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CLASSICAL AND QUANTUM SCATTERING BY A GRAVITATIONAL CENTER

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The corrections to the leading term of the small-angle deflection of a classical particle by the Schwarzschild field and its linear approximation were found. The corresponding cross sections were obtained. The comparison with known in Born approximation cross sections for quantum massless particles of spins 0, 1, and 2 shows that only the leading term in all cases is the same. As the conditions for classical treatment are well fulfilled, this means that the classical results are much more accurate than the quantum one in Born approximation. The fact that the photon cross section is always smaller than that of massless scalar particle (both in Born approximation) suggests that with small probability (at least of order of the difference of these cross sections) the photon can fly by the Sun without deflection. The deflection of light, observable at a final distance from the Sun, is also considered and it is shown that measurements at the distances of several Sun's radii will decide which coordinate system is the privileged one.

Найдены поправки к главному члену отклонения на малый угол классической частицы в поле Шварцшильда и в поле линейного приближения к нему. Получены соответствующие сечения. Сравнение с сечениями рассеяния квантовых частиц со спинами 0, 1, 2, полученными в борновском приближении, показывает, что только главные члены совпадают. Поскольку условия применимости классического приближения хорошо выполнены, это означает, что классические результаты в рассматриваемом случае существенно точнее борновского приближения квантовой теории. Так как в борновском приближении сечение рассеяния фотона всегда меньше сечения рассеяния безмассовой скалярной частицы, можно думать, что с малой вероятностью фотон может пролететь через поле без отклонения. Рассмотрено также отклонение света на конечных расстояниях от Солнца, и показано, что измерения на расстояниях порядка нескольких радиусов от Солнца решат вопрос о том, какая система координат является привилегированной.

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INTRODUCTION

When a photon is deflected by the Sun, we can know with good accuracy its impact parameter ρ, its frequency, and the coordinates of place where it is observed. We note that we always observe it at a finite distance from the Sun and in principle the place of observation is at our disposal. Classical theory permits one to predict the results of such observations. The classical approach is applicable when orbital quantum number is large: $l = k\rho \gg 1$ (ρ is impact parameter), i.e., $\rho/\lambda \gg 1$. For light passing the Sun $l \sim 10^{15}$, see Ch. 8 in Weinberg

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book [1] or § 20 in Schiff's book [2] for details. For a scattering with the momentum transfer q, the impact parameter ρ is of order of $\hbar/q$ and the formation length of this process is of order of several $\hbar/q$. Beyond this length the particle is unable to obtain the momentum transfer q. In quantum theory in Born approximation the particle obtains the required momentum transfer by interacting with only one graviton. In principle the quantum particle can pass the formation length without interaction or interacting with two or more gravitons in such a way that no momentum transfer is passed to it. It is very interesting to know the probability for this process which can be searched in photons passing the Sun. If it is nonzero, then the concept of curved space-time is of only limited validity. The classical particle cannot fly through the field without interaction. So the classical cross section should be greater than the quantum one and the difference should indicate the probability of flying without deflection. Unfortunately, the quantum cross sections are known only in Born approximation and there is very little hope to get more accurate expressions. The Born approximation turns out to be insufficient for comparing with more accurate (in this region of impact parameters) classical results. So we compare quantum cross sections for photon and massless scalar particle. (We note that similar effects should take place also in Coulomb scatterings [3].) According to heuristic approach to gravity [4–6] the curved space-time appears as a result of changing by the gravitational field of measuring rods and clocks. In principle these changes can be calculated and in the lowest approximation this was done by Thirring [5]. If Nature itself exactly calculates these changes, the result should be unique and it will fix the (privileged) coordinate system. It is expected that for a spherically symmetric body this system should be isotropic one [6]. In the privileged system the deflection of light at any point is given by the tangent to the trajectory. So the measurements of light deflection at several Sun's radii will decide which coordinate system is the privileged one.

In Sec. 1 we obtain the asymptotic expansions for the small-angle classical scattering by the Schwarzschild field. In Sec. 2 the same is done for linear approximation of that field. (This linear approximation is used in quantum calculations in Born approximation, so it is useful to compare these classical results with the quantum one.) The difference in exact and linearized versions indicates the role of nonlinearities in scatterings. We rimind that the nonlinear effects of general relativity are confirmed only in nonrelativistic region (precession of Mercury perihelion). In Sec. 3 we calculate in the lowest approximation the tangent to the trajectory in the standard Schwarzschild and isotropic coordinate systems. At the distances of order ρ, the tangents in these two systems are different and the experiment should decide which coordinate system has more chances to be the privileged one.

1. SMALL-ANGLE CLASSICAL SCATTERING BY THE SCHWARZSCHILD FIELD

1.1. Particle with Mass. The trajectories in the Schwarzschild field were studied in a number of papers [7–9], see also [10]. If the standard coordinate system

$$ds^2 = \left(1 - \frac{r_g}{r}\right)(dx^0)^2 - r^2(\sin^2\theta\, d\varphi^2 + d\theta^2) - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 \qquad (1)$$

is used, the equation governing the scattering trajectory has the simplest form

$$\frac{du}{d\vartheta} = \pm\sqrt{f(u)}, \quad u = \frac{\rho}{r}, \quad \delta = \frac{r_g}{\rho}, \quad r_g = \frac{2GM}{c^2},$$
$$f(u) = 1 - u^2 + \varkappa u\delta + u^3\delta = \delta(u-u_1)(u-u_2)(u-u_3), \quad \varkappa = \beta^{-2} - 1. \tag{2}$$

Here ρ is impact parameter; β is velocity at infinity in units of c; the signs «+» and «−» in front of the square root refer to the first and the second halves of trajectory, respectively. From (2) the half angle between the asymptotes is

$$\vartheta_{1/2} = \int\limits_0^{\vartheta_{1/2}} d\vartheta = \int\limits_0^{u_2} \frac{du}{\sqrt{f(u)}}. \tag{3}$$

This is the contribution from the first half of the trajectory. It is assumed that $u_1 < u_2 < u_3$. Similarly, for the second half,

$$2\vartheta_{1/2} - \vartheta_{1/2} = \int\limits_{\vartheta_{1/2}}^{2\vartheta_{1/2}} d\vartheta = -\int\limits_{u_2}^{0} \frac{du}{\sqrt{f(u)}} = \int\limits_0^{u_2} \frac{du}{\sqrt{f(u)}} = \vartheta_{1/2}. \tag{4}$$

So the angle between asymptotes is $2\vartheta_{1/2}$. The scattering angle is $\theta = 2\vartheta_{1/2} - \pi$. Each half of the trajectory contributes $\vartheta_{1/2}$.

Now the expressions for roots u_1, u_2, u_3 as functions of δ and $\varkappa$ can be obtained by the method of Newton: if $x^{(0)}$ is the zeroth-order approximation for the root of $f(x) = 0$, then in the first approximation we have $x^{(1)} = x^{(0)} - \dfrac{f(x^{(0)})}{f'(x^{(0)})}$. In the second approximation $x^{(2)} = x^{(1)} - \dfrac{f(x^{(1)})}{f'(x^{(1)})}$ and so on. Thus, for the root u_2 starting from the zeroth-order approximation $u_2^{(0)} = 1$, we have for the function $f(u)$ in (2) $f(1) = (\varkappa+1)\delta$. From $f'(u) = -2u + \varkappa\delta + 3u^2\delta$ we get $f'(1) = -2 + (\varkappa+3)\delta \approx -2$. So $u_2^{(1)} = 1 + \dfrac{1}{2}(1+\varkappa)\delta$. In the second approximation $f\left(u_2^{(1)}\right) = \left(\dfrac{5}{2^2} + \dfrac{3\varkappa}{2} + \dfrac{\varkappa^2}{2^2}\right)\delta$. As for $f'(u_2^{(1)})$, we may in the considered approximation still take $f'(1) = -2$. So we get $u_2^{(2)} = 1 + \dfrac{1}{2}(1+\varkappa)\delta + \left(\dfrac{5}{2^3} + \dfrac{3\varkappa}{2^2} + \dfrac{\varkappa^2}{2^3}\right)\delta^2$. Continuing this process, we find

$$u_2 = 1 + \frac{1}{2}(1+\varkappa)\delta + \left(\frac{5}{2^3} + \frac{3\varkappa}{2^2} + \frac{\varkappa^2}{2^3}\right)\delta^2 + \left(1 + \frac{3\varkappa}{2} + \frac{\varkappa^2}{2}\right)\delta^3 + \ldots \tag{5}$$

Now the root u_1 can be obtained from here if we note that according to (2) $f(u) \equiv f(u, \delta) = f(-u, -\delta)$. Hence,

$$u_1 \equiv u_1(\delta) = -u_2(-\delta) = -1 + \frac{1}{2}(1+\varkappa)\delta - \left(\frac{5}{2^3} + \frac{3\varkappa}{2^2} + \frac{\varkappa^2}{2^3}\right)\delta^2 + \left(1 + \frac{3\varkappa}{2} + \frac{\varkappa^2}{2}\right)\delta^3 + \ldots \tag{6}$$

The expansion for u_3 can be obtained in the same way as for u_2:

$$u_3 = \frac{1}{\delta} - (1+\varkappa)\delta - (2 + 3\varkappa + \varkappa^2)\delta^3 + \dots \tag{7}$$

It is easy to check that $u_1 + u_2 + u_3 = \delta^{-1}$ as it should be.

From (2) and (3) we have in agreement with [7–9]

$$\begin{gathered} \vartheta_{1/2} = \frac{1}{\sqrt{\delta}} \int\limits_0^{u_2} \frac{du}{\sqrt{(u-u_3)(u-u_2)(u-u_1)}} = \frac{2}{\sqrt{(u_3-u_1)\delta}} F(\phi, \kappa), \\ \kappa = \left(\frac{u_2 - u_1}{u_3 - u_1}\right)^{1/2}, \quad \sin^2\phi = \frac{1 - u_1 u_3^{-1}}{1 - u_1 u_2^{-1}}. \end{gathered} \tag{8}$$

Here $F(\phi, \kappa)$ is the elliptical integral. Using (5), (6) and (7) we get, retaining terms up to δ^2,

$$\kappa^2 = 2\delta(1 - \delta + \dots), \quad \sin^2\phi = \frac{1}{2}\left\{1 + \frac{3+\varkappa}{2}\delta + O(\delta^3)\right\}. \tag{9}$$

From here

$$\sin\phi = \frac{1}{\sqrt{2}}\left[1 + \frac{3+\varkappa}{2^2}\delta - \frac{(3+\varkappa)^2}{2^5}\delta^2 + \dots\right], \quad \cos\phi = \frac{1}{\sqrt{2}}\left[1 - \frac{3+\varkappa}{2^2}\delta + \dots\right]. \tag{10}$$

To get ϕ from the expression for $\sin^2\phi$ it is worthwhile to make the substitution $\phi = \frac{\pi}{4} + \psi$, $\sin^2\phi = \frac{1}{2} + \frac{1}{2}\sin 2\psi$. (This is especially useful when more terms are needed than we retain here, see the next subsection.) So for ψ we have

$$\sin 2\psi = \frac{3+\varkappa}{2}\delta + O(\delta^3). \tag{11}$$

Then $2\psi = \frac{3+\varkappa}{2}\delta + O(\delta^3)$. Hence,

$$\phi = \frac{\pi}{4} + \psi = \frac{\pi}{4} + \frac{3+\varkappa}{2^2}\delta + O(\delta^3). \tag{12}$$

As δ is assumed to be small, κ^2 is small and we can use the expansion, see Eq. (5) in Subsec. 13.6 in [11]

$$F(\phi, \kappa) = S_0 + \frac{1}{2}S_2\kappa^2 + \frac{3}{2^3}S_4\kappa^4 + \dots, \quad S_{2n} \equiv S_{2n}(\phi) = \int\limits_0^{\phi} (\sin t)^{2n}\, dt; \tag{13}$$

$$S_0 = \phi, \quad S_2 = \frac{1}{2}(\phi - \sin\phi\cos\phi); \quad S_4 = \frac{3}{8}\phi - \frac{3}{8}\sin\phi\cos\phi - \frac{1}{4}\sin^3\phi\cos\phi. \tag{14}$$

From (6) and (7) we have

$$(u_3 - u_1)\delta = 1 + \delta - \frac{3}{2}(1+\varkappa)\delta^2 + \dots \tag{15}$$

Using Eq. (3.6.19) in [12], we obtain from here

$$[(u_3 - u_1)\delta]^{-1/2} = 1 - \frac{1}{2}\delta + \left(\frac{3}{8} + \frac{\varkappa}{4}\right) 3\delta^2 + \ldots \tag{16}$$

Performing remaining calculation, we get for $\vartheta_{1/2}$ in (8)

$$\vartheta_{1/2} = \frac{\pi}{2} + \left(1 + \frac{\varkappa}{2}\right)\delta + \pi\left(\frac{15}{32} + \frac{3\varkappa}{8}\right)\delta^2 + \ldots \tag{17}$$

The scattering angle is $\theta = 2\vartheta_{1/2} - \pi$. So

$$\frac{\theta}{2} = \vartheta_{1/2} - \frac{\pi}{2} = \left(1 + \frac{\varkappa}{2}\right)\delta + \pi\left(\frac{15}{32} + \frac{3\varkappa}{8}\right)\delta^2 + \ldots \tag{18}$$

Using Eq. (3.6.25) in [12], we find

$$\delta = \frac{\theta}{2+\varkappa} - \frac{5+4\varkappa}{(2+\varkappa)^3}\frac{3\pi}{2^4}\theta^2 + \ldots \tag{19}$$

From here with the help of (3.6.25) in [12], we find

$$\delta^{-2} = \left(1 + \frac{\varkappa}{2}\right)^2 \left(\frac{2}{\theta}\right)^2 \left\{1 + \frac{5+4\varkappa}{(2+\varkappa)^2}\frac{3\pi}{8}\theta + \ldots\right\}. \tag{20}$$

This gives the classical integral cross section

$$\sigma_{\mathrm{cl}}(\theta) = \pi\rho^2(\theta), \quad \delta^{-2} \equiv \frac{\rho^2}{r_g^2}. \tag{21}$$

If we compare (18) with the corresponding equation (10.14) in [9], we find that only the leading term is the same. The preceding equation (10.13) in [9] contains errors in powers of the root of the cubic.

The differential cross section is obtained from (21) by differentiation, multiplying by π and changing the overall sign in the right-hand side:

$$d\sigma_{\mathrm{cl}}(\theta) = 2\pi r_g^2\left(1 + \frac{1}{2}\varkappa\right)^2 \frac{1}{y^3}\left\{1 + \frac{5+4\varkappa}{(2+\varkappa)^2}\frac{3\pi}{8}y + \ldots\right\}dy, \quad y = \frac{\theta}{2}. \tag{21a}$$

In quantum picture the first Born approximation for scalar particle cross section is given in Eq. (39) below. It contains only even powers of y in braces as in Coulomb scattering [3]. This should mean that corrections to the first Born approximation must contain classical terms independent of $\hbar$. In Coulomb scattering these classical terms in higher approximations may be expected only at unrealistically high $\alpha Z \gg 1$.

1.2. Massless Particle. In this subsection we obtain more terms of the expansions for the case of massless particle, i.e., for $\varkappa = 0$, see (2). Continuing the process described in getting (5), we find

$$u_2 = 1 + \frac{1}{2}\delta + \frac{5}{2^3}\delta^2 + \delta^3 + \frac{3\cdot 7\cdot 11}{2^7}\delta^4 + \frac{7}{2}\delta^5 + \ldots, \tag{22}$$

$$u_3 = \frac{1}{\delta} - \delta - 2\delta^3 - 7\delta^5 + \ldots, \tag{23}$$

$$u_1 \equiv u_1(\delta) = -u_2(-\delta) = -1 + \frac{1}{2}\delta - \frac{5}{2^3}\delta^2 + \delta^3 - \frac{231}{2^7}\delta^4 + \frac{7}{2}\delta^5 + \ldots \tag{24}$$

Using these expressions, we find for κ^2 and $\sin^2\phi$, see (7),

$$\kappa^2 = 2\delta\left\{1-\delta+\frac{5^2}{2^3}\delta^2-\frac{3\cdot 7}{2^2}\delta^3-\frac{7\cdot 281}{2^7}\delta^4+\dots\right\}, \tag{25}$$

$$\sin^2\phi = \frac{1}{2}\left\{1+\frac{3}{2}\delta+\frac{3\cdot 11}{2^4}\delta^3+\frac{3^2\cdot 5\cdot 37}{2^8}\delta^5+\dots\right\}. \tag{26}$$

From here

$$\sin\phi = \frac{1}{\sqrt{2}}\left\{1+\frac{3}{4}\delta-\frac{3^2}{2^5}\delta^2+\frac{3\cdot 53}{2^7}\delta^3-\frac{3^2\cdot 221}{2^{11}}\delta^4+\frac{3^2\cdot 3941}{2^{13}}\delta^5+\dots\right\}. \tag{27}$$

The term with δ^5 is needed only if we want to obtain ϕ from (27). But we proceed as in obtaining (11):

$$\sin 2\psi = \frac{3}{2}\delta\left\{1+\frac{11}{2^3}\delta^2+\frac{3\cdot 5\cdot 37}{2^7}\delta^4+\dots\right\}. \tag{28}$$

From here

$$2\psi = \frac{3}{2}\delta+\frac{3\cdot 7}{2^3}\delta^3+\frac{3^2\cdot 167}{2^5\cdot 5}\delta^5+\dots \tag{29}$$

Hence,

$$\phi = \frac{\pi}{4}+\psi = \frac{\pi}{4}+\frac{3}{2^2}\delta+\frac{3\cdot 7}{2^4}\delta^3+\frac{3^2\cdot 167}{2^6\cdot 5}\delta^5+\dots \tag{30}$$

From (27) we obtain

$$\cos\phi = \frac{1}{\sqrt{2}}\left\{1-\frac{3}{4}\delta-\frac{3^2}{2^5}\delta^2-\frac{3\cdot 53}{2^7}\delta^3-\frac{3^2\cdot 221}{2^{11}}\delta^4+\dots\right\}. \tag{31}$$

In Eq. (16) (with $\varkappa = 0$) we have now more terms:

$$[(u_3-u_1)\delta]^{-1/2} = 1-\frac{1}{2}\delta+\frac{3^2}{2^3}\delta^2-\frac{7}{2^2}\delta^3+\frac{5^2\cdot 23}{2^7}\delta^4-\frac{3^3\cdot 5}{2^4}\delta^5+\dots \tag{32}$$

Using expressions for $\sin\phi$ and $\cos\phi$ in (27) and (31), we obtain S_{2n} which are polynomials in $\sin\phi$ and $\cos\phi$. Then using also (25), we find similarly to (13)

$$F(\phi,\kappa) = S_0+\frac{1}{2}S_2\kappa^2+\frac{3}{2^3}S_4\kappa^4+\frac{5}{2^4}S_6\kappa^6+\frac{5\cdot 7}{2^7}S_8\kappa^8+\frac{3^2\cdot 7}{2^8}S_{10}\kappa^{10}+\dots = \frac{\pi}{4}+\left(\frac{\pi}{2^3}+\frac{1}{2}\right)\delta+ \tag{33}$$

$$+\left(\frac{\pi}{2^6}+\frac{1}{2^2}\right)\delta^2+\left(\frac{39\pi}{2^7}+\frac{43}{2^4\cdot 3}\right)\delta^3+\left(\frac{313\pi}{2^{12}}+\frac{5^2}{2^3\cdot 3}\right)\delta^4+\left(\frac{7\cdot 1487\pi}{2^{13}}+\frac{12689}{2^8\cdot 3\cdot 5}\right)\delta^5+\dots \tag{34}$$

Using (32), we get from (8) and (34)

$$\vartheta_{1/2} = \frac{\pi}{2}+\delta+\frac{3\cdot 5\pi}{2^5}\delta^2+\frac{2^3}{3}\delta^3+\frac{3^2\cdot 5\cdot 7\cdot 11\pi}{2^{11}}\delta^4+\frac{2^3\cdot 7}{5}\delta^5\dots, \tag{35}$$

and similarly to (19) we obtain

$$\delta = y\Bigg\{1 - \frac{3\cdot 5\pi}{2^5}y + \left(\frac{3^2\cdot 5^2\pi^2}{2^9} - \frac{2^3}{3}\right)y^2 + \left(\frac{5\cdot 1867\pi}{2^{11}} - \frac{3^3\cdot 5^4\pi^3}{2^{15}}\right)y^3 + \\ + \left(\frac{2^3\cdot 19}{3\cdot 5} - \frac{3^2\cdot 5^2\cdot 7\cdot 157\pi^2}{2^{15}} + \frac{3^4\cdot 5^4\cdot 7\pi^4}{2^{19}}\right)y^4 + \ldots\Bigg\}, \quad y = \frac{\theta}{2} = \vartheta_{1/2} - \frac{\pi}{2}. \quad (36)$$

As in (20), we find

$$\delta^{-2} \equiv \frac{\rho^2}{r_g^2} = y^{-2}\{1 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4 + \ldots\}, \quad (37)$$

$$c_1 = \frac{3\cdot 5\pi}{2^4}, \quad c_2 = \frac{2^4}{3} - \frac{3^2\cdot 5^2\pi^2}{2^{10}}, \\ c_3 = -\frac{5\cdot 331\pi}{2^{10}} + \frac{3^3\cdot 5^3\pi^3}{2^{14}}, \qquad c_4 = \frac{2^4}{3\cdot 5} + \frac{3^2\cdot 5^2\cdot 331\pi^2}{2^{15}} - \frac{3^4\cdot 5^5\pi^4}{2^{20}}. \quad (37a)$$

The approximate values for c_i, $i = 1, 2, 3, 4$, are

$$c_1 = 2.945, \quad c_2 = 3.165, \quad c_3 = 1.310, \quad c_4 = -0.016. \quad (37b)$$

The differential cross section is obtained from (37) by differentiation

$$d\sigma(\theta)_{\rm cl} = \pi r_g^2 \frac{2^3}{\theta^3}\left\{1 + c_1\frac{\theta}{2^2} - c_3\frac{\theta^3}{2^4} - c_4\frac{\theta^4}{2^4} + \ldots\right\}d\theta. \quad (38)$$

It is interesting to compare this classical cross section with the quantum one. For scalar particle we have in the first Born approximation [13]

$$d\sigma_{\rm sc} = 2\pi r_g^2\left(1 + \frac{\varkappa}{2}\right)^2 \frac{\cos y}{\sin^3 y}dy = 2\pi r_g^2\left(1 + \frac{\varkappa}{2}\right)^2 \frac{1}{y^3}\left\{1 - \frac{y^4}{15} + \ldots\right\}dy, \quad y = \frac{\theta}{2}. \quad (39)$$

$\varkappa$ is defined in (2); for massless particle $\varkappa = 0$. The substitution $y \to -y$ changes the sign of $\sin y$ and y. Hence, the expression in braces in (39) can contain only even powers of y. We note also that $-c_4$ in (38) is positive ($-c_4 = 0.016$) in contrast with the corresponding coefficient $-1/15$ in (39). So this term is larger in classical cross section than the corresponding term in the quantum cross section (39). The quantum cross section for photon contains an additional factor:

$$d\sigma_\gamma = d\sigma_{\rm sc}\cos^4 y, \quad (40)$$

see [13] and references therein. The factor $d\sigma_{\rm sc}$ in (40) and below is taken at $\varkappa = 0$, i.e., for massless particle. From (39) and (40) we get

$$d\sigma_{\rm sc} - d\sigma_\gamma = 2\pi r_g^2\frac{2 - \sin^2 y}{\sin y}\cos y\, dy.$$

We see that the difference of these two cross sections is always positive and even diverges for $y \to 0$ albeit only logarithmically:

$$\int_{\tilde{y}\to 0}^{\pi/2} (d\sigma_{\rm sc} - d\sigma_\gamma) = 2\pi r_g^2 \left\{ 2\ln\frac{1}{\sin\tilde{y}} - \frac{1}{2} \right\}.$$

This difference suggests that the flying through the field without interaction is easier for photon than for a massless scalar particle. As mentioned above, it would be interesting to obtain the differences of classical and quantum cross sections for photon, but Born approximation is insufficient for this. For graviton [14, 15],

$$d\sigma_g = d\sigma_{\rm sc}\frac{1}{8}(1 + 6\cos^2\theta + \cos^4\theta) = d\sigma_{\rm sc}(\cos^8 y + \sin^8 y), \tag{40a}$$

and

$$d\sigma_{\rm sc} - d\sigma_g = 2\pi r_g^2 \frac{\cos y}{\sin^3 y}\left\{4\sin^2 y - 6\sin^4 y + 4\sin^6 y - 2\sin^8 y\right\}\, dy;$$

$$\int_{\tilde{y}\to 0}^{\pi/2} (d\sigma_{\rm sc} - d\sigma_g) = 2\pi r_g^2 \left\{ 4\ln\frac{1}{\sin\tilde{y}} - \frac{7}{3} \right\}.$$

So for small-angle scattering the cross section is smaller for particle with higher spin. It seems reasonable to expect from these facts that spin facilitates forward scattering of a particle described by a wave packet.

2. SMALL-ANGLE CLASSICAL SCATTERING BY THE LINEAR APPROXIMATION OF THE SCHWARZSCHILD FIELD

To see the effects of the nonlinearity of the Schwarzschild field on scattering, we consider in this section the linear approximation of the isotropic coordinates. This approximation is used in the quantum treatment of scattering in [13–15]. Only massless particle is considered below.

Proceeding in the same way as in [8] and [9], we obtain instead of (2)

$$\frac{du}{d\vartheta} = \pm\sqrt{\frac{f(u)}{1-u\delta}}, \quad f(u) = 1 - u^2 + (u^3 + u)\delta = \delta(u-u_1)(u-u_2)(u-u_3). \tag{41}$$

So we use in this section the same notation as before, but with a somewhat different meaning. For $\vartheta_{1/2}$ we get a more complicated expression:

$$\vartheta_{1/2} = \frac{1}{\sqrt{\delta}}\int_0^{u_2}\sqrt{\frac{1-u\delta}{(u_3-u)(u_2-u)(u-u_1)}}du = \frac{1}{\sqrt{u_3\delta}}\int_0^{u_2}\sqrt{\frac{1-u\delta}{(1-uu_3^{-1})}}\frac{du}{\sqrt{R(u)}}. \tag{42}$$

Here $R(u) = (u_2 - u)(u - u_1)$.

From (41) we find instead of (22)–(24)

$$u_2 = 1 + \delta + \frac{3}{2}\delta^2 + 3\delta^3 + \frac{5\cdot 11}{2^3}\delta^4 + 17\delta^5 + \dots, \tag{43}$$

$$u_3 = \frac{1}{\delta} - 2\delta - 6\delta^3 - 34\delta^5 + \dots, \tag{44}$$

$$u_1 \equiv u_1(\delta) = -u_2(-\delta) = -1 + \delta - \frac{3}{2}\delta^2 + 3\delta^3 - \frac{5\cdot 11}{2^3}\delta^4 + 17\delta^5 + \dots, \tag{45}$$

$$u_3^{-1} = \delta\{1 + 2\delta^2 + 10\delta^4 + 66\delta^6 + \dots\}. \tag{44a}$$

For $\delta \ll 1$ we have $u_3 \gg 1$, hence $u/u_3 \ll 1$ in the r.h.s. of (42). So to evaluate (42) we can proceed as follows:

$$\frac{1-u\delta}{1-uu_3^{-1}} = \frac{(1-u\delta-2u\delta^3-10u\delta^5)+2u\delta^3(1+5\delta^2)}{1-u\delta(1+2\delta^2+10\delta^4\dots)} = \\ = 1 + \frac{2u\delta^3(1+5\delta^2)}{1-u\delta-2u\delta^3+\dots} = 1 + 2u\delta^3[1+u\delta+(u^2+5)\delta^2+\dots]. \tag{46}$$

From here

$$\left(\frac{1-u\delta}{1-uu_3^{-1}}\right)^{1/2} = 1 + u(\delta^3+5\delta^5) + u^2\delta^4 + u^3\delta^5 + \dots \tag{47}$$

Using this in (42) we have

$$\vartheta_{1/2} = \frac{1}{\sqrt{u_3\delta}}\{I_0 + (\delta^3+5\delta^5)I_1 + \delta^4 I_2 + \delta^5 I_3 + \dots\}, \quad I_n = \int\limits_0^{u_2} \frac{u^n du}{\sqrt{R(u)}}. \tag{48}$$

Evaluating these integrals, we find

$$I_0 = \frac{\pi}{2} + \arcsin\frac{u_2+u_1}{u_2-u_1}, \quad I_1 = \sqrt{-u_2u_1} + \frac{u_1+u_2}{2}I_0, \\ I_2 = 3\frac{u_1+u_2}{4}\sqrt{-u_1u_2} + \left\{\frac{3(u_1+u_2)^2}{8} - \frac{u_1u_2}{2}\right\}I_0, \quad R(0) = -u_1u_2, \quad R(u_2) = 0. \tag{49}$$

$R(u)$ is defined below (42). As for I_3, we need it only at $\delta = 0$. Then its value is $2/3$. Using (43)–(45), we get

$$I_0 = \frac{\pi}{2} + \delta + \frac{5}{3}\delta^3 + \frac{87}{10}\delta^5 + \dots, \quad I_1 = 1 + \frac{\pi}{2}\delta + 2\delta^2 + \dots, \quad I_2 = \frac{\pi}{4} + 2\delta + \dots, \tag{50}$$

$$\frac{1}{\sqrt{u_3\delta}} = 1 + \delta^2 + \frac{9}{2}\delta^4 + O(\delta^6).$$

Now for (42) we obtain

$$\vartheta_{1/2} = \frac{\pi}{2} + \delta + \frac{\pi}{2}\delta^2 + \frac{11}{3}\delta^3 + 3\pi\delta^4 + \frac{383}{15}\delta^5 \dots \tag{51}$$

(We have used this method to check Eq. (35).) From here

$$\delta = y\left\{1 - \frac{\pi}{2}y + \left(\frac{\pi^2}{2} - \frac{11}{3}\right)y^2 + \left(\frac{37\pi}{6} - \frac{5\pi^3}{8}\right)y^3 + \right.$$
$$\left. + \left(\frac{74}{5} - \frac{41\pi^2}{4} + \frac{7\pi^4}{2^3}\right)y^4 + \dots\right\}, \quad y = \frac{\theta}{2} = \vartheta_{1/2} - \frac{\pi}{2}. \quad (52)$$

As in (20), we obtain δ^{-2} in the form (37) where now

$$c_1 = \pi, \quad c_2 = \frac{22}{3} - \frac{\pi^2}{4}, \quad c_3 = -\frac{4\pi}{3} + \frac{\pi^3}{4}, \quad c_4 = \frac{161}{15} + 2\pi^2 - \frac{5\pi^4}{16}. \quad (53)$$

With these c_i Eq. (38) holds. Here the approximate values of c_i are

$$c_1 = 3.1416, \quad c_2 = 4.8660, \quad c_3 = 3.5627, \quad c_4 = 0.0322. \quad (53a)$$

Comparing with (37b), we see how the difference between c_i there and here increases with increase of i.

3. SMALL-ANGLE DEFLECTION BY AN INTERVAL OF GRAVITATIONAL FIELD

This problem is usually avoided because all coordinate systems are equivalent in general relativity and only the deflection by the whole gravitational field is independent of the coordinate system used. Here we assume that in the privileged system the observed deflection is given by the tangent to the trajectory. There are some reasons to think that the privileged system must be isotropic one [6] and we take it from general relativity.

For tangent we have

$$\tan\varphi = \frac{dy}{d\vartheta}\Big/\frac{dx}{d\vartheta} = \frac{\dfrac{dr}{d\vartheta}\sin\vartheta + r\cos\vartheta}{\dfrac{dr}{d\vartheta}\cos\vartheta - r\sin\vartheta}, \quad x = r\cos\vartheta, \quad y = r\sin\vartheta, \quad (54)$$

or in terms of $u = \rho/r$,

$$\tan\varphi = \frac{\dfrac{du}{d\vartheta}\sin\vartheta - u\cos\vartheta}{\dfrac{du}{d\vartheta}\cos\vartheta + u\sin\vartheta}. \quad (55)$$

So we have to know $u(\vartheta)$ or $\vartheta(u)$. For a massless particle in the isotropic system, we have

$$\frac{du}{d\vartheta} = \pm\sqrt{\frac{\left(1 + \frac{1}{4}u\delta\right)^6}{\left(1 - \frac{1}{4}u\delta\right)^2} - u^2}. \quad (56)$$

To get $u(\vartheta)$ from here is a much more difficult job than in the case of the standard Schwarzschild system utilized in (2).

In the following we assume for simplicity that $\delta \ll 1$ and $1 - u^2$ is of order of unity. (The latter assumption is not needed for the final result (60).) Then for the ingoing half of the trajectory we may write

$$\frac{du}{d\vartheta} = \sqrt{1-u^2}\left(1 + \frac{u\delta}{1-u^2} + O(\delta^2)\right). \tag{57}$$

From here we get

$$\vartheta = \arcsin u + \Delta, \quad \Delta = \left(1 - \frac{1}{\sqrt{1-u^2}}\right)\delta. \tag{58}$$

So

$$\sin\vartheta = u + \Delta\sqrt{1-u^2} = u + \left(\sqrt{1-u^2} - 1\right)\delta, \quad \cos\vartheta = \sqrt{1-u^2} - \Delta u. \tag{59}$$

Then, using (57) and (59), we find from (55)

$$\tan\varphi \approx \varphi = \left(1 - \sqrt{1-u^2}\right)\delta = \left(\frac{1}{2}u^2 + \frac{1}{8}u^4 + \dots\right)\delta. \tag{60}$$

This is the contribution from the part of the trajectory beginning at $u = 0$ and ending at u. For the second half of the trajectory the signs in front of square roots must be changed to the opposite ones.

We note here that from (59) we can obtain the trajectory in the form

$$u = \sin\theta + (1 - \cos\theta)\delta, \tag{57a}$$

which is Eq. (40.6) in § 40 in [10]. To that end in the term with δ in the second equation in (59) we replace $\sqrt{1-u^2}$ by its zeroth-order value $\cos\theta$.

As is well known, the leading term of the small-angle classical deflection can be obtained by simple mechanical considerations, see § 39, Problem 2 in [16], or Eq. (4.41) in Ch. 2 in [17]. In our case the contribution from the whole trajectory is

$$\varphi = r_g\rho\int\limits_{-\infty}^{\infty}\frac{dx}{(x^2+\rho^2)^{3/2}} = 2\delta. \tag{61}$$

In this approximation the trajectory is taken as a straight line: $r\sin\vartheta = \rho$. The contribution from the same part of the trajectory as in (60) is ($x = r\cos\vartheta$; $y = r\sin\vartheta$)

$$r_g\rho\int\limits_{x}^{\infty}\frac{dx}{(x^2+\rho^2)^{3/2}} = \frac{r_g}{\rho}\left(1 - \frac{x\rho^{-1}}{\sqrt{1+(x\rho^{-1})^2}}\right). \tag{62}$$

As

$$\frac{x}{\rho} = \frac{r}{\rho}\cos\vartheta = \frac{1}{u}\sqrt{1-u^2},$$

the r.h.s. of (62) is equal to that of (60). Only smallness of φ is used in obtaining (62) which is another form of (60). If $u \ll 1$, then δ may be of order of unity. If $\delta \ll 1$, then u may be of order of unity.

On the surface of the Sun $0 \leqslant \varphi \leqslant \delta$. Zero corresponds to the radial trajectory $\rho = 0$, δ corresponds to the trajectory touching the surface. For $u \ll 1$ we have from (62) $\varphi = (r_g\rho)/(2r^2)$.

If we use the standard Schwarzschild coordinate system, see (1) and (2), we get instead of (59)

$$\sin\vartheta = u + \left[\sqrt{1-u^2} + \frac{u^2}{2} - 1\right]\delta$$

and

$$\tan\varphi \approx \varphi = \left(1 - \sqrt{1-u^2}\left(1 + \frac{u^2}{2}\right)\right)\delta = \frac{3}{8}u^4\delta + \ldots \tag{63}$$

instead of (60). (As it should be, for $u = 1$ both (60) and (63) give $\tan\varphi = \vartheta_{1/2} = \delta$.) So the measurements of φ at $r \sim \rho$ can decide which coordinate system is the privileged one.

Finally, the contribution to φ from a finite interval of trajectory from x to $\tilde{x}$ is

$$r_g\rho \int\limits_x^{\tilde{x}} \frac{dx'}{(x'^2 + \rho^2)^{3/2}} = \frac{r_g}{\rho}\left[\frac{\tilde{x}}{\sqrt{\rho^2 + \tilde{x}^2}} - \frac{x}{\sqrt{\rho^2 + x^2}}\right], \tag{64}$$

see (61). Expression (64) is useful for evaluating the contribution to deflection angle from the almost rectilinear tails of the trajectory.

4. DISCUSSION AND CONCLUSION

We have shown that in the considered case the classical cross sections are much more accurate than the Born approximations in the quantum case. The contribution from nonlinearities of the gravitational field are given by differences in exact treatment of the gravitational field and its linearized version. It is interesting that the treatment of the linearized version turns out to be more difficult. (This is connected with the fact that the exact consideration in the isotropic system is much more difficult than in the standard Schwarzschild system.) So the new method was used to get the asymptotic expansions for integrals in this case. We note that the calculation of deflections is rather lengthy and tedious. For this reason we present some details to show that using some tricks considerably shortens the process. (See, for example, the symmetry relation (6), the using of ψ instead of φ, the preliminary rewritings in Eq. (46) showing that the terms with δ and δ^2 should not appear and so on.) The deflection of light, measured at finite distances from the Sun, is considered by two different methods. The result favors the isotropic coordinate system as the privileged one, thus giving an additional support to heuristic approach to gravity [4–6].

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