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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

ON UNIFIED FIELD THEORIES, DYNAMICAL TORSION AND GEOMETRICAL MODELS

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A new model of a nondualistic unified theory is proposed. This model is absolutely consistent from the mathematical and geometrical points of view and is based on a manifold equipped with an underlying hypercomplex structure and zero nonmetricity. Also we showed that interesting wormhole solutions, similar to the non-Abelian Born–Infeld theory of our previous work [14] can be obtained. The solution of this model is explicitly compared with our previous one and we find that the torsion plays in this unified theory a role similar to that of Yang–Mills type strength field coming from the non-Abelian Born–Infeld energy momentum tensor. The meaning of the Hosoya–Ogura ansatz (namely, the alignment of the isospin with the frame geometry of the space-time) is completely elucidated.

Предложена новая модель недуалистической единой теории. Эта модель, согласованная с математической и геометрической точек зрения, основывается на многообразии, оснащенном базисной гиперкомплексной структурой и нулевой неметричностью. Мы также показали, что могут быть получены интересные решения типа кротовой норы, аналогичные найденным нами ранее в неабелевой теории Борна–Инфельда [14]. Мы сравнили решения этих теорий и обнаружили, что кручение в единой теории играет ту же самую роль, что тензор напряженности Янга–Миллса, возникающий в тензоре энергии-импульса в неабелевой теории Борна–Инфельда. Выяснен физический смысл анзаца Хосоя–Огура, а именно отождествление изоспиновой структуры поля Янга–Миллса с базовой геометрией пространства-времени.

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1. MOTIVATION AND SUMMARY OF THE RESULTS

As is well known, spin-angular momentum and mass appear in very symmetric way in nongravitational physics. Moreover, the labels of the irreducible representations of the Poincare group [1] are precisely the mass and the spin. Then, in view of this fact, one is able to note that the Einstein theory is incomplete because only energy-momentum and not spin-angular momentum is given dynamical importance for the structure (geometrical properties) of the space-time.

The Einstein theory is deduced assuming a priori the Riemannian structure of the space-time, that is without torsion. Arguments have been given that the space-time should exhibit both curvature and torsion in the presence of the matter [2–6].

The coupling of spin density to torsion of space-time is natural when the R_4 geometry is extended to U_4, from a Riemannian to Riemannian–Cartan geometry [2–4, 6]. For instance, the Einstein–Cartan theory is the simplest generalization of Einstein's theory obtained in the

U_4 geometry. But, however, in the usual Einstein–Cartan geometry [2–4, 6] the spin-geometry coupling and the energy-geometry coupling still appear. The Christoffel connection depends upon the metric and its derivatives, but the torsion terms are regarded as independent fields. Then, the direct consequence is that we have upon variation with respect to the metric and the contorsion second-order differential equations for $g_{\mu\nu}$ and algebraic equations for $T_{\mu\nu\rho}$. This fact is unnatural and its meaning is obscure, indeed we can eliminate the torsion of the field equations and obtain an Einstein theory with a modified matter field Lagrangian. Thus, the theories involved are dynamically equivalent [7].

At this stage one suspects that a deeper question is involved in the same root of the problem: spin, energy-matter and space-time structure. The theories described above, besides the obvious difference of the spin-torsion coupling, that is both Einstein and Einstein–Cartan, are dualistic theories: we must include the fields (matter) by means of the addition of a (nongeometric) Lagrangian to the gravitational (geometrical) one. Einstein himself pointed out that this fact is «undesirable» and only has the status of some bridge towards the final unified theory. It seems reasonable, for instance, to continue these efforts in order to obtain the correct way to solve the important problem of the natural unification of the natural world (matter, energy, spin).

In this report we present a new model of a nondualistic unified theory. This model is absolutely consistent from the mathematical and geometrical points of view and is based on a manifold equipped with an underlying hypercomplex structure and zero nonmetricity, that lead to the important fact that the torsion of the space-time structure turns to be totally antisymmetric: this is the only important case that this type of affine geometrical frameworks are compatible with the physical «equivalence principle». Also we showed that interesting wormhole solutions, similarly to the previous reference with the non-Abelian Born–Infeld theory, can be obtained in this theory. The solution of this model is explicitly compared with our previous one and we find that the torsion plays in this unified theory a role similar to that of Yang–Mills type strength field coming from the non-Abelian Born–Infeld energy momentum tensor of our previous reference. Another important result is that the meaning of the Hosoya–Ogura ansatz (namely, the alignment of the isospin with the frame geometry of the space-time) is completely elucidated.

2. THE SPACE-TIME MANIFOLD AND THE GEOMETRICAL ACTION

The starting point is an hypercomplex construction of the (metric compatible) space-time manifold [8]. We list the main ingredients for this construction.

The metric is

$$g_{\mu\nu} = \overline{g}_{\mu\nu} = g_{\nu\mu} \in \mathbb{R} \text{ with } \nabla g = 0. \tag{1}$$

Also, we assume that the potential torsion exists and arises in a natural form, considering that the geometry is reductive (the ∇ for the covariant derivative with respect to the full connection Γ). This potential torsion has the following properties:

$$f_{\mu\nu} = \overline{f}_{\mu\nu} = -f_{\nu\mu} \in \mathbb{HC}, \quad \nabla_{[\rho} f_{\mu\nu]} = T_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} h^{\sigma}, \tag{2}$$

with the last equality coming from the full antisymmetry of the torsion field. Immediately we can see, as a consequence of the above statements, the following:

i) the torsion is the dual of an axial vector h^{σ};

ii) from i) follows the existence in the space-time of a completely antisymmetric tensor covariantly constant $\varepsilon_{\mu\nu\rho\sigma}$ $(\nabla\varepsilon = 0)$.

Notice that, as we will show in detail elsewhere [9], the choice for the real nature of the metric and the pure hypercomplex potential tensor comes from the Hermitian nature of the theory: if we assume (1), the condition (2) arises automatically.

The second important point is to consider the extended curvature [10]

$$\mathcal{R}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \Sigma^{ab}_{\mu\nu} \tag{3}$$

with

$$R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^{b}_{\nu c} - \omega^{ac}_\nu \omega^{b}_{\mu c}, \quad \Sigma^{ab}_{\mu\nu} = -\left(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu\right).$$

We assume here ω^{ab}_ν is a $SO(d-1,1)$ connection and e^a_μ is a vierbein field. Equation (3) can be obtained, for example, using the formulation that was first introduced in seminal works by E. Cartan long time ago [10]. It is well known that in such a formalism the gravitational field is represented as a connection one form associated with some group which contains the Lorentz group as subgroup. The typical example is provided by the $SO(d,1)$ de Sitter gauge theory of gravity. In this specific case, the $SO(d,1)$ gravitational gauge field $\omega^{AB}_\mu = -\omega^{BA}_\mu$ is broken into the $SO(d-1,1)$ connection ω^{ab}_μ and the $\omega^{da}_\mu = e^a_\mu$ vierbein field, with the dimension d fixed. Then, the de Sitter (anti-de Sitter) curvature

$$\mathcal{R}^{ab}_{\mu\nu} = \partial_\mu \omega^{AB}_\nu - \partial_\nu \omega^{AB}_\mu + \omega^{AC}_\mu \omega^{B}_{\nu C} - \omega^{AC}_\nu \omega^{B}_{\mu C} \tag{4}$$

splits into the curvature (3).

Now we define the following geometrical object:

$$\mathcal{R}^a_\mu = \lambda\left(e^a_\mu + f^a_\mu\right) + R^a_\mu\left(M^a_\mu \equiv e^{a\nu} M_{\nu\mu}\right). \tag{5}$$

The action will contain, as usual, $\mathcal{R} = \det\left(\mathcal{R}^a_\mu\right)$ as the geometrical object that defines the dynamics of the theory. The particularly convenient definition of $\mathcal{R}^a_\mu$ makes it easy to establish the equivalent expression in the spirit of the unified theories developed some time ago by Eddington, Einstein and Born and Infeld, for example:

$$\sqrt{\det \mathcal{R}^a_\mu \mathcal{R}_{a\nu}} = \sqrt{\det\left[\lambda^2\left(g_{\mu\nu} + f^a_\mu f_{a\nu}\right) + 2\lambda R_{(\mu\nu)} + 2\lambda f^a_\mu R_{[a\nu]} + R^a_\mu R_{a\nu}\right]}, \tag{6}$$

where $R_{\mu\nu} = R_{(\mu\nu)} + R_{[\mu\nu]}$.

The important point to consider in this simple Cartan inspired model is that, although a cosmological constant λ is required, the expansion of the action in four dimensions leads automatically to the Hilbert–Einstein part when $f^a_\mu = 0$. Explicitly $(R = g^{\alpha\beta} R_{\alpha\beta})$

$$\begin{aligned} S = \int d^4x\,(e+f)\Big\{ & \lambda^4 + \lambda^3\,(R + f^a_\mu R^\mu_a) + \\ & + \frac{\lambda^2}{2!}\,[R^2 - R^{\mu\nu}R_{\mu\nu} + (f^a_\mu R^\mu_a)^2 - f^{\mu\nu} f^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma}] + \\ & + \frac{\lambda}{3!}[R^3 - 3RR^{\mu\nu}R_{\mu\nu} + 2R^{\mu\alpha}R_{\alpha\beta}R^\beta_\mu + (f^a_\mu R^\mu_a)^3 - \\ & - 3(f^a_\mu R^\mu_a) f^{\mu\nu} f^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma} + 2 f^{\mu\nu} R^\alpha_\mu R_{\alpha\beta} R^\beta_\nu] + \det(R_{\mu\nu})\Big\}. \end{aligned} \tag{7}$$

3. THE DYNAMICAL EQUATIONS

Defining

$$\eta_{ab}\mathcal{R}^a_\mu\mathcal{R}^b_\nu \equiv G_{\mu\nu}, \tag{8}$$

the variation with respect to the metric $g_{\mu\nu}$ is straightforward:

$$\frac{\delta\sqrt{G}}{\delta g^{\alpha\beta}} = \frac{\sqrt{G}}{2}\left(G^{-1}\right)^{\mu\nu}\left[\lambda^2\left(-g_{\beta\nu}g_{\alpha\mu} + f_{\beta\nu}f_{\alpha\mu}\right) + 2\lambda f_{\alpha\mu}R_{[\beta\nu]}\right]. \tag{9}$$

In order to compute the variation with respect to f, it is useful to remind the structure of the Riemann tensor [12]

$$R_{\mu\nu} = \overbrace{\overset{\circ}{R}_{\mu\nu} - T^\alpha_{\mu\rho}T^\rho_{\alpha\nu}}^{R_{(\mu\nu)}} + \overbrace{\overset{\circ}{\nabla}_\alpha T^\alpha_{\mu\nu}}^{R_{[\mu\nu]}}, \tag{10}$$

where $\overset{\circ}{R}_{\mu\nu}$ and $\overset{\circ}{\nabla}_\alpha$ are the Riemann tensor and the covariant derivative computed from the Christoffel symbol $\left\{ {}^{\rho}_{\mu\nu} \right\}$. Then, using the last expression (10), we obtain for the f variation

$$\frac{\delta\sqrt{G}}{\delta f_{\sigma\tau}} = \nabla_\rho\left(\frac{\partial\sqrt{G}}{\partial T_{\rho\sigma\tau}}\right) - \frac{\partial\sqrt{G}}{\partial f_{\sigma\tau}} \equiv \nabla_\rho\mathbb{T}^{\rho\sigma\tau} - \mathbb{F}^{\sigma\tau} = 0. \tag{11}$$

From the above expressions it is not difficult to see that the full set of equations involved in our task are

$$R_{\mu\nu} = -2\lambda\left(g_{\mu\nu} + f_{\mu\nu}\right), \tag{12}$$

$$\nabla_\rho\left(\frac{\partial\sqrt{G}}{\partial T_{\rho\sigma\tau}}\right) - \frac{\partial\sqrt{G}}{\partial f_{\sigma\tau}} \equiv \nabla_\rho\mathbb{T}^{\rho\sigma\tau} - \mathbb{F}^{\sigma\tau} = 0. \tag{13}$$

4. THE DYNAMICAL EQUATIONS II: PHYSICAL AND GEOMETRICAL INTERPRETATION

The above variational equations (in Palatini's sense [10, 12]) (12) and (13), despite their simplest and compact form, contain the deep physical and geometrical meaning, which is necessary to clarify.

For expression (13) we have a highly nonlinear dynamical (propagating) equation for the torsion field, where the variation was performed with respect to their potential $f_{\mu\nu}$ and having a nonlinear term proportional to $f_{\mu\nu}$ playing the role of current for the $\mathbb{T}^{\rho\sigma\tau}$. Then, the potential two form is associated nonlinearly to the torsion field as its source regarding similar association between the electromagnetic field and the spin in particle physics.

For expression (12), firstly it is useful to split the equation into the symmetric and the antisymmetric parts using (10):

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T^\alpha_{\mu\rho}T^\rho_{\alpha\nu} = -2\lambda g_{\mu\nu}, \tag{14}$$

$$R_{[\mu\nu]} = \overset{\circ}{\nabla}_\alpha T^\alpha_{\mu\nu} = -2\lambda f_{\mu\nu} = \nabla_\alpha T^\alpha_{\mu\nu}, \tag{15}$$

the last equality coming from Eq. (2). The symmetric part (14) can be written in a «GR» suggestive fashion:

$$\overset{\circ}{R}_{\mu\nu} = -2\lambda g_{\mu\nu} + T^{\alpha}_{\mu\rho} T^{\rho}_{\alpha\nu}. \tag{16}$$

We can advertise that the equation has the aspect of the Einstein equations with the cosmological term modified by the torsion symmetric term $T^{\alpha}_{\mu\rho} T^{\rho}_{\alpha\nu}$. This can be interpreted by the energy of the gravitational field itself.

The second antisymmetric part (15) is more involved. In order to understand it, it will be necessary to use the language of differential forms to rewrite them which, beside their symbolic and conceptual simplicity, permit us to check consistency and covariance step by step:

$$\nabla_{\alpha} T^{\alpha}_{\mu\nu} = -2\lambda f_{\mu\nu}, \quad d\,^{*}T = -2\lambda\,^{*}f. \tag{17}$$

Now, using (2) $(T = {}^{*}h)$

$$dh = -2\lambda\,^{*}f \Rightarrow {}^{*}f = -\frac{1}{2\lambda} dh, \tag{18}$$

in more familiar form

$$\nabla_{\mu} h_{\nu} - \nabla_{\nu} h_{\mu} = -2\lambda\,^{*}f_{\mu\nu}, \tag{19}$$

then, using (2), follows again: $T = df = {}^{*}h$ and Eq. (17)

$$d\,^{*}f = 0 \tag{20}$$

and fundamentally

$$df = -\frac{1}{2\lambda} d\,^{*}dh = T = {}^{*}h, \tag{21}$$

$$d\,^{*}dh = -2\lambda\,^{*}h \tag{22}$$

that we can recognize the Laplace–de Rham operator that helps us to write the wave covariant equation

$$[(d\delta + \delta d) + 2\lambda]\,^{*}h = 0, \quad (\Delta + 2\lambda)\,^{*}h = 0. \tag{23}$$

If we start with the potential it is not difficult to see that an equivalent equation can be found:

$$(\Delta + 2\lambda)\,^{*}f = 0. \tag{24}$$

Notice that Eq. (23) comes from (18) and is a consequence of the Tfh-relation $(T = df = {}^{*}h)$, but (24) comes directly from (17). The geometric interplay between Tfh-relations is[1]

$$\begin{tikzcd}
& \boxed{T} \arrow[dl, shift right, "\int"'] \arrow[dr, shift left, "(-1)^{d+1}*"] & \\
\boxed{f} \arrow[ur, shift right, "d"'] \arrow[rr, shift right, "-2\lambda\int^{*}"'] & & \boxed{h} \arrow[ul, shift left, "*"'] \arrow[ll, shift right, "\frac{-1^{*}d}{2\lambda}"']
\end{tikzcd} \tag{25}$$

[1]In order to be consistent with the action of the Hodge operator $(*)$, in this section, we assume an even number of dimensions.

And finally, the explicit computation of the determinant in ($d = 4$) of expression (8) that will help us in comparing the unitarian model introduced here (in the sense of Eddington [13]) with the dualistic non-Abelian Born–Infeld model of [14], takes the familiar form [14]

$$S = \frac{b^2}{4\pi}\int \sqrt{-g}\,dx^4 \left\{ \overbrace{\sqrt{\gamma^4 - \frac{\gamma^2}{2}\overline{G}^2 - \frac{\gamma}{3}\overline{G}^3 + \frac{1}{8}\left(\overline{G}^2\right)^2 - \frac{1}{4}\overline{G}^4}}^{\equiv \mathbb{R}} \right\}, \tag{26}$$

$$G_{\mu\nu} \equiv \left[\lambda^2\left(g_{\mu\nu} + f^a_\mu f_{a\nu}\right) + 2\lambda R_{(\mu\nu)} + 2\lambda f^a_\mu R_{[a\nu]} + R^a_\mu R_{a\nu}\right], \tag{27}$$

$$G^\nu_\nu \equiv \left[\lambda^2\left(d + f_{\mu\nu} f^{\mu\nu}\right) + 2\lambda\left(R_S + R_A\right) + \left(R^2_S + R^2_A\right)\right], \tag{28}$$

where

$$\begin{gathered} R_S \equiv g^{\mu\nu} R_{(\mu\nu)}; \quad R_A \equiv f^{\mu\nu} R_{[\mu\nu]}; \quad \gamma \equiv \frac{G^\nu_\nu}{d}; \quad \overline{G}_{\mu\nu} \equiv G_{\mu\nu} - \frac{g_{\mu\nu}}{4} G^\nu_\nu; \\ \overline{G}^\nu_\rho \overline{G}^\rho_\nu \equiv \overline{G}^2, \quad \overline{G}^\nu_\lambda \overline{G}^\lambda_\rho \overline{G}^\rho_\nu \equiv \overline{G}^3 \left(\overline{G}^\nu_\rho \overline{G}^\rho_\nu\right)^2 \equiv \left(\overline{G}^2\right)^2 \overline{G}^\nu_\mu \overline{G}^\mu_\lambda \overline{G}^\lambda_\rho \overline{G}^\rho_\nu \equiv \overline{G}^4 \end{gathered} \tag{29}$$

and the relevant quantities involved into the dynamical equations (12) and (13) are

$$\mathbb{F}^{\mu\nu} \equiv \frac{\partial L_G}{\partial f_{\mu\nu}} = \frac{\lambda^2 N^{\mu\nu}\left(\delta^\sigma_\mu f^\rho_\nu + \delta^\sigma_\nu f^\rho_\mu\right)}{2\mathbb{R}}, \tag{30}$$

$$\mathbb{T}^{\epsilon\gamma\delta} \equiv \frac{\partial L_G}{\partial T_{\epsilon\gamma\delta}} = \frac{N^{\mu\nu} M^{\epsilon\cdot\gamma\cdot\delta}_{\cdot\alpha\cdot\beta}\left(2\lambda\delta^\alpha_\mu\delta^\beta_\nu + R^\alpha_\nu\delta^\beta_\mu + R^\alpha_\mu\delta^\beta_\nu\right)}{2\mathbb{R}}, \tag{31}$$

$$\begin{aligned} N^{\mu\nu} = g\Bigg[&-\gamma^2 G^{\mu\nu} - \gamma\left(G^2\right)^{\mu\nu} + \frac{\left(G^2\right)^\mu_\mu G^{\mu\nu}}{2} - \\ &- \left(G^3\right)^{\mu\nu} + \frac{4\gamma^3 g^{\mu\nu}}{d} - \frac{\gamma\left(G^2\right)^\mu_\mu g^{\mu\nu}}{d} - \frac{\left(G^3\right)^\mu_\mu g^{\mu\nu}}{3d}\Bigg], \end{aligned} \tag{32}$$

$$M^{\epsilon\cdot\gamma\cdot\delta}_{\cdot\alpha\cdot\beta} = \left(\delta^\epsilon_\mu T^{\delta\ \gamma}_\nu + T^{\delta\epsilon}_\mu \delta^\gamma_\nu\right). \tag{33}$$

5. WORMHOLE-INSTANTON SOLUTION IN UFT THEORY

The action in four dimensions is given by

$$S = -\frac{1}{16\pi G}\int d^4x \sqrt{\det\left|G_{\mu\nu}\right|}, \tag{34}$$

$$\mathbb{R} \equiv \sqrt{\gamma^4 - \frac{\gamma^2}{2}\overline{G}^2 - \frac{\gamma}{3}\overline{G}^3 + \frac{1}{8}\left(\overline{G}^2\right)^2 - \frac{1}{4}\overline{G}^4}. \tag{35}$$

Scalar curvature R and the torsion two-form field $T^a_{\mu\nu}$ with a $SU(2)$ — Yang–Mills structure are defined in terms of the affine connection $\Gamma^\lambda_{\mu\nu}$ and the $SU(2)$ potential torsion $f^{\ a}_{\mu}$ by

$$\begin{gathered} R = g^{\mu\nu}R_{\mu\nu}, \quad R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \quad R^\lambda_{\mu\lambda\nu} = \partial_\nu\Gamma^\lambda_{\mu\rho} - \partial_\rho\Gamma^\lambda_{\mu\nu} + \dots, \\ T^a_{\mu\nu} = \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \varepsilon^a_{bc} f^b_\mu f^c_\nu. \end{gathered} \tag{36}$$

G and Λ are the Newton gravitational constant and the cosmological constant, respectively. Notice the important fact that from the last equation for the torsion two-form, the potential f^a_μ must be proportional to the antisymmetric part of the affine connection $\Gamma^\lambda_{\mu\nu}$ as in the Strauss–Einstein UFT. As in the case of Einstein–Yang–Mills systems, for our new UFT model it can be interpreted as a prototype of gauge theories interacting with gravity (e.g., QCD, GUTs, etc.). Upon varying the action (31), we obtain the Einstein equation

$$R_{\mu\nu} = -2\lambda\left(g_{\mu\nu} + f_{\mu\nu}\right) \tag{37}$$

and the field equation for the torsion two-form in differential form

$$d^*\mathbb{T}^a + \frac{1}{2}\varepsilon^{abc}\left(f_b \wedge^* \mathbb{T}_c -^* \mathbb{T}_b \wedge f_c\right) = \mathbb{F}^a, \tag{38}$$

where we define as usual

$$\mathbb{T}^a_{bc} \equiv \frac{\partial L_{\mathrm{NBI}}}{\partial T^{bc}_a}, \quad \mathbb{F}^a_{bc} \equiv \frac{\partial L_{\mathrm{NBI}}}{\partial F_a}.$$

We are going to seek for a classical solution of Eqs. (33) and (34) with the following spherically symmetric ansatz for the metric and gauge connection:

$$ds^2 = d\tau^2 + a^2(\tau)\,\sigma^i \otimes \sigma^i \equiv d\tau^2 + e^i \otimes e^i, \tag{39}$$

here τ is the Euclidean time and the dreibein is defined by $e^i \equiv a^2(\tau)\,\sigma^i$. The gauge connection is

$$f^a \equiv f^a_\mu\, dx^\mu = h\sigma^a \tag{40}$$

for $a = 1, 2, 3$ and for $a = 0$

$$f^0 \equiv f^0_\mu\, dx^\mu = s\sigma^0, \tag{41}$$

this choice for the potential torsion is the most general and consistent from the physical and mathematical points of view, as we will show soon. The σ^i one-form satisfies the $SU(2)$ Maurer–Cartan structure equation

$$d\sigma^a + \varepsilon^a_{bc}\sigma^b \wedge \sigma^c = 0. \tag{42}$$

Notice that in the ansatz the frame and isospin indices are identified as for the case with the NBI Lagrangian of [14]. The torsion two-form

$$T^\gamma = \frac{1}{2}T^\gamma_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{43}$$

becomes

$$T^a = df^a + \frac{1}{2}\varepsilon^a_{bc} f^b \wedge f^c = \left(-h + \frac{1}{2}h^2\right)\varepsilon^a_{bc}\sigma^b \wedge \sigma^c. \tag{44}$$

Notice that f^0 plays no role here because we take simply $ds = 0$ (the $U(1)$ component of $SU(2)$, in principle, does not form part of the space spherical symmetry), and the expression for the torsion is analogous to the non-Abelian two-form strength field of [14]. Also the Levi–Civitta tensor is defined in order to have the T pure hyperimaginary in agreement with expression (2). Inserting T^a from Eq. (44) into the dynamical equation (38), we obtain

$$\begin{gathered} d^*\mathbb{T}^a + \frac{1}{2}\varepsilon^{abc}\left(f_b \wedge {}^*\mathbb{T}_c - {}^*\mathbb{T}_b \wedge f_c\right) = {}^*\mathbb{F}^a, \\ (-2h + h^2)(1-h)d\tau \wedge e^b \wedge e^c = -2\lambda d\tau \wedge e^b \wedge e^c, \end{gathered} \tag{45}$$

where

$${}^*\mathbb{T}^a \equiv \frac{\lambda\sqrt{|g|}}{\sqrt{3}} hA(-2h + h^2)d\tau \wedge \frac{e^a}{a^2}, \tag{46}$$

$${}^*\mathbb{F}^a = -\frac{2\lambda^2\sqrt{|g|}}{\sqrt{3}} hA \frac{d\tau \wedge e^b \wedge e^c}{a^3}, \tag{47}$$

$$A \equiv \lambda^4 \left[(1+\alpha)^2 + \alpha/2\right], \tag{48}$$

and

$$\alpha = \frac{1}{2}\left(s^2 + 3h^2\right), \tag{49}$$

from expression (45) we have an algebraic cubic equation for h

$$(-2h + h^2)(1-h) + 2\lambda = 0. \tag{50}$$

We can see that, in contrast with our previous work with a dualistic theory [14], for h there exist three nontrivial solutions depending on the cosmological constant λ. But, at this preliminary analysis of the problem, only the values of h that make the quantity $\left(-h + 1/2h^2\right) \in \mathbb{R}$ are relevant for our purposes: due to the pure imaginary character of T in the Euclidean framework (see Appendix 1) and mainly to compare with the NABI wormhole solution of our previous work (the question of the $h \in \mathbb{C}$ will be the focus of a further paper [9]). As the value of $h \in \mathbb{R}$ is -1 and in four space-time dimensions $\lambda = |1-d| = 3$,

$$T^a_{bc}|_{h_1} = 3\frac{\varepsilon^a_{bc}}{a^2}; \quad T^a_{0c} = 0. \tag{51}$$

Namely, only the magnetic field is nonvanishing, while the electric field vanishes. An analogous feature can be seen in the solution of Giddings and Strominger [15] and in our previous paper [14]. Substituting the expression for the torsion two-form (51) into the symmetric part of the variational equation, namely,[1]

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T^\alpha_{\mu\rho}T^\rho_{\alpha\nu} = -2\lambda g_{\mu\nu} \tag{52}$$

[1] In the tetrad: $\overset{\circ}{R}_{00} = -3\frac{\ddot{a}}{a}, \overset{\circ}{R}_{ab} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - \frac{2}{a^2}\right]$.

(in the tetrad: $\overset{\circ}{R}_{00} = -3\,\ddot{a}/a, \overset{\circ}{R}_{ab} = -1/a[\ddot{a}a + 2\dot{a}^2 - 2]$), we reduce Eq. (15) to an ordinary differential equation for the scale factor a,

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2}\right] = \frac{2\lambda}{3} - \frac{9}{2a^4}, \tag{53}$$

$$\frac{\ln\left[1 + 4a^2 + 2\sqrt{-9 + 2a^2 + 4a^4}\right]}{2\sqrt{2}} = \tau - \tau_0, \tag{54}$$

$$T^{\alpha}_{\mu\rho} T^{\rho}_{\alpha\nu} = \frac{\left(-h + 1/2h^2\right)^2}{a^4} 2\delta_{\mu\nu} = \frac{9}{2a^4}\delta_{\mu\nu}. \tag{55}$$

There are two values for the scale factor a: max. and min., respectively, namely,

$$a = \mp \frac{e^{-\sqrt{2}(\tau-\tau_0)}\sqrt{37 - 2\,e^{2\sqrt{2}(\tau-\tau_0)} + e^{4\sqrt{2}(\tau-\tau_0)}}}{2\sqrt{2}}. \tag{56}$$

Now we will need to see what happens with Eq. (17) in this particular case under consideration. Well, Eq. (17) takes the following form:

$$\begin{gathered} d\,{}^*T^a + \frac{1}{2}\varepsilon^{abc}\left(f_b \wedge {}^*T_c - {}^*T_b \wedge f_c\right) = -2\lambda^* f^a, \\ (-2h + h^2)(1 - h)d\tau \wedge e^b \wedge e^c = -2\lambda d\tau \wedge e^b \wedge e^c, \end{gathered} \tag{57}$$

$${}^*T^a \equiv h(-2h + h^2)\,d\tau \wedge \frac{e^a}{a^2}, \tag{58}$$

$${}^*f^a = -h\frac{d\tau \wedge e^b \wedge e^c}{a^3}. \tag{59}$$

Then we arrived at the same equation for λ as (50) corroborating the self-consistency of the procedure.

6. DISCUSSION

In the previous section we showed that the nondualistic unified model proposed here has, from the point of view of the obtained solutions, deep differences with the dualistic non-Abelian Born–Infeld model of our early reference. The first obvious difference comes from a conceptual framework: the geometrical action will provide, besides the space-time structure, the matter-energy spin distribution. This fact is the same basis of the unification: all the (apparently disconnected) theories and interactions of the natural world appear naturally as a consequence of the intrinsic space-time geometry. The second point to have account here is about the Hosoya and Ogura ansatz: Why does the identification of the isospin structure of the Yang–Mills field with the space frame lead to a physical situation similar to that in a nondualistic unified theory with torsion? The answer is: because at once such identification is implemented, a potential torsion is introduced and the solution of the set of equations is the consistency between the definition of the torsion tensor from the potential and the Cartan structure equations, namely,

$$df = T + f^{\alpha} \wedge T^{\beta}\eta_{\alpha\beta}, \tag{60}$$

$$D\omega^\alpha \equiv d\omega^\alpha + \omega^\alpha_\beta \wedge \omega^\beta = T^\alpha, \tag{61}$$

$$\mathcal{R}^\alpha_\beta = D\omega^\alpha_\beta. \tag{62}$$

Here, however, $f \equiv \frac{1}{2} f_{\alpha\beta}\omega^\alpha \wedge \omega^\beta$, $T \equiv T_{\alpha\beta\gamma}\omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$, $T^\alpha \equiv \frac{1}{2} T^\alpha_{\gamma\beta}\omega^\gamma \wedge \omega^\beta$ and $f^\alpha \equiv f^a_\mu \omega^\mu$. The set of equations (60), (61) is clearly self-consistent. The explanation from a pure algebraic and geometrical framework about what happens with the underlying structure of the manifold is given with details in Appendix 2.

The third point is about the obtained solutions for the scale factor a in the UFT and in the NABI model already introduced in [14]. The difference with our previous work comes precisely from the set of equations in both models that differ precisely in two points, namely,

i) the presence of the cosmological (dimension-dependent) constant λ that transforms the simple equation for h in [14] to a cubic equation in the UFT case;

ii) the form of the function A that comes from the particular form of the determinantal actions: from the geometrical fundamental Lagrangian here and the NABI energy momentum tensor in [14].

Beside these differences, both solutions describe a classical wormhole instanton, but this solution, Eq. (56), grows faster than the previous one of [14] due to its manifestly exponential behaviour. This characteristic of the solution can be analyzed in the context of inflationary cosmological models, issue that will be a focus in a future work [9]. By the way, it is interesting to note that N. Chernikov in [18] was able to find the link between the dynamical field equations of the standard (Abelian) Born–Infeld theory and the $T^\alpha_{\beta\alpha}$ covector torsion:

$$T^\beta_{\alpha\beta} = \frac{\delta^\mu_\alpha - f^\rho_\alpha f^\mu_\rho}{\sqrt{1+S-G^2}} \overset{\circ}{\nabla}_\gamma \frac{f^\gamma_\mu - G\widetilde{f}^\gamma_\mu}{\sqrt{1+S-G^2}}. \tag{63}$$

However, S and G are the scalar and pseudoscalar invariants of the antisymmetric part $\left(f_{[\mu\nu]}\right)$ of the fundamental nonsymmetric tensor $G_{\mu\nu} = g_{(\mu\nu)} + f_{[\mu\nu]}$ of the Einstein–Strauss unified theory («$\sim$» means «dual» in the common electromagnetic sense and the modern notation in (63) is from [19]). Notice from (63) that when the covector torsion is zero, the set of equations are precisely as in the Einstein–Born–Infeld model. This fact occurs in the model present here due to the full antisymmetry of the torsion tensor: $T^\alpha_{\beta\alpha} \equiv 0$, but a cosmological constant remains. Then, a slight discrepancy arises between the Einstein–Strauss theory with an asymmetric fundamental tensor and the theory presented due to the (still) existence of a cosmological term. This important issue must be discussed with greater care in the near future [9].

The advantages of this nice model are clearly exposed in all this paper. The things to improve are:

i) the dependence on the dimensions through the cosmological constant $\lambda = |1-d|$;

ii) the lack of a manifest fermionic structure;

iii) tetrad field depending on the breaking of symmetry of the underlying topological action, then the clear necessity of a reductive space-time structure from the geometrical point of view.

Acknowledgements. This paper is in the memory of the soviet scientist N. A. Chernikov, who was the first to remark the connection between the Born–Infeld field equations and the general form of the affine geometries.

Appendix 1

ON HYPERCOMPLEX AND COMPLEX QUANTITIES

In abstract algebra, the split-complex numbers (or hyperbolic numbers) are a two-dimensional commutative algebra over the real numbers different from the complex numbers. Every split-complex number has the form

$$x + yj,$$

where x and y are real numbers. The number j is similar to the imaginary unit i, except that

$$j^2 = 1.$$

As an algebra over the reals, the split-complex numbers are simply the same as the direct sum algebra $\mathbb{R} \oplus \mathbb{R}$ (under the isomorphism sending $x + yj$ to $(x + y,\ x - y)$). The name split comes from this characterization: as a real algebra, the split-complex numbers «split» into the direct sum $\mathbb{R} \oplus \mathbb{R}$.

Geometrically, multiplication of split-complex numbers preserves the (square) Minkowski norm $(x^2 - y^2)$ in the same way that multiplication of complex numbers preserves the (square) Euclidean norm $(x^2 + y^2)$. Unlike the complex numbers, the split-complex numbers contain nontrivial idempotents (other than 0 and 1), as well as zero divisors, and therefore they do not form a field.

The split-complex number is one of the concepts necessary to read a 2×2 real matrix.

Split-complex numbers (sometimes called hyperbolic hypercomplex numbers) are constructed from the bases with $j^2 = +1$ a nonreal root of 1.

Algebras that include such nonreal roots of 1 contain idempotents and zero divisors $(1 + j)(1 - j) = 0$, so such algebras cannot be division algebras. However, these properties can turn out to be very meaningful, for instance, in describing the Lorentz transformations of special relativity.

Appendix 2

HOLONOMIC AND ANHOLONOMIC «COORDINATES»

There is a confusion in the literature over the use of the word «coordinates». As a result, in the older literature influenced by J. A. Schouten [12], the terms «holonomic coordinate system» and «anholonomic system» are used. And for an anholonomic system an «anholonomic object» is employed. In the newer literature, exemplified by Bernard Schutz [16], the terms «coordinate system» and «noncoordinate system» are used. In this case the «anholonomic object» is replaced by the Lie algebra structure constant tensor. The key is to understand the relationships between manifolds and the vector fields which live on them. Also we must understand the difference between a commutative Lie group and a noncommutative Lie group and the effect which this difference makes on the vector fields on the respective Lie group manifolds. A coordinate system (= holonomic coordinate system) is characterized by the partial derivative nature of the vector fields associated with the coordinates. In symbols we can write that for coordinates, $x^1, x^2, \ldots$, we have the vector field basis:

$$\frac{\partial}{\partial x^1}, \quad \frac{\partial}{\partial x^2}.$$

Because a partial derivative is with respect to one variable and leaves all others fixed, the partial derivative operators are commutative. That is

$$\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right] \equiv \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2} = 0$$

(the same is true for any x^i, x^j of course).

On any manifold, however, our starting point could be to consider the set of vector fields which live on the manifold. These vector fields are characterized by the flow lines (or integral curves) on the manifold. These flow lines can be used to describe coordinate systems on the manifold. In this case we will describe the vector fields in terms of the parameters along the flow lines. If we write these parameters with Greek letters μ, λ, etc. (to distinguish them from coordinates x^i), then we can write these vector fields as

$$V = \frac{d}{d\mu}, \quad W = \frac{d}{d\lambda}.$$

Notice that these are total differential operators. These operators are appropriate in case the operators do not commute. In this case the parameters are not a parameterization appropriate to a coordinate system (or are «anholonomic coordinates» in the terminology of Schouten). As long as the vector fields V and W are independent, we can use them as a basis for a grid of parameters μ and λ. And, assuming V and W do not commute, this grid will not be a coordinate system (i.e., is «anholonomic»). Thus, it is clear that the «anholonomic object» must be equivalent to the Lie bracket structure constants for a Lie algebra. For a Lie algebra this is a tensor. How then is it possible for the «anholonomic object» of a geometry to be coordinatized away? To understand this we need a simple example. Take the ordinary Euclidean plane R^2, with coordinates x and y. We can define the X and Y vector fields as

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}.$$

This simply means that we fill up the x direction in the plane with a congruence of parallel flow lines for the vector field X, and similarly for the y direction. This is a perfectly commutative basis for R^2. However, we can also define polar «coordinates» (more correctly parameters) r and θ on R^2. In this case we can define the vector fields:

$$\widehat{r} = \cos\theta X + \sin\theta Y, \quad \widehat{\theta} = -\sin\theta X + \cos\theta\, Y$$

and the commutator of these vector fields is

$$\left[\widehat{r}, \widehat{\theta}\right] = -\frac{\widehat{\theta}}{r}.$$

Thus, $\widehat{r}$ and $\widehat{\theta}$ are a noncoordinate basis (cf. [16, p. 44]). It is clear, however, that we can revert to a coordinate basis with X and Y as basis vector fields. So in this case the commutator «anholonomic» object can be coordinatized away by changing to the x, y axes as coordinates. This is possible because the underlying manifold R^2 is a commutative Lie group. Other examples of commutative Lie group manifolds are the n-dimensional vector spaces R^n, C^n of real or complex numbers and the n-dimensional torus T^n (i.e., a direct product of n circles S^1).

By now it should be clear that if the underlying manifold is a noncommutative Lie group, then the (noncommutative) Lie algebra of left-invariant vector fields on the Lie group manifold will provide a vector field basis (equivalent in dimensionality to that of the Lie group) which is a noncoordinate basis (i.e., «anholonomic»). And in this case the commutator of these vector fields is nonzero and thus the Lie algebra structure constant tensor is nonzero. This tensor plays the role of the «anholonomic object» and there is no way to coordinatize away this tensor. Moreover, the connection provided by the left-invariant vector fields provides an absolute parallelism structure on the Lie group manifold. (Note: absolute parallelism provides parallel transport of tangent vectors independent of the path throughout the Lie group manifold.)

This connection is commonly called the Cartan connection because of his attempt to describe electromagnetism by way of the torsion tensor T associated with this asymmetric connection:

$$\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta} = T^{\alpha}_{\beta\gamma}.$$

This torsion tensor is equivalent to the Lie algebra structure constant tensor [11]:

$$[X_i, X_j] = T^k_{ij} Z_k$$

(where T is usually written as C: the structure constant or function, in the general case).

In summary, three cases must clearly be distinguished:

i) The underlying manifold is a commutative Lie group (for example, R^n, C^n, T^n). In this case, the Lie algebra (of left-invariant vector fields) is commutative and thus provides a coordinate basis («holonomic coordinates»). However, it is possible to set up a noncoordinate basis for vector fields, in which the basis fields do not commute. This sets up an artificial nonzero commutator, which plays the role of an «anholonomic object». But, clearly, it can be coordinatized away by reverting to the commutative Lie algebra basis structure of left-invariant vector fields. (Note that on any manifold there is an infinite dimensional basis of vector fields. However, on a Lie group manifold, the action of the Lie group on itself and its vector fields provides for a finite set of left-invariant basis fields, where the dimensionality of this basis is that of the Lie group itself. This is the canonical basis for the Lie algebra of the Lie group).

ii) The underlying manifold is a noncommutative Lie group (for example, $SU(n)$, $SO(n)$, $E6$, $E7$, $E8$). In this case, the Lie algebra (of left-invariant vector fields) is noncommutative, and thus provides a noncoordinate basis («anholonomic coordinates»). The Lie algebra structure constant tensor C^k_{ij} plays many roles: «anholonomic object»; torsion tensor (relative to the Cartan connection); and (for particle physics) gauge group eigenvalues.

iii) The underlying manifold is not a Lie group. (For example, spheres S^n of any dimension n, except 1 and 3, since $S^1 = U(1)$, and $S^3 = SU(2)$ are Lie groups.) This case may be of interest to certain applications of mechanics. However, it should be noted that according to the classification work of [17], only Lie group manifolds are capable of carrying an absolute parallelism connection. The one exception to this rule is the 7-sphere S^7, which gets its parallelization from the fact that it is the set of unit length vectors in the 8-d Cayley algebra (the octonions).

Thus, if one is attempting to model electromagnetism via torsion in an absolute parallelism geometry, one should consider only the noncommutative Lie group case. (The commutative Lie groups carry no (Cartan-type) torsion, of course.)

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