Письма в ЭЧАЯ. 2015. Т. 12, № 4(195). С. 787–810

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

APPROXIMATE ARBITRARY κ-STATE SOLUTIONS OF THE DIRAC EQUATION WITH THE SCHIÖBERG AND MANNING–ROSEN POTENTIALS WITHIN THE COULOMB-LIKE, YUKAWA-LIKE AND GENERALIZED TENSOR INTERACTIONS

A. N. Ikot [a,1], *H. Hassanabadi* [b],

H. P. Obong [a], *H. Mehraban* [c], *B. H. Yazarloo* [c]

[a] Theoretical Physics Group, Department of Physics, University of Port Harcourt, CHOBA, Port Harcourt, Nigeria

[b] Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran

[c] Physics Department, Semnan University, Semnan, Iran

The effects of the Coulomb-like tensor (CLT), Yukawa-like tensor (YLT) and generalized tensor (GLT) interactions are investigated in the Dirac theory with the Schiöberg and Manning–Rosen potentials within the framework of spin and pseudospin symmetries using the Nikiforov–Uvarov method. The bound state energy spectra and the radial wave functions have been approximately obtained in the case of spin and pseudospin symmetries. We have also reported some numerical results and figures to show the effects of these tensor interactions.

В работе исследуются эффекты кулонподобного, юкаваподобного и обобщенного тензорных взаимодействий в теории Дирака с потенциалами Шеберга и Маннинга–Розена в рамках спиновой и псевдоспиновой симметрий с помощью метода Никифорова–Уварова. Получены приближенные спектры энергии связанного состояния и радиальные волновые функции в случае спиновой и псевдоспиновой симметрий. Также приводятся некоторые численные результаты и рисунки, чтобы проиллюстрировать эффекты этих тензорных взаимодействий.

PACS: 03.65Ge; 03.65Pm; 03.65Db

INTRODUCTION

Within the framework of the Dirac theory, the spin symmetry occurs when the difference of the potential between the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant, that is, $\Delta(r) = V(r) - S(r) = \text{const}$. On the other hand, the pseudopsin symmetry arises when the sum of the potential of the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant,

[1] E-mail: ndemikotphysics@gmail.com

that is, $\Sigma(r) = V(r) + S(r) = \text{const}$ [1–6]. The solutions of the Dirac equation under pseudospin and spin symmetries with a number of potential models have been investigated by many researchers. These potentials include the Manning–Rosen [7], Eckart [8], Hylleraas [9], Deng–Fan [10], Möbious square [11], Tietz [12], hyperbolical [13], Yukawa and inversely quadratic Yukawa [14, 15] potentials. The spin and pseudospin symmetries under various phenomenological potentials have been investigated using various methods, such as the Nikiforov–Uvarov (NU) method [16], supersymmetric quantum mechanics (SUSYQM) [17], and others [18]. On the other hand, we are now almost sure that the spin and pseudospin symmetries of the Dirac equation play a significant role in nuclear and hadronic spectroscopy [19, 20]. The tensor interaction has attracted a great attention as it removes the degeneracy between the doublets [20]. In most of studies, due to the mathematical structure of the problem, the tensor interaction is considered as the Coulomb-like [19, 20] or Cornell interaction. Hassanabadi et al. were the first who introduced the Yukawa tensor interaction [21]. The investigation has shown that tensor interaction removes the degeneracy between two states in the pseudospin and spin doublets. The effect of tensor coupling under spin and pseudospin symmetries has been studied only for the Coulomb-like interaction until recently that Hassanabadi et al. [21] introduced the Yukawa tensor interaction.

In the present study, we obtain the approximate analytical solutions of the Dirac equation for the scalar and vector Schiöberg and Manning–Rosen potentials together with the Coulomb-like tensor (CLT), Yukawa-like tensor (YLT) and generalized tensor (GLT) potentials within the framework of spin and pseudospin symmetry limits.

The paper is organized as follows. In Sec. 1, we review the NU method. Section 2 is devoted to the Dirac equation for spin and pseudospin symmetries. We present the solutions of the Dirac equation under the Coulomb tensor interaction in Sec. 3. Solutions of the Dirac equation under the Yukawa tensor interaction are presented in Secs. 4 and 5. Special case of the potential is discussed in Sec. 6. Finally, we give a brief conclusion.

1. THE NIKIFOROV–UVAROV METHOD

The NU method can solve the second-order differential equation of the form [16]:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of the second degree, and $\tilde{\tau}(s)$ is the first-degree polynomial. To make the application of the NU method simpler and more direct, we introduce a more compact presentation of the idea. In order to do this, we rewrite Eq. (1) as follows [22]:

$$\psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1 - c_3 s)}\right)\psi_n'(s) + \left(\frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1 - c_3 s)^2}\right)\psi_n(s) = 0, \tag{2}$$

in which

$$\psi_n(s) = \phi(s) y_n(s). \tag{3}$$

Comparing Eq. (1) with Eq. (2), we obtain the following identifications:

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1 - c_3 s), \quad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \tag{4}$$

Following the NU method, we obtain the following required parameters:

(i) The relevant constant

$$c_4 = \frac{1}{2}(1-c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3), \quad c_6 = c_5^2 + \xi_1,$$
$$c_7 = 2c_4c_5 - \xi_2, \quad c_8 = c_4^2 + \xi_3, \quad c_9 = c_3c_7 + c_3^2c_8 + c_6,$$
$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \quad c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right), \quad c_{12} = c_4 + \sqrt{c_8},$$
$$c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right). \tag{5}$$

(ii) The essential polynomial functions

$$\pi(s) = c_4 + c_5 s - \left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right], \tag{6}$$

$$k = -(c_7 + 2c_3c_8) - 2\sqrt{c_8c_9}, \tag{7}$$

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2\left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right], \tag{8}$$

$$\tau'(s) = -2c_3 - 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) < 0. \tag{9}$$

(iii) The energy equation

$$c_2 n - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0. \tag{10}$$

(iv) The wave functions

$$\rho(s) = s^{c_{10}}(1 - c_3 s)^{c_{11}}, \tag{11}$$

$$\phi(s) = s^{c_{12}}(1 - c_3 s)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0, \tag{12}$$

$$y_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2c_3 s), \quad c_{10} > -1, \quad c_{11} > -1, \tag{13}$$

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}} (1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s), \tag{14}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$, and $x \in [-1, 1]$ are the Jacobi polynomials with

$$P_n^{(\alpha,\beta)}(1-2s) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1(-n, 1+\alpha+\beta+n; \alpha+1; s), \tag{15}$$

and $N_{n\kappa}$ is the normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function via

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}} (1 - c_3 s)^{c_{13}}\,{}_2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3 s), \tag{16}$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

2. THEORY OF THE DIRAC EQUATION

The Dirac equation for spin-$1/2$ particles moving in an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $U(r)$ in the relativistic unit $(\hbar = c = 1)$ is

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta(M + S(r)) - i\beta\boldsymbol{\alpha} \cdot \hat{r}U(r)]\, \psi(r) = [E - V(r)]\, \psi(r), \tag{17}$$

where E is the relativistic energy of the system; $\mathbf{p} = -i\boldsymbol{\nabla}$ is the three-dimensional momentum operator and M is the mass of the fermionic particle. $\boldsymbol{\alpha}, \beta$ are the 4×4 Dirac matrices given as

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{18}$$

where I is the 2×2 unitary matrix and $\boldsymbol{\sigma}_i$ are the Pauli three-vector matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

The eigenvalues of the spin-orbit coupling operator are $\kappa = \left(j + \frac{1}{2}\right) \succ 0, \kappa = -\left(j + \frac{1}{2}\right) \prec 0$ for the unaligned $j = l - \frac{1}{2}$ and the aligned $j = l + \frac{1}{2}$ spin, respectively. The set $\left(H, K, J^2, J_z\right)$ forms a complete set of conserved quantities. Thus, we can write the spinors as

$$\psi_{n\kappa}(r) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r)\, Y_{jm}^{l}(\theta, \varphi) \\ iG_{n\kappa}(r)\, Y_{jm}^{\tilde{l}}(\theta, \varphi) \end{pmatrix}, \tag{20}$$

where $F_{n\kappa}(r)$, $G_{n\kappa}(r)$ represent the upper and lower components of the Dirac spinors. $Y_{jm}^{l}(\theta, \varphi)$, $Y_{jm}^{\tilde{l}}(\theta, \varphi)$ are the spin and pseudospin spherical harmonics and m is the projection on the z-axis. Using the well-known identities,

$$(\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot (\mathbf{A} \cdot \mathbf{B}), \quad \boldsymbol{\sigma} \cdot \mathbf{p} = \boldsymbol{\sigma} \cdot \hat{r} \left(\hat{r} \cdot \mathbf{p} + i\frac{\boldsymbol{\sigma} \cdot \mathbf{L}}{r}\right), \tag{21}$$

as well as the relations

$$\begin{aligned} (\boldsymbol{\sigma} \cdot \mathbf{L})\, Y_{jm}^{\tilde{l}}(\theta, \varphi) &= (\kappa - 1)\, Y_{jm}^{\tilde{l}}(\theta, \varphi), \\ (\boldsymbol{\sigma} \cdot \mathbf{L})\, Y_{jm}^{l}(\theta, \varphi) &= -(\kappa - 1)\, Y_{jm}^{l}(\theta, \varphi), \\ (\boldsymbol{\sigma} \cdot \hat{r})\, Y_{jm}^{l}(\theta, \varphi) &= -Y_{jm}^{\tilde{l}}\,(\theta, \varphi), \\ (\boldsymbol{\sigma} \cdot \hat{r})\, Y_{jm}^{\tilde{l}}(\theta, \varphi) &= -Y_{jm}^{l}\,(\theta, \varphi), \end{aligned} \tag{22}$$

we find the following two coupled first-order Dirac equations:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right) F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))\, G_{n\kappa}(r), \tag{23}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right) G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))\, F_{n\kappa}(r), \tag{24}$$

where

$$\Delta(r) = V(r) - S(r), \tag{25}$$

$$\Sigma(r) = V(r) + S(r). \tag{26}$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ in Eqs. (23) and (24), we obtain the second-order Schrödinger-like equation

$$\left[\begin{array}{c} \dfrac{d^2}{dr^2} - \dfrac{\kappa(\kappa+1)}{r^2} + \dfrac{2\kappa U(r)}{r} - \dfrac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - \Delta(r)) \\ (M - E_{n\kappa} + \Sigma(r)) + \dfrac{\dfrac{d\Delta(r)}{dr}\left(\dfrac{d}{dr} + \dfrac{\kappa}{r} - U(r)\right)}{(M + E_{n\kappa} - \Delta(r))} \end{array}\right] F_{n\kappa}(r) = 0, \tag{27}$$

$$\left[\begin{array}{c} \dfrac{d^2}{dr^2} - \dfrac{\kappa(\kappa-1)}{r^2} + \dfrac{2\kappa U(r)}{r} + \dfrac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - \Delta(r)) \\ (M - E_{n\kappa} + \Sigma(r)) + \dfrac{\dfrac{d\Sigma(r)}{dr}\left(\dfrac{d}{dr} - \dfrac{\kappa}{r} + U(r)\right)}{(M - E_{n\kappa} + \Sigma(r))} \end{array}\right] G_{n\kappa}(r) = 0, \tag{28}$$

where $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1) = l(l+1)$.

3. PSEUDOSPIN AND SPIN SYMMETRY LIMITS UNDER CLT INTERACTION

In this section, we intend to investigate the Dirac equation with the Schiöberg and Manning–Rosen potentials in the presence of the Coulomb-like tensor interactions.

3.1. Pseudospin Symmetry in the Dirac Equation with CLT Interaction. The pseudospin symmetry occurs in the Dirac equation when $d\Sigma(r)/dr = 0$ or equivalently $\Sigma(r) = C_{\rm ps} = \rm const$. To investigate the approximate analytical solution of the Schiöberg and Manning–Rosen potentials, we define the Manning–Rosen potential [7] and the Schiöberg potential [23], respectively, as

$$\begin{aligned}\tilde{V}_{\rm Sc} = D_0\left(1 - \sigma\coth(\beta r)\right) + D_1\left(1 - \sigma\coth(\beta r)\right)^2 = \\ = D_0\left(\frac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right) + D_1\left(\frac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right)^2,\end{aligned} \tag{29}$$

$$\tilde{V}_{\rm MR} = \frac{\hbar^2}{2Mb^2}\left(\frac{\alpha(\alpha-1)\,{\rm e}^{-4\beta r}}{(1 - {\rm e}^{-2\beta r})^2} - \frac{A\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right), \tag{30}$$

where A and α are two constants and the parameter β characterizes the range of the potential. D_1, σ and $D_2 = (1-\sigma)$, $D_3 = (1+\sigma)$ are some parameters representing the molecular properties [24]. We consider the sum of the scalar and vector potentials as

$$\Delta(r) = \tilde{V}_{\mathrm{Sc}} + \tilde{V}_{\mathrm{MR}} = D_0 \left(\frac{D_2 + D_3 \,\mathrm{e}^{-2\beta r}}{1 - \mathrm{e}^{-2\beta r}} \right) + D_1 \left(\frac{D_2 + D_3 \,\mathrm{e}^{-2\beta r}}{1 - \mathrm{e}^{-2\beta r}} \right)^2 + \\ + \frac{\hbar^2}{2Mb^2} \left(\frac{\alpha(\alpha-1)\,\mathrm{e}^{-4\beta r}}{\left(1 - \mathrm{e}^{-2\beta r}\right)^2} - \frac{A\,\mathrm{e}^{-2\beta r}}{1 - \mathrm{e}^{-2\beta r}} \right) \quad (31)$$

in addition to the Coulomb tensor interaction term [25],

$$U(r) = -\frac{H_c}{r}, \quad r \geqslant R_e, \quad (32)$$

with

$$H = \frac{z_a z_b \,\mathrm{e}^2}{4\pi\varepsilon_0}, \quad (33)$$

where R_e is the Coulomb radius, z_a and z_b denote the charges of the projectile a and the target nuclei b, respectively.

Substituting the above equations into Eq. (28) yields

$$\left\{ \frac{d^2}{dr^2} - \frac{\delta}{r^2} - \varepsilon_{\mathrm{ps}}^2 + \beta_{\mathrm{ps}} \left[D_0 \left(\frac{D_2 + D_3 \,\mathrm{e}^{-2\beta r}}{1 - \mathrm{e}^{-2\beta r}} \right) + D_1 \left(\frac{D_2 + D_3 \,\mathrm{e}^{-2\beta r}}{1 - \mathrm{e}^{-2\beta r}} \right)^2 \right] \right\} G_{n,\kappa}^{\mathrm{ps}}(r) + \\ + \beta_{\mathrm{ps}} \left(\frac{\hbar^2}{2Mb^2} \left[\frac{\alpha(\alpha-1)\,\mathrm{e}^{-4\beta r}}{\left(1 - \mathrm{e}^{-2\beta r}\right)^2} - \frac{A\,\mathrm{e}^{-2\beta r}}{1 - \mathrm{e}^{-2\beta r}} \right] \right) G_{n,\kappa}^{\mathrm{ps}} = 0, \quad (34)$$

where

$$\delta = (\kappa(\kappa-1) + 2\kappa H_c + H_c^2 - H_c) = (\kappa + H_c)(\kappa + H_c - 1) = \eta_\kappa(\eta_\kappa - 1) \to \eta_\kappa = (\kappa + H_c), \quad (35)$$

$$\varepsilon_{\mathrm{ps}}^2 = \left(M + E_{n,\kappa}^{\mathrm{ps}}\right)\left(M - E_{n,\kappa}^{\mathrm{ps}} + C_{\mathrm{ps}}\right), \quad (36)$$

$$\beta_{\mathrm{ps}} = \left(M - E_{n,\kappa}^{\mathrm{ps}} + C_{\mathrm{ps}}\right). \quad (37)$$

It is well known that the above equation cannot be solved exactly due to the centrifugal term r^{-2}. In order to get rid of the centrifugal term, we make use of the following approximation [26]:

$$\frac{1}{r^2} \approx \frac{4\beta^2 \,\mathrm{e}^{-2\beta r}}{\left(1 - \mathrm{e}^{-2\beta r}\right)^2}. \quad (38)$$

Substituting Eq. (38) into Eq. (34) in view of the transformation, $y = \mathrm{e}^{-2\beta r}$, yields

$$\frac{d^2 G_{n,\kappa}^{\mathrm{ps}}}{dr^2} + \frac{(1-s)}{s\,(1-s)} \frac{dG_{n,\kappa}^{\mathrm{ps}}}{dr} + \frac{1}{s^2\,(1-s)^2} \left(-\rho_1^{\mathrm{ps}} s^2 + \rho_2^{\mathrm{ps}} s - \rho_3^{\mathrm{ps}}\right) G_{n,\kappa}^{\mathrm{ps}} = 0, \quad (39)$$

where

$$\rho_1^{\rm ps} = \left(\frac{\varepsilon_{\rm ps}^2}{4\beta^2} + \frac{\beta_{\rm ps} D_0 D_3}{4\beta^2} - \frac{\beta_{\rm ps} D_1 D_3^2}{4\beta^2} - \frac{\beta_{\rm ps}\hbar^2\alpha(\alpha-1)}{8\beta^2 M b^2} - \frac{\beta_{\rm ps}\hbar^2 A}{8\beta^2 M b^2}\right), \tag{40}$$

$$\rho_2^{\rm ps} = \left(\frac{\varepsilon_{\rm ps}^2}{2\beta^2} + \frac{\beta_{\rm ps} D_0 (D_3 - D_2)}{4\beta^2} + \frac{\beta_{\rm ps} D_1 D_2 D_3}{2\beta^2} - \delta + \frac{\beta_{\rm ps}\hbar^2 A}{8\beta^2 M b^2}\right), \tag{41}$$

$$\rho_3^{\rm ps} = \left(\frac{\varepsilon_{\rm ps}^2}{4\beta^2} - \frac{\beta_{\rm ps} D_0 D_2}{4\beta^2} - \frac{\beta_{\rm ps} D_1 D_2^2}{4\beta^2}\right). \tag{42}$$

3.2. Spin Symmetry in the Dirac Equation with CLT Interaction. In the spin symmetry limit case, we use the following scalar, vector and tensor potentials:

$$\Sigma(r) = D_0\left(\frac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right) + D_1\left(\frac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right)^2 + \\ + \frac{\hbar^2}{2Mb^2}\left(\frac{\alpha(\alpha-1)\,{\rm e}^{-4\beta r}}{(1 - {\rm e}^{-2\beta r})^2} - \frac{A\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right), \tag{43}$$

$$\Delta(r) = C_s, \quad U(r) = -\frac{H_c}{r}.$$

Substituting Eq. (43) into Eq. (27) yields,

$$\left\{\frac{d^2}{dr^2} - \frac{\eta_\kappa(\eta_\kappa - 1)}{r^2} - \varepsilon_s^2 - \beta_s\left[D_0\left(\frac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right) + D_1\left(\frac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right)^2\right]\right\} \times \\ \times F_{n,\kappa}^s(r) - \beta_s\left[\frac{\hbar^2}{2Mb^2}\left(\frac{\alpha(\alpha-1)\,{\rm e}^{-4\beta r}}{(1 - {\rm e}^{-2\beta r})^2} - \frac{A\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right)\right] F_{n\kappa}^s(r) = 0, \tag{44}$$

where

$$\gamma = \kappa(\kappa+1) + 2\kappa H_c + H_c + H_c^2 = \\ = (\kappa + H_c)(\kappa + H_c + 1) = \eta_\kappa(\eta_\kappa - 1) \to \eta_\kappa = (\kappa + H + 1), \tag{45}$$
$$\varepsilon_s^2 = \left(M - E_{n,\kappa}^s\right)\left(M + E_{n,\kappa}^s - C_s\right), \quad \beta_s = \left(M + E_{n,\kappa}^s - C_s\right).$$

By using the approximation of Eq. (38) for the centrifugal term in Eq. (44), we obtain the following second-order differential equation, in view of the $z = {\rm e}^{-2\alpha r}$ transformation,

$$\frac{d^2 F_{n,\kappa}^s}{dr^2} + \frac{(1-z)}{z(1-z)}\frac{dF_{n,\kappa}^s}{dr} + \frac{1}{z^2(1-z)^2}\left(-\rho_1^s z^2 + \rho_2^s z - \rho_3^s\right) F_{n,\kappa}^s(z) = 0, \tag{46}$$

where

$$\rho_1^s = \left(\frac{\varepsilon_s^2}{4\beta^2} + \frac{\beta_s D_1 D_3^2}{4\beta^2} - \frac{\beta_s D_0 D_3}{4\beta^2} + \frac{\beta_s\hbar^2\alpha(\alpha-1)}{8\beta^2 M b^2} + \frac{\beta_s\hbar^2 A}{8\beta^2 M b^2}\right), \tag{47}$$

$$\rho_2^s = \left(\frac{2\varepsilon_s^2}{4\beta^2} - \frac{\beta_s D_0 (D_3 - D_2)}{4\beta^2} - \frac{2\beta_s D_1 D_2 D_3}{4\beta^2} - \frac{\beta_s\hbar^2 A}{8\beta^2 M b^2} - \frac{\gamma}{4}\right), \tag{48}$$

$$\rho_3^s = \left(\frac{2\varepsilon_s^2}{4\beta^2} + \frac{\beta_s D_0 D_2}{4\beta^2} + \frac{\beta_s D_1 D_2^2}{4\beta^2}\right). \tag{49}$$

3.3. Pseudospin and Spin Symmetry Solutions with CLT Interaction. We will solve the solutions of Eqs. (39) and (46) by using the parametric generalization of the NU method in the subsequent subsection.

3.3.1. Pseudospin Symmetry Solution with CLT Interaction. Now, comparing Eq. (2) with Eq. (39), we obtain

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1, \quad \xi_1 = \rho_1^{\mathrm{ps}}, \quad \xi_2 = \rho_2^{\mathrm{ps}}, \quad \xi_3 = \rho_3^{\mathrm{ps}}. \tag{50}$$

Other parameters can be obtained from Eq. (5) as

$$\begin{gathered} c_4 = 0, \quad c_5 = -\frac{1}{2}, \quad c_6 = \frac{1}{4} + \rho_1^{\mathrm{ps}}, \quad c_7 = -\rho_2^{\mathrm{ps}}, \\ c_8 = \rho_3^{\mathrm{ps}}, \quad c_9 = \frac{1}{4} + \rho_1^{\mathrm{ps}} + \rho_3^{\mathrm{ps}} - \rho_2^{\mathrm{ps}}, \quad c_{10} = 1 + 2\sqrt{\rho_3^{\mathrm{ps}}}, \\ c_{11} = 2 + 2\left[\sqrt{\frac{1}{4} + \rho_1^{\mathrm{ps}} + \rho_3^{\mathrm{ps}} - \rho_2^{\mathrm{ps}}} + \sqrt{\rho_3^{\mathrm{ps}}}\right], \quad c_{12} = \sqrt{\rho_3^{\mathrm{ps}}}, \\ c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + \rho_1^{\mathrm{ps}} + \rho_3^{\mathrm{ps}} - \rho_2^{\mathrm{ps}}} + \sqrt{\rho_3^{\mathrm{ps}}}\right). \end{gathered} \tag{51}$$

Substituting Eqs. (50) and (51) into Eq. (10) yields

$$\begin{aligned} n^2 + \left(n + \frac{1}{2}\right) + (2n+1)\left(\sqrt{\frac{1}{4} + \rho_1^{\mathrm{ps}} + \rho_3^{\mathrm{ps}} - \rho_2^{\mathrm{ps}}} + \sqrt{\rho_3^{\mathrm{ps}}}\right) - \\ - \rho_2^{\mathrm{ps}} + 2\rho_3^{\mathrm{ps}} + 2\sqrt{\rho_3^{\mathrm{ps}}\left(\frac{1}{4} + \rho_1^{\mathrm{ps}} + \rho_3^{\mathrm{ps}} - \rho_2^{\mathrm{ps}}\right)} = 0. \end{aligned} \tag{52}$$

From Eqs. (14) and (15) the lower component of the wave functions is as follows:

$$\begin{aligned} G_{n,\kappa}^{\mathrm{ps}}(r) = N_{n,\kappa}^{\mathrm{ps}}\, \mathrm{e}^{-2\beta\sqrt{\rho_3^{\mathrm{ps}}}\,r} \times \\ \times \left(1 - \mathrm{e}^{-2\beta r}\right)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \rho_1^{\mathrm{ps}} + \rho_3^{\mathrm{ps}} - \rho_2^{\mathrm{ps}}}} P_n^{\left(2\sqrt{\rho_3^{\mathrm{ps}}},\, 2\sqrt{\frac{1}{4} + \rho_1^{\mathrm{ps}} + \rho_3^{\mathrm{ps}} - \rho_2^{\mathrm{ps}}}\right)}\left(1 - 2\,\mathrm{e}^{-2\beta r}\right) \end{aligned} \tag{53}$$

and the other component of the wave function can be obtained as

$$F_{n,\kappa}^{\mathrm{ps}}(r) = \frac{1}{M - E_{n,\kappa}^{\mathrm{ps}} + C_{\mathrm{ps}}}\left(\frac{d}{dr} - \frac{\kappa}{r} - \frac{H}{r}\right) G_{n,\kappa}^{\mathrm{ps}}(r), \tag{54}$$

where $N_{n,\kappa}^{\mathrm{ps}}$ is the normalization constant and $E_{n,\kappa}^{\mathrm{ps}} \neq M + C_{\mathrm{ps}}$.

3.3.2. Spin Symmetry Solution with CLT Interaction. Applying the same procedure to Eq. (46) for the spin symmetry limits, the energy eigenvalues equation and the corresponding upper wave function of the Dirac theory for the combined generalized Schiöberg and Manning–Rosen potentials in the presence of the Coulomb tensor interaction are obtained as

$$\begin{aligned} n^2 + \left(n + \frac{1}{2}\right) + (2n+1)\left(\sqrt{\frac{1}{4} + \rho_1^{s} + \rho_3^{s} - \rho_2^{s}} + \sqrt{\rho_3^{s}}\right) - \\ - \rho_2^{s} + 2\rho_3^{s} + 2\sqrt{\rho_3^{s}\left(\frac{1}{4} + \rho_1^{s} + \rho_3^{s} - \rho_2^{s}\right)} = 0, \end{aligned} \tag{55}$$

$$F^s_{n,\kappa}(r) = N^s_{n,\kappa}\,\mathrm{e}^{-2\beta\sqrt{\rho^s_3}r}\times$$
$$\times\left(1-\mathrm{e}^{-2\beta r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\rho^s_1+\rho^s_3-\rho^s_2}}\,P_n^{\left(2\sqrt{\rho^s_3},2\sqrt{\frac{1}{4}+\rho^s_1+\rho^s_3-\rho^s_2}\right)}(1-2\,\mathrm{e}^{-2\beta r}), \quad (56)$$

where $N^s_{n,\kappa}$ is the normalization constant.

The other component of the Dirac spinor can be found as

$$G^s_{n,\kappa}(r) = \frac{1}{M+E^s_{n,\kappa}-C_s}\left(\frac{d}{dr}+\frac{\kappa}{r}+\frac{H}{r}\right)F^s_{n,\kappa}(r). \quad (57)$$

4. PSEUDOSPIN AND SPIN SYMMETRY LIMITS UNDER YLT INTERACTION

In the following section, we intend to investigate the Dirac equation with the Schiöberg and Manning–Rosen potentials in the presence of the Yukawa-like tensor interactions. The present Yukawa potential as the tensor term in the Dirac equation also removes the degeneracies in addition to the Coulomb tensor interaction. Therefore, it is necessary to investigate this potential model with the Yukawa potential as tensor interaction term.

4.1. Pseudospin Symmetry in the Dirac Equation with YLT Interaction. The pseudospin symmetry occurs in the Dirac theory as $d\Sigma(r)/dr = 0$ or equivalently $\Sigma(r) = C_{\mathrm{ps}} = \mathrm{const}$. In order to find the approximate analytical solution of the Dirac equation under the pseudospin symmetry limit, we take the difference of the scalar and vector potentials as the combined generalized Schiöberg and Manning–Rosen potentials

$$\Delta(r) = D_0\left(\frac{D_2+D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)+D_1\left(\frac{D_2+D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)^2+$$
$$+\frac{\hbar^2}{2Mb^2}\left(\frac{\alpha(\alpha-1)\,\mathrm{e}^{-4\beta r}}{\left(1-\mathrm{e}^{-2\beta r}\right)^2}-\frac{A\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right) \quad (58)$$

in addition to the Yukawa tensor interaction

$$U(r) = -\frac{V_Y\,\mathrm{e}^{-\beta r}}{r}, \quad (59)$$

where V_Y is the Yukawa parameter. Inserting Eqs. (58) and (59) into Eq. (28) yields

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}-\frac{2\kappa V_Y\,\mathrm{e}^{-\beta r}}{r^2}+\frac{\beta V_Y\,\mathrm{e}^{-\beta r}}{r}+\frac{V_Y\,\mathrm{e}^{-\beta r}}{r^2}-\frac{V_Y^2\,\mathrm{e}^{-2\beta r}}{r^2}\right\}G^{\mathrm{ps}}_{n,\kappa}(r)+$$
$$+\left\{-\varepsilon^2_{\mathrm{ps}}+\beta_{\mathrm{ps}}\left[D_0\left(\frac{D_2+D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)+D_1\left(\frac{D_2+D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)^2+\right.\right.$$
$$\left.\left.+\frac{\hbar^2}{2Mb^2}\left(\frac{\alpha(\alpha-1)\,\mathrm{e}^{-4\beta r}}{\left(1-\mathrm{e}^{-2\beta r}\right)^2}-\frac{A\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)\right]\right\}G^{\mathrm{ps}}_{n,\kappa}(r)=0, \quad (60)$$

where $\varepsilon^2_{\mathrm{ps}} = \left(M+E^{\mathrm{ps}}_{n,\kappa}\right)\left(M-E^{\mathrm{ps}}_{n,\kappa}+C_{\mathrm{ps}}\right)$ and $\beta_{\mathrm{ps}} = \left(M-E^{\mathrm{ps}}_{n,\kappa}+C_{\mathrm{ps}}\right)$.

Again Eq. (60) cannot be solved exactly by any known method because of the centrifugal term $1/r^2$. In order to get the approximate solution of Eq. (60), we use Eq. (38) and the following approximation [27]:

$$\frac{1}{r^2} \approx \frac{4\beta^2\,\mathrm{e}^{-\beta r}}{\left(1-\mathrm{e}^{-2\beta r}\right)^2}. \tag{61}$$

Substituting Eq. (61) into Eq. (60) and applying the transformation $s = \mathrm{e}^{-2\beta r}$, we get

$$\frac{dG^{\mathrm{ps}}_{n,\kappa}}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dG^{\mathrm{ps}}_{n,\kappa}}{ds} + \frac{1}{s^2\,(1-s)^2}\left(-\gamma^{\mathrm{ps}}_1 s^2 + \gamma^{\mathrm{ps}}_2 s - \gamma^{\mathrm{ps}}_3\right) G^{\mathrm{ps}}_{n,\kappa}(s) = 0, \tag{62}$$

where

$$\gamma^{\mathrm{ps}}_1 = \left(\frac{\varepsilon^2_{\mathrm{ps}}}{4\beta^2} + V_Y\left(V_Y + \frac{1}{2}\right) + \frac{\beta_{\mathrm{ps}} D_0 D_3}{4\beta^2} - \frac{\beta_{\mathrm{ps}} D_1 D_3^2}{4\beta^2} - \right.$$
$$\left. - \frac{\beta_{\mathrm{ps}}\hbar^2 A}{8\beta^2 M b^2} - \frac{\beta_{\mathrm{ps}}\hbar^2\alpha(\alpha-1)}{8\beta^2 M b^2}\right), \tag{63}$$

$$\gamma^{\mathrm{ps}}_2 = \left(-\kappa(\kappa-1) - \left(2\kappa - \frac{3}{2}\right)V_Y + \right.$$
$$\left. + \frac{\varepsilon^2_{\mathrm{ps}}}{2\beta^2} + \frac{\beta_{\mathrm{ps}} D_0 (D_3 - D_2)}{4\beta^2} + \frac{\beta_{\mathrm{ps}} D_1 D_2 D_3}{2\beta^2} - \frac{\beta_{\mathrm{ps}}\hbar^2 A}{8\beta^2 M b^2}\right), \tag{64}$$

$$\gamma^{\mathrm{ps}}_3 = \left(\frac{\varepsilon^2_{\mathrm{ps}}}{4\alpha^2} - \frac{\beta_{\mathrm{ps}} D_0 D_2}{4\beta^2} - \frac{\beta_{\mathrm{ps}} D_1 D_2^2}{4\beta^2}\right). \tag{65}$$

4.2. Spin Symmetry in the Dirac Equation with YLT. In the spin symmetry limit case, we use the following scalar, vector and tensor potentials:

$$\Delta(r) = C_s, \quad \Sigma(r) = D_0\left(\frac{D_2 + D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right) +$$
$$+ D_1\left(\frac{D_2 + D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)^2 + \frac{\hbar^2}{2Mb^2}\left(\frac{\alpha(\alpha-1)\,\mathrm{e}^{-4\beta r}}{\left(1-\mathrm{e}^{-2\beta r}\right)^2} - \frac{A\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right), \tag{66}$$

$$U(r) = -\frac{V_Y\,\mathrm{e}^{-\alpha r}}{r}, \tag{67}$$

which transform Eq. (27) into the form

$$\frac{d^2F^s_{n,\kappa}}{dr^2} + \frac{(1-s)}{s(1-s)}\frac{dF^s_{n,\kappa}}{dr} + \frac{1}{s^2(1-s)^2}\left(-\sigma^s_1 s^2 + \sigma^s_2 s - \sigma^s_3\right) F^s_{n,\kappa}(s) = 0, \tag{68}$$

where

$$\sigma_1^s = \left(\frac{\varepsilon_s^2}{4\beta^2} + V_Y\left(V_Y - \frac{1}{2}\right) - \frac{\beta_s D_0 D_3}{4\beta^2} + \right.$$
$$\left. + \frac{\beta_s D_1 D_3^2}{4\beta^2} + \frac{\hbar^2 \beta_s \alpha(\alpha-1)}{8\beta^2 M b^2} + \frac{\hbar^2 \beta_s A}{8\beta^2 M b^2}\right), \quad (69)$$

$$\sigma_2^s = \left(\frac{2\varepsilon_s^2}{4\beta^2} - \kappa(\kappa+1) - \left(2\kappa + \frac{3}{2}\right) V_Y - \right.$$
$$\left. - \frac{\beta_s D_0 (D_3 - D_2)}{4\beta^2} - \frac{\beta_s D_1 D_2 D_3}{2\beta^2} + \frac{\hbar^2 \beta_s A}{8\beta^2 M b^2}\right), \quad (70)$$

$$\sigma_3^s = \left(\frac{\varepsilon_s^2}{4\beta^2} + \frac{\beta_s D_0 D_2}{4\beta^2} + \frac{\beta_s D_1 D_2^2}{4\beta^2}\right). \quad (71)$$

5. PSEUDOSPIN AND SPIN SYMMETRY SOLUTIONS WITH YLT

In this section, we intend to investigate the solutions of Eqs. (63) and (68) for pseudospin and spin symmetry using the parametric generalization of the NU method.

5.1. Pseudospin Symmetry Solution. Comparing Eqs. (63) and (2), we obtain

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1, \quad \xi_1 = \gamma_1^{\mathrm{ps}}, \quad \xi_2 = \gamma_2^{\mathrm{ps}}, \quad \xi_3 = \gamma_3^{\mathrm{ps}} \quad (72)$$

and other parameters are obtained as follows:

$$c_4 = 0, \quad c_5 = -\frac{1}{2}, \quad c_6 = \frac{1}{4} + \gamma_1^{\mathrm{ps}}, \quad c_7 = -\gamma_2^{\mathrm{ps}},$$
$$c_8 = \gamma_3^{\mathrm{ps}}, \quad c_9 = \frac{1}{4} + \gamma_1^{\mathrm{ps}} + \gamma_3^{\mathrm{ps}} - \gamma_2^{\mathrm{ps}},$$
$$c_{10} = 1 + 2\sqrt{\gamma_3^{\mathrm{ps}}}, \quad c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} + \gamma_1^{\mathrm{ps}} + \gamma_3^{\mathrm{ps}} - \gamma_2^{\mathrm{ps}}} + \sqrt{\gamma_3^{\mathrm{ps}}}\right), \quad (73)$$
$$c_{12} = \sqrt{\gamma_3^{\mathrm{ps}}}, \quad c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + \gamma_1^{\mathrm{ps}} + \gamma_3^{\mathrm{ps}} - \gamma_2^{\mathrm{ps}}} + \sqrt{\gamma_3^{\mathrm{ps}}}\right).$$

Substituting Eqs. (72) and (73) into Eq. (10) gives the energy eigenvalues as

$$n(n+1) + \frac{1}{2} + (2n+1)\left(\sqrt{\frac{1}{4} + \gamma_1^{\mathrm{ps}} + \gamma_3^{\mathrm{ps}} - \gamma_2^{\mathrm{ps}}} + \sqrt{\gamma_3^{\mathrm{ps}}}\right) -$$
$$- \gamma_2^{\mathrm{ps}} + 2\gamma_3^{\mathrm{ps}} + 2\sqrt{-\gamma_3^{\mathrm{ps}}\gamma_2^{\mathrm{ps}} + (\gamma_3^{\mathrm{ps}})^2 + \gamma_1^{\mathrm{ps}}\gamma_3^{\mathrm{ps}} + \frac{\gamma_3^{\mathrm{ps}}}{4}} = 0. \quad (74)$$

From Eq. (14) the lower and upper components of the wave function are obtained as follows:

$$G_{n,\kappa}^{\mathrm{ps}}(r) = N_{n,\kappa}^{\mathrm{ps}'} \,\mathrm{e}^{-2\beta\sqrt{\gamma_3^{\mathrm{ps}}}\, r} \times$$
$$\times \left(1 - \mathrm{e}^{-2\beta r}\right)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \gamma_1^{\mathrm{ps}} + \gamma_3^{\mathrm{ps}} - \gamma_2^{\mathrm{ps}}}} P_n^{\left(2\sqrt{\gamma_3^{\mathrm{ps}}},\, 2\sqrt{\frac{1}{4} + \gamma_1^{\mathrm{ps}} + \gamma_3^{\mathrm{ps}} - \gamma_2^{\mathrm{ps}}}\right)}(1 - 2\,\mathrm{e}^{-2\beta r}), \quad (75)$$

$$F^{\mathrm{ps}}_{n,\kappa}(r)=\frac{1}{M-E^{\mathrm{ps}}_{n,\kappa}+C_{\mathrm{ps}}}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G^{\mathrm{ps}}_{n,\kappa}(r), \tag{76}$$

where $E^{\mathrm{ps}}_{n,\kappa}\neq M+C_{\mathrm{ps}}$.

5.2. Spin Symmetry Solution. Also, comparing Eqs. (68) and (2), we obtained

$$c_1=1,\quad c_2=1,\quad c_3=1,\quad \xi_1=\sigma^s_1,\quad \xi_2=\sigma^s_2,\quad \xi_3=\sigma^s_3. \tag{77}$$

From Eq. (5) other parameters are obtained as follows:

$$\begin{gathered}
c_4=0,\quad c_5=-\frac{1}{2},\quad c_6=\frac{1}{4}+\sigma^s_1,\quad c_7=-\sigma^s_2,\quad c_8=\sigma^s_3,\\
c_9=\frac{1}{4}+\sigma^s_1+\sigma^s_3-\sigma^s_2,\quad c_{10}=1+2\sqrt{\sigma^s_3},\\
c_{11}=2+2\left(\sqrt{\frac{1}{4}+\sigma^s_1+\sigma^s_3-\sigma^s_2}+\sqrt{\sigma^s_3}\right),\quad c_{12}=\sqrt{\sigma^s_3},\\
c_{13}=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\sigma^s_1+\sigma^s_3-\sigma^s_2}+\sqrt{\sigma^s_3}\right).
\end{gathered} \tag{78}$$

Substituting Eqs. (77) and (78) into Eq. (10) gives the energy eigenvalues as

$$\begin{aligned}
n(n+1)+\frac{1}{2}+(2n+1)\left(\sqrt{\frac{1}{4}+\sigma^s_1+\sigma^s_3-\sigma^s_2}+\sqrt{\sigma^s_3}\right)-\\
-\sigma^s_2+2\sigma^s_3+2\sqrt{\sigma^s_3\left(\frac{1}{4}+\sigma^s_1+\sigma^s_3-\sigma^s_2\right)}=0.
\end{aligned} \tag{79}$$

Finally, we obtain the upper and lower wave functions as

$$\begin{aligned}
F^s_{n,\kappa}(r)=N^{s'}_{n,\kappa}\,\mathrm{e}^{-2\beta\sqrt{\sigma^s_3}r}\left(1-\mathrm{e}^{-2\beta r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\sigma^s_1+\sigma^s_3-\sigma^s_2}}\times\\
\times P_n^{\left(2\sqrt{\sigma^s_3},2\sqrt{\frac{1}{4}+\sigma^s_1+\sigma^s_3-\sigma^s_2}\right)}(1-2\,\mathrm{e}^{-2\beta r}),
\end{aligned} \tag{80}$$

$$G^s_{n,\kappa}(r)=\frac{1}{M+E^s_{n,\kappa}-C_s}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F^s_{n,\kappa}(r). \tag{81}$$

6. PSEUDOSPIN AND SPIN SYMMETRY LIMITS UNDER GLT INTERACTION

In the following section, we intend to investigate the Dirac equation with the Schiöberg and Manning–Rosen potentials in the presence of GLT interactions. The present GLT potential as the tensor term in the Dirac equation also removes the degeneracies in addition to the Coulomb-like and Yukawa-like tensor interactions. Thus, it is pertinent to investigate this potential under consideration with GLT as interaction term.

6.1. Pseudospin Symmetry in the Dirac Equation with GLT. The pseudospin symmetry occurs in the Dirac theory as $d\Sigma(r)/dr = 0$ or equivalently $\Sigma(r) = C_{\rm ps} = \text{const}$. In order to find the approximate analytical solution of the Dirac equation under the pseudospin symmetry limit, we take the difference of the scalar and vector potentials as the combined Schiöberg and Manning–Rosen potentials

$$\Delta(r) = D_0\left(\frac{D_2 + D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right) + D_1\left(\frac{D_2 + D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)^2 + \\ + \frac{\hbar^2}{2Mb^2}\left[\frac{\alpha(\alpha-1)\,\mathrm{e}^{-4\beta r}}{(1-\mathrm{e}^{-2\beta r})^2} - \frac{A\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right]. \quad (82)$$

In addition, we proposed a novel generalized tensor interaction of the form

$$U(r) = -\left(U_C(r) + U_Y(r)\right), \quad (83)$$

where, $U_C(r)$ and $U_Y(r)$ are the Coulomb-like and Yukawa-like potentials defined as

$$U_C(r) = -\frac{H_c}{r}, \quad U_Y(r) = -V_Y\frac{\mathrm{e}^{-\beta r}}{r}. \quad (84)$$

If we identify $U_C(r)$ as the standard Coulomb potential, the potential parameter H_c is the Coulomb parameter and V_Y is the Yukawa parameter.

Substituting Eq. (84) into Eq. (83), we obtained our proposed Generalized Tensor Interaction (GTI) as

$$U(r) = -\frac{1}{r}\left(H_c + V_Y\,\mathrm{e}^{-\beta r}\right). \quad (85)$$

Substituting the above equations into Eq. (28) yields

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa H_c}{r^2} + \frac{H_c}{r^2} - \frac{H_c^2}{r^2} - \left(M + E_{n,\kappa}^{\rm ps}\right)\left(M - E_{n,\kappa}^{\rm ps} + C_{\rm ps}\right)\right] G_{n,\kappa}^{\rm ps}(r)\times \\ \times\left(-\frac{2V_Y\kappa\,\mathrm{e}^{-\beta r}}{r^2} + \frac{\beta V_Y\,\mathrm{e}^{-\beta r}}{r} + \frac{V_Y\,\mathrm{e}^{-\beta r}}{r^2} - \frac{2H_cV_Y\,\mathrm{e}^{-\beta r}}{r^2} - \frac{V_Y^2\,\mathrm{e}^{-2\beta r}}{r^2}\right) G_{n\kappa}^{\rm ps}(r) + \\ + \left(M - E_{n,\kappa}^{\rm ps} + C_{\rm ps}\right)\left[\begin{matrix} D_0\left(\dfrac{D_2 + D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right) + D_1\left(\dfrac{D_2 + D_3\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)^2 + \\ \dfrac{\hbar^2}{2Mb^2}\left(\dfrac{\alpha(\alpha-1)\,\mathrm{e}^{-4\beta r}}{(1-\mathrm{e}^{-2\beta r})^2} - \dfrac{A\,\mathrm{e}^{-2\beta r}}{1-\mathrm{e}^{-2\beta r}}\right)\end{matrix}\right] G_{n,\kappa}^{\rm ps}(r) = 0. \quad (86)$$

It is well known that the above equation cannot be solved exactly due to the centrifugal term r^{-2}. In order to get rid of the centrifugal term, we make use of Eq. (38) and Eq. (61). By using approximation (38) for $\left(\frac{\kappa(\kappa-1)}{r^2}, \frac{2\kappa H_c}{r^2}, \frac{H_c}{r^2}, \frac{H_c^2}{r^2}, \frac{\beta V_0\,\mathrm{e}^{-\alpha r}}{r} \text{ and } \frac{V_Y^2\,\mathrm{e}^{-2\beta r}}{r^2}\right)$, and approximation (61) for $\left(\frac{2V_Y\kappa\,\mathrm{e}^{-\beta r}}{r^2}, \frac{V_Y\,\mathrm{e}^{-\beta r}}{r^2} \text{ and } \frac{2H_cV_Y\,\mathrm{e}^{-\beta r}}{r^2}\right)$, as well as applying the transformation $s = \mathrm{e}^{-2\beta r}$, we obtain

$$\frac{d^2G_{n,\kappa}^{\rm ps}}{ds^2} + \frac{(1-s)}{s\,(1-s)}\frac{dG_{n,\kappa}^{\rm ps}}{dr} + \frac{1}{s^2\,(1-s)^2}\left[-\chi_1^{\rm ps}s^2 + \chi_2^{\rm ps}s - \chi_3^{\rm ps}\right] G_{n,\kappa}^{\rm ps} = 0, \quad (87)$$

where

$$\chi_1^{\rm ps} = \frac{\varepsilon_{\rm ps}^2}{4\beta^2} + \frac{\beta_{\rm ps}D_0D_3}{4\beta^2} - \frac{\beta_{\rm ps}D_1D_3^2}{4\beta^2} + V_Y\left(V_Y + \frac{1}{2}\right) - \frac{\hbar^2\beta_{\rm ps}\alpha(\alpha-1)}{8Mb^2\beta^2} - \frac{\hbar^2\beta_{\rm ps}A}{8Mb^2\beta^2}, \quad (88)$$

$$\chi_2^{\rm ps} = \frac{2\varepsilon_{\rm ps}^2}{4\beta^2} + \frac{2\beta_{\rm ps}D_1D_2D_3}{4\beta^2} + \frac{\beta_{\rm ps}D_0(D_3-D_2)}{4\beta^2} - \\ - \frac{\hbar^2\beta_{\rm ps}A}{8Mb^2\beta^2} - V_Y\left(2\kappa + 2H_c - \frac{3}{2}\right) - \eta_\kappa(\eta_\kappa - 1), \quad (89)$$

$$\chi_3^{\rm ps} = \frac{\varepsilon_{\rm ps}^2}{4\beta^2} - \frac{\beta_{\rm ps}D_0D_2}{4\beta^2} - \frac{\beta_{\rm ps}D_1D_2^2}{4\beta^2}, \quad (90)$$

$$\varepsilon_{\rm ps}^2 = M^2 - E_{n\kappa}^2 + (M + E_{n\kappa})\,C_{\rm ps}, \quad (91)$$

$$\beta_{\rm ps} = (M - E_{n\kappa} + C_{\rm ps}), \quad (92)$$

$$H^2 + 2\kappa H_c + \kappa(\kappa-1) - H_c = \eta_\kappa(\eta_\kappa - 1) \to \eta_\kappa = (\kappa + H_c). \quad (93)$$

6.2. Spin Symmetry in the Dirac Equation with GLT. In the spin symmetry limit condition, we take the sum potential $\Sigma(r)$ as the hyperbolical potential, the difference potential $\Delta(r)$ as the constant and the tensor potential $U(r)$ as the GTI term. Thus, we have the following:

$$\Sigma(r) = D_0\left(\frac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right) + D_1\left(\frac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right)^2 + \\ + \frac{\hbar^2}{2Mb^2}\left(\frac{\alpha(\alpha-1)\,{\rm e}^{-4\beta r}}{(1 - {\rm e}^{-2\beta r})^2} - \frac{A\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right), \quad \Delta(r) = C_s, \quad (94)$$

$$U(r) = -\frac{1}{r}\left(H_c + V_Y\,{\rm e}^{-\alpha r}\right).$$

Substituting Eq. (94) into Eq. (27) yields

$$\left\{\begin{array}{l} \dfrac{d^2}{dr^2} - \dfrac{\kappa(\kappa+1)}{r^2} - \dfrac{2\kappa H_c}{r^2} - \dfrac{H_c}{r^2} - \dfrac{H_c^2}{r^2} - (M + E_{n\kappa} - C_s)\,(M - E_{n\kappa}) - \\ - (M + E_{n\kappa} - C_s)\left[D_0\left(\dfrac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right) + D_1\left(\dfrac{D_2 + D_3\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right)^2 + \right. \\ \left. + \dfrac{\hbar^2}{2Mb^2}\left(\dfrac{\alpha(\alpha-1)\,{\rm e}^{-4\beta r}}{(1 - {\rm e}^{-2\beta r})^2} - \dfrac{A\,{\rm e}^{-2\beta r}}{1 - {\rm e}^{-2\beta r}}\right)\right] \end{array}\right\} F_{n,\kappa}^s(r) - \\ - \left(\frac{2\kappa V_Y\,{\rm e}^{-\beta r}}{r^2} + \frac{\beta V_Y\,{\rm e}^{-\beta r}}{r} + \frac{V_Y\,{\rm e}^{-\beta r}}{r^2} + \frac{2H_cV_Y\,{\rm e}^{-\beta r}}{r^2} + \frac{V_Y^2\,{\rm e}^{-2\beta r}}{r^2}\right) F_{n\kappa}^s(r) = 0. \quad (95)$$

By using approximation (38) for $\left(\frac{\kappa(\kappa+1)}{r^2}, \frac{2\kappa H_c}{r^2}, \frac{H_c}{r^2}, \frac{H_c^2}{r^2}, \frac{\beta V_Y\,{\rm e}^{-\beta r}}{r} \text{ and } \frac{V_Y^2\,{\rm e}^{-2\beta r}}{r^2}\right)$, and approximation (62) for $\left(\frac{2V_Y\kappa\,{\rm e}^{-\beta r}}{r^2}, \frac{V_Y{\rm e}^{-\beta r}}{r^2} \text{ and } \frac{2H_cV_Y\,{\rm e}^{-\beta r}}{r^2}\right)$, the above second-order differential equation via the transformation $y = \mathrm{e}^{-2\beta r}$ appears as

$$\frac{d^2F^s_{n,\kappa}}{dy^2} + \frac{(1-y)}{y(1-y)}\frac{dF^s_{n,\kappa}}{dy} + \frac{1}{y^2(1-y)^2}\left(-f^s_1 y^2 + f^s_2 y - f^s_3\right)F^s_{n,\kappa}(y) = 0, \tag{96}$$

where

$$f^s_1 = \frac{\varepsilon_s^2}{4\beta^2} + \frac{\beta_s D_1 D_3^2}{4\beta^2} - \frac{\beta_s D_0 D_3}{4\beta^2} + V_Y\left(V_Y - \frac{1}{2}\right) + \frac{\hbar^2\beta_s\alpha(\alpha-1)}{8Mb^2\beta^2} + \frac{\hbar^2\beta_s A}{8Mb^2\beta^2}, \tag{97}$$

$$f^s_2 = \frac{2\varepsilon_s^2}{4\beta^2} - \frac{2\beta_s D_1 D_2 D_3}{4\beta^2} - \frac{\beta_s D_0(D_3 - D_2)}{4\beta^2} + \\ + \frac{\hbar^2\beta_s A}{8Mb^2\beta^2} - V_Y\left(2\kappa + 2H_c + \frac{3}{2}\right) - \Lambda_\kappa(\Lambda_\kappa - 1), \tag{98}$$

$$f^s_3 = \frac{\varepsilon_s^2}{4\beta^2} + \frac{\beta_s D_1 D_2^2}{4\beta^2} + \frac{\beta_s D_0 D_2}{4\beta^2}, \tag{99}$$

$$\varepsilon_s^2 = M^2 - (E^s_{n\kappa})^2 + (E^s_{n\kappa} - M)C_s, \tag{100}$$

$$\beta_s = (M + E^s_{n\kappa} - C_s), \tag{101}$$

$$H^2 + 2\kappa H_c + \kappa(\kappa - 1) + H_c = \Lambda_\kappa(\Lambda_\kappa - 1) \to \Lambda_\kappa = (\kappa + H_c + 1). \tag{102}$$

6.3. Pseudospin and Spin Symmetry Solutions. We will solve Eqs. (87) and (96) by using the parametric generalization of the NU method in the subsequent subsection.

6.4. Pseudospin Symmetry Solution. Now, comparing Eq. (2) with Eq. (89), we obtain

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1, \quad \xi_1 = \chi_1^{\mathrm{ps}}, \quad \xi_2 = \chi_2^{\mathrm{ps}}, \quad \xi_3 = \chi_3^{\mathrm{ps}}. \tag{103}$$

Other parameters can be obtained from Eq. (5) as

$$\begin{gathered} c_4 = 0, \quad c_5 = -\frac{1}{2}, \quad c_6 = \frac{1}{4} + \chi_1^{\mathrm{ps}}, \quad c_7 = -\chi_2^{\mathrm{ps}}, \\ c_8 = \chi_3^{\mathrm{ps}}, \quad c_9 = \frac{1}{4} + \chi_1^{\mathrm{ps}} + \chi_3^{\mathrm{ps}} - \chi_2^{\mathrm{ps}}, \quad c_{10} = 1 + 2\sqrt{\chi_3^{\mathrm{ps}}}, \\ c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} + \chi_1^{\mathrm{ps}} + \chi_3^{\mathrm{ps}} - \chi_2^{\mathrm{ps}}} + \sqrt{\chi_3^{\mathrm{ps}}}\right), \quad c_{12} = \sqrt{\chi_3^{\mathrm{ps}}}, \\ c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + \chi_1^{\mathrm{ps}} + \chi_3^{\mathrm{ps}} - \chi_2^{\mathrm{ps}}} + \sqrt{\chi_3^{\mathrm{ps}}}\right). \end{gathered} \tag{104}$$

Substituting Eqs. (103) and (104) into Eq. (10) yields

$$n^2 + \left(n + \frac{1}{2}\right) + (2n+1)\left(\sqrt{\frac{1}{4} + \chi_1^{\mathrm{ps}} + \chi_3^{\mathrm{ps}} - \chi_2^{\mathrm{ps}}} + \sqrt{\chi_3^{\mathrm{ps}}}\right) - \chi_2^{\mathrm{ps}} + 2\chi_3^{\mathrm{ps}} + \\ + 2\sqrt{\chi_3^{\mathrm{ps}}\left(\frac{1}{4} + \chi_1^{\mathrm{ps}} + \chi_3^{\mathrm{ps}} - \chi_2^{\mathrm{ps}}\right)} = 0. \tag{105}$$

This equation determines the energy eigenvalues. From Eqs. (14) and (15) the lower component of the wave functions is as follows:

$$G^{\mathrm{ps}}_{n,\kappa}(r) = N^{\mathrm{ps}}_{n,\kappa}\,\mathrm{e}^{-2\beta\sqrt{\chi_3^{\mathrm{ps}}}r}\times$$
$$\times\left(1-\mathrm{e}^{-2\beta r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\chi_1^{\mathrm{ps}}+\chi_3^{\mathrm{ps}}-\chi_2^{\mathrm{ps}}}}\,P_n^{\left(2\sqrt{\chi_3^{\mathrm{ps}}},2\sqrt{\frac{1}{4}+\chi_1^{\mathrm{ps}}+\chi_3^{\mathrm{ps}}-\chi_2^{\mathrm{ps}}}\right)}(1-2\,\mathrm{e}^{-2\beta r}) \quad (106)$$

and the other component of the wave function can be obtained as

$$F^{\mathrm{ps}}_{n,\kappa}(r) = \frac{1}{M-E^{\mathrm{ps}}_{n,\kappa}+C_{\mathrm{ps}}}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G^{\mathrm{ps}}_{n,\kappa}(r), \quad (107)$$

where $N^{\mathrm{ps}}_{n,\kappa}$ is the normalization constant and $E^{\mathrm{ps}}_{n,\kappa}\neq M+C_{\mathrm{ps}}$.

6.5. Spin Symmetry Solution. We applied the same procedure of Eq. (87) to Eq. (96) for the spin symmetry limits to avoid repetition. The energy eigenvalues equation and the corresponding upper wave function of the Dirac theory for the Schiöberg and Manning–Rosen potentials in the presence of generalized tensor interaction are obtained as

$$n^2+\left(n+\frac{1}{2}\right)+(2n+1)\left(\sqrt{\frac{1}{4}+f_1^s+f_3^s-f_2^s}+\sqrt{f_3^s}\right)-f_2^s+2f_3^s+$$
$$+2\sqrt{f_3^s\left(\frac{1}{4}+f_1^s+f_3^s-f_2^s\right)}=0, \quad (108)$$

$$F^{s}_{n,\kappa}(r) = N^{s}_{n,\kappa}\,\mathrm{e}^{-2\alpha\sqrt{f_3^s}r}\times$$
$$\times\left(1-\mathrm{e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+f_1^s+f_3^s-f_2^s}}\,P_n^{\left(2\sqrt{f_3^s},2\sqrt{\frac{1}{4}+f_1^s+f_3^s-f_2^s}\right)}(1-2\,\mathrm{e}^{-2\alpha r}). \quad (109)$$

Equation (109) determines the energy eigenvalues of the spin symmetry, and $N^s_{n,\kappa}$ is the normalization constant. The other component of the Dirac spinor can be found as

$$G^{s}_{n,\kappa}(r) = \frac{1}{M+E^{s}_{n,\kappa}-C_s}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F^{s}_{n,\kappa}(r). \quad (110)$$

7. NUMERICAL RESULTS

In this section, we discuss the effect of the tensor interactions on the wave functions and energy of the Dirac equation. In our calculations, we have taken $M=5$, $\beta=0.1$, $D_0=-1$, $D_1=-0.5$, $D_2=-0.3$, $D_3=-0.9$, $\alpha=2$, $b=1$, $A=-0.5$, $C_{\mathrm{ps}}=-5$ for the pseudospin symmetry limit and $M=5$, $\beta=0.1$, $D_0=-1$, $D_1=-0.5$, $D_2=-0.3$, $D_3=0.9$, $\alpha=2$, $b=1$, $A=-0.5$, $C_s=5$ for the spin symmetry limit.

7.1. The Effect of the Coulomb-Like Tensor Interaction. To show the effect of the Coulomb-like Tensor Interaction (CTI) on the energy eigenvalues and the wave functions of the system, we have calculated numerical results for different states both in pseudospin symmetry and spin symmetry limits in Tables 1 and 2, respectively. We can see that there are degeneracies among energy levels in the absence of CTI such as ($1p_{3/2}$, $0f_{5/2}$), ($1s_{1/2}$, $0d_{3/2}$), ... in pseudospin symmetry and ($0p_{3/2}$, $0p_{1/2}$), ($0d_{5/2}$, $0d_{3/2}$), ... in the spin symmetry limits and, when the CTI appears, these degeneracies remove. Figure 1 shows the effect of CTI on the components of the Dirac spinors. The effect of the parameters H and β on the energy of the pseudospin symmetry limits for $1p_{3/2}$, $0f_{5/2}$, $1d_{5/2}$, $0g_{7/2}$ and spin symmetry limits for $0d_{5/2}$, $0d_{3/2}$, $0f_{7/2}$, $0f_{5/2}$ is plotted in Figs. 2 and 3. It is clear that when $H = 0$, ($1p_{3/2}$, $0f_{5/2}$) and ($1d_{5/2}$, $0g_{7/2}$) (($0d_{5/2}$, $0d_{3/2}$) and ($0f_{7/2}$, $0f_{5/2}$)) are degenerated in the pseudospin symmetry (spin symmetry).

Table 1. **The energy of the pseudospin symmetry in the presence and absence of CTI**

$\tilde{l}$	n, κ	State	$E^{\rm ps}_{n,\kappa(H_c=0)}$	$E^{\rm ps}_{n,\kappa(H_c=0.5)}$	$n-1, \kappa$	State	$E^{\rm ps}_{n,\kappa(H_c=0)}$	$E^{\rm ps}_{n,\kappa(H_c=0.5)}$
1	1, –1	$1s_{1/2}$	–4.608653315	–4.600841083	0, 2	$0d_{3/2}$	–4.608653315	–4.618940601
2	1, –2	$1p_{3/2}$	–4.631144681	–4.618940601	0, 3	$0f_{5/2}$	–4.631144681	–4.644655207
3	1, –3	$1d_{5/2}$	–4.658858604	–4.644655207	0, 4	$0g_{7/2}$	–4.658858604	–4.673175985
4	1, –4	$1f_{7/2}$	–4.687088176	–4.673175985	0, 5	$0h_{9/2}$	–4.687088176	–4.700148773

Table 2. **The energy of the spin symmetry in the presence and absence of CTI**

l	n, κ	State	$E^{s}_{n,\kappa(H_c=0)}$	$E^{s}_{n,\kappa(H_c=0.5)}$	n, κ	State	$E^{s}_{n,\kappa(H_c=0)}$	$E^{s}_{n,\kappa(H_c=0.5)}$
1	0, –2	$0p_{3/2}$	4.789457980	4.621780059	0, 1	$0p_{1/2}$	4.789457980	4.925299124
2	0, –3	$0d_{5/2}$	5.027073504	4.925299124	0, 2	$0d_{3/2}$	5.027073504	5.101155481
3	0, –4	$0f_{7/2}$	5.154405031	5.101155481	0, 3	$0f_{5/2}$	5.154405031	5.192262398
4	0, –5	$0g_{9/2}$	5.218681837	5.192262398	0, 4	$0g_{7/2}$	5.218681837	5.236461226

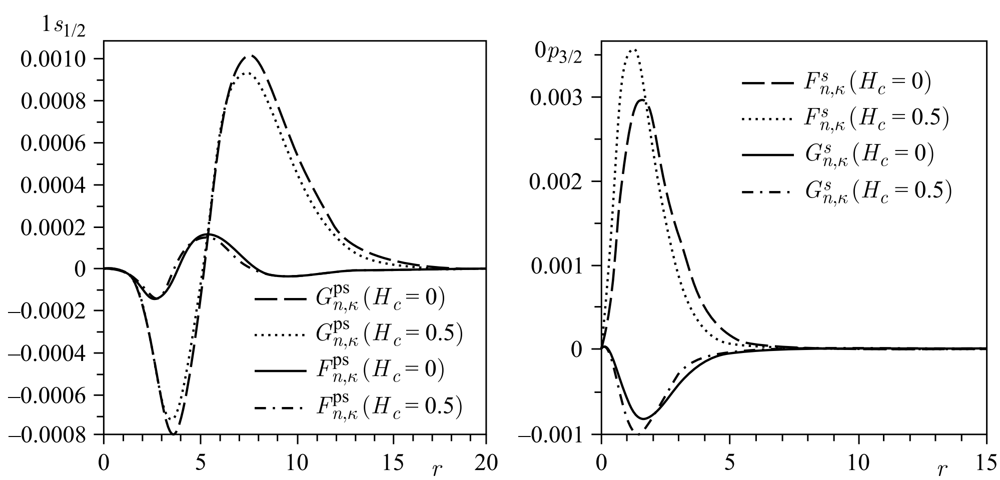

Fig. 1. Wave functions in the pseudospin and spin symmetry limits in the presence and absence of CTI

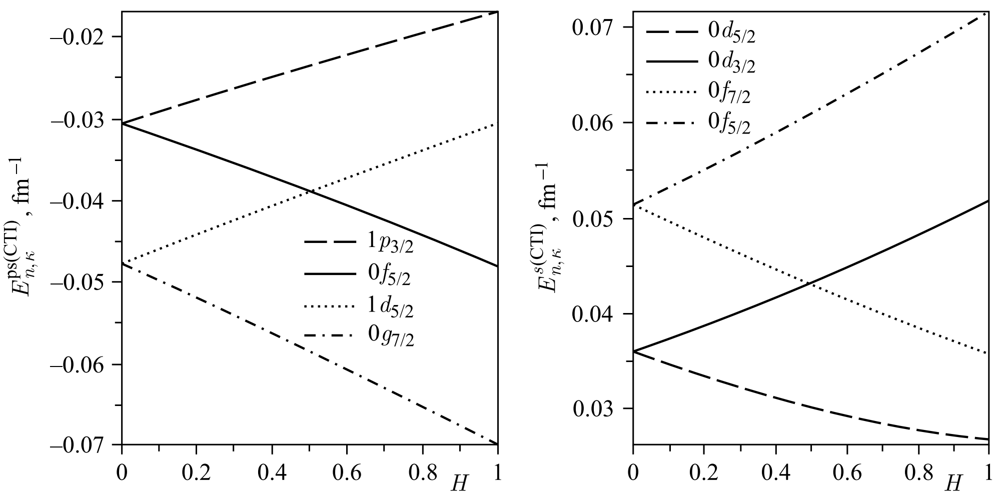

Fig. 2. Energy spectra in the pseudospin and spin symmetries versus H for CTI

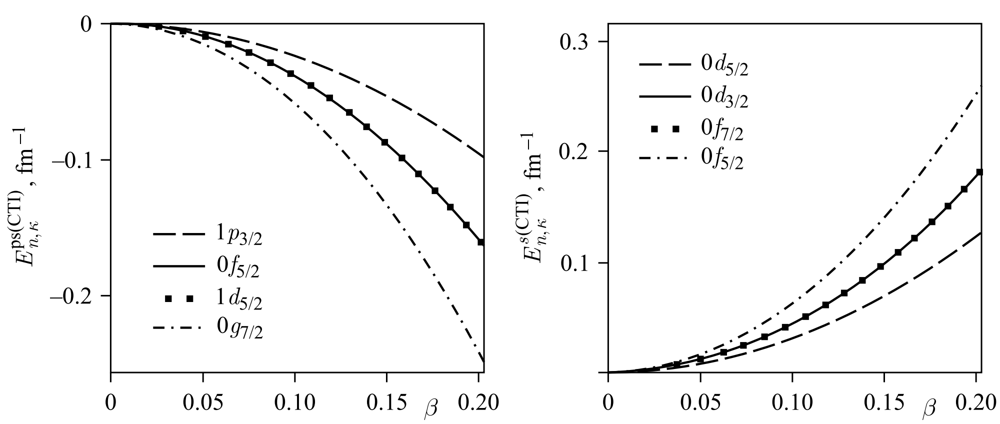

Fig. 3. Energy spectra in the pseudospin and spin symmetries versus β for CTI

7.2. The Effect of the Yukawa-Like Tensor Interaction. In Tables 3 and 4, the energy of the Dirac equation in the absence and presence of the Yukawa-like Tensor Interaction (YTI) is reported. It is obvious that when $V_Y = 0.5$, we have degeneracies between ($1p_{3/2}$, $0d_{3/2}$), ($1d_{5/2}$, $0f_{5/2}$) and ($0g_{7/2}$,$1f_{7/2}$) in the pseudospin symmetry and between ($0p_{1/2}$, $0d_{5/2}$),

Table 3. **The energy of the pseudospin symmetry in the presence of YTI**

$\tilde{l}$	n, κ	State	$E^{\mathrm{ps}}_{n,\kappa(V_Y=0.5)}$	$E^{\mathrm{ps}}_{n,\kappa(V_Y=1)}$	$n-1, \kappa$	State	$E^{\mathrm{ps}}_{n,\kappa(V_Y=0.5)}$	$E^{\mathrm{ps}}_{n,\kappa(V_Y=1)}$
1	1, –1	$1s_{1/2}$	–4.598964392	–4.590151550	0, 2	$0d_{3/2}$	–4.617230119	–4.626318669
2	1, –2	$1p_{3/2}$	–4.617230119	–4.603198793	0, 3	$0f_{5/2}$	–4.643186449	–4.654834634
3	1, –3	$1d_{5/2}$	–4.643186449	–4.626318669	0, 4	$0g_{7/2}$	–4.671989498	–4.683941811
4	1, –4	$1f_{7/2}$	–4.671989498	–4.654834634	0, 5	$0h_{9/2}$	–4.699256858	–4.709724270

Table 4. **The energy of the spin symmetry in the presence of YTI**

l	n,κ	State	$E^s_{n,\kappa(V_Y=0.5)}$	$E^s_{n,\kappa(V_Y=1)}$	n,κ	State	$E^s_{n,\kappa(V_Y=0.5)}$	$E^s_{n,\kappa(V_Y=1)}$
1	0, –2	$0p_{3/2}$	4.621780059	4.436480125	0, 1	$0p_{1/2}$	4.925299124	5.021208859
2	0, –3	$0d_{5/2}$	4.925299124	4.778475420	0, 2	$0d_{3/2}$	5.101155481	5.151282885
3	0, –4	$0f_{7/2}$	5.101155481	5.021208859	0, 3	$0f_{5/2}$	5.192262398	5.217118910
4	0, –5	$0g_{9/2}$	5.192262398	5.151282885	0, 4	$0g_{7/2}$	5.236461226	5.246962866

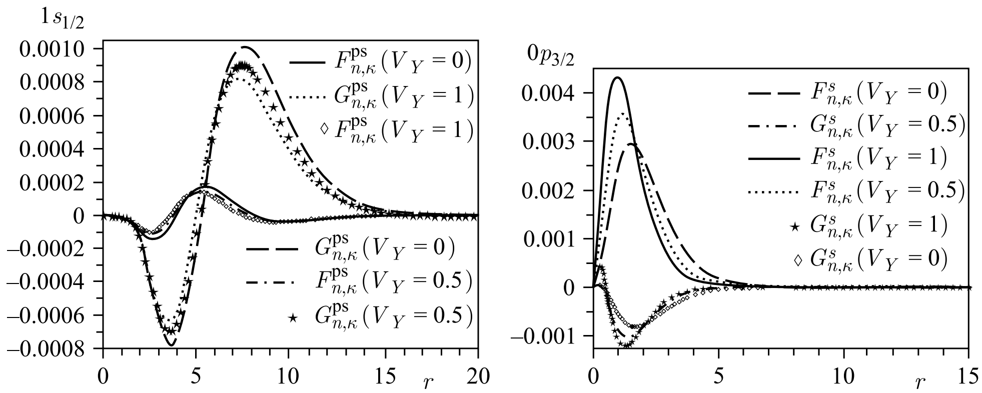

Fig. 4. Wave functions in the pseudospin and spin symmetry limits in the presence and absence of YTI

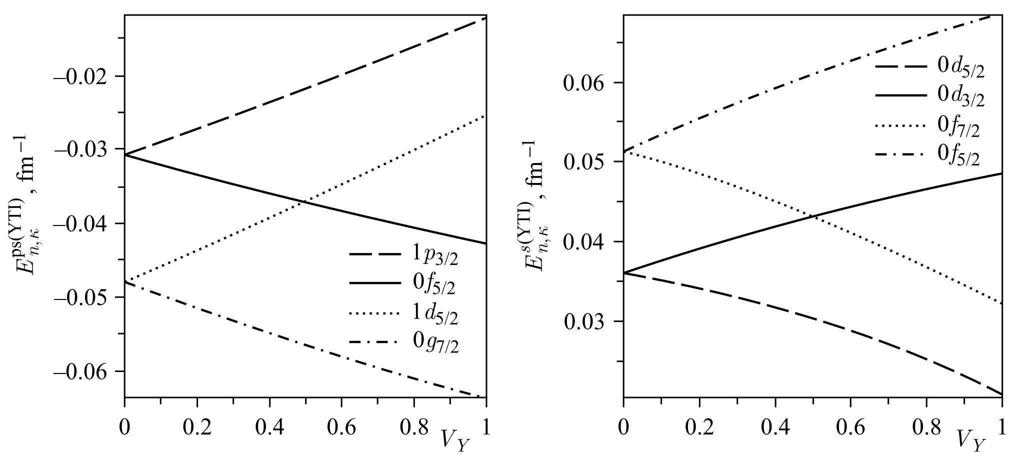

Fig. 5. Energy spectra in the pseudospin and spin symmetries versus V_Y for YTI

($0d_{3/2}$, $0f_{7/2}$) and ($0f_{5/2}$, $0g_{9/2}$) in the spin symmetry. For $V_Y = 1$ there are degeneracies between ($1d_{5/2}$, $0d_{3/2}$) and ($0f_{5/2}$, $1f_{7/2}$) in the pseudospin symmetry and between ($0p_{1/2}$, $0f_{7/2}$) and ($0d_{3/2}$, $0g_{9/2}$) in the spin symmetry. For $V_Y = 0$, 0.5, 1 we have plotted the wave functions of the pseudospin and spin symmetries of the Dirac equation in Fig. 4. To show the effect of parameters V_Y and β on the energy of the system, we have presented Figs. 5 and 6.

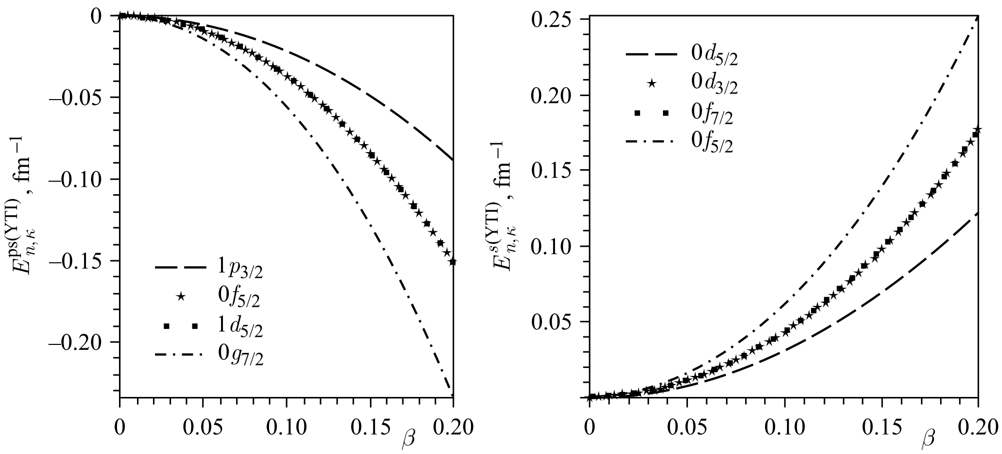

Fig. 6. Energy spectra in the pseudospin and spin symmetries versus β for YTI

Table 5. **The energy of the pseudospin symmetry in the presence of GTI**

$\tilde{l}$	$n,\ \kappa$	State	$E^{\mathrm{ps}}_{n,\kappa(H_c=V_Y=0.5)}$	$E^{\mathrm{ps}}_{n,\kappa(H_c=V_Y=0.75)}$	$n-1,\kappa$	State	$E^{\mathrm{ps}}_{n,\kappa(H_c=V_Y=0.5)}$	$E^{\mathrm{ps}}_{n,\kappa(H_c=V_Y=0.75)}$
1	1, –1	$1s_{1/2}$	–4.594034713	–4.590651427	0, 2	$0d_{3/2}$	–4.629547878	–4.641892062
2	1, –2	$1p_{3/2}$	–4.606848084	–4.597311362	0, 3	$0f_{5/2}$	–4.657527906	–4.670943114
3	1, –3	$1d_{5/2}$	–4.629547878	–4.615723251	0, 4	$0g_{7/2}$	–4.686048811	–4.698469156
4	1, –4	$1f_{7/2}$	–4.657527906	–4.641892062	0, 5	$0h_{9/2}$	–4.711241559	–4.721170141

Table 6. **The energy of the spin symmetry in the presence of GTI**

l	n,κ	State	$E^{s}_{n,\kappa(H_c=V_Y=0.5)}$	$E^{s}_{n,\kappa(H_c=V_Y=0.75)}$	n,κ	State	$E^{s}_{n,\kappa(H_c=V_Y=0.5)}$	$E^{s}_{n,\kappa(H_c=V_Y=0.75)}$
1	0, –2	$0p_{3/2}$	4.455746153	4.370885519	0, 1	$0p_{1/2}$	5.027073504	5.099553937
2	0, –3	$0d_{5/2}$	4.789457980	4.616217022	0, 2	$0d_{3/2}$	5.154405031	5.191426720
3	0, –4	$0f_{7/2}$	5.027073504	4.922303782	0, 3	$0f_{5/2}$	5.218681837	5.236077381
4	0, –5	$0g_{9/2}$	5.154405031	5.099553937	0, 4	$0g_{7/2}$	5.247572613	5.253316145

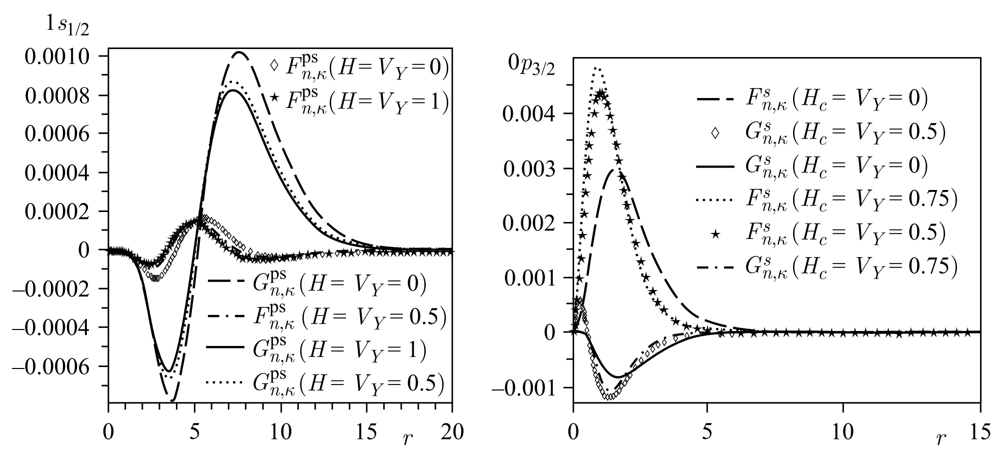

Fig. 7. Wave functions in the pseudospin and spin symmetry limits in the presence and absence of GTI

7.3. The Effect of Generalized Tensor Interaction. Equations (52), (55), (74), (79), (105) and (108) determine the energy of the pseudospin and spin symmetries of the Dirac equations in the presence of GTI. For $H_c = V_Y = 0.5$, 0.75 we have calculated the energy of the system and presented it in Tables 5 and 6. It is illustrated that there are no degeneracies between the energy levels. Figure 7 shows the lower and upper components of the Dirac spinors under the pseudospin and spin symmetry limits. By taking $H_c = 0.5$, we have plotted the behavior of the energy of the system versus V_Y in Fig. 8 for both the pseudospin symmetry and spin symmetry. When $V_Y = 0$, $(1d_{5/2},\ 0f_{5/2})$ in the pseudospin symmetry and $(0d_{3/2},\ 0f_{7/2})$ in the spin symmetry are degenerated. In Fig. 9, by choosing $V_Y = 0.5$, we have presented the effect of H_c, on the energy of the pseudospin symmetry and spin symmetry. We can see that in the case of $H_c = 0$, we have degeneracies between $(1d_{5/2},\ 0f_{5/2})$ in the pseudospin

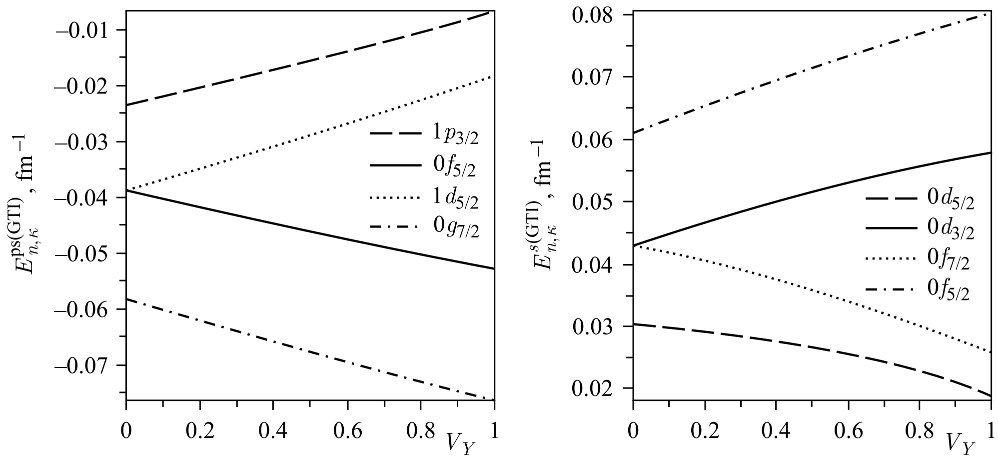

Fig. 8. Energy spectra in the pseudospin and spin symmetries versus V_Y for GTI

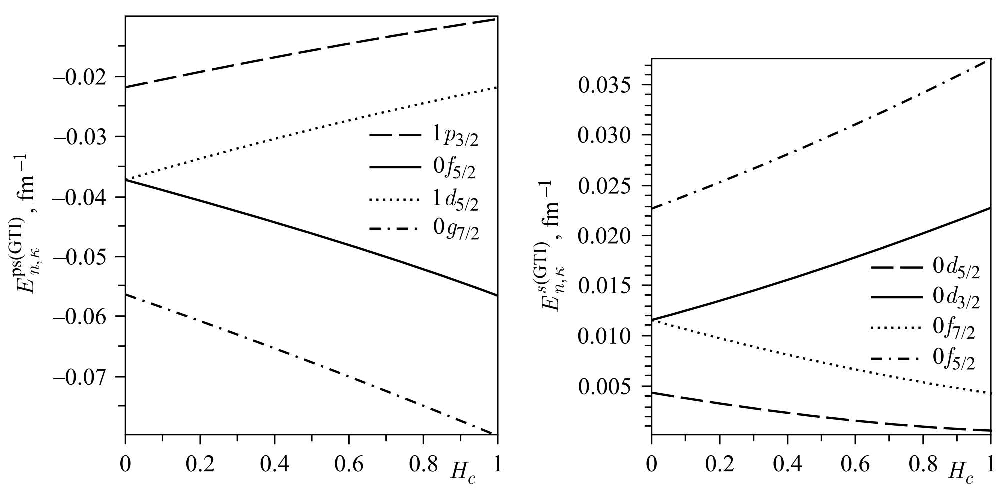

Fig. 9. Energy spectra in the pseudospin and spin symmetries versus H_c for GTI

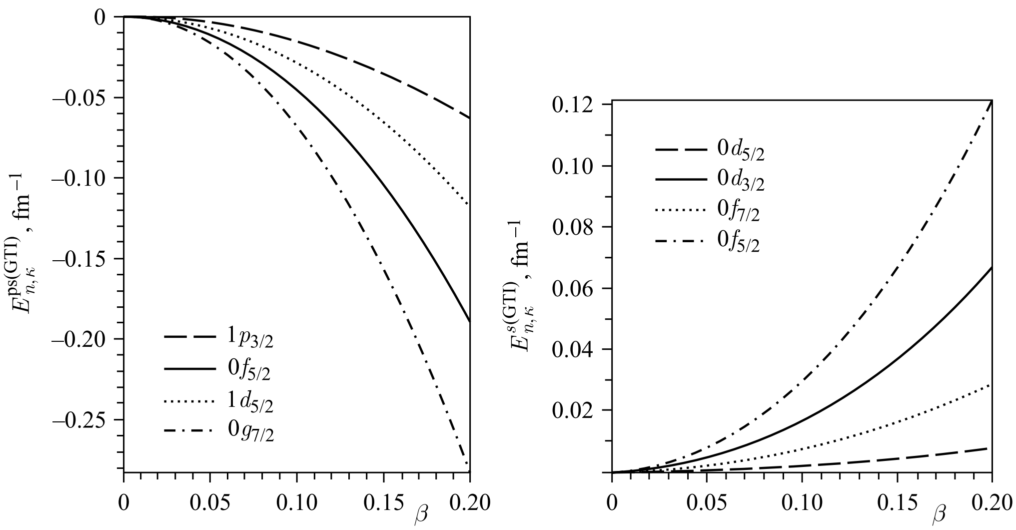

Fig. 10. Energy spectra in the pseudospin and spin symmetries versus β for GTI

symmetry and ($0d_{3/2}$, $0f_{7/2}$) in the spin symmetry. And, finally, the effect of parameter β on the energy of the system is shown in Fig. 10. It is seen that as β increases, the energy of the pseudospin (spin) symmetry decreases (increases).

CONCLUSIONS

We have used the NU method to obtain the approximate solutions of the Dirac equation for the combined Schiöberg and Manning–Rosen potentials within the framework of spin and pseudospin symmetry limits. Based on the knowledge, it should be noted that the Dirac equation with these potentials under the Coulomb, Yukawa and generalized tensor interactions had not been considered before using the NU method. We have obtained explicitly the energy levels in a closed form and the corresponding wave functions expressed in terms of the Jacobi polynomials for these potentials besides the Coulomb-like, Yukawa-like and generalized tensor interactions within the spin and pseudospin symmetry limits. We have also computed the numerical results of our work and it shows that the present of the Coulomb, Yukawa and generalized tensor terms removed the degeneracies between two states in spin and pseudospin doublets. Finally, the results of our work find many applications in both nuclear and hadron physics, and therefore provide more general solutions compared to other previous works in [28] and [29].

REFERENCES

1. *Ginocchio J. N.* Relativistic Symmetries in Nuclei and Hadrons // Phys. Rep. 2005. V. 414. P. 165–261.
2. *Troltenier D., Bahri C., Draayer J. P.* Generalized Pseudo-$SU(3)$ Model and Pairing // Nucl. Phys. A. 1995. V. 586. P. 53.

3. *Dudek J. et al.* Abundance and Systematics of Nuclear Superdeformed States; Relation to the Pseudospin and Pseudo-$SU(3)$ Symmetries // Phys. Rev. Lett. 1987. V. 59. P. 1405.
4. *Page P. R., Goldman T., Ginocchio J. N.* Relativistic Symmetry Suppresses Quark Spin-Orbit Splitting // Phys. Rev. Lett. 2001. V. 86. P. 204.
5. *Ginocchio J. N.* Relativistic Harmonic Oscillator with Spin Symmetry // Phys. Rev. C. 2004. V. 69. P. 034318.
6. *Ginocchio J. N.* Pseudospin as a Relativistic Symmetry // Phys. Rev. Lett. 1997. V. 78. P. 436.
7. *Falaye B. J. et al.* Bound State Solutions of the Manning–Rosen Potential // Can. J. Phys. 2013. V. 91. P. 98.
8. *Jia C. S., Guo P., Peng X. L.* Exact Solution of the Dirac–Eckart Problem with Spin and Pseudospin Symmetries // J. Phys. A: Math. Gen. 2006. V. 39. P. 7737.
9. *Hassanabadi H. et al.* Approximate Any l-State Solutions of the Dirac Equation for Modified Deformed Hylleraas Potential by Using the Nikiforov–Uvarov Method // Chin. Phys. B. 2012. V. 21. P. 120302.
10. *Maghsoodi E., Hassanabadi H., Zarrinkamar S.* Spectrum of Dirac Equation under Deng–Fan Scalar and Vector Potentials and a Coulomb Tensor Interaction by SUSYQM // Few-Body Syst. 2012. V. 53. P. 525.
11. *Maghsoodi E. et al.* Arbitrary-State Solutions of the Dirac Equation for a Möbius Square Potential Using the Nikiforov–Uvarov Method // Chin. Phys. C. 2013. V. 37. P. 04105.
12. *Hassanabadi H., Maghsoodi M., Zarrinkamar S.* Relativistic Symmetries of Dirac Equation and the Tietz Potential // Eur. Phys. J. Plus. 2012. V. 127. P. 31.
13. *Ikot A. N. et al.* Approximate Relativistic κ-State Solutions to the Dirac-Hyperbolic Problem with Generalized Tensor Interactions // Intern. J. Mod. Phys. E. 2013. V. 22. P. 1350048.
14. *Ikhdair S. M., Falaye B. J.* Approximate Relativistic Bound States of a Particle in Yukawa Field with Coulomb Tensor Interaction // Phys. Scripta. 2013. V. 87. P. 035002.
15. *Hamzavi M., Ikhdair S. M., Ita B. I.* Approximate Spin and Pseudospin Solutions to the Dirac Equation for the Inversely Quadratic Yukawa Potential and Tensor Interaction // Phys. Scripta. 2012. V. 85. P. 045009.
16. *Nikiforov A. F., Uvarov V. B.* Special Functions of Mathematical Physics. Basel: Birkhauser, 1988.
17. *Cooper F., Khare A., Sukhatme U.* Supersymmetry and Quantum Mechanics // Phys. Rep. 1995. V. 251. P. 267.
18. *Hassanabadi H., Yazarloo B. H., Zarrinkamar S.* Exact Solution of Klein–Gordon Equation for Hua Plus Modified Eckart Potentials // Few-Body Syst. 2013. V. 54. P. 2017.
19. *Aydogdu O., Sever R.* Exact Pseudospin Symmetric Solution of the Dirac Equation for Pseudoharmonic Potential in the Presence of Tensor Potential // Few-Body Syst. 2010. V. 47. P. 193.
20. *Ikot A. N.* Solutions of Dirac Equation for Generalized Hyperbolical Potential Including Coulomb-Like Tensor Potential with Spin Symmetry // Few-Body Syst. 2012. V. 53. P. 549.
21. *Hassanabadi H., Maghsoodi E., Zarrinkamar S.* Spin and Pseudospin Symmetries of Dirac Equation and the Yukawa Potential as the Tensor Interaction // Commun. Theor. Phys. 2012. V. 58. P. 807.
22. *Tezcan C., Sever R.* A General Approach for the Exact Solution of the Schrödinger Equation // Intern. J. Theor. Phys. 2009. V. 48. P. 337.
23. *Schiöberg D.* The Energy Eigenvalues of Hyperbolical Potential Functions // Mol. Phys. 1986. V. 59. P. 1123.
24. *Arda A., Sever R., Tezcan C.* Approximate Pseudospin and Spin Solutions of the Dirac Equation for a Class of Exponential Potentials // Chin. J. Phys. 2010. V. 48. P. 27.

25. *Ikhdair S. M., Sever R.* Approximate Bound State Solutions of Dirac Equation with Hulthén Potential Including Coulomb-Like Tensor Potential // Appl. Math. & Comp. 2010. V. 216. P. 911.
26. *Greene R. L., Aldrich C.* Variational Wave Functions for a Screened Coulomb Potential // Phys. Rev. A. 1976. V. 14. P. 2363.
27. *Hassanabadi H., Yazarloo B. H., Salehi N.* Pseudospin and Spin Symmetries of Dirac Equation under Deng–Fan Potential and Yukawa Potential as a Tensor Interaction // Ind. J. Phys. 2014. V. 88. P. 405.
28. *Hassanabadi H. et al.* Approximate Solutions of the Dirac Equation for Hyperbolic Scalar and Vector Potentials and a Coulomb Tensor Interaction by SUSYQM // Mod. Phys. Lett. A. 2011. V. 26. P. 2703.
29. *Hassanabadi H. et al.* Actual and General Manning–Rosen Potentials under Spin and Pseudospin Symmetries of the Dirac Equation // Can. J. Phys. 2012. V. 90. P. 633.

Received on October 30, 2014.