirokov, is proposed. A new realization of the Poincaré algebra $p(1,3)$ is presented as an example.

PACS: 02.20.Sv

INTRODUCTION

Realizations of Lie algebras by vector fields are widely applicable, e.g., in integration of ordinary differential equations, in group classification of partial differential equations, theory of differential invariants, general relativity and other physical problems such as classification of gravity fields of a general form under the motion groups and groups of conformal transformations, or quantization based on Noether symmetries, see also [1, 2].

There are many papers devoted to the problem of construction of realizations of Lie algebras, but here we concentrate only on two of them, and for more details the reader is referred to the references in [3] and [4].

The paper is arranged as follows. In the first section we give the definition of realization, explain the direct method and establish connection between linear realizations and representations of Lie algebras. The second and the third sections are devoted to the algebraic method, its properties and construction of a new realization of the Poincaré algebra $p(1,3)$.

1. REALIZATIONS AND REPRESENTATIONS

Let $\mathfrak{g}$ be an n-dimensional Lie algebra over a field $\mathbb{R}$ or $\mathbb{C}$. We denote an open subset of $\mathbb{R}^m$ as M and the Lie algebra of vector fields on it as $\mathrm{Vect}(M)$. In this paper we consider vector fields in a form of linear first-order differential operators with analytical coefficients and the Lie product of vector fields is given by their commutator. The groups of all automorphisms of $\mathfrak{g}$ and its inner automorphisms are denoted by $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g})$, respectively. Then, in contrast to the classical definition of the representation of a Lie algebra, the notion of realization of a Lie algebra is defined as follows.

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A *realization* of a Lie algebra $\mathfrak{g}$ in vector fields on M is a homomorphism $R\colon \mathfrak{g} \to \mathrm{Vect}(M)$. The realization is *faithful* if $\ker R = \{0\}$ and *unfaithful* otherwise.

Note that realizations of a Lie algebra are usually constructed within $\mathrm{Int}(\mathfrak{g})$-equivalence (strong equivalence) or within $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$-equivalence (weak equivalence).

If the realizations with linear coefficients do exist for a given Lie algebra, then the connection between these realizations and representations can be established in the following way.

Let an n-dimensional Lie algebra $\mathfrak{g}$ is realized by n linearly independent vector fields of the form

$$e_i = \sum_{k=1}^{m} \left(\sum_{l=1}^{m} a_{ik}^{l} x_l \right) \partial_k,$$

hereafter $\partial_k = \partial/\partial x_k$, $a_{ik}^l \in \mathbb{R}$ (or $\mathbb{C}$) and $k, l, p, q = 1, 2, \ldots, m$.

Using the matrices

$$A_i = \begin{pmatrix} a_{i1}^1 & a_{i2}^1 & \cdots & a_{im}^1 \\ a_{i1}^2 & a_{i2}^2 & \cdots & a_{im}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}^m & a_{i2}^m & \cdots & a_{im}^m \end{pmatrix}, \quad D = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \vdots \\ \partial_m \end{pmatrix} \quad \text{and} \quad X = (x_1, x_2, \ldots, x_m),$$

we can rewrite the basis elements as $e_i = X A_i D$. Then, as far as $[e_i, e_j] = C_{ij}^k e_i$, the matrices A_i satisfy the same commutation relations $[A_i, A_j] = C_{ij}^k A_i$ and, therefore, form a representation of the initial Lie algebra.

To classify realizations of an n-dimensional Lie algebra $\mathfrak{g}$ in the most direct way, we have to take n linearly independent vector fields of the general form $e_i = \xi^{ik}(x)\partial_k$, $x = (x_1, x_2, \ldots, x_m) \in M$, and require them to satisfy the commutation relations of $\mathfrak{g}$. This method was developed in [4] and results in the necessity to solve a complicated nonlinear system of PDEs and to study the inequivalence of the obtained realizations.

2. ALGEBRAIC METHOD

The alternative purely algebraic method of construction of realizations was developed by I. Shirokov et al. [3]. Below we skip all the geometric details and propose an extraction that gives a short and practical scheme for the direct construction of vector fields.

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ with a complementary part $e_1, \ldots, e_m$, then, before the calculation, the basis of Lie algebra is to be rearranged in the following way: all the subalgebra basis elements should be written (enumerated) first and the arbitrary chosen complementary part should be written (enumerated) after that.

Therefore, the coefficients $\xi_k^i(x)$ of the vector fields can be recovered from the one-forms using the equation $\omega_i^j(x)\xi_k^i(x) = \delta_k^j$.

The differential one-forms are calculated as follows:

$$\omega_i^j(x) = \left(A^{(1)}(x^1) A^{(2)}(x^2) \cdots A^{(i-1)}(x^{i-1}) \right)_i^j,$$

where $i = 2, 3, \ldots, n$, $\omega_1^j = \delta_1^j$.

The matrices A are calculated from the system: $\begin{cases} \dot{A}^{(l)}(t) = -\mathrm{ad}\, e_l A^{(l)}(t), \\ A^{(l)}(0) = I. \end{cases}$

The proposed algorithm produces polynomial coefficients in the partial differential operators only in case matrices of the adjoint representation of the basis elements are nilpotent. So, according to Engel's theorem, in order to obtain the realization in the simplest form the nilpotent linear combinations of basis elements are to be chosen.

The other important fact that allows us to improve realizations is the basis which gives the simplest appearance of the Killing form.

Suppose that, using the algebraic approach for a subalgebra $\mathfrak{h}$, we have constructed a realization given by the following basis elements:

$$e_i = \xi_{i1}(x_1, x_2, \ldots, x_m)\partial_1 + \xi_{i2}(x_1, x_2, \ldots, x_m)\partial_2 + \ldots + \xi_{im}(x_1, x_2, \ldots, x_m)\partial_m,$$

then the number of necessary variables $x_1, \ldots, x_m$ coincides with the dimension of the space complementary to $\mathfrak{h}$, namely, $m = \dim(\mathfrak{g}) - \dim(\mathfrak{h})$. In particular, this means that the transitive realization (the realization that corresponds to zero subalgebra) can always be realized in $n = \dim(\mathfrak{g})$ variables.

The structure of realizations constructed by means of the algebraic method has one more useful property: a realization corresponding to a subalgebra $\mathfrak{h}_1$ can be constructed by means of projection from a realization corresponding to a subalgebra $\mathfrak{h}_2$ if $\mathfrak{h}_2 \subset \mathfrak{h}_1$.

3. EXAMPLE OF NEW REALIZATION

Consider the Poincaré algebra $p(1,3)$ generated by the operators P_α, $J_{\alpha\beta}$ $(\alpha < \beta;\ \alpha, \beta = 0, 1, 2, 3)$ with the following commutation relations:

$$[P_\alpha, P_\beta] = 0, \quad [P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta,$$

$$[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta} J_{\beta\gamma} + g_{\beta\gamma} J_{\alpha\delta} - g_{\alpha\gamma} J_{\beta\delta} - g_{\beta\delta} J_{\alpha\gamma},$$

where $-g_{00} = g_{11} = g_{22} = g_{33} = -1$, $g_{\alpha\beta} = 0$, if $\alpha \neq \beta$; $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3$.

Realization that corresponds to the subalgebra

$$\mathfrak{h} = \langle P_0 + P_3, P_1, P_2, P_0 - P_3 + J_{12}, J_{01} - J_{13}, J_{02} - J_{23} \rangle$$

with the complementary part $\{J_{01} + J_{13}, J_{02} + J_{23}, J_{03}, P_0 - P_3\}$ requires four variables x_1, x_2, x_3, x_4 and has the form

$$\begin{gathered} P_0 = \frac{1}{2}(1 + x_1^2 + x_2^2)e^{x_3}\partial_4, \quad P_1 = -x_1 e^{x_3}\partial_4, \quad P_2 = -x_2 e^{x_3}\partial_4, \\ P_3 = \frac{1}{2}(x_1^2 + x_2^2 - 1)e^{x_3}\partial_4, \quad J_{01} = J'_{01} + x_1\partial_3 - x_2\partial_4, \\ J_{02} = J'_{02} + x_2\partial_3 + x_1\partial_4, \quad J_{03} = J'_{03} + \partial_3, \quad J_{12} = J'_{12} - \partial_4, \\ J_{13} = J'_{13} - x_1\partial_3 + x_2\partial_4, \quad J_{23} = J'_{23} - x_2\partial_3 - x_1\partial_4, \end{gathered}$$

where

$$J'_{01}=\frac{1}{2}\left(1-x_1^2+x_2^2\right)\partial_1-x_1x_2\partial_2,\quad J'_{02}=-x_1x_2\partial_1+\frac{1}{2}\left(1+x_1^2-x_2^2\right)\partial_2,$$
$$J'_{03}=-x_1\partial_1-x_2\partial_2,\quad J'_{12}=x_2\partial_1-x_1\partial_2,$$
$$J'_{13}=\frac{1}{2}\left(1+x_1^2-x_2^2\right)\partial_1+x_1x_2\partial_2,\quad J'_{23}=x_1x_2\partial_1+\frac{1}{2}\left(1-x_1^2+x_2^2\right)\partial_2.$$

Note that the above elements J'_{01}, J'_{02}, J'_{03}, J'_{12}, J'_{13}, J'_{23} form the unique transitive realization of the Lorentz algebra and this realization together with the zero operators $P_i=0$, $i=0,1,2,3$, corresponds to the eight-dimensional subalgebra of $p(1,3)$. It is obvious that this realization can be obtained by means of projection. The realization of $p(1,3)$ corresponding to its (unique!) eight-dimensional subalgebra is necessarily unfaithful, which is caused by the ideal $\langle P_0,P_1,P_2,P_3\rangle$ that is contained in the eight-dimensional subalgebra.

Acknowledgements. The author is grateful to the organizers of the Workshop SQS'2013 for the invitation and hospitality.

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