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ON LOCAL SMOOTH SOLUTIONS FOR THE VLASOV EQUATION WITH THE POTENTIAL OF INTERACTIONS $\pm r^{-2}$

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1 Introduction. Notation. Results

Vlasov equations appear in the mean-field approximations of the dynamics of a large number of interacting classical particles (molecules). Currently, there is a numerous literature devoted to its mathematical treatments. In particular, in [1-4] a well-posedness for this equation supplied with initial data and its derivation from a molecular dynamics is considered in the case when the potential of interactions between particles is smooth and bounded. In [5-11], this equation is studied for the singular Coulomb potential $U(r) = \pm r^{-1}$ (in [8], the Vlasov-Maxwell system is considered). In [12], the local existence of smooth solutions in the case $U(r) = \pm r^{-2}$ is studied. We also mention paper [13] where a well-posedness of this equation supplied with a joint distribution of particles at two moments of time is proved.

In the present article, we consider the problem

$$\frac{\partial}{\partial t} f + v \cdot \nabla_x f + \nabla_v f \cdot E(x,t) = 0, \quad f = f(t,x,v),\ t \in \mathbb{R},\ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{1}$$

$$E(x,t) = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla U(x-y) f(t,y,v)\,dy\,dv, \quad U(x) = \kappa |x|^{-2},\ \kappa = \pm 1, \tag{2}$$

$$f(0,x,v) = f_0(x,v), \tag{3}$$

where all quantities are real, $x, v \in \mathbb{R}^3$, κ is a constant, ∇_x and ∇_v are the gradients in x and v, respectively, $v \cdot \nabla_x f$ and $\nabla_v f \cdot E(x,t)$ are the scalar products in $\mathbb{R}^3$, and f is an unknown function. For any fixed t, $f(t,x,v)$ regarded as a function of (x,v) has the sense of a distribution function of particles in $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$. Therefore, the following requirements are natural:

$$f(t,\cdot,\cdot) \geq 0, \quad \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f(t,x,v)\,dx\,dv = 1, \quad \forall t \in \mathbb{R}. \tag{4}$$

Generally speaking, it is known that proving the existence of a solution for problem (1)-(4) is more difficult for a singular potential U than for a more regular one. Also, though the Vlasov equation appeared for the first time with the Coulomb potential $U(r) = \pm r^{-1}$ for a description of plasma, it is well known that in statistical physics potentials with higher singularities occur: for example, the following one, the so-called

Lennard-Jones potential, is known: $U(r) = Ar^{-12} - Br^{-6}$. So, the author of the present article believes that considerations of Vlasov equations for potentials with singularities of the degree higher than r^{-1} make a sense. Here we consider the case $U(r) = \pm r^{-2}$ proving the existence and uniqueness of a local smooth solution of the problem (1)-(4). This case is critical in a sense. A treatment of the problem in the case $U(r) = r^{2-a}$ with $a > 0$ is still left open. As for the case $U(r) = r^{-a}$ with $a \in (1, 2)$, here the problem becomes simpler, and we do not study this case.

Now, we introduce some notation. Let

$$S = \Big\{ g(x,v) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) : \ \forall k_i, m_i = 0, 1, 2, \ldots :$$
$$\sup_{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3} |x|^{k_1}|v|^{k_2} \left| \frac{\partial^{m_1+m_2} g(x,v)}{\partial x^{m_1}\partial v^{m_2}} \right| < \infty \Big\}.$$

The linear space S is equipped with a topology of open subsets becoming a complete topological space. This topology is generated by the system of seminorms

$$p_{k,m}(g) = \left\{ \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} (1+|x|^2)^{k_1}(1+|v|^2)^{k_2} \sum_{i=1}^{3} \left(\frac{\partial^m g(x,v)}{\partial x_i^m} \right)^2 dx\, dv \right\}^{1/2}$$

and

$$q_{k,m}(g) = \left\{ \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} (1+|x|^2)^{k_1}(1+|v|^2)^{k_2} \sum_{i=1}^{3} \left(\frac{\partial^m g(x,v)}{\partial v_i^m} \right)^2 dx\, dv \right\}^{1/2}$$

where $k = (k_1, k_2)$ with $k_i, m = 0, 1, 2, \ldots$. By $C(I; S)$, where $I \subset \mathbb{R}$ is an interval, we denote the linear space of all continuous functions $g : I \to S$ such that each seminorm $p_{k,m}(g(t))$ and $q_{k,m}(g(t))$ is bounded uniformly in $t \in I$.

Our main result here is the following.

Theorem *Let $f_0 \in S$ and satisfy (4). Then, there exists $T > 0$ such that problem (1)-(4) has a unique local solution $f(t,x,v)$ satisfying $f(t,\cdot,\cdot), f'_t(t,\cdot,\cdot) \in C([-T,T]; S)$.*

Remark As it is shown in [14], if $\kappa > 0$, then problem (1)-(4) possesses solutions blowing up in finite intervals of time. So, in this case, generally speaking, solutions we consider can be not continuable onto the entire real line $t \in \mathbb{R}$.

2 Proof of the theorem

Let $\omega(\cdot)$ be a nonnegative even C^∞-function in $\mathbb{R}^3$ with a compact support satisfying $\int\limits_{\mathbb{R}^3} \omega(x)\, dx = 1$ and let $\omega_n(x) = n^3\omega(nx)$, $n = 1, 2, 3, \ldots$. Set $U_n(x) = (U \star \omega_n)(x)$ where the star means the convolution. Consider the following sequence of regularizations of problem (1)-(4):

$$f_t^n + v \cdot \nabla_x f^n + \nabla_v f^n \cdot E_n(x,t) = 0, \quad f^n = f^n(t,x,v), \tag{5}$$

$$E_n(x,t) = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U_n(x-y) f^n(t,y,v)\, dy\, dv, \tag{6}$$

$$f^n\Big|_{t=0} = f_0 \in S. \tag{7}$$

$$f^n(t,\cdot,\cdot) \geq 0, \quad \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} f^n(t,x,v)\, dx\, dv \equiv 1, \quad n = 1, 2, 3, \ldots \tag{8}$$

The following statement is a corollary of results in [1-4].

Proposition *For each n problem (5)-(8) has a unique global solution $f^n(t,x,v)$ that for any $t_0 > 0$ belongs to $C(-t_0, t_0); S)$ together with $f_t^{n\prime}(t,x,v)$.*

Denote $(T_n g)(x) = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U_n(x-y) g(y,v)\, dy\, dv$ where $g \in S$. Note that

$$\frac{\partial^m (T_n g)(x)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \equiv (T_n(\frac{\partial^m g(x,\cdot)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}))(x), \quad m = m_1 + m_2 + m_3.$$

Lemma 1 *There exists $C > 0$ such that $\|(T_n g)\|_{L_2(\mathbb{R}^3)} \leq C p_{(0,2),0}(g)$ for all $g \in S$ and n.*

Proof. As well known, there exists $C_1 > 0$ such that

$$\left\| \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x-y)h(y)\, dy \right\|_{L_2(\mathbb{R}^3)} \leq C_1 \|h\|_{L_2(\mathbb{R}^3)} \quad \forall h \in S(\mathbb{R}^3).$$

Also,

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U_n(x-y)g(y,v)\, dy\, dv = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x-y)(\omega_n \star_x g)(y,v)\, dy\, dv$$

and $\|\omega_n \star h\|_{L_2(\mathbb{R}^3)} \leq \|h\|_{L_2(\mathbb{R}^3)}$ $\forall h \in S(\mathbb{R}^3)$. Hence, we have for $g \in S$:

$$\left\| \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U_n(x-y)g(y,v)\, dy\, dv \right\|_{L_2(\mathbb{R}^3)} \leq C_1 \left\| \int\limits_{\mathbb{R}^3} g(\cdot,v)\, dv \right\|_{L_2(\mathbb{R}^3)} =$$

$$= C_1 \left\| \int\limits_{\mathbb{R}^3} \frac{(1+|v|^2)g(\cdot,v)\, dv}{(1+|v|^2)} \right\|_{L_2(\mathbb{R}^3)} \leq$$

$$\leq C \left\{ \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} (1+|v|^2)^2 g^2(x,v)\, dx\, dv \right\}^{1/2} = C p_{(0,2),0}(g). \square$$

Let $[a]$ be the maximal integer not larger than a real a and let an integer $m_0 > 0$ be such that each Sobolev space $H^{l+m_0}(\mathbb{R}^3 \times \mathbb{R}^3)$ with a positive integer l is embedded into $C^l(\mathbb{R}^3 \times \mathbb{R}^3)$ (in fact, $m_0 > 3$). Let $m_1 = 2m_0 + 2$. Everywhere $\overline{k}$ denotes $k_1 + k_2$. In what follows, we exploit the following embeddings ($g \in S$):

$$\left\{ \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, (1+|x|^2)^{k_1}(1+|v|^2)^{k_2} \left(\frac{\partial^l g}{\partial x_i^n \partial v_j^r} \right)^2 \right\}^{1/2} \leq$$

$$\leq C[p_{k,0}(g) + p_{k,s}(g) + q_{k,s}(g)],$$

$$p_{k,m}(g) \leq C[p_{(\overline{k},0),m}(g) + p_{(0,\overline{k}),m}(g)], \quad q_{k,m}(g) \leq C[q_{(\overline{k},0),m}(g) + q_{(0,\overline{k}),m}(g)],$$

$$m = 0, 1, 2, \ldots,$$

$$\sup_{(x,v)}(1+|x|^2)^{k_1/2}(1+|v|^2)^{k_2/2}\left|\frac{\partial^l g(x,v)}{\partial x_i^n \partial v_j^r}\right| \le C[p_{k,0}(g)+p_{k,m+l}(g)+q_{k,m+l}(g)], \tag{9}$$

$$\sup_x \left\{ \int_{\mathbb{R}^3} (1+|x|^2)^{k_1}(1+|v|^2)^{k_2} \left(\frac{\partial^l g(x,v)}{\partial x_i^n \partial v_j^r}\right)^2 dv \right\}^{1/2} \le$$

$$\le C[p_{0,k}(g) + p_{k,l+m}(g) + q_{k,l+m}(g)],$$

where $0 \le l = n + r \le s$, $i,j = 1,2,3$ and $m = m_0, m_0+1, m_0+2, \ldots$.

Lemma 2 *There exist $T > 0$ and $C > 0$ such that*

$$p^2_{(0,2),0}(f^n(t,\cdot,\cdot)) + p^2_{(0,2),m_1}(f^n(t,\cdot,\cdot)) + q^2_{(0,2),m_1}(f^n(t,\cdot,\cdot)) \le C$$

for all $t \in [-T,T]$ and all $n = 1,2,3,\ldots$.

Proof. We derive our estimates only for $t > 0$ because the case $t < 0$ can be treated by analogy. We have from (5):

$$\frac{1}{2}\frac{d}{dt}p^2_{(0,2),0}(f^n(t,\cdot,\cdot)) = -\int_{\mathbb{R}^3\times\mathbb{R}^3} (1+|v|^2)^2 f^n(t,x,v) f_v^{n\prime}(t,x,v)\cdot E_n(x,t)\,dx\,dv$$

$$\le Cp^2_{(0,2),0}(f^n(t,\cdot,\cdot))[p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot))]. \tag{10}$$

By analogy, using lemma 1 and integration by parts, we obtain:

$$\frac{1}{2}\frac{d}{dt}p^2_{(0,2),m_1}(f^n(t,\cdot,\cdot)) = -\int_{\mathbb{R}^3\times\mathbb{R}^3} dx\ dv\ (1+|v|^2)^2 \sum_{i,j=1}^{3} f_{x_i}^{n(m_1)}(t,x,v)\times$$

$$\times \sum_{l=0}^{m_1} \binom{m_1}{l} \frac{\partial^{l+1} f^n(t,x,v)}{\partial v_j \partial x_i^l} \frac{\partial^{m_1-l}}{\partial x_i^{m_1-l}} \times$$

$$\times E_{n,j}(x,t) \le Cp^2_{(0,2),m_1}(f^n(t,\cdot,\cdot))[p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot))] +$$

$$+\left|\int_{\mathbb{R}^3\times\mathbb{R}^3} dx\ dv \left(\sum_{l=0}^{m_0+1} + \sum_{l=m_0+2}^{m_1-1}\right)\times\right.$$

$$\left.\times \sum_{i,j=1}^{3}(1+|v|^2)^2 \binom{m_1}{l} f_{x_i}^{n(m_1)}(t,x,v) \frac{\partial^{l+1} f^n(t,x,v)}{\partial v_j \partial x_i^l} \frac{\partial^{m_1-l} E_{n,j}(x,t)}{\partial x_i^{m_1-l}}\right| \le$$

$$\leq C p_{(0,2),m_1}(f^n(t,\cdot,\cdot)) \cdot [p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot))] \times$$
$$\times [p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot)) + q_{(0,2),m_1}(f^n(t,\cdot,\cdot))]. \qquad (11)$$

Finally, we deduce

$$\frac{1}{2}\frac{d}{dt} q^2_{(0,2),m_1}(f^n(t,\cdot,\cdot)) \leq C[p^2_{(0,2),0}(f^n(t,\cdot,\cdot)) +$$
$$+ p^2_{(0,2),m_1}(f^n(t,\cdot,\cdot)) + q^2_{(0,2),m_1}(f^n(t,\cdot,\cdot))] +$$
$$+ \left| \int_{\mathbb{R}^3\times\mathbb{R}^3} (1+|v|^2)^2 \sum_{i,j=1}^{3} \partial_{v_j} f^{n(m_1)}_{v_i}(t,x,v) f^{n(m_1)}_{v_i}(t,x,v) E_{n,j}(x,t)\, dx\, dv \right| \leq$$
$$\leq C[p^2_{(0,2),0}(f^n(t,\cdot,\cdot)) + p^2_{(0,2),m_1}(f^n(t,\cdot,\cdot)) + q^2_{(0,2),m_1}(f^n(t,\cdot,\cdot))] \times$$
$$\times [p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot)) + 1]. \qquad (12)$$

Now, in view of (10)-(12), denoting $A = p^2_{(0,2),0}(f^n) + p^2_{(0,2),m_1}(f^n)$ and $B = p^2_{(0,2),0}(f^n) + q^2_{(0,2),m_1}(f^n)$, where $p_{(0,2),0}(f^n) \equiv q_{(0,2),0}(f^n)$, we obtain

$$\dot{A}(t) \leq C_1(1 + A(t) + B(t))^{3/2},$$
$$\dot{B}(t) \leq C_2(1 + A(t) + B(t))^{3/2},$$

and our claim is proved.□

Lemma 3 *There exist* $C_k > 0$ *such that* $p_{k,m_1}(f^n(t,\cdot,\cdot)) + q_{k,m_1}(f^n(t,\cdot,\cdot)) + p_{k,0}(f^n(t,\cdot,\cdot)) \leq C_k$ *for all* $t \in [-T,T]$ *and all* $k_1, k_2, n = 1, 2, 3, \ldots$.

Proof. Again, we consider only the case $t > 0$. We have from (5) and the embeddings (9):

$$\frac{1}{2}\frac{d}{dt} p^2_{(k,0),m_1}(f^n(t,\cdot,\cdot)) \leq Ck[p^2_{(k,0),m_1}(f^n(t,\cdot,\cdot)) + p^2_{(0,k),m_1}(f^n(t,\cdot,\cdot))] +$$
$$+ \left| \int_{\mathbb{R}^3\times\mathbb{R}^3} (1+|x|^2)^k \sum_{i,j=1}^{3} f^{n(m_1)}_{x_i}(t,\cdot,\cdot)) \times \right.$$
$$\times \left(\sum_{l=0}^{m_0+1} + \sum_{l=m_0+2}^{m_1-1} \right) \binom{m_1}{l} \frac{\partial^{l+1} f^n(t,\cdot,\cdot)}{\partial v_j \partial x_i^l} \times$$

$$\times \frac{\partial^{m_1-l} E_{n,j}(x,t)}{\partial x_i^{m_1-l}}\, dx\, dv\Big| \leq Ck[p^2_{(k,0),m_1}(f^n(t,\cdot,\cdot)) + p^2_{(0,k),m_1}(f^n(t,\cdot,\cdot))]+$$

$$+Cp_{(k,0),m_1}(f^n(t,\cdot,\cdot))[p_{(k,0),0}(f^n(t,\cdot,\cdot)) + p_{(k,0),m_1}(f^n(t,\cdot,\cdot))+$$

$$+q_{(k,0),m_1}(f^n(t,\cdot,\cdot))]\cdot[p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot))]. \quad (13)$$

By analogy,

$$\frac{1}{2}\frac{d}{dt}p^2_{(0,k),m_1}(f^n(t,\cdot,\cdot)) \leq$$

$$\leq C^1_k p_{(0,k),m_1}(f^n(t,\cdot,\cdot))[p_{(0,k),0}(f^n(t,\cdot,\cdot)) + p_{(0,k),m_1}(f^n(t,\cdot,\cdot))+$$

$$+q_{(0,k),m_1}(f^n(t,\cdot,\cdot))]\cdot[p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot))], \quad (14)$$

$$\frac{1}{2}\frac{d}{dt}q^2_{(k,0),m_1}(f^n(t,\cdot,\cdot)) \leq C^2_k[p^2_{(k,0),0}(f^n(t,\cdot,\cdot)) + p^2_{(0,k),0}(f^n(t,\cdot,\cdot))+$$

$$+q^2_{(k,0),m_1}(f^n(t,\cdot,\cdot)) + q^2_{(0,k),m_1}(f^n(t,\cdot,\cdot))], \quad (15)$$

$$\frac{1}{2}\frac{d}{dt}q^2_{(0,k),m_1}(f^n(t,\cdot,\cdot)) \leq$$

$$\leq C^3_k[p^2_{(0,k),0}(f^n(t,\cdot,\cdot)) + p^2_{(0,k),m_1}(f^n(t,\cdot,\cdot)) + q^2_{(0,k),m_1}(f^n(t,\cdot,\cdot))]+$$

$$+C^3_k q^2_{(0,k),m_1}(f^n(t,\cdot,\cdot))[p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot))] \quad (16)$$

and

$$\frac{1}{2}\frac{d}{dt}p^2_{k,0}(f^n(t,\cdot,\cdot)) \leq C^4_k[p^2_{(\overline{k},0),0}(f^n(t,\cdot,\cdot)) + p^2_{(0,\overline{k}),0}(f^n(t,\cdot,\cdot))]\times$$

$$\times[p_{(0,2),0}(f^n(t,\cdot,\cdot)) + p_{(0,2),m_1}(f^n(t,\cdot,\cdot)) + 1]. \quad (17)$$

Now, in view of (13)-(17), lemma 2 and the embeddings (9), our result follows.□

Lemma 4 *There exist* $C_{k,m} > 0$ *such that* $p_{k,m}(f^n(t,\cdot,\cdot)) + q_{k,m}(f^n(t,\cdot,\cdot)) \leq C_{k,m}$ *for all* n, k, m *and* $t \in [-T,T]$.

Proof. Again, we establish all our estimates only for $t > 0$. Let $m \geq 2m_0 + 3$. It follows from equation (5) that:

$$\frac{1}{2}\frac{d}{dt}p^2_{k,m}(f^n(t)) \leq C\overline{k}[p^2_{(\overline{k},0),0}(f^n(t,\cdot,\cdot)) + p^2_{(0,\overline{k}),0}(f^n(t,\cdot,\cdot))+$$

$$+p^2_{(\overline{k},0),m}(f^n(t,\cdot,\cdot)) + p^2_{(0,\overline{k}),m}(f^n(t,\cdot,\cdot))]+$$

$$+\Big|\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x|^2)^{k_1}(1+|v|^2)^{k_2}\times$$

$$\times\sum_{i,j=1}^{3} f_{x_i}^{n(m)}(t,x,v)\left[\frac{\partial^{m+1}f^n}{\partial v_j\partial x_i^m}E_{n,j}(x,t)+\left(\sum_{l=0}^{[m/2]}+\sum_{l=[m/2]+1}^{m-1}\right)\times\right.$$

$$\left.\times\binom{m}{l}\frac{\partial^{l+1}f^n(t,x,v)}{\partial v_j\partial x_i^l}\frac{\partial^{m-l}E_{n,j}(x,t)}{\partial x_i^{m-l}}\,dx\,dv\right]\Big|\leq$$

$$\leq C_1[p^2_{(\overline{k},0),0}(f^n(t,\cdot,\cdot))+p^2_{(0,\overline{k}),0}(f^n(t,\cdot,\cdot))+$$

$$+p^2_{(\overline{k},0),m}(f^n(t,\cdot,\cdot))+p^2_{(0,\overline{k}),m}(f^n(t,\cdot,\cdot))]+$$

$$+C_2p^2_{k,m}(f^n(t,\cdot,\cdot))[p_{(0,2),m_0}(f^n(t,\cdot,\cdot))+p_{(0,2),0}(f^n(t,\cdot,\cdot))]+$$

$$+C_2p_{k,m}(f^n(t,\cdot,\cdot))[p_{k,0}(f^n(t,\cdot,\cdot))+p_{k,m-1}(f^n(t,\cdot,\cdot))+q_{k,m-1}(f^n(t,\cdot,\cdot))]\times$$

$$\times[p_{(0,2),0}(f^n(t,\cdot,\cdot))+p_{(0,2),m}(f^n(t,\cdot,\cdot))]+$$

$$+C_2p_{k,m}(f^n(t,\cdot,\cdot))[p_{k,0}(f^n(t,\cdot,\cdot))+p_{k,m}(f^n(t,\cdot,\cdot))+q_{k,m}(f^n(t,\cdot,\cdot))]\times$$

$$\times[p_{(0,2),0}(f^n(t,\cdot,\cdot))+p_{(0,2),m-1}(f^n(t,\cdot,\cdot))]. \qquad (18)$$

By analogy,

$$\frac{1}{2}\frac{d}{dt}q^2_{k,m}(f^n(t,\cdot,\cdot))\leq C[p^2_{(\overline{k},0),0}(f^n(t,\cdot,\cdot))+p^2_{(0,\overline{k}),0}(f^n(t,\cdot,\cdot))+$$

$$+q^2_{(\overline{k},0),m}(f^n(t,\cdot,\cdot))+q^2_{(0,\overline{k}),m}(f^n(t,\cdot,\cdot))]+$$

$$+C[q^2_{(\overline{k},0),m}(f^n(t,\cdot,\cdot))+q^2_{(0,\overline{k}),m}(f^n(t,\cdot,\cdot))]\times$$

$$\times[p_{(0,2),0}(f^n(t,\cdot,\cdot))+p_{(0,2),m_0}(f^n(t,\cdot,\cdot))]. \qquad (19)$$

Now, the statement of our lemma follows from (18),(19) and lemmas 2 and 3 by induction.□

In view of lemmas 1-4 and equation (5), the sequences $\{f^n\}$ and $\{df^n/dt\}$ are relatively compact in $C([-T,T];S)$. Without the loss of generality we can accept that these sequences converge and let $f(t,x,v)$ and $f_1(t,x,v)$ be their limit points in this sense. Clearly, $f'_t(t,x,v)\equiv f_1(t,x,v)$ and f satisfies problem (1)-(4). Let us prove the uniqueness of this solution. Suppose the existence of two solutions f_1 and f_2 of the above class and set $f = f_1 - f_2$. As earlier, we establish our estimates only for $t > 0$. One can obtain as when proving lemmas 2 and 3:

$$\frac{d}{dt} p^2_{(0,2),0}(f(t,\cdot,\cdot)) \leq C p^2_{(0,2),0}(f(t,\cdot,\cdot)),$$

therefore $f(t,x,v) \equiv 0$ for all $t \in [0,T]$, which completes our proof of the theorem.

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Жидков П. Е. E5-2003-218
О локальных гладких решениях уравнения Власова с потенциалом взаимодействий $\pm r^{-2}$

Для задачи Коши для уравнения Власова с потенциалом взаимодействий $\pm r^{-2}$ доказаны существование и единственность локального решения со значениями в пространстве Шварца S бесконечно дифференцируемых функций, которые быстро убывают на бесконечности.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2003

Zhidkov P. E. E5-2003-218
On Local Smooth Solutions for the Vlasov Equation with the Potential of Interactions $\pm r^{-2}$

For the initial value problem for the Vlasov equation with the potential of interactions $\pm r^{-2}$ we prove the existence and uniqueness of a local solution with values in the Schwartz space S of infinitely differentiable functions rapidly decaying at infinity.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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