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GROUP ANALYSIS OF VARIABLE COEFFICIENT GENERALIZED FIFTH-ORDER KdV EQUATIONS

O. Kuriksha [a, 1], *S. Pošta* [b, 2], *O. Vaneeva* [c, 3]

[a] Petro Mohyla Black Sea State University, Mykolaiv, Ukraine

[b] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Prague

[c] Institute of Mathematics of NAS of Ukraine, Kyiv

We carry out group analysis of a class of generalized fifth-order Korteweg–de Vries (KdV) equations with time-dependent coefficients. Admissible transformations, Lie symmetries and similarity reductions of equations from the class are classified exhaustively. A criterion of reducibility of variable coefficient fifth-order KdV equations to their constant coefficient counterparts is derived. Some exact solutions are presented.

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INTRODUCTION

The classical Korteweg–de Vries (KdV) equation and its generalizations model various physical systems, including gravity waves, plasma waves and waves in lattices [1]. In particular, the KdV equation arises in the modeling of one-dimensional plane waves in cold quasi-neutral collision-free plasma propagating along the x-direction under the presence of a uniform magnetic field [2]. It appeared that, when the propagation angle of the wave relative to the external magnetic field becomes critical, the third-order (dispersion) term in the model equation should be replaced by the fifth-order one [3]. Namely, magneto-acoustic waves propagating along this critical direction are modeled by the simplest fifth-order KdV (fKdV) equation (called also quintic KdV equation),

$$u_t + uu_x + \mu u_{xxxxx} = 0, \quad \mu = \mathrm{const}. \tag{1}$$

In [4] Eq. (1) with $\mu = -1$ was shown to describe solitary waves in the nonlinear transmission line of an LC ladder type.

Later, Eq. (1) and its generalizations were studied in a number of papers. Thus, an exact solitary wave solution of Eq. (1) in terms of Jacobi elliptic function cn was found

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[1] E-mail: oksana.kuriksha@gmail.com

[2] E-mail: severin.posta@fjfi.cvut.cz

[3] E-mail: vaneeva@imath.kiev.ua

in [5, 6]. Pulsating multiplet solutions of this equation were examined in [7]. Local conservation laws with the densities u, u^2 and $u^3 + (3/2)(u_{xx})^2$ were indicated therein. Note that the fKdV equation is not integrable by the inverse scattering transform method in contrast to the classical KdV equation [8]. Lie symmetries and the corresponding reductions of (1) to ordinary differential equations (ODEs) were found in [9].

In the last decades there has been a great interest in variable coefficient models that in many cases describe the real world phenomena with more accuracy. Classifications of Lie symmetries are usual tasks in studies of such models. This is due to the fact that Lie symmetries allow one not only to reduce a model PDE to a PDE with fewer number of independent variables or to an ODE but also to derive cases that are potentially more interesting for applications [10].

An attempt of Lie symmetry classification of the generalized fKdV equations with time-dependent coefficients, $u_t + u^n u_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0$, was made in [11]. However, the results presented therein are incorrect in general. In the present paper we perform the correct and complete group classification of the class

$$u_t + uu_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0, \quad \beta \neq 0, \tag{2}$$

where α and β are smooth functions of the variable t. To be able to reduce the number of variable coefficients and to proceed with Lie symmetry analysis in an optimal way, we at first find the admissible transformations [12] (called also allowed [13] or form-preserving [14] ones) in class (2). Classifications of Lie symmetries and similarity reductions are presented in Secs. 2 and 3, respectively.

1. ADMISSIBLE TRANSFORMATIONS

Roughly speaking, an admissible transformation is a triple consisting of two fixed equations from a class and a point transformation linking these equations. The set of admissible transformations of a class of DEs possesses the groupoid structure with respect to the standard composition of transformations [15]. More details and examples on finding and utilizing admissible transformations for fKdV-like equations as well as definitions of different kinds of equivalence groups are given in [16, 17].

To classify admissible transformations in class (2), we suppose that an equation from (2) is connected with an equation from the same class,

$$\tilde u_{\tilde t} + \tilde u\tilde u_{\tilde x} + \tilde\alpha(\tilde t)\tilde u + \tilde\beta(\tilde t)\tilde u_{\tilde x\tilde x\tilde x\tilde x\tilde x} = 0, \tag{3}$$

via a nondegenerate point transformation in the space of variables (t, x, u). Without loss of generality the consideration can be restricted to point transformations of the special form

$$\tilde t = T(t), \quad \tilde x = X^1(t)x + X^0(t), \quad \tilde u = U^1(t, x)u + U^0(t, x), \tag{4}$$

where T, X^i, and U^i, $i = 0, 1$, are arbitrary smooth functions of their variables with $T_t X^1 U^1 \neq 0$. This restriction is true for any subclass of the class of evolution equations of the form $u_t = F(t)u_n + G(t, x, u, u_1, \dots, u_{n-1})$, with $F \neq 0$, and $G_{u_i u_{n-1}} = 0$, $i = 1, \dots, n-1$. Here $n \geqslant 2$, $u_n = \partial^n u/\partial x^n$, F and G are arbitrary smooth functions of their variables [17]. We make the change of variables (4) in (3) and further substitute $u_t = -uu_x - \alpha(t)u - \beta(t)u_{xxxxx}$ to the obtained equation in order to confine it to the manifold defined by (2) in the fifth-order jet space with the independent variables (t, x) and the dependent variable u. Splitting the obtained identity with respect to the derivatives of u leads to the determining equations for the functions T, X^1, X^0, U^1 and U^0. After certain simplifications the equations become

$$\tilde{\beta}T_t = \beta(X^1)^5, \quad U^1_x = 0, \quad U^1T_t = X^1, \quad U^0T_t = X^1_t x + X^0_t,$$
$$U^0_t = -\tilde{\alpha}T_tU^0, \quad U^1_tX^1 - \alpha U^1X^1 + U^1U^0_xT_t + \tilde{\alpha}U^1T_tX^1 = 0.$$

We solve these equations and get the following assertion.

Theorem 1. *The generalized extended equivalence group $\hat{G}^{\sim}$ of class* (2) *is formed by the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = (x + \delta_1)X^1 + \delta_2, \quad \tilde{u} = \frac{1}{T_t}\left(X^1u + X^1_t(x + \delta_1)\right),$$
$$\tilde{\alpha}(\tilde{t}) = \frac{1}{T_t}\left(\alpha(t) - 2\frac{X^1_t}{X^1} + \frac{T_{tt}}{T_t}\right), \quad \tilde{\beta}(\tilde{t}) = \frac{(X^1)^5}{T_t}\beta(t),$$

where $X^1 = \left(\delta_3 \int \exp\left(-\int \alpha(t)\,dt\right)dt + \delta_4\right)^{-1}$, δ_j, $j = 1, \dots, 4$, are arbitrary constants with $(\delta_3, \delta_4) \neq (0, 0)$ and $T = T(t)$ is a smooth function with $T_t \neq 0$.

The entire set of admissible transformations of class (2) *is generated by the transformations from the group $\hat{G}^{\sim}$.*

Using this theorem, we can formulate a criterion of reducibility of variable coefficient fKdV equations to constant coefficient ones.

Theorem 2. *A variable coefficient equation from class* (2) *is reducible to the constant coefficient fKdV equation* (1) *if and only if its coefficients α and β are related by the formula*

$$\beta = \exp\left(-\int \alpha(t)\,dt\right)\left(c_1 \int \exp\left(-\int \alpha(t)\,dt\right)dt + c_2\right)^3, \tag{5}$$

where c_1 and c_2 are arbitrary constants with $(c_1, c_2) \neq (0, 0)$.

Using the equivalence transformation

$$\hat{t} = \int \exp\left(-\int \alpha(t)\,dt\right)dt, \quad \hat{x} = x, \quad \hat{u} = \exp\left(\int \alpha(t)\,dt\right)u \tag{6}$$

from the group $\hat{G}^{\sim}$, we can set the arbitrary element α to the zero value. Indeed, this transformation maps class (2) to its subclass with $\hat{\alpha} = 0$. The arbitrary element $\hat{\beta}$ of a mapped equation is expressed in terms of α and β as $\hat{\beta} = \exp\left(\int \alpha(t)\,dt\right)\beta$. Without loss of generality we can restrict ourselves to the investigation of the class

$$u_t + uu_x + \beta(t)u_{xxxxx} = 0, \tag{7}$$

since all results on symmetries, classical solutions, conservation laws and other related objects for equations from class (2) can be found using the similar results obtained for equations from class (7).

We derive equivalence transformations in class (7) setting $\tilde{\alpha} = \alpha = 0$ in transformations presented in Theorem 1.

Corollary 1. *The usual equivalence group $G^{\sim}_{\alpha=0}$ of class* (7) *consists of the transformations*

$$\tilde{t} = \frac{at+b}{ct+d}, \quad \tilde{x} = \frac{e_2x + e_1t + e_0}{ct+d},$$
$$\tilde{u} = \frac{e_2(ct+d)u - e_2cx - e_0c + e_1d}{\Delta}, \quad \tilde{\beta} = \frac{{e_2}^5}{(ct+d)^3}\frac{\beta}{\Delta},$$

where a, b, c, d, e_0, e_1 and e_2 are arbitrary constants with $\Delta = ad - bc \neq 0$ and $e_2 \neq 0$, the tuple $(a, b, c, d, e_0, e_1, e_2)$ is defined up to a nonzero multiplier and hence without loss of generality we can assume that $\Delta = \pm 1$.

The entire set of admissible transformations of class (7) *is generated by the transformations from the group $G^{\sim}_{\alpha=0}$.*

The transformation components for t, x and u coincide with those obtained for the class of Burgers equations $u_t + uu_x + \beta(t)u_{xx} = 0$ [18] and the class of KdV equations $u_t + uu_x + \beta(t)u_{xxx} = 0$ [19].

Corollary 2. *A variable coefficient equation from class* (7) *is reducible to the constant coefficient fKdV equation* (1) *if and only if $\beta = (c_1t + c_2)^3$, where c_1 and c_2 are arbitrary constants with $(c_1, c_2) \neq (0, 0)$.*

2. LIE SYMMETRIES

To perform the group classification of class (7), we use the classical technique [21]. Namely, we look for Lie symmetry operators of the form $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ that generate one-parametric Lie groups of transformations leaving equations from class (7) invariant. The Lie invariance criterion is written as

$$Q^{(5)}(u_t + uu_x + \beta(t)u_{xxxxx})\big|_{u_t = -uu_x - \beta(t)u_{xxxxx}} = 0, \tag{8}$$

where $Q^{(5)}$ is the fifth prolongation of the operator Q [20, 21]. Equation (8) leads to the determining equations for the coefficients τ, ξ and η, the simplest of which result in

$$\tau = \tau(t), \quad \xi = \xi(t,x), \quad \eta = \eta^1(t,x)u + \eta^0(t,x),$$

where τ, ξ, η^1 and η^0 are arbitrary smooth functions of their variables. The rest of the determining equations have the form

$$\tau\beta_t = (5\xi_x - \tau_t)\beta, \quad \eta^1_x = 2\xi_{xx}, \quad \eta^1_{xx} = \xi_{xxx}, \quad 2\eta^1_{xxx} = \xi_{xxxx},$$
$$\eta^1_x u^2 + (\eta^0_x + \eta^1_t + \eta^1_{xxxxx}\beta)u + \eta^0_t + \eta^0_{xxxxx}\beta = 0,$$
$$(\tau_t - \xi_x + n\eta^1)u + (5\eta^1_{xxxx} - \xi_{xxxxx})\beta - \xi_t + \eta^0 = 0.$$

The latter two equations can be split with respect to the variable u. After splitting we solve those equations that do not involve arbitrary element β and get the general form of the infinitesimal generator,

$$Q = (c_2t^2 + c_1t + c_0)\partial_t + ((c_2t + c_3)x + c_4t + c_5)\partial_x + ((c_3 - c_1 - c_2t)u + c_2x + c_4)\partial_u,$$

where c_i, $i = 0, \dots 5$, are arbitrary constants. The single classifying equation is

$$(c_2t^2 + c_1t + c_0)\beta_t = (3c_2t - c_1 + 5c_3)\beta.$$

If the arbitrary element β varies, then we can split the latter equation with respect to β and its derivative β_t. As a result, we obtain that $c_0 = c_1 = c_2 = c_3 = 0$ and the kernel A^{ker} of the maximal Lie invariance algebras of equations from class (7) coincides with the two-dimensional algebra $\langle\partial_x,\, t\partial_x+\partial_u\rangle$. To exhaustively describe cases of Lie symmetry extension, we should integrate the classifying equation with respect to β up to $G^{\sim}_{\alpha=0}$-equivalence. Since the procedure is quite similar to that of the Lie symmetry classification for KdV equations, $u_t + uu_x + \beta(t)u_{xxx} = 0$, we omit details of calculations and refer the interested reader to [19]. The following assertion is true.

Theorem 3. *The kernel of the maximal Lie invariance algebras of equations from class* (7) *is the two-dimensional Abelian algebra* $A^{\mathrm{ker}} = \langle\partial_x,\, t\partial_x + \partial_u\rangle$. *All possible* $G^{\sim}_{\alpha=0}$*-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by Cases* 1–4 *of Table 1.*

Table 1. **The group classification of the class** $u_t + uu_x + \beta(t)u_{xxxxx} = 0$

No.	$\beta(t)$	Basis of $A^{\max}$
0	$\forall$	$\partial_x,\ t\partial_x + \partial_u$
1	t^{ρ}	$\partial_x,\ t\partial_x + \partial_u,\ 5t\partial_t + (\rho+1)x\partial_x + (\rho-4)u\partial_u$
2	e^{t}	$\partial_x,\ t\partial_x + \partial_u,\ 5\partial_t + x\partial_x + u\partial_u$
3	$(t^2+1)^{3/2}e^{5\nu\arctan t}$	$\partial_x,\ t\partial_x + \partial_u,\ (t^2+1)\partial_t + (t+\nu)x\partial_x + ((\nu-t)u + x)\partial_u$
4	1	$\partial_x,\ t\partial_x + \partial_u,\ \partial_t,\ 5t\partial_t + x\partial_x - 4u\partial_u$

Note. Here ρ and ν are real constants, $\rho \neq 0$. Up to $G^{\sim}_{\alpha=0}$-equivalence we can assume that $\rho \leqslant 3/2$, $\nu \geqslant 0$.

Remark 1. A group classification list for class (2) up to $\hat{G}^{\sim}$-equivalence coincides with the list presented in Table 1.

Remark 2. An equation of the form (2) admits a four-dimensional Lie symmetry algebra if and only if it is point-equivalent to the constant coefficient fKdV equation (1).

In Table 2 we present also the complete list of Lie symmetry extensions for class (2), where arbitrary elements are not simplified by point transformations. This is achieved using the equivalence-based approach [22].

The cases presented in Table 2 give all equations (2) for which the classical method of Lie reduction can be effectively used.

Table 2. **The group classification of class $u_t + uu_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0$ using no equivalence**

No.	$\beta(t)$	Basis of $A^{\max}$
0	$\forall$	$\partial_x,\ T\partial_x + T_t\partial_u$
1	$\lambda T_t(aT+b)^\rho(cT+d)^{3-\rho}$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 5T_t^{-1}(aT+b)(cT+d)\partial_t + [5acT+$ $+ad(\rho+1)+bc(4-\rho)]x\partial_x + (5acxT_t - [5acT+$ $+5\alpha T_t^{-1}(aT+b)(cT+d) + bc(\rho+1) + ad(4-\rho)]u)\partial_u$
2	$\lambda T_t(cT+d)^3\exp\left(\dfrac{aT+b}{cT+d}\right)$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 5T_t^{-1}(cT+d)^2\partial_t+$ $+(5c(cT+d)+\Delta)x\partial_x + [5c^2xT_t+$ $+(\Delta - 5(cT+d)(c+\alpha(cT+d)T_t^{-1}))u]\partial_u$
3	$\lambda T_t\exp\left[5\nu\arctan\left(\dfrac{aT+b}{cT+d}\right)\right]\mathcal{H}^3$	$\partial_x,\ T\partial_x + T_t\partial_u,\ T_t^{-1}\mathcal{H}^2\partial_t + [a(aT+b)+$ $+c(cT+d)+\nu\Delta]x\partial_x + [(a^2+c^2)xT_t - (a(aT+b)+$ $+c(cT+d) - \nu\Delta + \alpha T_t^{-1}\mathcal{H}^2)u]\partial_u$
4a	λT_t	$\partial_x,\ T\partial_x + T_t\partial_u,\ T_t^{-1}(\partial_t - \alpha u\partial_u),\ 5TT_t^{-1}\partial_t + x\partial_x-$ $-(4+5TT_t^{-1}\alpha)u\partial_u$
4b	$T_t(cT+d)^3$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 5T_t^{-1}(cT+d)\partial_t + 4cx\partial_x-$ $-(c+5T_t^{-1}(cT+d)\alpha)u\partial_u,\quad T_t^{-1}(cT+d)^2\partial_t+$ $+c(cT+d)x\partial_x + [c^2xT_t-$ $-(cT+d)(c+T_t^{-1}(cT+d)\alpha)u]\partial_u$

Note. Here a, b, c, d, λ, ν, and ρ are arbitrary constants with $\lambda \neq 0$ and $\rho \neq 0, 3$, $\Delta = ad - bc \neq 0$; $\mathcal{H} = \sqrt{(aT+b)^2 + (cT+d)^2}$. The function $\alpha(t)$ is arbitrary in all cases, $T = \int \exp\left(-\int \alpha(t)\,dt\right)dt$.

3. LIE SYMMETRY REDUCTIONS

Lie symmetries provide one with the powerful tool for finding solutions of nonlinear PDEs reducing them to PDEs with fewer number of independent variables or even to ODEs. If a $(1+1)$-dimensional PDE admits a Lie symmetry operator, $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$, then the ansatz reducing this PDE to an ODE is found as a solution of the invariant surface condition $Q[u] := \tau u_t + \xi u_x - \eta = 0$ [20,21]. In practice, one has to solve the corresponding characteristic system $dt/\tau = dx/\xi = du/\eta$. To get inequivalent reductions, one should use subalgebras from an optimal system (see Subsec. 3.3 in [20]).

We have constructed optimal systems of one-dimensional subalgebras for all the maximal Lie invariance algebras presented in Table 1. The results are summarized in Table 3.

The reductions with respect to the subalgebra $\mathfrak{g}$ lead to constant solutions only. The reduction with respect to the subalgebra $\mathfrak{g}_{4.3}$ is not presented since it coincides with that performed using $\mathfrak{g}_{1.1}$ for $\rho = 0$. Other reductions are listed in Table 4.

Solving the first-order reduced equation from Table 4 and subsequently applying to it transformation (6), we get a "degenerate" solution of Eq. (2),

$$u = \frac{x+b}{\int \exp\left(-\int \alpha(t)\,dt\right)dt + a}\exp\left(-\int \alpha(t)\,dt\right),$$

which is valid for any smooth function α. Here a and b are arbitrary constants.

Table 3. **Optimal systems of one-dimensional subalgebras of $A^{\max}$ presented in Table 1**

Case	Optimal system
0	$\mathfrak{g}=\langle\partial_x\rangle,\quad \mathfrak{g}^a=\langle(t+a)\partial_x+\partial_u\rangle$
$1_{\rho\neq-1,4}$	$\mathfrak{g}=\langle\partial_x\rangle,\quad \mathfrak{g}^\sigma=\langle(t+\sigma)\partial_x+\partial_u\rangle,\quad \mathfrak{g}_{1.1}=\langle 5t\partial_t+(\rho+1)x\partial_x+(\rho-4)u\partial_u\rangle$
$1_{\rho=-1}$	$\mathfrak{g}=\langle\partial_x\rangle,\quad \mathfrak{g}^\sigma=\langle(t+\sigma)\partial_x+\partial_u\rangle,\quad \mathfrak{g}^a_{1.2}=\langle t\partial_t+a\partial_x-u\partial_u\rangle$
2	$\mathfrak{g}=\langle\partial_x\rangle,\quad \mathfrak{g}^0=\langle t\partial_x+\partial_u\rangle,\quad \mathfrak{g}_2=\langle 5\partial_t+x\partial_x+u\partial_u\rangle$
3	$\mathfrak{g}=\langle\partial_x\rangle,\quad \mathfrak{g}_3=\langle(t^2+1)\partial_t+(t+\nu)x\partial_x+(x+(\nu-t)u)\partial_u\rangle$
4	$\mathfrak{g}=\langle\partial_x\rangle,\quad \mathfrak{g}_{4.1}=\langle\partial_t\rangle,\quad \mathfrak{g}^\sigma_{4.2}=\langle\sigma\partial_t+t\partial_x+\partial_u\rangle,\quad \mathfrak{g}_{4.3}=\langle 5t\partial_t+x\partial_x-4u\partial_u\rangle$

Note. Here a is a real constant, $\sigma\in\{-1,0,1\}$. Up to $G^{\sim}_{\alpha=0}$-equivalence we can assume that $\rho\leqslant 3/2$, $\nu\geqslant 0$.

Table 4. **Similarity reductions of the equations** $u_t+uu_x+\beta(t)u_{xxxxx}=0$

Case	$\mathfrak{g}$	ω	Ansatz, $u=$	Reduced ODE
0	$\mathfrak{g}^a$	t	$\varphi(\omega)+\dfrac{x}{t+a}$	$(\omega+a)\varphi'+\varphi=0$
$1_{\rho\neq-1,4}$	$\mathfrak{g}_{1.1}$	$xt^{-\frac{\rho+1}{5}}$	$t^{\frac{\rho-4}{5}}\varphi(\omega)$	$\varphi'''''+\left(\varphi-\dfrac{\rho+1}{5}\omega\right)\varphi'+\dfrac{\rho-4}{5}\varphi=0$
$1_{\rho=-1}$	$\mathfrak{g}^a_{1.2}$	$x-a\ln t$	$t^{-1}\varphi(\omega)$	$\varphi'''''+(\varphi-a)\varphi'-\varphi=0$
2	$\mathfrak{g}_2$	$x\exp\left(-\dfrac{1}{5}t\right)$	$\exp\left(\dfrac{1}{5}t\right)\varphi(\omega)$	$\varphi'''''+\left(\varphi-\dfrac{1}{5}\omega\right)\varphi'+\dfrac{1}{5}\varphi=0$
3	$\mathfrak{g}_3$	$\dfrac{x\,\mathrm{e}^{-\nu\arctan t}}{\sqrt{t^2+1}}$	$\dfrac{\mathrm{e}^{\nu\arctan t}}{\sqrt{t^2+1}}\varphi(\omega)+\dfrac{xt}{t^2+1}$	$\varphi'''''+(\varphi-\nu\omega)\varphi'+\nu\varphi+\omega=0$
4.1	$\mathfrak{g}_{4.1}$	x	$\varphi(\omega)$	$\varphi'''''+\varphi\varphi'=0$
4.2	$\mathfrak{g}^\sigma_{4.2}$	$x\pm\dfrac{t^2}{2}$	$\varphi(\omega)\mp t$	$\varphi'''''+\varphi\varphi'\mp 1=0$

Note. Here a is an arbitrary constant.

Using equivalence transformations it is possible to construct an exact solution for Eqs. (2) that are reducible to their constant coefficient counterparts, i.e., whose coefficients are related by (5). We take the known solution in terms of the Jacobi elliptic function cn from [6] for Eq. (1) and get the exact solution

$$u=\frac{\dfrac{105}{16}a\,\mathrm{cn}^4\left\{\dfrac{\sqrt{2}}{4}a^{1/4}\left[\dfrac{x+d}{Z}-\dfrac{21}{8}a\displaystyle\int\dfrac{\exp\left(-\int\alpha(t)\,dt\right)}{Z^2}\,dt\right]+b;\dfrac{\sqrt{2}}{2}\right\}+c_1(x+d)}{\exp\left(\int\alpha(t)\,dt\right)Z}$$

for the variable coefficient fKdV equation,

$$u_t+uu_x+\alpha(t)u-\exp\left(-\int\alpha(t)\,dt\right)Z^3u_{xxxxx}=0,$$

where $Z = c_1 \int \exp\left(-\int \alpha(t)\,dt\right) dt + c_2$, a is a positive constant; c_1, c_2, b and d are arbitrary constants with $(c_1, c_2) \neq (0, 0)$.

CONCLUSIONS

In the present paper, the group classification problem for class (2) of variable coefficient fKdV equations, which appear in various gravity and plasma wave models, is completely solved. The use of the generalized extended equivalence group $\hat{G}^{\sim}$ has allowed us to present the classification list in a rather simple form (Table 1). For the sake of convenience in further applications, we also write down the classification list extended by equivalence transformations (Table 2). The Lie symmetry algebra of an equation from class (2) is of maximal dimension (which is equal to four) if this equation has constant coefficients or is point-equivalent to one with constant coefficients.

One-dimensional subalgebras of the Lie symmetry algebras admitted by equations from class (2) are classified in Table 3 and all inequivalent reductions with respect to such subalgebras are summarized in Table 4. Performed reductions can be used for the construction of exact and/or numerical solutions. Examples of such constructions were given in [16] for the generalized Kawahara equations. Two simple solutions are also constructed in the present paper.

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