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ON THE FOURIER TRANSFORMATION
OF RENORMALIZATION INVARIANT COUPLING

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1 Invariant coupling in different representations

Current practice of the QFT calculations, as a rule, employs expression of observables and of some other intermediate renormalization-group (RG) invariant, or covariant, objects in terms of the invariant coupling function (also referred to as the “invariant, or running coupling constant”¹). This invariant coupling $\bar{\alpha}(Q^2)$ is a real function of a real positive argument $Q^2 \equiv \mathbf{Q}^2 - Q_0^2$, momentum transfer squared.

The notion of invariant coupling (IC) has initially been introduced² — see, e.g., refs.[1, 2] on renormalization group — in terms of a product of *real* constants z_i entering into finite Dyson renormalization transformations

$$D_i(\dots, \alpha) \rightarrow D'_i(\dots, \alpha') = z_i^{-1} D_i(\dots, z_3 \alpha); \quad \alpha \rightarrow \alpha' = z_3 \alpha, \quad (1)$$

with particle propagators $D_i(\dots, \alpha) = D_i(Q, m, \mu; \alpha)$ taken in the energy-momentum representation. Here, the IC is expressed in terms of scalar (*i.e.*, Lorentz-invariant) QFT amplitudes taken also in the momentum representation. For instance, in QED one has

$$\bar{\alpha}(Q^2) \equiv \alpha d \left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}; \alpha \right)$$

with d , the transverse photon propagator amplitude which is real in the Euclidean domain. In QCD, IC $\bar{\alpha}_s(Q^2)$ is usually defined in a similar way as a product of the expansion parameter α_s with an appropriate scalar vertex and propagator amplitudes, see, e.g., [3] and references therein.

This IC is commonly used for parameterization of the renormalization-invariant quantities, in particular, of observables. To this goal, one should take an invariant quantity of interest in the appropriate representation. In particular, in terms of $\bar{\alpha}_s(Q^2)$ there could be presented only an object taken in the momentum representation. Moreover, this object to be expressible in terms of the real function $\bar{\alpha}_s(Q^2)$ should be real itself. In the QFT case, this corresponds to the space-like Euclidean domain with $Q^2 > 0$.

Meanwhile, some observables, like effective cross-sections, are functions of another Lorentz-invariant argument s , the center of mass energy

¹In view of semantic absurdity of the last term, we use the expression “invariant coupling function” or “*invariant coupling*”.

²Under the name of “invariant charge” (of electron), the natural one in the QED context.

squared. As it is well known, the polarization operator $\Pi(Q^2)$, being represented in the form of the Cauchy type integral, provides us with a technical means for the integral connection of invariant functions of the Q^2 and s arguments. Its logarithmic derivative, known as the Adler function, is defined via integral

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s + Q^2)^2} R(s), \quad (2)$$

close to the Källén–Lehmann spectral representation for $\Pi(Q^2)$. Here, $R(s)$ being the imaginary part of the polarization operator, due to the optical theorem, is proportional to the total cross-section which is an observable quantity.

Further on, we shall take relation (2) as the base for definition of the integral operation \mathbb{D} transforming a real function $M(s)$ of the positive (time-like) argument into another real function $E(Q^2)$ of the positive real (space-like) argument Q^2 — see below eq.(7).

More generally, we consider a linear integral transformation with kernel $K(x)$, depending on one argument

$$f(x) \rightarrow F(y) = \mathbb{L}[f](y) = \int_0^\infty d(xy) K(xy) f(x) \quad (3)$$

that comprises various kinds of one-dimensional Fourier transformations (usual, Laplace, cosine–Fourier, sine–Fourier and some others, *e.g.*, with a kernel in the form of the Bessel function that could be derived from the four-dimensional Fourier one) and, with $K(z) = (1 + z^2)^2$, just mentioned the “Adler transformation” \mathbb{D} defined below by eq.(7).

In Section 2, we start with the discussion of the interrelation between the RG invariance and integral transformation. Our first explicit example taken from the so-called Analytic Perturbation Theory (APT) concerns a transition from the Minkowskian energy–momentum domain to the corresponding Euclidean one. We continue with the transition from the “distance” (*i.e.*, three-dimensional space) representation to the common momentum (transfer) one using the one-dimensional sine–Fourier transformation.

Further on, in Section 3, we shall pay special attention to the relation between a *long-range distance* and *infrared momentum* asymptotic behaviors in QCD, and comment on the interpretation of some recent results of the ALPHA collaboration.

2 Integral transformations

2.1 RG invariance through integral transformation

First of all, note that transformation \mathbb{L} , eq.(3), is compatible with the RG transformation. For example, in the massless case, let some initial function $f(x/\mu; \alpha)$ be invariant with respect to the RG transformation

$$R_\tau : \{ \mu^2 \rightarrow (\mu')^2 = \mu^2 \tau; \quad \alpha \rightarrow \alpha' = \bar{\alpha}_{\text{tr}}(\tau, \alpha) \} . \quad (4)$$

Then, its integral image $F(y\mu; \alpha) = \mathbb{L}[f](y)$ will be invariant with respect to the same RG transformation (4) — and vice versa — with *the same transformation function* $\bar{\alpha}_{\text{tr}}$ which satisfies the functional equation

$$\bar{\alpha}(\theta\tau, \alpha) = \bar{\alpha}(\theta, \bar{\alpha}(\tau, \alpha)) \quad (5)$$

that follows from the group composition law $R_\theta R_\tau = R_{\theta\tau}$. Note here, that eq.(5) implies the “canonical normalization condition” for the transformation function $\bar{\alpha}_{\text{tr}}$

$$\bar{\alpha}(1, \alpha) = \alpha \quad (6)$$

that corresponds to the identity transformation R_1 .

Relation (2) is commonly used to define the Adler function D . However, we shall treat it instead as a definition of the integral operation \mathbb{D} transforming a function $M(s)$ of the positive real argument s into another function $E(Q^2)$ of the positive real argument Q^2

$$M(s) \rightarrow \mathbb{D}[M](Q^2) \equiv Q^2 \int_0^\infty \frac{ds}{(s + Q^2)^2} M(s) = E(Q^2) \quad (7)$$

with the reverse operation $\mathbb{R} = [\mathbb{D}]^{-1}$ that can be expressed in terms of the contour integral; for more details about these operations, see, *e.g.*, ref.[4, 5].

The transformations \mathbb{D} and \mathbb{R} preserve the RG invariance; for instance, they transform a renormalization invariant function $M(s/\mu^2; \alpha)$ into another RG invariant $E(Q^2/\mu^2; \alpha)$ and vice versa. The first of these invariants M should be representable as a function of an adequate “Minkowskian invariant coupling” $\alpha(s)$, while the other E — via the common invariant coupling $\bar{\alpha}(Q^2)$. In particular, this means that starting with the usual QCD coupling $\bar{\alpha}_s(Q^2)$, by the operation \mathbb{R} it is possible to define ³ the QCD

³However, in practice, the nonphysical singularity in perturbative $\bar{\alpha}_s(Q^2)$ prevents one from a straightforward performance of the integration procedure. In more detail, this issue has been discussed in [5].

effective coupling in the time-like Minkowskian region $\alpha_M(s) = \mathbb{R}[\bar{\alpha}_s](s)$ in terms of $\bar{\alpha}_s(Q^2)$, a QCD invariant coupling initially introduced in the Euclidean domain.

Another example is related to the physical amplitude A depending on a couple of Lorentz invariant kinematic arguments s and t . Let it be renormalization invariant

$$A(s, t; \mu, \alpha) = A(s, t; \mu', \alpha').$$

Here, there are two possibilities. The first one deals with an integral transformation with respect to one of the two kinematic arguments like an eikonal transformation $F(s, t) \rightarrow \Phi(s, b)$. A new amplitude Φ has the RG transformation property similar to that of F . Like in the general case of several kinematic arguments, the RG transformation properties of F and Φ are not very useful.

However, this situation changes for an integral transformation of the second type involving some function of ratio of both the variables $\varphi(s/t)$. Such a transformation

$$A(s, t) \rightarrow A_k(s) = \int_0^\infty A(s, t) K_{(k)}(s/t) d\varphi(s/t); \quad k = 0, 1, 2, \dots$$

projects the initial function of two (kinematic) arguments onto a (set of) function(s) $A_k(s)$ of one argument which is(are) RG invariant. The transition to partial waves and to moments of structure functions provides us with examples.

One more example of an integral transformation of the “one-argument” function compatible with the RG invariance is given by the transformation of the Fourier type which follows the general linear form, eq.(3). It relates the function $f(r)$ of the space-time Lorentz invariant argument $r = \sqrt{\mathbf{r}^2 - t^2}$ with the function $F(Q)$ of an energy-momentum invariant argument $Q = \sqrt{Q^2} = \sqrt{\mathbf{Q}^2 - Q_0^2}$.

As it is well known, the Dyson transformation (1), being “a self-similarity property of the Schwinger-Dyson equations” [6], can be, on equal footing, considered in both the (energy-)momentum — like in eq.(1) — and (time-)space representation, that is for functions like

$$D(\dots, \alpha) = \tilde{D}(x, m, \mu; \alpha); \quad x = \{\mathbf{r}, t\}.$$

In the QED context, this latter picture⁴ was used by Dirac while discussing polarization of vacuum in terms of *effective charge of electron* first introduced by him[7] as a space distribution

$$\bar{e}_D(r) = e_0 \left\{ 1 - \frac{\alpha_0}{3\pi} \ln \frac{r}{r_e} \right\}; \quad \alpha_0 = \frac{e_0^2}{4\pi} \simeq \frac{1}{137.0360} \quad r_e = \frac{\hbar}{m_e c} \quad (8)$$

of the electric charge around the point “bare” electron.

More recently, an analogous object, the QCD effective coupling $\bar{g}^2(L)$, which is a function of the spatial size L of a lattice, has been introduced[8] and used[9] – [12] by ALPHA collaboration.

Some other examples of the relation between integral transformations and renormalization group symmetries in problems of classical and mathematical physics can be found in a fresh review paper [13] (see, especially, Example 2 on page 358 and references therein).

The issue of correlation between the RG formulations in different pictures has two aspects. The first one deals with the basic notions and objects of RG transformations. The second one is that of these objects correlation via an appropriate integral transformation.

According to the terminology formulated in Refs.[14, 15], an invariant function satisfying the functional equation (5) and canonical normalization condition should be named *effective coupling* (EC). To the case of an invariant function $\bar{\alpha}_N$ with a more general normalization

$$\bar{\alpha}_N(1, \alpha) \stackrel{\perp}{=} N(\alpha) \neq \alpha$$

there corresponds a term *invariant coupling* (IC). Contrary to EC, it can not be used as a function transforming a coupling constant in (4). Generally, integral transformation (3) maps an EC $\bar{\alpha}(x)$ onto some other RG-invariant function $A(y)$ which is IC rather than EC.

A few questions are in order:

a) How to define $\bar{\alpha}$ and its integral image A explicitly for the given QFT model?

b) How to relate them?

c) Which of them can be used as the transformation function α_{tr} ?

“Immediate” answers —

a) Use a common algorithm with perturbative beta-functions to define $\bar{\alpha}$ and A ,

⁴We shall refer to it as to the “distance representation”.

- b) Relate them by an appropriate transformation (3)
 — turn out to be incompatible with the each other.

2.2 From the c.m. energy to the momentum picture

This situation can be illuminated by the known answers to the same questions for the integral transformation (7). Let us write it down for the particular case of the coupling function transformation

$$\alpha(s) \rightarrow \mathbb{D}[\alpha](Q^2) \equiv Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} \cdot \alpha(s) = \bar{\alpha}(Q^2). \quad (9)$$

It turns out to be impossible to use here a common perturbative QCD coupling $\bar{\alpha}_s(Q^2)$ as $\bar{\alpha}(Q^2)$ because of its unphysical singularities, like the Landau pole at $Q^2 = \Lambda^2$, which contradicts the integral expression (9). The latter implies that function $\bar{\alpha}(Q^2)$ should be free of any singularities outside a cut $0 > Q^2 > -\infty$. The same is true for the candidature of $\bar{\alpha}_s(Q^2)$ for the role of function $\alpha(s)$. One of the possible solutions that was proposed in the so-called Analytic Perturbation Theory (APT) (see, refs. [17, 5]) consists in using of $\bar{\alpha}_s(Q^2)$ only as a prototype for both the $\bar{\alpha}(Q^2)$ and $\alpha(s)$ which in the one-loop case with

$$\bar{\alpha}_s^{(1)}(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}; \quad \Lambda^2 = \mu^2 \exp\left(-\frac{1}{\beta_0 \alpha_s}\right) \quad (10)$$

turn out to be

$$\alpha(s) = \frac{1}{\pi\beta_0} \arccos \frac{L_s}{\sqrt{L_s^2 + \pi^2}} \Big|_{L_s > 0} = \frac{\arctan(\pi/L_s)}{\pi\beta_0}; \quad L_s = \ln \frac{s}{\Lambda^2}, \quad (11)$$

and

$$\bar{\alpha}(Q^2) = \frac{1}{\beta_0} \left[\frac{1}{L} - \frac{\Lambda^2}{Q^2 - \Lambda^2} \right]; \quad L = \ln \frac{Q^2}{\Lambda^2}. \quad (12)$$

Both these two functions, corresponding to (10) in the weak coupling limit, are ghost-free monotonous functions related by the transformation (9) $\bar{\alpha}(Q^2) = \mathbb{D}[\alpha](Q^2)$ and its reverse.

As it has been noticed in Ref.[5], transitions from expression (10) for $\bar{\alpha}_s^{(1)}(Q^2)$ to $\alpha(s)$ and $\bar{\alpha}(Q^2)$ can be represented as a consequence of a

transition from the usual QCD coupling constant α_s to the new ones

$$\alpha_s \rightarrow \alpha_M = \frac{1}{\pi\beta_0} \arccos \frac{1}{\sqrt{1 + \pi^2\beta_0^2\alpha_s^2}} = \frac{1}{\pi\beta_0} \arctan(\pi\beta_0\alpha_s),$$

and

$$\alpha_s \rightarrow \alpha_E = \alpha_s + \frac{1}{\beta_0} (1 - e^{1/\beta_0\alpha_s})^{-1}.$$

Hence, a transition from the Minkowskian coupling $\alpha(s)$ to the Euclidean one $\bar{\alpha}(Q^2)$ is equivalent to that one induced by the following coupling constant transformation :

$$\alpha_M \rightarrow \alpha_E(\alpha_M) = \frac{1}{\pi\beta_0} \tan(\pi\beta_0\alpha_M) + \frac{1}{\beta_0} \cdot \frac{1}{1 - e^{\pi \cot(\pi\beta_0\alpha_M)}} \quad (13)$$

at $0 < \alpha_M \leq 1/\beta_0$ and $0 < \alpha_E \leq 1/\beta_0$.

In turn, this implies that the integral transformation (9), generally changes the normalization of coupling functions and, in particular, maps the EC onto IC.

2.3 Momentum and distance representations

Turn now to the Fourier transformation. It will be convenient to use the modified sine-Fourier transformation⁵ of the RG-compatible form

$$\mathbb{F}_{\sin/r}[f](Q) \equiv \frac{2}{\pi} \int_0^\infty \frac{dr}{r} \sin(Qr) f(r) = F(Q) \quad (14)$$

and its reverse

$$\mathbb{F}_{\sin/r}^{-1}[F](r) \equiv r \int_0^\infty dQ \sin(Qr) F(Q) = f(r).$$

Now, rewrite expression (8) squared in a more contemporary and general notation

$$\alpha_D^{pt} \left(\frac{r^2}{r_\mu^2}, \alpha_\mu \right) = \alpha_\mu \left\{ 1 - \frac{\alpha_\mu}{3\pi} \ln \frac{r^2}{r_\mu^2} \right\} \quad (15)$$

⁵that is a particular case of (3) with $K(z) = 2 \sin(z)/\pi z$ and follows from the usual 3-dimensional one

$$\bar{\psi}(Q) = (2\pi)^{-2} \int dr \psi(r) e^{i\mathbf{Q}\mathbf{r}} \quad \text{with} \quad F(Q) = Q^2 \bar{\psi}(Q), \quad f(r) = r \psi(r).$$

(r_μ being a reference distance) with the correspondence relation $\alpha_\mu = \alpha_0$ at $r_\mu = r_e$. This *distance-representation perturbative EC* $\alpha_D^{pt}(\rho^2, \alpha)$ can be connected by the transformation

$$\bar{\alpha}(Q^2/\mu^2, \alpha) = \mathbb{F}_{\sin/r}[\alpha_D^{pt}](Q^2)$$

with a more common perturbative QED coupling in the momentum representation

$$\bar{\alpha}_{pt}\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = \alpha_\mu \left\{ 1 + \frac{\alpha_\mu}{3\pi} \ln \frac{Q^2}{\mu^2} \right\} \quad \text{with} \quad \mu = \frac{c_E}{r_\mu} \quad (16)$$

and $c_E = e^{-C} = 0.5614$. Expression (16) can also be obtained from (15) by the argument substitution

$$r \rightarrow c_E/Q, \quad (17)$$

that is close to the quantum-mechanical correspondence relation $r \rightarrow 1/Q$ being slightly modified by changing the Q scale. By the way, the positive effect of the c_F factor is clearly seen in our Figure 2 — see below.

Quite analogously with the presented example, both the functions $\bar{\alpha}^{RG}$ and α_D^{RG} , being explicitly defined (in the familiar form of the geometric progression sum) by the common RG perturbation-based procedure with (15) and (16) as inputs, are not related by eq.(14). Instead, they are connected by a more involved relation $\bar{\alpha}^{RG}(Q) = \Psi \{ \mathbb{F}_{\sin} [\alpha_D^{RG}] (Q) \}$.

Moreover, generally, Fourier transformation maps an effective coupling onto some invariant coupling that cannot be used as α_{tr} in (5). For example, if one starts with the perturbative distance-representation EC of the form

$$\alpha_D^{pt}(\rho^2, \alpha) = \alpha - \beta_0(\alpha) \ln \rho^2 + \alpha \left(\frac{\alpha}{3\pi} \right)^2 \ln^2 \rho^2; \quad \rho^2 = \frac{r^2}{r_\mu^2}; \quad \alpha = \alpha_\mu \quad (18)$$

then, as a result of the sine-Fourier transformation (14), one arrives at

$$\bar{\alpha}_{pt}(q^2, \alpha) = \alpha + \beta(\alpha) \ln q^2 + \alpha \left(\frac{\alpha}{3\pi} \right)^2 \{ \ln^2 q^2 + \Delta_2 \} \quad (19)$$

with

$$q^2 = \frac{Q^2}{\mu^2} = \left(\frac{Q r_\mu}{c_E} \right)^2; \quad \Delta_2 = \frac{\pi^2}{12}.$$

Hence, $\bar{\alpha}(1, \alpha) = \alpha + \alpha(\alpha/3\pi)^2 \Delta_2 = \alpha' \neq \alpha$ and $\bar{\alpha}(q^2, \alpha)$ is an invariant coupling⁶. By transition to another coupling constant $\alpha \rightarrow \alpha'$ it can be transformed (see Section 3 in [15]) into the EC $\bar{\alpha}'(q^2, \alpha')$. The transformation is relevant to the $\Psi(\alpha)$ dependence. This could be important at the strong coupling case in the IR region.

Note also that the logarithmic terms in both (18) and (19) yield the sums of geometric progression with arguments of logarithms related by eq.(17).

3 Long-range and infrared asymptotics

3.1 Tauberian theorem

Turn now to the particular issue of correlation between asymptotic behaviors of the functions f and \bar{f} related by the Fourier transformation.

This correlation is popular in quantum mechanics where, quite often, one uses the so-called “quantum-mechanical correspondence” relation

$$r \rightarrow 1/Q, \quad (QMC)$$

which in the IR case is equivalent to

$$F(Q) \sim f(Q^{-1}) \quad \text{as } Q \rightarrow 0. \quad (20)$$

Heuristically, this last feature could be simply understood by a change of the integration variable $r \rightarrow x = rQ$ in the general linear transformation (3)

$$F(Q) = \int_0^\infty \frac{dx}{x} K(x) f\left(\frac{x}{Q}\right). \quad (3a)$$

However, for a more rigorous derivation of (20) one needs to specify some asymptotic property of the function $f(r)$ as $r \rightarrow \infty$. In short, this can be formulated as the **Tauberian theorem**⁷: (Here, the symbol “ \sim ”

⁶Meanwhile, this feature is not essential (like for transformation (13)) in the weak coupling case at the one- and two-loop levels.

⁷Originally, under the name of Tauberian theorems one implied statements concerning the relation between *summability* and *convergence* of series. More recently, in the middle of XX century, this term started to be used in the context of asymptotic properties of integral transformations. Here, we give only a crude outline of this important theorem for the Fourier transformation, the sketch that is sufficient for our application. For a more complete and rigorous exposition of this matter the reader is referred to refs.[26, 27].

line's one $F_{\{2\}} = \ln(Q^{-1}) - C$, (with C being the Euler constant) satisfying the Tauberian condition as $Q \rightarrow 0$, generally differs from $f_{\{2\}}(1/Q)$, especially in the region $Q \simeq 1$. The same is true for the line $\{3\}$; here the constant Δ_2 had been introduced before in eq. (19).

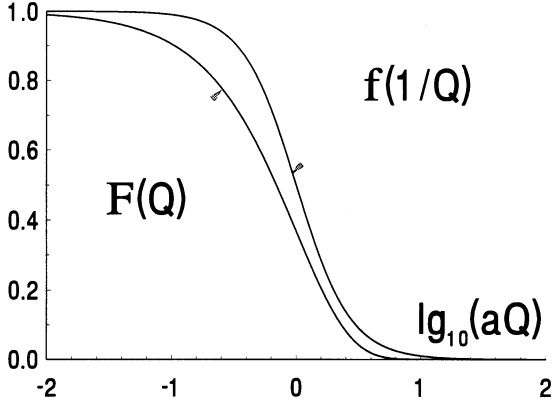


Figure 1: Behavior of the “QMC-substituted” function $f_{\{4\}}(1/Q) = (1 + a^2Q^2)^{-1}$ and of its sine-Fourier transform $F_{\{4\}}(Q) = \exp(-aQ)$.

The next line $\{4\}$ provides us with an example that satisfies the Tauberian condition (S), but severely violates the (QMC) rule for the whole function $F_{\{4\}}$. It is instructive to compare the behavior of the Fourier image $F_{\{4\}}(Q) = \exp(-aQ)$ with the “QMC-substituted” initial function $f_{\{4\}}(1/Q) = (1 + a^2Q^2)^{-1}$. As it can be seen in Figure 1, the relative error being at $aQ \geq 0.2$ of an order of 15%, quickly increases and reaches the level of 30% at $aQ \simeq 1$.

Only last example $f_{\{5\}}$ does not satisfy the Tauberian condition, while the corresponding limiting values $f_{\{5\}}(\infty) = F_{\{5\}}(0) = 0$ coincide.

To give some example more close to the realistic QCD case, consider the sine-Fourier transformation for the class of functions $f(Q)$ that satisfy the Källén–Lehmann spectral representation

$$f(q) \rightarrow F(r) = \frac{2}{\pi} \int_0^\infty \frac{dQ}{Q} \sin(Qr) \int_0^\infty \frac{d\sigma \rho(\sigma)}{\sigma + Q^2} \quad (22)$$

with the density $\rho(\sigma)$. Changing the order of integration and performing the integration over Q with the help of line $\{4\}$ from the Table we arrive

at

$$F(r) = \int_0^\infty \frac{\rho(\sigma) d\sigma}{\sigma} \left(1 - e^{-r\sqrt{\sigma}}\right). \quad (23)$$

In the case of positive spectral density, this expression defines a monotonically rising function of r with possible singularity as $r \rightarrow \infty$.

As an explicit example, we consider the analyticized invariant QCD coupling $\bar{\alpha}(Q^2)$, eq.(12) with the spectral density $\rho_{\text{APT}}(\sigma) \sim [(\ln \sigma)^2 + \pi^2]^{-1}$ taken from the one-loop APT[18]. For $a_{\text{APT}} = \beta_0 \alpha_{\text{APT}}$ this yields

$$a_{\text{APT}}(r) = \int_0^\infty \frac{d\sigma(1 - e^{-r\sqrt{\sigma}})}{\sigma[(\ln \sigma)^2 + \pi^2]} = 1 - I(r); \quad I(r) = \int_0^\infty \frac{e^{-r\sqrt{\sigma}} d\sigma}{\sigma[(\ln \sigma)^2 + \pi^2]}. \quad (24)$$

The r.h.s. integral $I(r)$ resembles the Ramanujan ones[19]

$$R(r) \equiv \int_0^\infty \frac{e^{-rx} dx}{x[(\ln x)^2 + \pi^2]} = 2 \int_0^\infty \frac{e^{-r\sqrt{\sigma}} d\sigma}{\sigma[(\ln \sigma)^2 + 4\pi^2]} = \nu(r) \cdot e^r$$

expressible in terms of the special transcendental function $\nu(x)$. We shall use this proximity for analysis of the $I(r)$ behavior as $r \rightarrow 0$. The function ν asymptotics is well known [19]. One has

$$R(r) \rightarrow -1 + \frac{1}{\ln r^2} + O\left(\frac{1}{\ln^2 r^2}\right) \quad \text{as } r \rightarrow 0. \quad (25)$$

The difference

$$\Delta(r) = I(r) - \frac{R(r)}{2} = 3\pi^2 \int_0^\infty \frac{d\sigma}{\sigma} \frac{e^{-r\sqrt{\sigma}}}{[(\ln \sigma)^2 + \pi^2][(\ln \sigma)^2 + 4\pi^2]}$$

being positive, vanishes $\Delta(\infty) = 0$ at infinity. At the same time, $\Delta(0) = 1/4$. Hence

$$a_{\text{APT}}(r) \rightarrow \frac{1}{\ln r^2} \quad \text{as } r \rightarrow 0 \quad \text{and} \quad a_{\text{APT}}(\infty) = 1. \quad (26)$$

It is instructive to compare α_{APT} with the monotonous initial function

$$\bar{a}_{an}(Q^2) = \beta_0 \bar{\alpha}(Q^2) = \int_0^\infty \frac{\rho_{\text{APT}}(w^2) dw^2}{w^2 + Q^2} = \frac{1}{\ln(Q^2)} + \frac{1}{1 - Q^2},$$

$$a_{an}(0) = 1; \quad a_{an}(Q^2) \rightarrow \frac{1}{\ln Q^2} \quad \text{as } Q^2 \rightarrow \infty \quad (27)$$

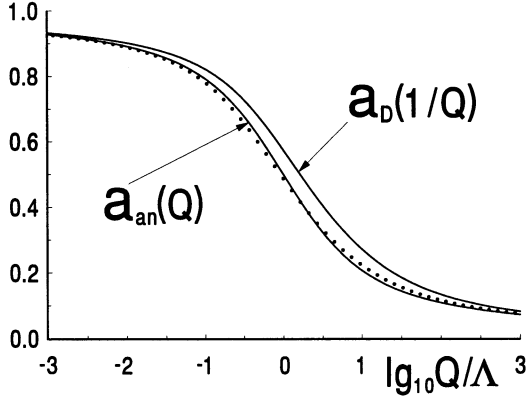


Figure 2: Behavior of the APT-analytized $a_{an}(Q)$ and of its Fourier transform $a_D(1/Q)$ with the argument modified by the quantum-mechanical rule” (QMC) and of its Fourier image $a_D(c_F/Q)$ (dotted line) with the scale modified by (17).

in the coordinates $r = 1/Q$.

Such a comparison (see Fig. 2) reveals a puzzling similarity of both the functions $a_D(1/Q)$ and $a_{an}(Q^2)$, that could be essentially improved by changing $a_D(1/Q)$ for $a_D(c_F/Q)$. In particular, the relative error in the region $4 \lesssim Q/\Lambda \lesssim 70$ is reduced from 30% to 10%. This produces an impression that one has a strong argument for supporting the use of the correspondence rules (QMC) and (17) for the qualitative estimate of Fourier image of functions with an “adequate” — to eq.(21) — asymptotic behavior. Unhappily, this impression is erroneous. For example, the analysis of the family of functions

$$f(q^2) = \frac{1}{\ln^2(q^2) + b^2}; \quad q^2 = Q^2/\Lambda^2$$

with adequate asymptotics reveals⁸ that similarity like in Figure 2 represents a “puzzling exclusion” of usual case, which, typically, is far even from very approximate similarity like the one in Figure 1.

Hence, the (QMC) rule, generally, does not provide us with some reasonable means for qualitative estimation of the Fourier image behavior in the region far from the asymptotic one even for functions satisfying the Tauberian condition.

⁸Private communication by Dr. A.V. Nesterenko.

3.3 The “Schrödinger functional QCD coupling”

To analyze by nonperturbative means the infrared behavior of QCD, the “ALPHA collaboration” uses the functional integral approach (both in the quenched QCD version and with two massless flavors). It works with the *Schrödinger functional* (SF) defined in the Euclidean space–time manifold in a specific way: all three space dimensions are subject to periodic boundary conditions, while the “time” one is singled out — the gauge field values on the “upper” and “bottom” lids differ by a phase factor with the parameter η . Then, the renormalized coupling α_{SF} is defined via the derivative $\Gamma' = \partial\Gamma/\partial\eta$ of the effective action $\Gamma = \alpha^{-1}\Gamma_0 + \Gamma_1 + \alpha_s\Gamma_2 + \dots$ as (cf. eq.(8.3) in ref.[10]) a function in *the distance representation* $\alpha_{\text{SF}}(L) = \Gamma'_0/\Gamma'$ with L being the spatial size of the above–mentioned manifold.

Quite fresh results reveal the steep rise of the SF $\alpha_{\text{SF}}(L)$ coupling with L in the region $\bar{\alpha}_{\text{SF}} \simeq 1$. Here, the analytic fit [12] to the numerically calculated behavior of $\bar{\alpha}_{\text{SF}}$ has an exponential form

$$\bar{\alpha}_{\text{SF}}(L) \simeq e^{mL} \quad (28)$$

with $m \simeq 2.3/L_{\text{max}}$ and L_{max} (a reference distance in the region of sufficiently weak coupling) indirectly defined via the condition $\bar{\alpha}_{\text{SF}}(L_{\text{max}}) = 0.275$. For a discussion of the momentum–transfer QCD behavior, in the papers of ALPHA collaboration the “quantum–mechanical correspondence” rule is used in the form

$$L \rightarrow 1/\mu \quad (QMC')$$

with μ practically treated as a reference momentum transfer value. This way of transition from the distance to the momentum transfer picture is equivalent to the one discussed above in Section 3.1.. It works quite well in the UV region as far as $\sim \ln^{-1}Q^2$ behavior is compatible with the Tauberian theorem. Nevertheless, in some precise consideration, like in numerical relation between various scales[9]⁹, one should take into account the modification of scale according to eq.(17).

At the same time, the steep rising function of the type (28) does not fit to the (S) condition of this theorem. Due to this the IR (i.e., as $Q^2 \rightarrow 0$) behavior needs a more elaborate analysis.

⁹This important paper was brought to our attention by Dr. U.Wolff. His friendly assistance is gratefully acknowledged.

3.4 On the representation of the ALPHA results in the IR region

Generally, there are two different standpoints for discussing this transition for lattice simulation results obtained on a finite lattice of size L .

The first, simple-minded one treats the obtained numerical results and their analytic description (28) as an approximation to some limit with the continuous function $\bar{\alpha}_{\text{SF}}(L)$ that is subject to an integral Fourier transformation. On the other hand, as far as a lattice boundary condition is a periodical one, it is possible to use a Fourier series instead of the Fourier integral.

In what follows, we shall try to discuss the possible Fourier image of $\bar{\alpha}_{\text{SF}}(L)$ having in mind both the possibilities.

To start with the integral Fourier transformation, we represent the ALPHA “distance running coupling” $\bar{\alpha}_{\text{SF}}$ by the sum of two terms:

$$\bar{\alpha}_{\text{SF}}(L) = \alpha_{\text{PT}}(L) + \alpha_{\text{AL}}(L)$$

with α_{PT} , a perturbation contribution, and α_{AL} , an essentially nonperturbative part.

As far as results of *all* nonperturbative calculations — by both the numerical lattice simulation ones and solving the Schwinger–Dyson equations (SDE) — reveal no traces of unphysical singularities, we, on the one hand, change the first perturbative term α_{PT} for some regular expression α_{PTF} (like the “frozen” one (24) in the Analytic Perturbation Theory or some other smooth ones like those emerging from the “effective massive glueball model” [20] or from the SDE solving[21]).

On the other hand, we approximate the second, essentially nonperturbative, term by expression of type

$$\alpha_{\text{AL}}(L) \simeq \alpha_k(L) = C_k \frac{\pi}{2} \left(\frac{L}{L_m} \right)^k e^{m(L-L_m)}. \quad (29)$$

It is close to (28) and admits further explicit integration. Here, $k \geq 1$ is small integer, and the $mL_m \simeq 2.0$ and C_k parameter values follow from results of numerical lattice simulations. For example, $C_1 \simeq 0.013$.

That is

$$\alpha_{\text{SF}}(L) = \alpha_{\text{PTF}}(L) + \alpha_k(L). \quad (30)$$

For the Fourier image of regularized $\alpha_{\text{PT-R}}$ we assume, qualitatively, the momentum transfer behavior related by the (QMC) rule

$$\bar{\alpha}_{\text{PT-R}}(Q) \equiv \mathbb{F}_{\text{sin}}[\alpha_{\text{PTF}}](Q) = \alpha_{\text{PT-R}}(1/Q) \quad (31)$$

with a finite IR limit like in (27)

$$\bar{\alpha}_{\text{PT-R}}(0) = C; \quad 0 < C < \infty \quad (32)$$

which is supported by the second illustration of Section 3.2 — see Figure 2.

To perform transformation (14) for $\alpha_k(L)$, one needs some regularization. To this goal, we shall insert the cutoff factor¹⁰

$$\exp\{(m + \mu)\theta(L - \xi L_m)(\xi L_m - L)\}; \quad \xi = 2.5 \div 3, \quad \mu \geq 0$$

into the integrand of the r.h.s in eq. (14). Here, parameters ξ and μ , regulate “the place of switching-on” and the intensity of the cutoff.

Then, the “ALPHA coupling function” in the momentum picture can be represented as a sum of three terms

$$\bar{\alpha}_{\text{SF}}(Q) = \bar{\alpha}_{\text{PTF}}(Q) + \bar{\alpha}_{k,\text{reg}}(Q) + \bar{\alpha}_{k,\text{sing}}(Q; \mu) \quad (33)$$

with the first one being defined by (31) and two others can be obtained from expressions (29) taken at $k = 1$:

$$\bar{\alpha}_{1,\text{reg}}(Q) = \frac{C_1}{L_m} \int_{L_m}^{\xi L_m} dL \sin(QL) e^{mL} = \frac{C_1}{L_m} [f_1(Q, \xi L_m) - f_1(Q, L_m)]; \quad (34)$$

$$\bar{\alpha}_{1,\text{sing}}(Q; \mu) = e^{m\xi L_m} \frac{C_1}{L_m} \int_{\xi L_m}^{\infty} dL \sin(QL) e^{\mu(\xi L_m - L)} = \frac{C_1}{L_m} \varphi_1(Q, \xi L_m; \mu) \quad (35)$$

by appropriate differentiation with respect to m . Here,

$$f_1(Q, L) = e^{mL} \frac{m \sin(QL) - Q \cos(QL)}{m^2 + Q^2},$$

$$\varphi_1(Q, L; \mu) = e^{mL} \frac{Q \cos(QL) + \mu \sin(QL)}{\mu^2 + Q^2}.$$

¹⁰Strictly speaking, this exponential regularization corresponds to the *generalized Fourier transformation* — see, e.g., Section 1.3. in ref.[22].

The most important qualitative feature of the functions $f_k(Q, L)$ and $\varphi_k(Q, L; \mu)$ is their IR behavior. All the functions generally tend to zero linearly with $Q \rightarrow 0$. The only exception is the case of $\mu = 0$ when one has a power singularity $\varphi_k(Q \rightarrow 0, L; 0) \rightarrow 1/Q$.

By combining this result with (32) we conclude that $\bar{\alpha}_{\text{SF}}(Q)$ in the IR region can have a finite limit or the first order pole (the latter — only for the case of integral Fourier transformation), contrary to the exponential growth $\sim \exp(m/Q)$ that could be anticipated from some results of the ALPHA collaboration (we mean, *e.g.*, Fig.3 and Fig.4 in ref.[12]).

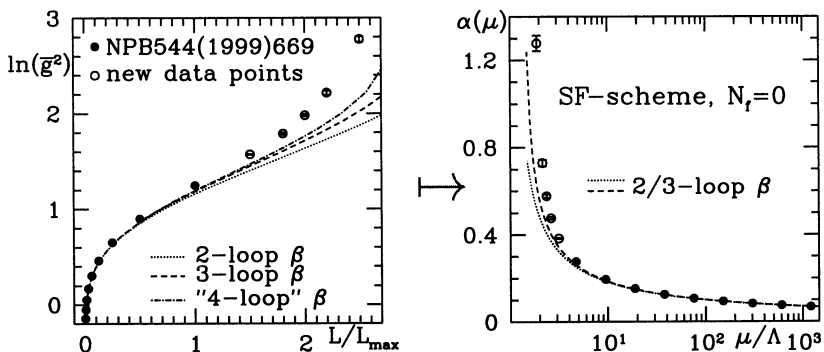


Figure 3: (left part) L -dependence of $\ln(g^2)$ in Fig.4 in [12]; (right part) μ -dependence of α in Fig.3 in [12]

On a more general ground, one can argue that the exponential growth of the QCD coupling with L is the utmost steep possible one. In particular, for the $\ln \alpha(L) \sim L^\nu$, $\nu > 1$ regime there is no known mathematical means for defining a Fourier transformation. Vice versa, for the $\sim \exp\{m/Q\}$ IR asymptotic behavior it is impossible to construct any Fourier transformation and "return" to the L -picture.

4 Discussion

Our first observation concerns the issue of unphysical singularities in the QCD effective coupling and related QFT quantities like the propagator amplitudes. These singularities are inconsistent with integral transformations. In particular, the well-known unphysical singularity in the region $Q \sim \Lambda \simeq 300 - 400 \text{ GeV}$ prevents the common QCD perturbative effective coupling $\bar{\alpha}_s(Q^2)$ from a straightforward performing of the integration procedure necessary for transition to the distance representation by the Fourier transformation. This gives us an additional theoretical *argument against of existence of unphysical singularities in the physically reasonable theories* like QED and QCD. Remind here, that *all* the QCD nonperturbative calculations — by both the numerical lattice simulation and the SDE solving — reveal no traces of unphysical singularities.

The second result based on the analysis of the Fourier transformation deals with the quantum–mechanical correspondence rule

$$r \rightarrow 1/Q \quad (\text{QMC})$$

relating the asymptotic behavior of a function $f(r)$ as $r \rightarrow \infty$ and of its Fourier transform $F(Q)$ as $Q \rightarrow 0$. It has been confirmed that this rule, being a reasonable guide for some class of asymptotics (the power and logarithmic type), has its rigid limits of applicability.

First, even for the function with admissible asymptotic behavior, in the region not very close to the singularity the (QMC) rule yields only a qualitative correspondence, as it follows from our Figure 1.

Second, it is not valid at all for wide class of asymptotic behaviors violating the so-called Tauberian conditions[27], like the exponential ones.

In particular, the exponentially rising long-range behavior of QCD coupling in the distance representation

$$\alpha_{\text{SF}}(L) \sim e^{mL}$$

observed by ALPHA collaboration on the basis of the lattice simulation of the Schrödinger functional, according to our analysis, can correspond in the momentum (transfer) picture to the

$$\text{a) finite or} \quad \text{b) "slightly singular"} \sim 1/Q \quad (\text{ALPHA-IR})$$

IR asymptotics.

This means that these long-distance results, being properly translated to the IR region, qualitatively, will not be so far from the results of other groups¹¹ that perform lattice simulation calculations (partially supported by solution of appropriate truncated Schwinger–Dyson equations) for the functional integral defined in the momentum representation.

From the physical point of view, in our opinion, there are at least two issues that should be mentioned in connection with the QCD infrared asymptotic behavior.

First, we have to remember that the region of $Q \lesssim 500 \text{ MeV}$, physically, corresponds to distances $r \gtrsim 10^{-13} \text{ cm}$, that is to the hadronic scales. Here, all the QCD notions, like gluonic and quark propagators, seem to be meaningless. Even in a more moderate region $0.5 \text{ MeV} \lesssim Q \lesssim 3 \text{ GeV}$ of strong QCD coupling there arises a question of physical meaning of nonperturbative QCD functions, including the effective coupling one.

The mentioned above results for the effective QCD coupling obtained[21, 23, 24] by a numerical lattice simulation of the path QCD integral in the momentum representation are formulated by the “common QFT language” using the vertices with fixed dynamics, like, *e.g.*, in [23, 25] for the QCD model with two massive quarks. There, an invariant coupling $\bar{g}(Q^2)$ is defined on the basis of the gluon–quark vertex $\Gamma_{q-g}(0; Q^2, Q^2)$ in the particular MOM scheme with gluon momentum equal to zero. The invariant coupling function thus defined suffers from the usual drawback of MOM schemes — the gauge dependence. At the same time, it can be, in principle, considered as directly corresponding to some definite physical situation (being incorporated into a series for some observable with gauge–dependent coefficients).

Second, it is reasonable to relate the IR behavior of the QCD functions with the confinement phenomenon. Here, it is attractive to appeal to the so-called Kugo-Ojima condition[28] that, physically, corresponds to the absence of “open colour” in the asymptotic states. In the QCD language, this yields the vanishing of the gluon and quarks fields renormalization constants, that, in turn, is equivalent to the zero IR limit of corresponding QCD propagators. Such a behavior has been observed in the most of the “lattice–simulation QCD papers” [21, 23, 24]. Quite recently it has been supported by Orsay group[29] on the basis of an instanton liquid model and, in a sense, by analysis[30] of the τ decay data.

¹¹See, *e.g.*, refs.[21, 23, 24] and short discussion of their difference in [15].

In this context, the (ALPHA-IR,_a) IR behavior seems to be a quite reasonable possibility to correlate all above-mentioned lattice simulation results with the each others and with the physics of confinement.

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Ширков Д. В.
О фурье-преобразовании
ренорм-инвариантной функции связи

E2-2003-1
hep-th/0210013

Рассматриваются интегральные преобразования ренорм-инвариантной (бегущей) функции связи КХД. Особое внимание уделено преобразованию Фурье, т. е. переходу от пространственно-временной картины к представлению энергии-импульса.

Наше первое заключение состоит в том, что условие возможности такого перехода представляет еще один аргумент против реальности существования нефизических сингулярностей, наблюдаемых в пертурбативной КХД.

Второе заключение сводится к способу «перевода» некоторых сингулярных «длинноволновых» асимптотик в инфракрасную область переданных импульсов, который следует проводить с учетом теоремы Таубера. Это замечание имеет отношение к недавним результатам, полученным коллаборацией АЛЬФА, по асимптотическому поведению эффективной функции связи КХД на основе численных решеточных симуляций.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

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Shirkov D. V.
On the Fourier Transformation
of Renormalization Invariant Coupling

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Integral transformations of the QCD invariant (running) coupling and of some related objects are discussed. Special attention is paid to the Fourier transformation, that is to transition from the space-time to the energy-momentum representation.

The conclusion is that the condition of possibility of such a transition provides us with one more argument against the real existence of unphysical singularities observed in the perturbative QCD.

The second one relates to the way of «translation» of some singular long-range asymptotic behaviors to the infrared momentum region. Such a transition has to be performed with due account of the Tauberian theorem. This comment relates to the recent ALPHA collaboration results on the asymptotic behavior of the QCD effective coupling obtained by lattice simulation.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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