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ON GLOBAL $L_1 \cap L_\infty$ SOLUTIONS FOR THE VLASOV–POISSON SYSTEM

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1 Introduction. Result

Consider the classical Vlasov-Poisson system

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_v f \cdot E(x,t) = 0, \quad (t,x,v) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3, \ f = f(t,x,v), \tag{1}$$

$$E(x,t) = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla U(x-y) f(t,y,v) dy\, dv, \quad U(x) = \kappa |x|^{-1}, \tag{2}$$

$$f(0,x,v) = f_0(x,v), \tag{3}$$

where all quantities are real, $a \cdot b$ means the usual scalar product of $a, b \in \mathbb{R}^3$, $\kappa = \pm 1$ is a constant, and f is an unknown function that has the sense of a distribution function of particles in the (x,v)-space. In view of the sense of f, we require:

$$f \geq 0 \quad \text{and} \quad \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f(t,x,v) dx\, dv \equiv 1. \tag{4}$$

Everywhere L_p denotes the standard Lebesgue space $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ with the standard norm (here $1 \leq p \leq \infty$). In what follows, we look for weak solutions of (1)-(4) that belong to $L_1 \cap L_\infty$ for each fixed $t \in \mathbb{R}$.

There is a numerous literature devoted to studies of Vlasov equations. Here we mention the following papers. In [1-4], the Vlasov equation with a smooth bounded potential U is considered; the existence and uniqueness of a weak solution with values in the space of normalized nonnegative measures is proved. In [5-7], the Vlasov-Poisson system is investigated (see also [4]). In [6], the existence and uniqueness of radial solutions is proved. In [4,5,7], weak solutions of this system are studied (we note that in these papers the question about the uniqueness of weak solutions similar to ours is left open). We also mention paper [8] where weak solutions of the Vlasov-Maxwell system are considered. In [9], the existence of a global smooth solution to (1)-(4) is demonstrated. We also mention paper [10], where the Vlasov equation with potentials of higher singularities is considered, and paper [11] where a two-time problem for the equation with a smooth bounded potential is treated.

Here we prove in particular the uniqueness of a weak solution of (1)-(4). We accept the following.

Definition *Let $f(t,\cdot,\cdot) \in C(I; L_p)$ for all $1 \le p < \infty$ where $I \subset \mathbb{R}$ is an interval containing 0 and $\|f(t,\cdot,\cdot)\|_{L_\infty} \le C$ for all $t \in I$. Then, we call f a weak solution of (1)-(4) if (2)-(4) are satisfied and if for any function $\eta = \eta(t,x,v)$ in $I \times \mathbb{R}^3 \times \mathbb{R}^3$ continuously differentiable and equal to zero from outside of a compact set one has for all $t \in I$:*

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, [\eta(t,x,v)f(t,x,v) - \eta(0,x,v)f_0(x,v)] - \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, \times$$

$$\times f(s,x,v)\{\eta_s(s,x,v) + v\cdot\nabla_x\eta(s,x,v) + \nabla_v\eta(s,x,v)\cdot E(x,s)\} = 0. \tag{5}$$

In the present paper, our main result is the following.

Theorem 1 *For any $f_0 \in L_1 \cap L_\infty$ with a compact support problem (1)-(4) has a unique weak solution $f(t,x,v)$ global in t such that its (x,v)-support is bounded uniformly in t from an arbitrary finite interval. The energy of the system,*

$$E(f) = 1/2 \int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)\, dx\, dv -$$

$$- \int_{\mathbb{R}^{12}} dx\, dx'\, dv\, dv'\, f(t,x,v)U(x-x')f(t,x',v'),$$

does not depend on t.

We prove this result in the next section.

2 Proof of Theorem

Associate with (1)-(4) the following system:

$$\dot{x}(t,x_0,v_0) = v(t,x_0,v_0), \tag{6}$$

$$\dot{v}(t,x_0,v_0) = w(x(t,x_0,v_0),t) := \int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x(t,x_0,v_0)-y)f(t,y,v)\, dy\, dv, \tag{7}$$

$$(x(0,x_0,v_0), v(0,x_0,v_0)) = (x_0,v_0), \tag{8}$$

$$f(t,x(t,x_0,v_0),v(t,x_0,v_0)) = f_0(x_0,v_0) \text{ for almost all } (x_0,v_0)\in\mathbb{R}^3\times\mathbb{R}^3 \text{ for any fixed } t, \tag{9}$$

where (x_0, v_0) runs over the entire $\mathbb{R}^3\times\mathbb{R}^3$. Formally, if $(x(t,x_0,v_0), v(t,x_0,v_0))$ is a solution of system (6)-(9) and if $f(t,x,v)$ is given by (9), then f satisfies (1)-(4). Here, we are aimed in particular to justify this fact.

Set for $g\in L_1\cap L_\infty$

$$(Tg)(x) = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x-y)g(y,v)dy\,dv.$$

Let $\omega(\cdot)$ be a nonnegative even C^∞-function with a compact support in $\mathbb{R}^3$ satisfying $\int\limits_{\mathbb{R}^3}\omega(x)\,dx = 1$ and let $U_n(x) = (U(\cdot)\star n^3\omega(n\cdot))(x)$ where the star means the convolution and $n=1,2,3,\ldots.$ Consider the sequence of approximations of system (6)-(9) that occur by substitutions of U_n in place of U in (6)-(9). We denote these approximations by (6_n)-(9_n). Let also T_n be the integral operator which is defined by analogy with T with the change of U by U_n.

It is the well known result proved in fact in [1-4] that for each n system (6_n)-(9_n) possesses a unique global solution $(x_n(t,x_0,v_0), v_n(t,x_0,v_0))$; also, for any fixed t the map S_n^t transforming (x_0,v_0) into $(x_n(t,x_0,v_0), v_n(t,x_0,v_0))$ is a diffeomorphism of $\mathbb{R}^3\times\mathbb{R}^3$ onto itself (i. e. it is a one-to-one map continuously differentiable with its inverse), and the corresponding function $f_n(t,x_1,v_1)\equiv f_0(x_n(-t,x_1,v_1), v_n(-t,x_1,v_1))$ is finite for each fixed t and it is a weak solution of the problem arising from(1)-(4) by replacing U by U_n; in addition, diam (supp $f_n(t,\cdot,\cdot)$) is continuous in t. Also, for each n the corresponding energy $E_n(f_n)$, which occurs by replacing U by U_n in the representation for E, and the norms $\|f_n(t,\cdot,\cdot)\|_{L_p}$ with $1\le p\le\infty$ do not depend on t. In addition, according to [12] det $J_n(t,x_0,v_0)\equiv 1$ where $J_n = \frac{\partial(x_n(t,x_0,v_0),v_n(t,x_0,v_0))}{\partial(x_0,v_0)}$ is the Jacobi matrix. Denote $D_n^x(t) = \sup\{p\in[0,\infty):\ \text{ess}\sup\limits_{|x|>p} f_n(t,x,v)>0\}$ and $D_n^v(t) = \sup\{q\in[0,\infty):\ \text{ess}\sup\limits_{|v|>q} f_n(t,x,v)>0\}$.

Lemma 1 *For any finite $f_0\in L_1\cap L_\infty$ there exist $D_0>0$ and $T=T(D_0)>0$, where $T(s)$ is a nonincreasing function of $s>0$, such that $D_n^x(t)+D_n^v(t)\le D_0$ for all n and all $t\in[-T,T]$.*

Proof. We consider only the case $t > 0$ because for $t < 0$ all our estimates can be made by analogy. First of all, we have the estimate:

$$|(T_n f_n)(x,t)| \le C([D_n^v(t)]^3 \|f_n(t,\cdot,\cdot)\|_{L_\infty} + \|f_n(t,\cdot,\cdot)\|_{L_1}). \qquad (10)$$

' Further, it can be easily derived from (6_n),(7_n) and (10) that one has for any $(x_0, v_0) \in$ supp (f_0):

$$|x_n(t,x_0,v_0)| \le |x_0| + \int_0^t |v_n(s,x_0,v_0)|\, ds \le |x_0| + \int_0^t D_n^v(s)\, ds$$

and

$$|v_n(t,x_0,v_0)| \le C_1 + |v_0| + C_2 \int_0^t [D_n^v(s)]^3\, ds$$

and hence,

$$D_n^x(t) \le D_n^x(0) + \int_0^t D_n^v(s)\, ds$$

and

$$D_n^v(t) \le C_3 + D_n^v(0) + C_4 \int_0^t [D_n^v(s)]^3\, ds$$

with constants $C_3, C_4 > 0$ independent of $t \in [0,1]$ and n, which easily implies our claim.□

Corollary 1 *There exists $C > 0$ such that $|(T_n f_n)(x,t)| \le C$ for all x and $t \in [-T,T]$.*

Proof follows from (10) and lemma 1.□

Lemma 2 *There exists $C > 0$ such that*

$$|(T_n g)(x_1) - (T_n g)(x_2)| \le -C|x_1 - x_2| \ln |x_1 - x_2|$$

for all $g \in L_1 \cap L_\infty$ satisfying $\|g\|_{L_1} + \|g\|_{L_\infty} \le 1$ and $g(x,v) = 0$ if $|x|+|v| > D_0$, all $n = 1,2,3,...$ and for all $x_1, x_2 \in \mathbb{R}^3$ such that $|x_1 - x_2| \le 1/2$.

Proof. Take arbitrary $x, h \in \mathbb{R}^3$, where $|h| \le 1/2$, and g. Then, we have:

$$(T_n g)(x+h) - (T_n g)(x) = \left(\int\limits_{B_{2|h|}(x)} + \int\limits_{B_{D_0}(0)\setminus B_{2|h|}(x)} \right) dy (\nabla U(x+h-y) -$$

$$-\nabla U(x-y)) \int\limits_{B_{D_0}(0)} g(y,v)\, dv = I_1 + I_2.$$

Since $\|g\|_{L_\infty} \le 1$, we have for I_1:

$$|I_1| \le C_1 D_0^3(T) \int\limits_{B_{2|h|}(0)} |y|^{-2}\, dy \le C'|h|.$$

For I_2 we deduce:

$$|I_2| \le C_2 |h| D_0^3(T) \int\limits_{B_{D_0}(0)\setminus B_{2|h|}(0)} |y|^{-3}\, dy \le -C_3 |h| \ln|h|. \Box$$

Corollary 2 *One has $|(Tg)(x+h) - (Tg)(x)| \le -C|h| \ln|h|$ for all x and h : $|h| \le 1/2$ and for all $g \in L_1 \cap L_\infty$ satisfying $\|g\|_{L_1} + \|g\|_{L_\infty} \le 1$ and $g(x,v) = 0$ if $|x| + |v| > D_0$.*

Lemma 3 *For any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|(x_n(t,x_0,v_0), v_n(t,x_0,v_0)) - (x_n(t,x_1,v_1), v_n(t,x_1,v_1))| < \epsilon$$

for all n and all $t \in [-T,T]$ if $|(x_0,v_0) - (x_1,v_1)| < \delta$.

Proof. We consider only the case $t > 0$ because for $t < 0$ the proof can be made by analogy. We have by lemma 2:

$$|x_n(t,x_0,v_0) - x_n(t,x_1,v_1)| \le |x_0 - x_1| + \int_0^t |v_n(s,x_0,v_0) - v_n(s,x_1,v_1)|\, ds$$

and

$$|v_n(t,x_0,v_0) - v_n(t,x_1,v_1)| \le |v_0 - v_1| - C \int_0^t |x_n(s,x_0,v_0) - x_n(s,x_1,v_1)| \times$$

$$\times \ln |x_n(s, x_0, v_0) - x_n(s, x_1, v_1)| \, ds$$

until $|x_n(t, x_0, v_0) - x_n(t, x_1, v_1)| \leq 1/2$. Now our claim follows by standard arguments similar to those used when proving the Gronwell's lemma (see also [13]).□

Now, applying the Arzéla-Ascoli theorem, in view of lemmas 1-3 and (10) we deduce that the sequence $\{(x_n(t, x_0, v_0), v_n(t, x_0, v_0))\}_{n=1,2,3,...}$ of functions from $[-T, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^3 \times \mathbb{R}^3$ contains a subsequence still denoted $\{(x_n(t, x_0, v_0), v_n(t, x_0, v_0))\}_{n=1,2,3,...}$ which converges to a pair of continuous functions $(x(t, x_0, v_0), v(t, x_0, v_0))$ uniformly in (t, x_0, v_0) from an arbitrary compact subset of $[-T, T] \times \mathbb{R}^3 \times \mathbb{R}^3$.

Lemma 4 *For any $t \in [-T, T]$ there exists $f(t, \cdot, \cdot) \in L_1 \cap L_\infty$ such that for any $p \in [1, \infty)$ the sequence $\{f_n(t, \cdot, \cdot)\}_{n=1,2,3,...}$ converges to $f(t, \cdot, \cdot)$ strongly in L_p.*

Proof. Take a sequence $h^k(\cdot, \cdot)$ of continuous functions converging to f_0 strongly in L_p and almost everywhere, bounded in L_∞ and such that $h^k(x, v) = 0$ if $|x| + |v| > D_0 + 1$ for all k. Then, we have

$$\|f_n(t, \cdot, \cdot) - f_m(t, \cdot, \cdot)\|_{L_p} \leq \|h^k(x_n(-t, \cdot, \cdot), v_n(-t, \cdot, \cdot)) -$$

$$- h^k(x_m(-t, \cdot, \cdot), v_m(-t, \cdot, \cdot))\|_{L_p} +$$

$$+ \|h^k(x_n(-t, \cdot, \cdot), v_n(-t, \cdot, \cdot)) - f_0(x_n(-t, \cdot, \cdot), v_n(-t, \cdot, \cdot))\|_{L_p} +$$

$$+ \|h^k(x_m(-t, \cdot, \cdot), v_m(-t, \cdot, \cdot)) - f_0(x_m(-t, \cdot, \cdot), v_m(-t, \cdot, \cdot))\|_{L_p}.$$

Then, obviously, for any $\epsilon > 0$ the second and third terms in the right-hand side of this inequality are smaller than $\epsilon/3$ for all sufficiently large k and for all n and m, and the first term is smaller than $\epsilon/3$ for the same (fixed) values of k and for all sufficiently large n and m.□

Corollary 3 *One has $\|f(t, \cdot, \cdot)\|_{L_p} \equiv \|f_0\|_{L_p}$ for all $p \geq 1$ and all t.*

Proof follows from lemma 4 and the relations $\frac{d}{dt}\|f_n(t, \cdot, \cdot)\|_{L_p} \equiv 0$, holding for all $t \in \mathbb{R}$ and for all $p \in [1, \infty)$, which are well known.□

Lemma 5 *Let $\{g^n\}_{n=1,2,3,...} \subset L_1 \cap L_\infty$, each $g^n = 0$ if $|x| + |v| > D_0$, this sequence is bounded in L_∞, and let for any $p \in [1, \infty)$ $g^n \to g$ strongly in L_p. Then, $(T_n g^n)(x) \to (Tg)(x)$ uniformly in $x \in \mathbb{R}^3$.*

Proof. First, we have:

$$|(T_n g^n)(x) - (Tg)(x)| \leq |((T_n - T)g^n)(x)| + |(T(g^n - g))(x)|.$$

The first term in the right-hand side of this inequality tends to 0 as $n \to \infty$ because

$$|((T_n - T)g^n)(x)| \leq \int\limits_{B_{D_0}(0)} dy\, |\nabla U_n(x-y) - \nabla U(x-y)| \int\limits_{B_{D_0}(0)} g^n(y,v)\, dv \leq$$

$$\leq C \int\limits_{B_{D_0}(0)} |\nabla U_n(x-y) - \nabla U(x-y)|\, dy \to 0$$

uniformly in $x \in \mathbb{R}^3$.

As for the second term, we have:

$$|T(g^n - g)(x)| \leq \int\limits_{B_{D_0}(0)} dy\, |\nabla U(x-y)| \int\limits_{B_{D_0}(0)} |g^n(y,v) - g(y,v)|\, dv \to 0.\square$$

Taking now the limit $n \to \infty$ in (6_n),(7_n), we obtain by lemmas 4 and 5:

$$x(t, x_0, v_0) = x_0 + \int\limits_0^t v(s, x_0, v_0)\, ds, \tag{11}$$

$$v(t, x_0, v_0) = v_0 + \int\limits_0^t w(x(s, x_0, v_0), s)\, ds, \tag{12}$$

where the function w is given by (7) and (11) and (12) hold for all (x_0, v_0) and $t \in [-T, T]$. Now, it follows by the known uniqueness theorem for ODEs (see, for example, [13]) and by corollary 3 that system (11),(12) may have at most one solution. It is also easy to see from (11),(12) that for any fixed t the transformation $(x_0, v_0) \to (x(t, x_0, v_0), v(t, x_0, v_0))$ is a one-to-one map of $\mathbb{R}^3 \times \mathbb{R}^3$ onto itself continuous with the inverse (to see this, it suffices to consider the initial value problem for $(x(t, x_0, v_0), v(t, x_0, v_0))$ with initial data given at an arbitrary time $t_0 \in [-T, T]$).

Theorem 2 *Denote $S^t(x_0, v_0) = (x(t, x_0, v_0), v(t, x_0, v_0))$. Then, for any fixed $t \in [-T, T]$ S^t is a one-to-one map continuous with its inverse* *of* $\mathbb{R}^3 \times \mathbb{R}^3$ *onto itself so that in particular it transforms Borel subsets of* $\mathbb{R}^3 \times \mathbb{R}^3$ *into Borel ones. For any Borel set* $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ *one has* $m(A) = m(S^t(A))$ *where* $m(\cdot)$ *is the Lebesgue measure in* $\mathbb{R}^3 \times \mathbb{R}^3$.

Proof. Take an arbitrary open bounded set $A \subset \mathbb{R}^3 \times \mathbb{R}^3$. As well known, for any $\epsilon > 0$ there exists compact $K_\epsilon \subset A$ such that $m(A \setminus K_\epsilon) < \epsilon$. Let $\alpha = \text{dist}\ (K_\epsilon, \partial A) > 0$ and $A_\beta = \{z \in S^t(A) :\ \text{dist}\ (z, \partial S^t(A)) \geq \beta\}$. Let also $\beta = \text{dist}\ (S^t(K_\epsilon), \partial S^t(A)) > 0$. For any $z \in K_\epsilon$ take a ball $B_r(z) \subset A$ such that $S^t(B_r(z)) \in A_{\frac{\beta}{2}}$. Let $B_{r_1}(z_1), ..., B_{r_l}(z_l)$ be a finite covering of K_ϵ by these balls. Then, by construction, there exists a number N such that $S^t_n(K_\epsilon) \subset S^t(A_{\frac{\beta}{4}})$ for all $n \geq N$. Now, we have:

$$m(A) - \epsilon \leq m(K_\epsilon) \leq m\left(\bigcup_{k=1}^{l} B_{r_k}(z_k)\right) = m\left(S^t_n\left(\bigcup_{k=1}^{l} B_{r_k}(z_k)\right)\right) \leq$$

$$\leq m(A_{\frac{\beta}{4}}) \leq m(S^t(A))$$

so that $m(A) - \epsilon \leq m(S^t(A))$. The inequality $m(S^t(A)) - \epsilon \leq m(A)$ can be obtained by the complete analogy by considering the inverse map S^{-t}. So, $m(A) = m(S^t(A))$.

For an unbounded open set A the same equality follows in view of representations

$$A = \bigcup_{k=1}^{\infty} A \cap B_k(0) \quad \text{and} \quad S^t(A) = \bigcup_{k=1}^{\infty} S^t(A \cap B_k(0)).$$

This also implies the same equality for closed sets. For an arbitrary Borel set $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ the equality $m(A) = m(S^t(A))$ now can be obtained by approximations of A by open sets from outside.□

Lemma 6 *For any fixed* $t \in [-T, T]$ *one has* $f(t, x(t, x_0, v_0), v(t, x_0, v_0)) = f_0(x_0, v_0)$ *for almost all* $(x_0, v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Proof. We have that, over a subsequence, $f_n(t, \cdot, \cdot) \to f(t, \cdot, \cdot)$ almost everywhere and $f_n(t, x_1, v_1) \equiv f_0(x_n(-t, x_1, v_1), v_n(-t, x_1, v_1))$. So, in view of theorem 2 to prove lemma, it suffices to show that $f_0(x_n(-t, \cdot, \cdot), v_n(-t, \cdot, \cdot)) \to f_0(x(-t, \cdot, \cdot), v(-t, \cdot, \cdot))$ almost everywhere. Set

$$(x_n(-t, x_1, v_1), v_n(-t, x_1, v_1)) =$$

$$= (x(-t, x_1, v_1) + \delta_n(x_1, v_1), v(-t, x_1, v_1) + \gamma_n(x_1, v_1)),$$

where $(\delta_n, \gamma_n) \to 0$ as $n \to \infty$ uniformly in an arbitrary compact set, and show that

$$f_0(x(-t, x_1, v_1) + \delta_n, v(-t, x_1, v_1) + \gamma_n) \to f_0(x(-t, x_1, v_1), v(-t, x_1, v_1))$$

almost everywhere over a subsequence. Let φ_k be a sequence of continuous functions, uniformly bounded in L_∞ and supports of which are uniformly bounded, converging to f_0 almost everywhere. Then, for $p \in [1, \infty)$:

$$\|f_0(x(-t,\cdot,\cdot) + \delta_n, v(-t,\cdot,\cdot) + \gamma_n) - f_0(x(-t,\cdot,\cdot), v(-t,\cdot,\cdot))\|_{L_p} \le$$

$$\le \|\varphi_k(x(-t,\cdot,\cdot) + \delta_n, v(-t,\cdot,\cdot) + \gamma_n) - \varphi_k(x(-t,\cdot,\cdot), v(-t,\cdot,\cdot))\|_{L_p} +$$
$$+\|f_0(x(-t,\cdot,\cdot)+\delta_n, v(-t,\cdot,\cdot)+\gamma_n)-\varphi_k(x(-t,\cdot,\cdot)+\delta_n, v(-t,\cdot,\cdot)+\gamma_n)\|_{L_p}+$$
$$+\|f_0(x(-t,\cdot,\cdot), v(-t,\cdot,\cdot)) - \varphi_k(x(-t,\cdot,\cdot), v(-t,\cdot,\cdot))\|_{L_p}.$$

The third term in the right-hand side tends to 0 as $k \to \infty$ in view of theorem 2. As for the second one, since the map $(x_1, v_1) \to (x(-t, x_1, v_1) + \delta_n(x_1, v_1), v(-t, x_1, v_1) + \gamma_n(x_1, v_1))$ is one-to-one continuous with the inverse and preserving the Lebesgue measure, we have that it is equal to $\|f_0 - \varphi_k\|_{L_p} \to 0$ as $k \to \infty$. So, for a given $\epsilon > 0$ the second and third terms can be made smaller than $\epsilon/3$ by taking a sufficiently large k, uniformly in n. As for the first term, it can be made smaller than $\epsilon/3$ by taking the same sufficiently large fixed k and sufficiently large n.□

Proposition 1 *System (6)-(9) has a unique solution $x(t, x_0, v_0), v(t, x_0, v_0))$ in the interval of time $t \in [-T, T]$ such that the (x, v)-support of the corresponding function $f(t, x, v)$ is bounded uniformly in t from an arbitrary bounded interval.*

Proof. We have to prove only the uniqueness of a solution. Suppose the opposite. Without the loss of generality we can accept that there exists $t_0 \in [0, T)$ such that two different solutions $(x_i, v_i)(t, x_0, v_0)$, $i = 1, 2$, coincide for $t \in [0, t_0]$ and for all (x_0, v_0) and that in an arbitrary small right half-neighborhood of t_0 there are points t where $(x_1, v_1)(t, x_0, v_0) \neq (x_2, v_2)(t, x_0, v_0)$ for some (x_0, v_0). Let also $f = f_i(t, x, v)$, $i = 1, 2$, be the corresponding functions given by (9). Without the loss of generality we accept that $f_i(t, x_i(t, x_0, v_0), v_i(t, x_0, v_0)) = f_0(x_0, v_0)$ for all (x_0, v_0) and t. Set also $(x, v)(t, x_0, v_0) = [(x_1, v_1) - (x_2, v_2)](t, x_0, v_0)$, $h(t) =$

$\max_{(x_0,v_0)\in \text{supp }(f_0)} |x(t,x_0,v_0)|$ and $r(t) = \max_{(x_0,v_0)\in \text{supp }(f_0)} |v(t,x_0,v_0)|$. Then, applying lemma 2, we obtain for $t > t_0$ sufficiently close to t_0:

$$h(t) \le \int_{t_0}^{t} r(s)\ ds, \tag{13}$$

$$|v(t,x_0,v_0)| \le -C\int_{t_0}^{t} h(s)\ln h(s)\ ds + \int_{t_0}^{t} ds \Big| \int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x_1(s,x_0,v_0)-y)\times$$

$$\times [f_1(s,y,v)-f_2(s,y,v)]\ dy\ dv \Big| = I + II. \tag{14}$$

Let us estimate II. Denote by $(x_i,v_i)(t,s,x_0,v_0)$ the values of those functions (x_i,v_i) at the moment of time t (where $i = 1,2$) which satisfy $(x_i,v_i)(s,x_0,v_0) = (x,v)$. We have that $f_i(t,x,v)$ are solutions of the linear transport equations with the exterior forces

$$E_i(x,t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x-y) f_i(t,y,v)\ dy\ dv$$

and so, again the mappings $S_i(s,t_0) :\ (x_0,v_0) \to (x_i,v_i)(s,t_0,x_0,v_0)$ preserve the Lebesgue measure in $\mathbb{R}^3\times\mathbb{R}^3$. Then, we have for $t > t_0$ sufficiently close to t_0:

$$II = \int_{t_0}^{t} ds \Bigg| \left(\int_{B_{4h(s)}(x_1(s,x_0,v_0))} + \int_{\mathbb{R}^3\setminus B_{4h(s)}(x_1(s,x_0,v_0))} \right) dy \times$$

$$\times \int_{\mathbb{R}^3} dv\ \nabla U(x_1(s,x_0,v_0)-y)\times[f_1(s,y,v)-f_2(s,y,v)] \Bigg| \le$$

$$\le \int_{t_0}^{t} ds \Big\{ C_1 h(s) + \Big| \int_{S_1(t_0,s)(P_1(4,s))} \nabla U(x_1(s,x_0,v_0)-x(s,t_0,y,v)-$$

$$-x_2(s,t_0,y,v)) f_2(t_0,y,v)\ dy\ dv +$$

$$+\int\limits_{P_2(4,s)} \nabla U(x_1(s,x_0,v_0)-y)f_2(s,y,v)\,dy\,dv\Big|\Big\} \le \int\limits_{t_0}^{t} ds\Big\{C_1 h(s)+$$

$$+C_2 \int\limits_{B_{5h(s)}(x_1(s,x_0,v_0))\setminus B_{3h(s)}(x_1(s,x_0,v_0))} |\nabla U(x_1(s,x_0,v_0)-y)|\,dy+$$

$$+\int\limits_{P_2(3,s)} |\nabla U(x_1(s,x_0,v_0)-x(s,t_0,x_2(t_0,s,y,v),v_2(t_0,s,y,v))-y)-$$

$$-\nabla U(x_1(s,x_0,v_0)-y)|f_2(s,y,v)\,dy\,dv\Big\} \le$$

$$\le \int\limits_{t_0}^{t}\Big\{C_3 h(s)+C_4 h(s)\int\limits_{D_2^x(s)\setminus B_{3h(s)}(0)} |y|^{-3}\,dy \le -C_5\int\limits_{t_0}^{t} h(s)\ln h(s)\,ds, \tag{15}$$

where $P_i(k,s) = \text{supp }(f_i(s,\cdot,\cdot)) \setminus (B_{kh(s)}(x_1(s,x_0,v_0)) \times \mathbb{R}^3)$, $D_2^x(t) = \sup\{p \in [0,\infty) : \text{ ess } \sup\limits_{|x|>p} f_2(t,x,v) > 0\}$ and we exploited the fact that if $(y',v') = S_2(s,t_0)(S_1(t_0,s)(y,v))$, then $|y'-y| \le h(s)$ for all $(y,v) \in$ supp $(f_1(s,\cdot,\cdot))$. Estimates (13)-(15) yield:

$$h(t) \le \int\limits_{t_0}^{t} r(s)\,ds \quad \text{and} \quad r(t) \le -C_3\int\limits_{t_0}^{t} h(s)\ln h(s)\,ds,$$

which easily imply that $h(t) \equiv r(t) \equiv 0$ in a right half-neighborhood of t_0.□

Proposition 2 *The function $f(t,x,v)$ is a weak solution of (1)-(4).*
Proof. Obviously, $f(t,\cdot,\cdot) \in C([-T,T];L_p)$ for each $p \in [1,\infty)$. Let $\eta(t,x,v)$ be an admissible function for (5). Then, equality (5) for f can be obtained by writing it for $f_n(t,x,v)$ with the further passing to the limit $n \to \infty$.□

Proposition 3 *Let $f(t,\cdot,\cdot) \in C(I;L_p)$ for each $p \in [1,\infty)$, where $0 \ni I \subset [-T,T]$ and I is an interval, and let it be a weak solution of (1)-(4). Then, $f(t,x(t,x_0,v_0),v(t,x_0,v_0)) = f_0(x_0,v_0)$ for almost all (x_0,v_0)*

(here $x(t,x_0,v_0), v(t,x_0,v_0))$ is the solution of (6)-(8) corresponding to this f).

<u>Proof.</u> Take an arbitrary finite continuously differentiable function $\eta(x_0,v_0)$ in $\mathbb{R}^3\times\mathbb{R}^3$ and let $\eta_n(t,x_n(t,x_0,v_0),v_n(t,x_0,v_0)) \equiv f(x_0,v_0)$ where now $(x_n(t,\cdot,\cdot),v_n(t,\cdot,\cdot))$ is the solution of system (6)-(8) taken with this function f and with U_n in place of U. Then, $\frac{\partial}{\partial t}\eta_n(t,x,v) = -v\cdot\eta_{n,x} - \eta_{n,v}\cdot w_n(x,t)$ where

$$w_n(x,t) = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U_n(x-y) f(t,y,v)\, dy\, dv.$$

Substitute this function η_n in (5). Then, we have:

$$0 \equiv \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, (\eta_n(t,x,v)f(t,x,v) - \eta(x,v)f_0(x,v)) -$$

$$-\int\limits_0^t ds \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, f(s,x,v)\{\eta_{n,v}(s,x,v)\cdot w(x,s) - \eta_{n,v}(s,x,v)\cdot w_n(x,s)\}.$$

The second term in the right-hand side of this identity goes to 0 as $n\to\infty$ by the arguments above, and so, we arrive at the relation

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, (\eta_n(t,x,v)f(t,x,v) - \eta(x,v)f_0(x,v)) \to 0 \quad \text{as} \quad n\to\infty.$$

Make in the first term here the change of variables: $(x,v)\to(x',v') = (x_n(-t,x,v), v_n(-t,x,v))$. Then, we obtain:

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, \eta(x,v)[f(t,x_n(t,x,v),v_n(t,x,v)) - f_0(x,v)] \to 0 \quad \text{as} \quad n\to\infty.$$

Passing here to the limit $n\to\infty$, we deduce as when proving lemma 6:

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx_0\, dv_0\, \eta(x_0,v_0)[f(t,x(t,x_0,v_0),v(t,x_0,v_0)) - f_0(x_0,v_0)] = 0,$$

and hence, due to the arbitrariness of η, proposition 3 is proved.□

So, we have proved the existence and uniqueness of a local solution to (1)-(4) finite for any fixed t. The relation $\frac{d}{dt}E(f) \equiv 0$ is also obvious. According to the result proved in fact in [9], for any $p > \frac{33}{17}$ there exists $C > 0$ such that $D^x(t) + D^v(t) \leq C(1+|t|)^p$ for all t from an arbitrary interval of the existence of our solution, where $D^x(t) = \sup\{p \in [0,\infty) : \operatorname{ess\,sup}_{|x|>p} f(t,x,v) > 0\}$ and $D^v(t) = \sup\{q \in [0,\infty) : \operatorname{ess\,sup}_{|v|>q} f(t,x,v) > 0\}$. This immediately yields that our solution $f(t,x,v)$ can be uniquely continued onto the entire real line $t \in \mathbb{R}$ and that it is finite for any fixed t. So, our proof of Theorem 1 is complete.

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О глобальных $L_1 \cap L_\infty$ -решениях системы Власова–Пуассона

Для классической системы Власова–Пуассона с начальными данными из $L_1 \cap L_\infty$ доказаны существование и единственность слабого глобального решения со значениями в $L_1 \cap L_\infty$.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

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Zhidkov P. E. E5-2003-197
On Global $L_1 \cap L_\infty$ Solutions for the Vlasov–Poisson System

For the classical Vlasov–Poisson system with initial data in $L_1 \cap L_\infty$, we prove the existence and uniqueness of a weak global solution with values in $L_1 \cap L_\infty$.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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