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КОМПЬЮТЕРНЫЕ ТЕХНОЛОГИИ ФИЗИКИ

PHASE REPRESENTATION OF QUANTUM-OPTICAL SYSTEMS VIA NONNEGATIVE QUANTUM DISTRIBUTION FUNCTION

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We propose a new method for describing phase distributions of nonclassical states in optical systems based on the nonnegative quantum distribution function. A comparison of the proposed method with other known methods such as the Pegg–Barnett and operational ones is given.

Предлагается новый метод для описания фазовых распределений неклассических состояний в оптических системах, основанный на неотрицательных квантовых функциях распределения. Дается сравнение предложенного метода с другими известными методами, такими как метод Пегга–Барнетта и операционный метод.

INTRODUCTION

The problem of quantum description of the phase was first considered by Dirac [1]. However, this problem has become the actual one after wide use of laser technique in optical experiments and generation of squeezed states which reveal nonuniform phase distributions. There were many attempts to describe phase properties of quantum states both via construction of the phase operators and in terms of quasi-distributions, as well as in the operational approach (see, e.g., the review [2] and references therein). All those methods show good agreement with each other in the quasi-classical approximation, but have principal discord in the case of apparent quantum states.

From the point of view of quantum mechanics, the root of the phase problem lies in the correspondence rules prescribing the way to establish the corresponding operator for the classical phase variable. It was shown that in the framework of the conventional quantum mechanics, based on the von Neumann correspondence rules, it is impossible to construct the Hermitian phase operator as canonical conjugate to the number operator [3].

Another approach to the problem of correspondence rules was proposed by V. V. Kuryshkin [4]. In the framework of this approach, a nonnegative quantum distribution function could be introduced that defines quantum mean values of the observables.

In this Letter, we use the nonnegative quantum distribution function of Kuryshkin to get the corresponding phase distribution as its marginal that approximates the phase distributions of previous approaches such as phase-difference distributions of Pegg and Barnett and of Mandel.

1. QUANTUM PHASE

An idea to construct a (Hermitian) phase operator being conjugated to a number operator,

$$[\hat{N}, \hat{\varphi}] = i, \tag{1}$$

is the old one in quantum mechanics and belongs to Dirac [1]. However, this problem cannot be solved in the ordinary infinite Hilbert space because of the bounded spectrum of the number operator [3]. Nevertheless, some approaches to define phase operators were proposed, among which the Pegg–Barnett [5] and Mandel [6] ones have got wide popularity. The former approach is related to the definition of the phase operator in the finite $(s+1)$-dimensional Hilbert space in the form

$$\hat{\varphi}_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| \,, \tag{2}$$

with the eigenvectors and eigenvalues

$$|\theta_m\rangle = 1/\sqrt{s+1} \sum_{n=0}^{s} \exp{(in\theta_m)}|n\rangle, \quad \theta_m = \theta_0 + 2\pi m/(s+1), \tag{3}$$

respectively. Then, the corresponding phase distribution of the quantum state allowing the Fock decomposition $|\psi\rangle = \sum_{n=0}^{s} c_n|n\rangle$ turns out to be

$$P^{\mathrm{PB}}(\theta) = \lim_{s\to\infty} (s+1)/(2\pi)|\langle\theta_m|\psi\rangle|^2 = (2\pi)^{-1} \left[1 + 2\sum_{n>k} c_n c_k \cos{((n-k)\theta)}\right]. \tag{4}$$

This allows one to generalize the approach for two-mode states of the form

$$|\psi\rangle = \sum_{n_1,n_2} c_{n_1,n_2}|n_1\rangle|n_2\rangle$$

to become

$$P^{\mathrm{PB}}(\theta_1, \theta_2) = (2\pi)^{-2} \left| \sum_{n_1,n_2} c_{n_1,n_2} \exp{[-i(n_1\theta_1 + n_2\theta_2)]} \right|^2, \tag{5}$$

which leads to the definition of the phase-difference distribution

$$\begin{aligned} P^{\mathrm{PB}}(\theta_-) = \int P^{\mathrm{PB}}(\theta_1, \theta_1 + \theta_-) d\theta_1 = \\ = (2\pi)^{-1} \sum_{n_1,n_2,n_3} c^*_{n_1,n_2} c_{n_1,n_1+n_2-n_3} \exp{[-i(n_3 - n_1)]}, \end{aligned} \tag{6}$$

which is usually implied in the phase-sensitive experiments.

The latter (operational) phase approach is based on analogy with classical optics for the definition of sine and cosine of phase difference of input modes in the eight-port scheme, when the corresponding phase operators take the form

$$\hat{\mathcal{C}}^M = \hat{\mathcal{N}}^{-1}(\hat{n}_4 - \hat{n}_3), \quad \hat{\mathcal{S}}^M = \hat{\mathcal{N}}^{-1}(\hat{n}_6 - \hat{n}_5) \tag{7}$$

with the normalizing factor $\mathcal{N} = [(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2]^{1/2}$, where $\hat{n}_i$ is the number operator in the ith port.

In addition to the operator approach, there were attempts to obtain quantum phase distributions as marginals of quasi-distribution functions, such as the Husimi Q function [7] and Wigner function [8]. In view of the fact that quasi-distributions have no intrinsic properties of genuine distributions, these phase distributions fail to describe phase properties of all nonclassical states.

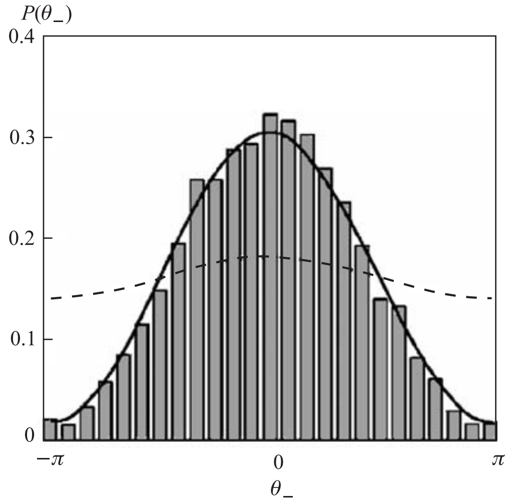

Fig. 1. Mandel's phase distribution (solid line) and the Pegg–Barnett one (6) (dashed line) together with the experimental bar chart of the phase difference between two coherent states $|\alpha_1|^2 = 0.047$ and $|\alpha_1|^2 = 0.076$

In quasi-classical limit, all these phase approaches come to an agreement with each other. However, for low-intensity quantum states there appear fundamental differences between them. Thus, L. Mandel [9] demonstrated the big discrepancy in the Pegg–Barnett phase-difference distribution compared with the distribution obtained in the operational approach for two weak coherent states (see Fig. 1), which apparently indicates the very problem of the phase in quantum mechanics. In the next section, we will show how one can treat the problem of the phase definition in the framework of another approach to the corresponding rules in the quantum theory.

2. NONNEGATIVE DISTRIBUTION FUNCTION

One of consequences of realization of new correspondence rules proposed by Kuryshkin [4] was the introduction of the nonnegative distribution function in the phase space

$$F^K(q,p) = (2\pi)^{-1} \sum_k \left| \int \phi_k^*(q-\xi)\, \mathrm{e}^{i\xi p} \psi(\xi) d\xi \right|^2, \tag{8}$$

so that a quantum mean value for the state ψ of any physical observable $A(q,p)$ is represented as a function of position and momentum to be calculated as

$$\langle A\rangle = \int A(q,p)F^K(q,p)dqdp. \tag{9}$$

Here the functions ϕ_k obey the relation

$$\sum_k \int |\phi_k(q)|^2 dq = 1 \tag{10}$$

to ensure the normalization condition

$$\int F^K(q,p)dqdp = 1. \tag{11}$$

The functions ϕ_k are not set by the quantum state, but may have a meaning of window functions [10].

In the phase space, instead of the Cartesian coordinates, position q and momentum p, one can introduce the polar ones, radius r and phase θ, by the relations $q = r\cos\theta$, $p = r\sin\theta$. These polar coordinates, r and θ, correspond to the intensity and phase variables. Therefore, the nonnegative distribution (8) can be used for the definition of the phase distribution of the quantum state as the marginal distribution in the polar coordinates when the radius is integrated over

$$P^K(\theta) = \int_0^\infty F^K(r\cos\theta, r\sin\theta)rdr. \tag{12}$$

It means that a quantum average of any function $A(\theta)$ of phase variable is given by

$$\langle A\rangle = \int_{-\pi}^{\pi} A(\theta)P^K(\theta)d\theta. \tag{13}$$

In the case of two modes, the nonnegative quantum distribution in the state $\psi(q_1,q_2)$ takes the form

$$F^K(q_1,p_1;q_2,p_2) = \\ = (2\pi)^{-2}\sum_k \left|\int \phi_k^*(q_1-\xi_1, q_2-\xi_2)\exp\left[-i(\xi_1 p_1 + \xi_2 p_2)\right]\psi(\xi_1,\xi_2)d\xi_1 d\xi_2\right|^2 \tag{14}$$

with the window functions satisfying the condition

$$\sum_k \int |\phi_k(q_1,q_2)|^2 dq_1 dq_2 = 1. \tag{15}$$

It is straightforward to generalize the definition of the phase distribution in the two-mode case when the phase-difference distribution is worth to be determined:

$$P^K(\theta_-) = \int F^K(r_1,\theta_1;r_2,\theta_1+\theta_-)r_1 r_2 dr_1 dr_2 d\theta_1, \tag{16}$$

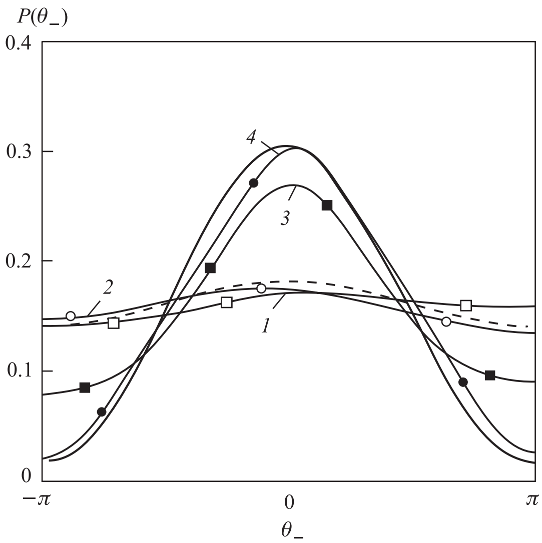

Fig. 2. Phase distributions (16) imposed upon the plot of Fig. 1 for: *1* — $\beta_1 = 0.55$, $\varepsilon_1 = 1$, $\beta_2 = 0.55$, $\varepsilon_2 = 1$; *2* — $\beta_1 = 0.55$, $\varepsilon_1 = 5$, $\beta_2 = 0.55$, $\varepsilon_2 = 5$; *3* — $\beta_1 = 0.10$, $\varepsilon_1 = 4$, $\beta_2 = 0$, $\varepsilon_2 = 4$; *4* — $\beta_1 = 0$, $\varepsilon_1 = 5$, $\beta_2 = 0$, $\varepsilon_2 = 5$

where the corresponding change of variables is done in the nonnegative quantum distribution function, $F^K(q_1, p_1; q_2, p_2) \to F^K(r_1, \theta_1; r_2, \theta_2)$.

This approach to the phase problem can serve to a certain extent to approximate phase distributions discussed in Sec. 1 by means of the choice of the window functions ϕ_k. To demonstrate it, let us consider the problem of Mandel's experiment concerning the dissimilarity in the phase-difference distributions between the experimental data and the Pegg–Barnett one. For this, we consider the input quantum state as the two-mode coherent state $|\alpha_1, \alpha_2\rangle$:

$$\Psi(q_1, q_2) = \psi_1^{\text{coh}}(q_1)\psi_2^{\text{coh}}(q_2), \quad \psi_i^{\text{coh}}(q) = (2\pi)^{-1/4} \exp\left[-(q - \sqrt{2}\alpha_i)^2\right] \quad (i = 1, 2) \tag{17}$$

and choose the window function in the form

$$\Phi(q_1, q_2) = \phi_1(q_1)\phi_2(q_2), \quad \phi_i(q) = (2\pi\varepsilon_i)^{-1/4} \exp\left[-(q - \sqrt{2}\beta_i)^2/\varepsilon_i\right] \quad (i = 1, 2), \tag{18}$$

where β_i and ε_i are thought to be the adjustment parameters. In Fig. 2, the phase distributions (16) are plotted for a number of sets of parameters which simulate the behavior of the Mandel distribution as well as the Pegg–Barnett one.

CONCLUSIONS

In the Letter, we have shown how the phase distributions could be defined via the nonnegative quantum distribution function of the system. The proposed method was applied to the case of two-mode states for describing the phase-difference distribution.

Our distribution by appropriate choice of window functions was shown to approximate Mandel's operational distribution and the Pegg–Barnett one.

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REFERENCES

1. *Dirac P. A. M.* // Proc. Roy. Soc. (London). Ser. A. 1927. V. 114. P. 243.
2. *Murzakhmetov B. K., Chizhov A. V.* // Part. Nucl. 1996. V. 27, No. 3. P. 309.
3. *Carruthers P., Nieto M. M.* // Rev. Mod. Phys. 1968. V. 40. P. 411.
4. *Kuryshkin V. V.* // Intern. J. Theor. Phys. 1972. V. 7. P. 451.
5. *Pegg D. T., Barnett S. M.* // Europhys. Lett. 1988. V. 6. P. 483.
6. *Noh W., Fougéres A., Mandel L.* // Phys. Rev. Lett. 1991. V. 67. P. 1426.
7. *Tanaś R., Gantsog Ts.* // Phys. Rev. A. 1992. V. 45. P. 5031.
8. *Tanaś R. et al.* // Quant. Opt. 1992. V. 4. P. 1.
9. *Torgerson J. R., Mandel L.* // Phys. Rev. Lett. 1996. V. 76. P. 3939.
10. *Cohen L.* // IEEE Proc. 1989. V. 77. P. 941.