ONE- AND TWO-RANK SEPARABLE KERNELS OF THE
TWO-NUCLEON SYSTEM IN THE BETHE–SALPETER
APPROACH
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We propose a completely covariant separable kernel for the nucleon–nucleon (NN) interaction in $J = 0$ ($^1S_0$, $^3P_0$) and $J = 1$ ($^3S_1$, $^1P_1$, $^3P_1$) states. We calculate the nucleon–nucleon (NN) $T$ matrix in the framework of the covariant Bethe–Salpeter approach for a system of two spin-one-half particles with one-rank, extended one-rank, and two-rank Yamaguchi-type separable kernel of interaction. The explicit connection between parameters of the separable kernel and low-energy scattering and between parameters of deuteron binding energy and phase shifts is established. The obtained kernel is used for calculation of phase shifts. This approach can be applied to higher partial waves for NN scattering and other reactions.

INTRODUCTION

Relativistic investigations of the reactions with light nuclei (both with electromagnetic and hadron probes) in the Bethe–Salpeter approach [1] require constructing the amplitudes for bound states and for states of scattering. To construct them, one should solve Bethe–Salpeter equation for partial-wave decomposed $T$ matrix. The separable kernels of interaction are very useful [2] in this case. The separable form of interaction is good not only for the technical reason but, in our opinion, also for the reflection of the nonlocal type of elementary nucleon–nucleon interaction (it was stressed first in paper [2]). This allows us not only to avoid the divergence peculiar to local theories but also to take into account the internal structure of the nucleon.

In the paper, we consider the BS equation for $T$ matrix and solve it using the separable kernel of interaction for a spin-one-half particle system. This allows us to solve the BS equation without referring to the ladder approximation.

The paper is organized as follows: after describing the used formalism in Sec. 1, the result of constructing one- and two-rank separable kernels for partial states of the $np$ system is given in Sec. 2. The summary is in Sec. 3.
1. FORMALISM

We start with the partial-wave decomposed Bethe–Salpeter equation for the nucleon–nucleon $T$ matrix (in the rest frame of two-nucleon system):

$$T_{l'l}(p'_0, p'_1, p_0, p, s) = V_{l'l}(p'_0, p'_1, p_0, p; s) + \frac{i}{4\pi^3} \sum_{l''} \int dk_0 \int k^2 dk \frac{V_{l'l''}(p'_0, p'_1, k; s)T_{l''l}(k, k_0, p_0, p; s)}{(\sqrt{s}/2 - e_k + i\epsilon)^2 - k_0^2}. \quad (1)$$

Here $T_{l'l}$ is the partial-wave decomposed $T$ matrix and $V_{l'l}$ is the kernel of the $NN$ interaction, $e_k = \sqrt{k^2 + m^2}$. There is only one term in the sum for the singlet case ($L = J$), and there are two terms for the coupled triplet case ($L = J \mp 1$). We introduce square of the total momentum $s = P^2 = (p_1 + p_2)^2$ and the relative momentum $p = (p_1 - p_2)/2$ [$p' = (p'_1 - p'_2)/2$] (for details, see Ref. [1]).

Assuming the separable form (rank $N$) for the partial-wave decomposed kernels of $NN$ interactions,

$$V_{l'l}(p'_0, p'_1, p_0, p; s) = \sum_{i,j=1}^{N} \lambda_{ij} g_i^{(l)}(p'_0, p'_1)g_j^{(l)}(p_0, p), \quad (2)$$

we can solve Eq. (1) and write for the $T$ matrix:

$$T_{l'l}(p'_0, p'_1, p_0, p; s) = \sum_{i,j=1}^{N} \tau_{ij}(s)g_i^{(l)}(p'_0, p'_1)g_j^{(l)}(p_0, p). \quad (3)$$

Here the function $\tau(s)$ is

$$\tau^{-1}(s)_{ij} = \lambda^{-1}_{ij} + H_{ij}(s). \quad (4)$$

Function $H_{ij}(s)$ has the following form:

$$H_{ij}(s) = \sum_l \tilde{H}_{ij}^{l}(s) = -\frac{i}{4\pi^3} \int dk_0 \int k^2 dk \sum_l \frac{[g_i^{(l)}(k_0, k)][g_j^{(l)}(k_0, k)]}{(\sqrt{s}/2 - e_k + i\epsilon)^2 - k_0^2}. \quad (5)$$

We use the following normalization condition for the on-mass-shell $T$ matrix for the singlet case:

$$T_{ll}(s) = T_{ll}(0, \bar{p}, 0, \bar{p}, s) = -\frac{16\pi}{\sqrt{s}\sqrt{s-4m^2}} e^{i\delta} \sin \delta; \quad (6)$$

and for the coupled triplet case:

$$T_{l'l'}(s) = \frac{8\pi}{\sqrt{s}\sqrt{s-4m^2}} \begin{pmatrix} \cos 2\epsilon e^{2i\delta_0} - 1 & i \sin 2\epsilon e^{i(\delta_0 + \delta_0)} \\ i \sin 2\epsilon e^{i(\delta_0 + \delta_0)} & \cos 2\epsilon e^{2i\delta_0} - 1 \end{pmatrix}. \quad (7)$$

with $\bar{p} = \sqrt{s/4 - m^2} = \sqrt{mT_{lab}/2}$. We introduce phase shifts $\delta \equiv \delta_{L=J}$, $\delta_0 \equiv \delta_{L=J\mp 1}$, and mixing parameter $\epsilon$. 
A bound state, if exists, is described by a simple pole in the $T$ matrix. Using Eq. (4), we can write ($M_b = 2m - E_b$, $E_b$ is the energy of the bound state)

$$\det |τ^{-1}(s = M_b^2)| = 0.$$  \hspace{1cm} (8)

We also introduce the low-energy parameters — scattering length $a_l$ and effective range $r_l$ — by the following equation:

$$\bar{p}^{2l+1} \cot \delta_l(s) = -\frac{1}{a_l} + \frac{r_l}{2} \bar{p}^2 + O(\bar{p}^3).$$  \hspace{1cm} (9)

At this point by using Eqs. (6) and (7) and calculating $T$ matrix on the mass shell ($p_0 = p'_0 = 0, p = p' = \bar{p}$), we can connect internal parameters of the $NN$ kernel and observables — phase shifts, bound state energy and low-energy parameters.

2. CALCULATIONS AND RESULTS

2.1. One-Rank Yamaguchi-Type Kernel. We use covariant generalization of the Yamaguchi [2] functions for $g^{[\sigma]}(k_0, k)$:

$$g^{[\sigma]}(k_0, k) = \frac{1}{k_0^2 - k^2 - \beta_0^2 + i\epsilon},$$  \hspace{1cm} (10)

$$g^{[P]}(k_0, k) = \sqrt{\frac{1 - k_0^2 + k^2}{(k_0^2 - k^2 - \beta_1^2 + i\epsilon)^2}},$$  \hspace{1cm} (11)

$$g^{[D]}(k_0, k) = \frac{C(k_0^2 - k^2)}{(k_0^2 - k^2 - \beta_2^2 + i\epsilon)^2}. $$  \hspace{1cm} (12)

Now we can calculate internal parameters of the $NN$ kernel by using the above equations to reproduce experimental values for the phase shifts (data from SAID program http://gwdac.phys.gwu.edu/), deuteron energy and quadropole moment, and low-energy parameters (data from Ref. [5]).

1. To find parameters $\lambda$ and $\beta$ in $^1S_0^+$ channel, we solve a system of nonlinear equations (exp stands for experimental, $s$ — for singlet):

$$a_s^{\exp} = a_s(\lambda, \beta), \quad r_s^{\exp} = r_s(\lambda, \beta).$$  \hspace{1cm} (13)

2. To find parameters $\lambda$, $\beta_0$, $\beta_2$, and $C$ in $^3S_1^+-^3D_1^+$ coupled channel, we solve a system of the nonlinear equations ($t$ stands for triplet):

$$a_t^{\exp} = a_t(\lambda, \beta_0, \beta_2, C), \quad E_d^{\exp} = r_0(\lambda, \beta_0, \beta_2, C),$$

$$p_d = p_d(\lambda, \beta_0, \beta_2, C), \quad q_d^{\exp} = q_d(\lambda, \beta_0, \beta_2, C).$$  \hspace{1cm} (14)

Here we introduce $D$-wave pseudoprobability $p_d$.

3. To find parameters $\lambda$ and $\beta$ in uncoupled $^3P_0^+$, $^1P_1^+$, and $^3P_1^+$ channels, we use procedure to minimize function:

$$\chi^2 = \sum_{i=1}^{n} \frac{(\delta^{\exp}(s_i) - \delta(s_i))^2}{(\Delta \delta^{\exp}(s_i))^2},$$  \hspace{1cm} (15)

where $n$ is the number of the experimental points. The results of calculations are given in Tables 1 and 2 and Fig. 1.
Table 1. Parameters for $^1S_0^+$ and $^3S_1^-^3D_1^+$ channels. One-rank Yamaguchi-type functions

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$^1S_0^+$</th>
<th>$^3S_1^-^3D_1^+$ ($p_d = 4%$)</th>
<th>$^3S_1^-^3D_1^+$ ($p_d = 5%$)</th>
<th>$^3S_1^-^3D_1^+$ ($p_d = 6%$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$, GeV$^2$</td>
<td>-0.285549</td>
<td>-0.502690</td>
<td>-0.429637</td>
<td>-0.364905</td>
</tr>
<tr>
<td>$\beta_0$, GeV</td>
<td>0.221858</td>
<td>0.251241</td>
<td>0.246706</td>
<td>0.242285</td>
</tr>
<tr>
<td>$\beta_2$, GeV</td>
<td>1.6471</td>
<td>2.4071</td>
<td>3.2735</td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Parameters for $^3P_0^+$, $^1P_1^+$ and $^3P_1^+$ channels. One-rank Yamaguchi-type functions

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$^3P_0^+$, $n = 3$</th>
<th>$^3P_0^+$, $n = 5$</th>
<th>$^1P_1^+$, $n = 4$</th>
<th>$^1P_1^+$, $n = 5$</th>
<th>$^3P_1^+$, $n = 7$</th>
<th>$^3P_1^+$, $n = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$, GeV$^2$</td>
<td>-0.029420</td>
<td>-0.016123</td>
<td>0.091535</td>
<td>0.195130</td>
<td>0.312630</td>
<td>0.657010</td>
</tr>
<tr>
<td>$\beta_1$, GeV</td>
<td>0.238337</td>
<td>0.218617</td>
<td>0.276730</td>
<td>0.308910</td>
<td>0.338900</td>
<td>0.381910</td>
</tr>
</tbody>
</table>

Fig. 1. $^1S_0^+$ and $^3S_1^-$ (a), $^3P_0^+$ (b), $^1P_1^+$ (c), and $^3P_1^+$ (d) channel phase shifts. One-rank Yamaguchi-type functions

2.2. Extended One-Rank Yamaguchi-Type Functions. Let us now extend the form of functions $g^{[l]}(k_0, k)$ and add one more term

$$g^{[S]}(k_0, k) = \frac{1}{(k_0^2 - k^2 - \beta_{01}^2 + i\epsilon)} + \frac{C_{02}(k_0^2 - k^2)}{(k_0^2 - k^2 - \beta_{02}^2 + i\epsilon)^2},$$

(16)
Table 3. Parameters for $^3P^+_0$, $^1P^+_1$ channels. Extended one-rank Yamaguchi-type functions

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$^3P^+_0$, $n = 4$</th>
<th>$^3P^+_0$, $n = 5$</th>
<th>$^1P^+_1$, $n = 6$</th>
<th>$^1P^+_1$, $n = 8$</th>
<th>$^1P^+_1$, $n = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$, GeV$^2$</td>
<td>-2.09</td>
<td>-0.0108</td>
<td>0.416$\cdot$10$^{-2}$</td>
<td>0.8323$\cdot$10$^{-2}$</td>
<td>0.706$\cdot$10$^{-2}$</td>
</tr>
<tr>
<td>$C_{12}$, GeV</td>
<td>0.91</td>
<td>9.74</td>
<td>-25.7</td>
<td>-31.549</td>
<td>-28.92</td>
</tr>
<tr>
<td>$\beta_{11}$, GeV</td>
<td>0.401</td>
<td>0.209</td>
<td>0.172</td>
<td>0.18716</td>
<td>0.183</td>
</tr>
<tr>
<td>$\beta_{12}$, GeV</td>
<td>0.31</td>
<td>0.495</td>
<td>0.388</td>
<td>0.44545</td>
<td>0.428</td>
</tr>
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\[
g_{[P]}(k_0, k) = \frac{\sqrt{|-k_0^2 + k^2|}}{(k_0^2 - k^2 - \beta_{11}^2 + i\epsilon)^2} + \frac{C_{12} \sqrt{|-k_0^2 + k^2|^3}}{(k_0^2 - k^2 - \beta_{12}^2 + i\epsilon)^3}, \tag{17}
\]

\[
g_{[D]}(k_0, k) = \frac{C_{21}(k_0^2 - k^2)}{(k_0^2 - k^2 - \beta_{21}^2 + i\epsilon)^2} + \frac{C_{22}(k_0^2 - k^2)^2}{(k_0^2 - k^2 - \beta_{22}^2 + i\epsilon)^3}. \tag{18}
\]

Now we can calculate internal parameters of the $NN$ kernel by using above equations to reproduce experimental values for the phase shifts (data from SAID program http://gwdac.phys.gwu.edu/), deuteron energy and quadrupole moment, and low-energy parameters (data from Ref. [5]).

Fig. 2. $^3P^+_0$ (a), $^1P^+_1$ (b), and $^3P^+_1$ (c) channel phase shifts. Extended one-rank Yamaguchi-type functions.
1. To find parameters $\lambda$, $C_{12}$, $\beta_{11}$, and $\beta_{12}$ in uncoupled $^3P_0^+$, $^1P_1^+$, and $^3P_1^+$ channels, we use procedure to minimize the $\chi^2$ value:

$$
\chi^2 = \sum_{i=1}^{n} \frac{(\delta_{\text{exp}}(s_i) - \delta(s_i))^2}{(\Delta \delta_{\text{exp}}(s_i))^2},
$$

(19)

The results of calculations are given in Table 3 and Fig. 2. We also show former results (with two parameters) for comparison, where $n$ is the number of the experimental points.

2.3. Two-Rank Yamaguchi-Type Kernel. We use covariant generalization of the Yamaguchi [2] functions for $g^{[S]}(k_0, k)$:

$$
g^{[S]}_1(k_0, k) = \frac{1}{(k_0^2 - k^2 - \beta_1^2 + i\epsilon)^2},
$$

$$
g^{[S]}_2(k_0, k) = \frac{(k_0^2 - k^2)}{(k_0^2 - k^2 - \beta_2^2 + i\epsilon)^2},
$$

(20)

$$
g^{[P]}_1(k_0, k) = \frac{\sqrt{|-k_0^2 + k^2|}}{(k_0^2 - k^2 - \beta_1^2 + i\epsilon)^2},
$$

$$
g^{[P]}_2(k_0, k) = \frac{\sqrt{|-k_0^2 + k^2|^3}}{(k_0^2 - k^2 - \beta_2^2 + i\epsilon)^3}.
$$

1. To find parameters $\beta_1$, $\beta_2$, $\lambda_{11}$, $\lambda_{12}$, and $\lambda_{22}$ in $^1S_0^+$ channel, we use procedure to minimize function:

$$
\chi^2 = \sum_{i=1}^{n} \frac{(\delta_{\text{exp}}(s_i) - \delta(s_i))^2}{(\Delta \delta_{\text{exp}}(s_i))^2} + (a_{0s}^{\text{exp}} - a_{0s}^{\text{cal}})^2/(\Delta a_{0s}^{\text{exp}})^2.
$$

(21)

2. To find parameters $\beta_1$, $\beta_2$, $\lambda_{11}$, $\lambda_{12}$, and $\lambda_{22}$ in uncoupled $^3S_1^+$ channels, we use procedure to minimize function:

$$
\chi^2 = \sum_{i=1}^{n} \frac{(\delta_{\text{exp}}(s_i) - \delta(s_i))^2}{(\Delta \delta_{\text{exp}}(s_i))^2} + (a_{0t}^{\text{exp}} - a_{0t}^{\text{cal}})^2/(\Delta a_{0t}^{\text{exp}})^2 +
$$

$$
+ (E_{d}^{\text{exp}} - E_{d}^{\text{cal}})^2/(\Delta E_{d}^{\text{exp}})^2.
$$

(22)

3. To find parameters $\beta_1$, $\beta_2$, $\lambda_{11}$, $\lambda_{12}$, and $\lambda_{22}$ in uncoupled $^3P_0^+$, $^1P_1^+$, and $^3P_1^+$ channels, we use procedure to minimize function:

$$
\chi^2 = \sum_{i=1}^{n} \frac{(\delta_{\text{exp}}(s_i) - \delta(s_i))^2}{(\Delta \delta_{\text{exp}}(s_i))^2}.
$$

(23)

The results of calculations are given in Tables 4, 5 and Fig. 3.
One- and Two-Rank Separable Kernels of the Two-Nucleon System

Fig. 3. $^1S_0^+$ and $^3S_1^+$ (a), $^3P_0^+$ (b), $^1P_1^+$ (c), and $^3P_1^+$ (d) channel phase shifts. Two-rank Yamaguchi-type functions

Table 4. Parameters for $^3P_1^+$ channel. Extended one-rank Yamaguchi-type functions

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$^3P_1^+$, $n = 10$</th>
<th>$^3P_1^+$, $n = 15$</th>
<th>$^3P_1^+$, $n = 19$</th>
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</thead>
<tbody>
<tr>
<td>$\lambda$, GeV$^2$</td>
<td>0.861·10$^{-2}$</td>
<td>0.0342</td>
<td>0.0632</td>
</tr>
<tr>
<td>$C_{12}$, GeV</td>
<td>-22.7</td>
<td>-24.3</td>
<td>-26.7</td>
</tr>
<tr>
<td>$\beta_{11}$, GeV</td>
<td>0.200</td>
<td>0.235</td>
<td>0.250</td>
</tr>
<tr>
<td>$\beta_{12}$, GeV</td>
<td>0.409</td>
<td>0.503</td>
<td>0.535</td>
</tr>
</tbody>
</table>

Table 5. The binding energy and low-energy parameters for singlet and triplet channels. Two-rank Yamaguchi-type functions

<table>
<thead>
<tr>
<th>$^1S_0$</th>
<th>$a_{01}$, Fm</th>
<th>$^3S_1$</th>
<th>$a_{01}$, Fm</th>
<th>$E_{dt}$, MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculation</td>
<td>-23.745</td>
<td>Calculation</td>
<td>5.419</td>
<td>2.224606</td>
</tr>
<tr>
<td>Experiment</td>
<td>-23.748±0.010</td>
<td>Experiment</td>
<td>5.424±0.004</td>
<td>2.224644±0.000046</td>
</tr>
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</table>

CONCLUSION

We constructed completely covariant separable kernels for the nucleon–nucleon interaction in singlet and triplet channels. We have found that the use of the one-rank and the extended
one-rank kernels can reproduce the $^1S_0$, $^3S_1$, and $^3P_0$ phase shifts till $T_{\text{lab}} = 200–300$ MeV. The two-rank Yamaguchi-type kernels are able to reproduce the deuteron static properties and the phase shifts up to $T_{\text{lab}} = 600$ MeV.

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**REFERENCES**