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QUANTUM FIDELITY OF GAUSSIAN STATES IN OPEN SYSTEMS

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Using the expression of the fidelity for the most general Gaussian quantum states, the behaviour of the quantum fidelity is described for the states of a harmonic oscillator interacting with an environment, in particular with a thermal bath. By taking a correlated squeezed Gaussian state as initial state, we calculate the quantum fidelity for both kinds of undisplaced and displaced states, and for different values of the squeezing and correlation parameters and of the environment temperature.

С использованием выражения точности для наиболее общих гауссовых квантовых состояний описано поведение квантовой точности для состояний гармонического осциллятора, взаимодействующего с окружением, в частности с тепловым резервуаром. Если взять коррелированное сжатое гауссово состояние как начальное состояние, можно вычислить квантовую точность как для несмещенных, так и смещенных состояний при различных значениях окружающей температуры и параметров сжатия и корреляции.

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INTRODUCTION

In recent years there has been an increased interest in the study of Gaussian states of continuous variable systems used in quantum information processing. Usually, the quantum fidelity for quantum optics experiments, in particular quantum teleportation experiments, is calculated for pure coherent states as input states. However, in real experiments the input quantum states have some non-negligible degree of mixedness, achieved mainly due to the decoherence phenomenon which takes place during the interaction of the system with its environment. In the framework of the theory of open systems based on quantum dynamical semigroups, we consider a one-dimensional harmonic oscillator interacting with an environment, in particular with a thermal bath. Our purpose is to study the influence of the environment on the time evolution of the quantum fidelity for Gaussian states of the considered system. The structure of the paper is the following. In Sec. 1 we briefly review the basic formalism for the calculation of the quantum fidelity for a general pair of single-mode Gaussian states. The time evolution of the harmonic oscillator in the theory of open quantum systems is described in Sec. 2. By taking a correlated squeezed Gaussian state as initial state, the quantum fidelity of this initial pure state and an arbitrary time mixed state of the considered system is calculated in Sec. 3. A summary is given in the final section.

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1. FIDELITY OF GAUSSIAN STATES

For two quantum states, described by the density operators ρ_1 and ρ_2, the fidelity F is defined as Uhlmann's transition probability [1] by $F(\rho_1,\rho_2) = [\mathrm{Tr}(\sqrt{\rho_1}\,\rho_2\sqrt{\rho_1})^{1/2}]^2$. A general single-mode Gaussian state ρ is completely characterized by its first and second moments and can be represented by a correlated squeezed state [2]. The fidelity of two displaced thermal states was calculated in [3] and the fidelity of a large class of single-mode Gaussian states was also obtained in [4].

Introducing the matrices $\mathbf{A}_i$ $(i = 1, 2)$ of the form

$$\mathbf{A} = \begin{pmatrix} a_{qq} & a_{pq} \\ a_{pq} & a_{pp} \end{pmatrix}, \tag{1}$$

whose elements are connected with the variances σ_{qq}, σ_{pp} and covariance σ_{pq} of canonical position q and momentum p operators through the relations $a_{qq} = 2\sigma_{qq}$, $a_{pp} = 2\sigma_{pp}/\hbar^2$, $a_{pq} = 2\sigma_{pq}/\hbar$ and denoting the mean amplitudes of ρ_1 and ρ_2 by $\alpha_i \equiv \begin{pmatrix} \alpha_{qi} \\ \alpha_{pi} \end{pmatrix}$ $(i = 1, 2)$, we obtain the following formula for the quantum fidelity of two general Gaussian quantum states:

$$F = \frac{2}{\sqrt{\Delta+\delta}-\sqrt{\delta}}\,\exp\left[-\beta^{\mathrm{T}}(\mathbf{A}_1+\mathbf{A}_2)^{-1}\beta\right], \tag{2}$$

where $\beta = \alpha_2 - \alpha_1$ and $\Delta = \det(\mathbf{A}_1 + \mathbf{A}_2)$, $\delta = (\det\mathbf{A}_1 - 1)(\det\mathbf{A}_2 - 1)$. If ρ_1 is a pure state, then $\det\mathbf{A}_1 = 1$ and the fidelity (2) becomes

$$F = \frac{1}{\sqrt{\det((\mathbf{A}_1+\mathbf{A}_2)/2)}}\exp\left[-\beta^{\mathrm{T}}(\mathbf{A}_1+\mathbf{A}_2)^{-1}\beta\right]. \tag{3}$$

2. MASTER EQUATION FOR THE HARMONIC OSCILLATOR

In the axiomatic formalism based on quantum dynamical semigroups, the irreversible time evolution of a damped harmonic oscillator is described by the following quantum Markovian master equation for the density operator $\rho(t)$ [5,6]:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0,\rho] - \frac{i}{2\hbar}(\lambda+\mu)[q,\rho p+p\rho] + \frac{i}{2\hbar}(\lambda-\mu)[p,\rho q+q\rho] - \\ - \frac{D_{pp}}{\hbar^2}[q,[q,\rho]] - \frac{D_{qq}}{\hbar^2}[p,[p,\rho]] + \frac{D_{pq}}{\hbar^2}([q,[p,\rho]]+[p,[q,\rho]]). \tag{4}$$

The harmonic oscillator Hamiltonian H is chosen of the most general quadratic form

$$H = H_0 + \frac{\mu}{2}(qp+pq), \quad H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 \tag{5}$$

and the quantum diffusion coefficients D_{pp}, D_{qq}, D_{pq} and the dissipation constant λ satisfy the following fundamental constraints: $D_{pp} > 0, D_{qq} > 0$ and $D_{pp}D_{qq} - D_{pq}^2 \geqslant \dfrac{\lambda^2\hbar^2}{4}$. In the particular case when the asymptotic state is a Gibbs state $\rho_G(\infty) = \exp\left(-\dfrac{H_0}{kT}\right)\Big/\mathrm{Tr}\exp\left(-\dfrac{H_0}{kT}\right)$,

these coefficients have the form [6]

$$D_{pp} = \frac{\lambda+\mu}{2}\hbar m\omega \coth\frac{\hbar\omega}{2kT}, \quad D_{qq} = \frac{\lambda-\mu}{2}\frac{\hbar}{m\omega}\coth\frac{\hbar\omega}{2kT}, \quad D_{pq} = 0, \tag{6}$$

where T is the temperature of the thermal bath. From the master equation (4) we can obtain the equations of motion for the expectation values σ_q and σ_p of coordinate and momentum and the equations of motion for the variances σ_{qq}, σ_{pp} and covariance σ_{pq} [6]. In the underdamped case ($\omega > \mu$) we obtain $\sigma_q(\infty) = \sigma_p(\infty) = 0$ and the asymptotic values of $\sigma_{qq}(t)$, $\sigma_{pp}(t)$, $\sigma_{pq}(t)$ in the case of a thermal bath with coefficients (6) reduce to [6]

$$\sigma_{qq}(\infty) = \frac{\hbar}{2m\omega}\coth\frac{\hbar\omega}{2kT}, \quad \sigma_{pp}(\infty) = \frac{\hbar m\omega}{2}\coth\frac{\hbar\omega}{2kT}, \quad \sigma_{pq}(\infty) = 0. \tag{7}$$

3. CALCULATION OF QUANTUM FIDELITY

We consider a harmonic oscillator with the initial Gaussian wave function

$$\Psi(q) = \left(\frac{1}{2\pi\sigma_{qq}(0)}\right)^{1/4}\exp\left[-\frac{1}{4\sigma_{qq}(0)}\left(1-\frac{2i}{\hbar}\sigma_{pq}(0)\right)(q-\sigma_q(0))^2 + \frac{i}{\hbar}\sigma_p(0)q\right], \tag{8}$$

where $\sigma_{qq}(0)$ is the initial spread; $\sigma_{pq}(0)$ the initial covariance, and $\sigma_q(0)$ and $\sigma_p(0)$ are the initial averaged position and momentum of the wave packet. The initial state (8) represents a correlated coherent state [2] with the variances and covariance of coordinate and momentum

$$\sigma_{qq}(0) = \frac{\hbar\delta}{2m\omega}, \quad \sigma_{pp}(0) = \frac{\hbar m\omega}{2\delta(1-r^2)}, \quad \sigma_{pq}(0) = \frac{\hbar r}{2\sqrt{1-r^2}}. \tag{9}$$

Here, δ is the squeezing parameter which measures the spread in the initial Gaussian packet and r, with $|r| < 1$, is the correlation coefficient at time $t = 0$. For $\delta = 1$ and $r = 0$ the correlated coherent state becomes a Glauber coherent state.

If the initial wave function is Gaussian, then the density matrix remains Gaussian for all times [7]. We take the initial Gaussian wave function (8) as the pure state 1 with the corresponding matrix $\mathbf{A}_1$ and the state at an arbitrary time t described by the density matrix $\rho(t)$ as the state 2 with the corresponding matrix $\mathbf{A}_2$. When the initial Gaussian wave function (8) is not displaced, $\alpha_1 = \alpha_2 = 0$ and the exponential factor in expression (3) of the quantum fidelity becomes 1. In this case we obtain ($\sigma \equiv \sigma_{qq}\sigma_{pp} - \sigma_{pq}^2$)

$$F(t) = \frac{\hbar}{\sqrt{\sigma(0)+\sigma(t)+\sigma_{qq}(0)\sigma_{pp}(t)+\sigma_{pp}(0)\sigma_{qq}(t)-2\sigma_{pq}(0)\sigma_{pq}(t)}}. \tag{10}$$

Introducing the expressions obtained in [5,6] for the variances of the coordinate and momentum, expression (10) takes the following explicit form when the initial state is a squeezed

state ($r = 0$) ($\Omega^2 \equiv \omega^2 - \mu^2, \epsilon \equiv \hbar\omega/2kT$):

$$F(t) = 2\Bigg\{ \mathrm{e}^{-4\lambda t}\left[1 - \left(\delta + \frac{1}{\delta}\right)\coth\epsilon + \coth^2\epsilon\right] + \frac{\mathrm{e}^{-2\lambda t}}{2\Omega^2}\Bigg[\Bigg(\omega^2\left(2 + \delta^2 + \frac{1}{\delta^2} - 4\coth^2\epsilon\right) +$$

$$+ 4\mu\Omega\left(\delta - \frac{1}{\delta}\right)\sin(2\Omega t)\coth\epsilon + \left[\omega^2\left(2 - \delta^2 - \frac{1}{\delta^2}\right) - 4\mu^2(1 - \coth^2\epsilon)\right]\cos(2\Omega t)\Bigg)\Bigg] +$$

$$+ 1 + \left(\delta + \frac{1}{\delta}\right)\coth\epsilon + \coth^2\epsilon\Bigg\}^{-1/2}. \quad (11)$$

The time evolution of the quantum fidelity (11) as a function of the temperature, dissipation constant and squeezing parameter of the initial Gaussian state is represented in Figs. 1–3. In the particular case of an initial coherent state ($\delta = 1$) and for $T = 0$, the quantum fidelity is constant, $F(t) = 1$. For large times we get the following expression of the asymptotic fidelity

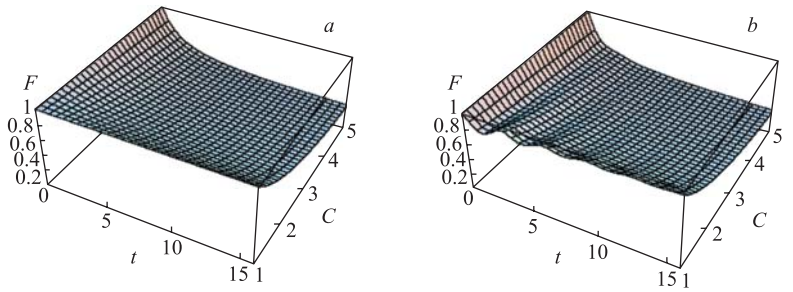

Fig. 1. Dependence of quantum fidelity F (11) on time t and temperature T via $C \equiv \coth\dfrac{\hbar\omega}{2kT}$, for $\omega = 1$, $\lambda = 0.1$, $\mu = 0$, $r = 0$. *a*) $\delta = 1$; *b*) $\delta = 2$

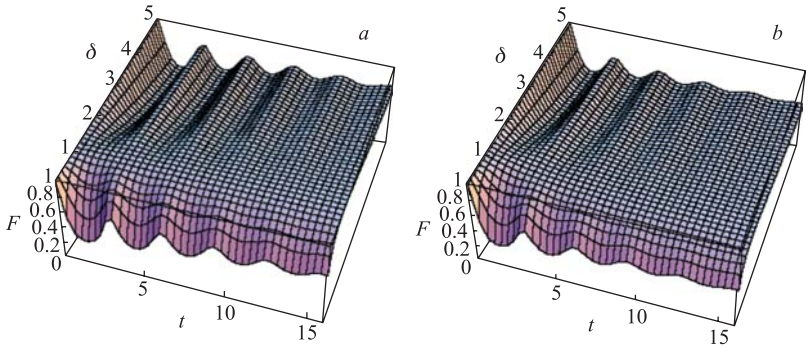

Fig. 2. Dependence of quantum fidelity F (11) on time t and squeezing coefficient δ, for $\omega = 1$, $\lambda = 0.1$, $r = 0$. *a*) $\mu = 0$, $C = 1$ ($T = 0$); *b*) $\mu = 0.08$, $C = 5/3$

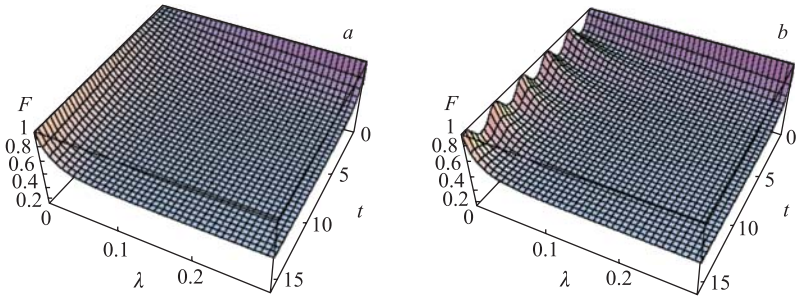

Fig. 3. Dependence of quantum fidelity F (11) on time t and dissipation constant λ, for $\omega = 1$, $\mu = 0$, $r = 0$, $C = 3$. *a*) $\delta = 1$; *b*) $\delta = 2$

when the initial state is a correlated squeezed state:

$$F(\infty) = \frac{2}{\sqrt{1 + \left(\delta + \dfrac{1}{\delta(1-r^2)}\right)\coth\epsilon + \coth^2\epsilon}}. \tag{12}$$

It depends on the environment temperature and parameters of the initial state, but does not depend on the dissipation constant. For a coherent state ($\delta = 1$) we obtain $F(\infty) = \dfrac{2}{1+\coth\epsilon}$. When the temperature of the environment is $T = 0$, expression (12) becomes, for $r = 0$, $F(\infty) = \dfrac{2\sqrt{\delta}}{\delta+1}$ and $F(\infty) = 1$, if the initial state is a coherent state. For $\delta \neq 1$, we get always $F(\infty) < 1$.

When the initial Gaussian wave function (8) is displaced, in the previous expressions of the asymptotic fidelity the exponential factor $\exp(-E)$ of Eq. (3) is present, where E for large times ($t \to \infty$) and for a coherent state has the form

$$E(\infty) = \frac{m^2\omega^2\sigma_q^2(0) + \sigma_p^2(0)}{\hbar m\omega(1+\coth\epsilon)}. \tag{13}$$

CONCLUSION

Using the expression of the fidelity for the most general Gaussian quantum states, we have investigated the behaviour of the quantum fidelity for the states of a harmonic oscillator interacting with an environment, in particular with a thermal bath. The time evolution of the considered system was described in the framework of the theory of open systems based on quantum dynamical semigroups. By taking a correlated squeezed Gaussian state as the initial state, we calculated the quantum fidelity for both kinds of undisplaced and displaced states. We have also described the behaviour of the time evolution of the quantum fidelity in dependence on the squeezing and correlation parameters characterizing the initial Gaussian state and on the dissipation constant and temperature characterizing the environment.

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