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Č. Burdík[1], O. Navrátil[2]

SOLUTION OF 2×2 MATRIX THREE-BODY CALOGERO MODEL

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[1]Department of Mathematics, Czech Technical University,
Faculty of Nuclear Sciences and Physical Engineering, Trojanova 13,
120 00 Prague 2, Czech Republic
[2]Department of Mathematics, Faculty of Transportation Sciences,
Czech Technical University, Na Florenci 25, 110 00 Prague,
Czech Republic

Бурдик Ч., Навратил О. E5-2004-2
Решение 2×2-матричной трехчастичной модели Калоджеро

Дано определение трехчастичной 2×2-матричной точно решаемой модели. Рассматриваемая модель по форме очень близка трехчастичной модели Калоджеро. Найдены основное собственное состояние и спектр данной модели.

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Burdík Č., Navrátil O. E5-2004-2
Solution of 2×2 Matrix Three-Body Calogero Model

We define a three-body 2×2 matrix exactly solvable model. This model has a very similar form to the Calogero three-body model. We find the ground-state eigenvector and give the spectrum of this model.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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In the famous papers [1] the Calogero model was defined. In the three-body case the Hamiltonian is given as

$$H = -\frac{1}{2}\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2}\sum_{j=1}^{3} x_j^2 + g\sum_{1\le j<k\le 3}\frac{1}{(x_j-x_k)^2}\,, \tag{1}$$

where $g = \nu(\nu-1)$, i.e. $\nu = \dfrac{1+\sqrt{1+4g}}{2}$, which is the coupling constant associated with a long-range interaction. One can exactly solve this Calogero model and find out the complete set of energy eigenvalues as

$$E_{n_1,n_2,n_3} = \left(\frac{3}{2} + 3\nu + n_1 + n_2 + n_3\right)\omega\,, \tag{2}$$

where n_js are nonnegative integer valued quantum numbers with $n_j \le n_{j+1}$.

The ground state eigenfunction is given by

$$\mathrm{e}^{a(x)} = \exp\left(-\frac{\omega}{2}X^2\right)\big|(x_1-x_2)(x_1-x_3)(x_2-x_3)\big|^{\nu}\,,$$

$X^2 = x_1^2 + x_2^2 + x_3^2$. It was shown by Calogero that the eigenfunctions for this model can be expressed as

$$\Psi(x) = \mathrm{e}^{a(x)}\widehat{\Psi}(x)\,,$$

where $\widehat{\Psi}(x)$ is a polynomial symmetric under permutations of any two x_i's. The operator having these polynomials as eigenfunctions can be obtained by performing on (1) the gauge rotation

$$\widehat{H} = \mathrm{e}^{-a(x)}H\mathrm{e}^{a(x)}\,.$$

The aim of our paper is to study the 2×2 matrix model which seems very similar to the Calogero model (1). The model is given by means of the Hamiltonian

$$\mathbf{H}_{2\times 2} = -\frac{1}{2}\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2}\sum_{j=1}^{3} x_j^2 + \gamma^2\sum_{1\le j<k\le 3}\frac{1}{(x_j-x_k)^2} - \gamma\mathbf{V}\,, \tag{3}$$

where $x_3 < x_2 < x_1$ and $\mathbf{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & -V_{11}\end{pmatrix}$,

$$V_{11} = \frac{1}{2}\left(\frac{1}{(x_1-x_2)^2} - \frac{2}{(x_1-x_3)^2} + \frac{1}{(x_2-x_3)^2}\right),$$
$$V_{12} = \frac{\sqrt{3}}{2}\left(\frac{1}{(x_1-x_2)^2} - \frac{1}{(x_2-x_3)^2}\right).$$

This Hamiltonian is not explicitly symmetric. To obtain a self-adjoint Hamiltonian we have to define the Hilbert space of the square integrable vector functions $f(x_1,x_2,x_3)$ on M, where $M=\{(x_1,x_2,x_2)\,;\ x_3<x_2<x_1\}$. In this paper we only solve the equation $\mathbf{H}_{2\times 2}\psi=\lambda\psi$ by algebraic means and we do not deal with the problem of the operator (3) domain. This problem is briefly mentioned at the end of the paper.

If we introduce the center-of-mass coordinate

$$
\begin{aligned}
X &= x_1+x_2+x_3\,, & x_1 &= y_1+\frac{1}{3}\,X\,,\\
y_1 &= x_1-\frac{1}{3}\,X=\frac{2x_1-x_2-x_3}{3}\,, & x_2 &= \frac{1}{3}\,X-y_1-y_2\,,\\
y_2 &= x_3-\frac{1}{3}\,X=\frac{-x_1-x_2+2x_3}{3}\,, & x_3 &= y_2+\frac{1}{3}\,X\,,
\end{aligned}
$$

where $X\in\mathbf{R}$, $2y_1+y_2>0$ and $y_1+2y_2<0$, the Hamiltonian (3) in new variables is separable, consisting of two parts

$$\mathbf{H}=\mathbf{H}_0+\mathbf{H}_{\rm rel}$$

where

$$
\begin{aligned}
\mathbf{H}_0 &= -\frac{3}{2}\partial_{XX}+\frac{\omega^2}{6}\,X^2,\\
\mathbf{H}_{\rm rel} &= -\frac{1}{3}\left(\partial_{11}-\partial_{12}+\partial_{22}\right)+\omega^2\left(y_1^2+y_1y_2+y_2^2\right)+\mathbf{V}
\end{aligned}
$$

and

$$\mathbf{V}=\gamma^2\left(\frac{1}{(y_1-y_2)^2}+\frac{1}{(y_1+2y_2)^2}+\frac{1}{(2y_1+y_2)^2}\right)-\gamma\begin{pmatrix} V_{11} & V_{12}\\ V_{12} & -V_{11}\end{pmatrix}$$

with

$$
\begin{aligned}
V_{11} &= \frac{1}{2}\left(\frac{1}{(y_1+2y_2)^2}+\frac{1}{(2y_1+y_2)^2}-\frac{2}{(y_1-y_2)^2}\right),\\
V_{12} &= \frac{\sqrt{3}}{2}\left(\frac{1}{(2y_1+y_2)^2}-\frac{1}{(y_1+2y_2)^2}\right).
\end{aligned}
$$

For the eigenvalue problem $\mathbf{H}\psi(X,y_1,y_2)=\lambda\psi(X,y_1,y_2)$ we can use the ansatz $\psi=\psi_0(X)\psi_{\rm rel}(y_1,y_2)$, $\lambda=\lambda_0+\lambda_{\rm rel}$, where

$$\mathbf{H}_0\psi_0(X) = \lambda_0\psi_0(X), \tag{4}$$

$$\mathbf{H}_{\rm rel}\psi_{\rm rel}(y_1,y_2) = \lambda_{\rm rel}\psi_{\rm rel}(y_1,y_2). \tag{5}$$

The equation (4) is the harmonic oscillator problem for which the ground state is $\mathrm{e}^{-\omega X^2/6}$ and excitation states are given by Hermit's polynomials. The spectrum of this operator is $\left(m+\frac{1}{2}\right)\omega$, where $m=0,\,1,\,2,\,\ldots$

To solve the equation (5) is more complicated. First we introduce 2×2 matrices

$$\mathrm{e}^{\mathbf{a}} = \mathrm{e}^{-\omega\left(y_1^2+y_1y_2+y_2^2\right)}\left|(y_2-y_1)(2y_1+y_2)(y_1+2y_2)\right|^{\gamma}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{pmatrix},$$

where

$$X_{11} = y_2 - y_1\,, \qquad X_{12} = \sqrt{3}\big(y_1^2 - y_2^2\big)\,,$$
$$X_{21} = \sqrt{3}\big(y_1 + y_2\big)\,, \qquad X_{22} = y_1^2 + 4y_1y_2 + y_2^2\,.$$

By direct calculation it is possible to check that

$$\mathbf{H}_{\mathrm{rel}}\mathrm{e}^{\mathbf{a}} = \mathrm{e}^{\mathbf{a}}\mathbf{K}\,,$$

where

$$\mathbf{K} = \begin{pmatrix} (3\gamma+2)\omega & 0 \\ 0 & (3\gamma+3)\omega \end{pmatrix}.$$

We use this $\mathrm{e}^{\mathbf{a}}$ for gauge transformation $\psi_{\mathrm{rel}} = \mathrm{e}^{\mathbf{a}}\widehat{\psi}$ and we obtain

$$\widehat{\mathbf{H}}_{\mathbf{rel}}\widehat{\psi}\big(y_1,y_2\big) = \lambda_{\mathrm{rel}}\widehat{\psi}\big(y_1,y_2\big)\,,$$

where

$$\widehat{\mathbf{H}}_{\mathbf{rel}} = \mathrm{e}^{-\mathbf{a}}\mathbf{H}\mathrm{e}^{\mathbf{a}} = -\frac{1}{3}\big(\partial_{11} - \partial_{12} + \partial_{22}\big) + \\ + \frac{\mathbf{B}_1}{\big(y_1-y_2\big)\big(2y_1+y_2\big)}\,\partial_1 + \frac{\mathbf{B}_2}{\big(y_1-y_2\big)\big(y_1+2y_2\big)}\,\partial_2 + \mathbf{K},$$

$$\mathbf{B}_1 = \begin{pmatrix} y_1\Big((y_1-y_2)(2y_1+y_2)\omega - 3\gamma - 1\Big) & \dfrac{2}{\sqrt{3}}\left(y_1^2+y_1y_2+y_2^2\right) \\ \dfrac{1}{\sqrt{3}} & y_1\Big((y_1-y_2)(2y_1+y_2)\omega - 3\gamma - 2\Big)\end{pmatrix},$$

$$\mathbf{B}_2 = \begin{pmatrix} y_2\Big((y_1-y_2)(y_1+2y_2)\omega + 3\gamma + 1\Big) & -\dfrac{2}{\sqrt{3}}\left(y_1^2+y_1y_2+y_2^2\right) \\ -\dfrac{1}{\sqrt{3}} & y_2\Big((y_1-y_2)(y_1+2y_2)\omega + 3\gamma + 2\Big)\end{pmatrix}.$$

After the transformation [2]

$$z_1 = -y_1^2 - y_1y_2 - y_2^2,$$
$$z_2 = -y_1y_2\big(y_1+y_2\big)\,,$$

we finally obtain

$$
\begin{aligned}
\widehat{\mathbf{H}}_{\rm rel} = \ & z_1\partial_{11} + 3z_2\partial_{12} - \frac{1}{3}\, z_1^2\partial_{22} + \big(2\omega z_1 + 3\gamma + 2\big)\partial_1 + 3\omega z_2\partial_2 + \\
& + \left(\begin{array}{cc} (3\gamma+2)\omega & -\dfrac{2}{\sqrt{3}}\, z_1\partial_2 \\ \dfrac{1}{\sqrt{3}}\,\partial_2 & \partial_1 + 3(\gamma+1)\omega \end{array} \right) .
\end{aligned} \tag{6}
$$

The set of vector polynomials

$$
\widehat{\psi}(z_1, z_2) = \sum_{\substack{r,\ s\ \geq\ 0 \\ r\,+\,s\ \leq\ N}} \left(\begin{array}{c} A_{r,s} \\ B_{r,s} \end{array} \right) z_1^r z_2^s \tag{7}
$$

is a finite-dimensional subspace $\mathcal{V}_N$.

On the space $\mathcal{V}_N$ we will solve the equation

$$
\widehat{\mathbf{H}}_{\rm rel}\widehat{\psi}(z_1, z_2) = \lambda_{\rm rel}\widehat{\psi}(z_1, z_2)\,, \tag{8}
$$

where $\widehat{\psi} \in \mathcal{V}_N$. The equation (8) together with (6) and (7) gives the system of the difference equations for $A_{r,s}$ and $B_{r,s}$. If $r+s = N$ and $\lambda_{\rm rel} = (\mu+3\gamma+2)\omega$ we obtain

$$
\begin{aligned}
& (3N-\mu)\omega A_{0,N} = 0\,, \\
& (3N-\mu+1)\omega B_{0,N} = 0\,, \\
& (3N-\mu-1)\omega A_{1,N-1} - \frac{2}{\sqrt{3}}\, N B_{0,N} = 0\,, \\
& (3N-\mu)\omega B_{1,N-1} = 0\,, \\
& (2N+s-\mu)\omega A_{N-s,s} - \frac{2}{\sqrt{3}}\,(s+1) B_{N-s-1,s+1} - \\
& \qquad\qquad\qquad\qquad - \frac{1}{3}\,(s+2)(s+1) A_{N-s-2,s+2} = 0\,, \\
& (2N+s-\mu+1)\omega B_{N-s,s} - \frac{1}{3}\,(s+2)(s+1) B_{N-s-2,s+2} = 0\,,
\end{aligned}
$$

where $s < N-1$. It is easy to show that this system has nonzero solutions iff $\mu = 2N+n$, where $n = 0,\, 1,\, \ldots,\, N+1$. Linearly-independent solutions of this system are:

The nonzero coefficients are for even $n = 2r$

$$A_{N-2k,2k} = \frac{r!}{(r-k)!\,(2k)!}\,(-6\omega)^k \tag{9}$$

for $k = 0,\,1,\,\ldots,\,r$, or

$$\begin{aligned} B_{N-2k+1,2k-1} &= \frac{r!}{(r-k)!\,(2k-1)!}\,(-6\omega)^k, \\ A_{N-2k,2k} &= -\frac{\sqrt{3}\cdot r!}{(r-k)!(2k-1)!}\,(-6\omega)^k \end{aligned} \tag{10}$$

for $k = 1,\,2,\,\ldots,\,r$, and for even $n = 2r+1$

$$A_{N-2k-1,2k+1} = \frac{r!}{(r-k)!\,(2k+1)!}\,(-6\omega)^k \tag{11}$$

for $k = 0,\,1,\,\ldots,\,r$, or

$$\begin{aligned} B_{N-2k,2k} &= \frac{r!}{(r-k)!\,(2k)!}\,(-6\omega)^k, \\ A_{N-2k-1,2k+1} &= -\frac{\sqrt{3}\cdot r!}{(r-k)!(2k)!}\,(-6\omega)^k \end{aligned} \tag{12}$$

for $k = 0,\,1,\,\ldots,\,r$.

We do not write the systems of the difference equations for $A_{r,s}$ and $B_{r,s}$, where $r + s = M < N$. We only note that there are the solutions of the systems. The constants (9) – (12) are the initial conditions for these solutions.

In this way we obtain the eigenfunctions $\widehat{\psi}_{(N,n,1)}$, where $n = 0,\,1,\,\ldots,\,N$, and $\widehat{\psi}_{(N,n,2)}$, where $n = 1,\,2,\,\ldots,\,N+1$, which correspond to the coefficients $A_{N-n,n}$ and $B_{N-n,n}$ given in (9) – (12). These functions are the solution of the equation

$$\widehat{\mathbf{H}}_{\mathrm{rel}}\widehat{\psi}_{(N,n,k)} = (2N + n + 3\gamma + 2)\widehat{\psi}_{(N,n,k)} = \lambda_{\mathrm{rel}}\widehat{\psi}_{(N,n,k)}\,.$$

The eigenvalues of the Hamiltonian (3) are $\lambda = \lambda_0 + \lambda_{\mathrm{rel}}$ and the eigenvalues $\lambda_0 = \left(m + \dfrac{1}{2}\right)\omega$. Therefore the spectrum of (3) is

$$E_{m,N,n} = \left(m + 2N + n + 3\gamma + \frac{5}{2}\right)\omega\,, \tag{13}$$

where $m = 0,\,1,\,2,\,\ldots,\,N = 0,\,1,\,2,\,\ldots$ and $n = 0,\,1,\,\ldots,\,N+1$. Moreover, the eigenvalues with $n = 1,\,2,\,\ldots,\,N$ have multiplicity 2.

If we compare this spectrum with the spectrum of the Calogero model (2), we see that the energies of the ground state are different. If we take in the Calogero limit $g \to 0$, we obtain energy of the ground state $E_0 = \frac{9}{2}\,\omega$. However, if we formally take this limit in our model, we obtain $E_0 = \frac{5}{2}\,\omega$. This contradiction arises from the fact that the transformation $\widehat{\mathbf{H}}_{\mathrm{rel}} = \mathrm{e}^{-\mathbf{a}}\mathbf{H}_{\mathrm{rel}}\mathrm{e}^{\mathbf{a}}$ affects to $\gamma > 0$ only. For $\gamma \le 0$ there are problems on the boundary, i.e. for $2y_1 + y_2 = 0$ and $y_1 + 2y_2 = 0$. It is easy to see that the substitution $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix}$ leads to the exchange $\gamma \leftrightarrow -\gamma$. Therefore, the spectrum (13) can be written for $\gamma \ne 0$ in the form

$$E_{m,N,n} = \left(m + 2N + n + 3|\gamma| + \frac{5}{2}\right)\omega\,.$$

But for $\gamma = 0$ it is not true, because the ground state does not vanish on the boundary.

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141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@pds.jinr.ru
www.jinr.ru/publish/