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THE RIEMANN SURFACE OF STATIC LIMIT DISPERSION RELATION AND PROJECTIVE SPACES

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Риманова поверхность статического предела
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Строгое доказательство Боголюбовым дисперсионных соотношений (ДС) для пион-нуклонного рассеяния обеспечивает надежный фундамент для статических моделей. ДС содержат малый параметр (отношение масс пиона и нуклона). Статические модели возникают, когда этот параметр стремится к нулю. S-матрица в статическом пределе имеет блочную структуру. Каждый блок S-матрицы имеет конечный порядок $N \times N$ и состоит из мероморфных функций энергии легкой частицы ω в комплексной плоскости с разрезами $(-\infty, -1]$, $[+1, +\infty)$. В упругом случае он сводится к N функциям $S_i(\omega)$, связанным матрицей перекрестной симметрии A размерности $N \times N$. Унитарность и перекрестная симметрия приводят к системе нелинейных краевых задач. Она определяет аналитическое продолжение функций $S_i(\omega)$ с физического листа на нефизические и может рассматриваться как система нелинейных разностных уравнений. Задача решается для любой двухрядной матрицы A, что позволяет найти траектории Редже статической $SU(2)$-модели. Показано, что глобальный анализ этой системы может быть эффективно проведен в проективных пространствах P_{N-1} и P_N. Обсуждается соотношение между этими пространствами. Найдено несколько частных решений системы.

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The Riemann Surface of Static Limit Dispersion Relation
and Projective Spaces

The rigorous Bogoliubov's prove of the dispersion relations (DR) for pion–nucleon scattering is a good foundation for the static models. DR contain the small parameter (ratio of the pion–nucleon masses). The static models arise when this parameter goes to zero. The S-matrix in the static models has a block structure. Each block of the S-matrix has a finite order $N \times N$ and is a matrix of meromorphic functions of the light particle energy ω in the complex plane with cuts $(-\infty, -1]$, $[+1, +\infty)$. In the elastic case, it reduces to N functions $S_i(\omega)$ connected by $N \times N$ the crossing-symmetry matrix A. The unitarity and the crossing symmetry are the base for the system of nonlinear boundary value problems. It defines the analytical continuation of $S_i(\omega)$ from the physical sheet to the unphysical ones and can be treated as a system of nonlinear difference equations. The problem is solvable for any 2×2 crossing-symmetry matrix A that permits one to calculate the Regge trajectories for $SU(2)$ static model. It is shown that global analyses of this system can be carried out effectively in projective spaces P_{N-1} and P_N. The connection between spaces P_{N-1} and P_N is discussed. Some particular solutions of the system are found.

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1. INTRODUCTION

The prove of the dispersion relation for πN scattering given by N. N. Bogoliubov [1] has, at least, two consequences. In mathematics it gives rise to investigation on the analytic cotinuation of distributions of several complex variables (the so-called edge of the wedge theorem by Bogoliubov [2]). In physics, in essence, it introduces the concept of scattering amplitude for several processes regarded as the single analytic function of several variables, whose different boundary values with respect to the corresponding variables discribe these processes. Particularly, it gives the solid foundation for the static models [3]. The low-energy hadron scattering problem remains in the focus of attention [4]. The successful development of QCD poses the question of the validity of the analytic properties of hadron-hadron process amplitudes previously proved for strong interactions. In the series of works by Oehme [5], it was recently shown that they remain valid in QCD as well. We consider the nonrelativistic limit of the dispersion relations, which is known as static equations [6], and confine ourselves to study the equations of this type by reducing them to a nonlinear boundary-value problem [7]. It has the form of the series of conditions on the S-matrix elements S_i.

Conditions 1

A) $S_i(z)$ — are meromorphic functions in the complex plane z with the cuts $(-\infty,-1],\ [+1,+\infty),$ i. e. the only singularities of these functions in this domain are their poles.

B) $S_i^*(z) = S_i(z^*)$; (1)

C) $\mid S_i(\omega+i0)\mid^2 = 1$ for $\omega \geq 1$ $S_i(\omega+i0) = \lim_{\epsilon\to+0} S_i(\omega+i\epsilon)$;

D) $S_i(-z) = \sum_{j=1}^{N} A_{ij} S_j(z)$.

The real values of the variable z are the total energy ω of a relativistic particle scattered by a fixed center. The meromorphy requirement for the functions $S_i(z)$ arises as a consequence of the static limit of the scattering problem [8]. Elastic unitarity condition (1)C holds only on the right cut in the z plane. On the left cut, the functions $S_i(z)$ are determined by crossing-symmetry conditions (1)D. The crossing-symmetry matrix A is determined by the group that leaves the S-matrix invariant; the matrix A is known for some groups [7]. The aim of this paper is to formulate a method for studying the Riemann surfaces of some static dispersion models.

2. ANALYTIC CONTINUATION OF THE S-MATRIX TO NONPHYSICAL SHEETS

We write Conditions 1 in a matrix form. For this, we introduce the column

$$S^{(0)}(z) = [S_1(z), S_2(z), \cdots, S_N(z)]^T,$$

where the upper index denotes the physical sheet of the S-matrix Riemann surface. Conditions (1)A, (1)B, and (1)D hold on the physical sheet, and unitarity condition (1)C can be extended to the complex values of ω, and, just like condition (1)C, the extension has the component form

$$S_i^{(0)}(z)S_i^{(1)}(z) = 1$$

and analytically continues the S-matrix to the first unphysical sheet of the Riemann surface. To rewrite unitarity conditions (1)C in the matrix form, we introduce the nonlinear inversion transformation I by the formula

$$IS(z) = [1/S_1(z), 1/S_2(z), \cdots, 1/S_N(z)]^T.$$

As a result, Conditions 1 take the following form.

Conditions 2

A) $S^{(0)}(z)$ — is a column of N meromorphic functions in the complex plane z with the cuts $(-\infty, -1], [+1, +\infty)$, i. e. the only singularities of these functions in this domain are their poles.

B) ${S^{(0)}}^*(z) = S^{(0)}(z^*)$; (2)

C) $S^{(1)}(z) = IS^{(0)}(z)$;

D) $S^{(0)}(-z) = AS^{(0)}(z)$.

We define the analytic continuation to unphysical sheets as

$$S^{(p)}(z) = (IA)^p S^{(0)}(z(-1)^p). \tag{3}$$

By definition (3), unitarity condition (2)C and crossing-symmetry condition (2)D are easily extended to unphysical sheets:

$$IS^{(p)}(z) = S^{(1-p)}(z), AS^{(p)}(z) = S^{(-p)}(-z), \tag{4}$$

and we have the formula

$$(IA)^q S^{(p)}(z) = S^{(q+p)}(z(-1)^q). \tag{5}$$

Definition (3) is motivated by the well-known solution [8] of the problem defined by Conditions 1 for the two-row matrix

$$A = \frac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}.$$

This solution for the S-matrix $S(z)$ is given by

$$S(z) = \begin{pmatrix} W(W-2)/(W^2-1) \\ W(W+1)/(W^2-1) \end{pmatrix} D(z), \tag{6}$$

where $W = w + i\sqrt{z^2-1}\beta(z),\ w = 1/\pi \arcsin z,\ \beta(z) = -\beta(-z)$ is a meromorphic function, and $D(z) = D(-z)$ is the Blaschke function of the variable $\zeta = \dfrac{1+i\sqrt{z^2-1}}{z}$. The Blaschke function is given by

$$D(\zeta[z]) = \zeta^\lambda \prod_n \frac{|\zeta_n|}{\zeta_n}\frac{\zeta_n - \zeta}{1-\zeta_n^*\zeta},$$

where λ is the order of zero, and the set of zeros $\{\zeta_n\}$, $|\zeta_n| < 1$ is symmetric with respect to the origin and the axes $Im\zeta = 0, Re\zeta = 0$. In addition to solution (6), Conditions 1 allow a trivial solution: the column of identical Blaschke functions

$$S(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} D(z).$$

Therefore, Conditions 2 do not determine the form of the Riemann surface of $S(z)$ uniquely. For solution (6), the Riemann surface of $S(z)$ is infinite-sheeted because of the function w, and the equalities

$$S^{(0)}(z) = S(W)_{|w|\le 1/2}, \quad S^{(\pm n)}(z(-1)^{(\pm n)}) = S(W)_{|w\pm n|\le 1/2}$$

hold, which allow rewriting Eq. (5) as

$$\begin{aligned} (IA)^n S(W) &= S(W+n), \\ (AI)^n S(W) &= S(W-n). \end{aligned} \tag{7}$$

Equations (7) are a system of nonlinear autonomous difference equations and can naturally be called the dynamic form of the static dispersion relations. The same term can, therefore, be used for Eq. (5) as well. Unlike Eqs. (7), they form a system of nonlinear functional equations in which the number of a sheet of the Riemann surface and the energy variable z serve as arguments.

3. FORMULATION OF THE PROBLEM IN PROJECTIVE SPACES

The example of two-row solution (6) shows that, in general, the solution of the problem defined by Conditions 1 is determined by $N+1$ entire functions, among which N functions satisfy crossing-symmetry condition (1)D, and the last one is symmetric with respect to z and ensures the validity of unitarity condition (1)C. Conditions (1)A, (1)B, and (1)D are homogeneous and can be considered in the projective spaces P_{N-1} and P_N. We define the nonlinear inversion transformation I_p such that it is correct in these spaces [9]:

$$I_p S_i = \Pi^m_{j=1, j\neq i} S_j,$$

$$m = N-1, N.$$

We reformulate the problem defined by Conditions 1 for these spaces. For the space P_{N-1}, the crossing-symmetry matrix has the form specified by Conditions 1; for the space P_N, its dimensionality increases by one, i. e.

$$A_{N-1} = A, A_N = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where A_N is a block matrix. As a result, instead of Conditions 1, we obtain the following set of requirements on a column of m functions.

Conditions 3

A) $S^{(o)}(z)$ — is a column of m meromorphic functions in the complex plane z with the cuts $(-\infty, -1], [+1, +\infty)$, i. e. the only singularities of these functions in this domain are their poles.

B) $S^{(0)^*}(z) = S^{(0)}(z^*)$; (8)

C) $S^{(1)}(z) = I_p S^{(0)}(z)$;

D) $S^{(0)}(-z) = A_m S^{(0)}(z)$.

We illustrate the scheme of the solution for the two-row case in terms of the projective spaces P_1, P_2. We let $(x_o, x_1) = (S_1, S_2)$ denote the coordinates of the point (x) in the space P_1. We introduce the affine coordinate $X = x_0/x_1$ on the projective line P_1. Setting $z = 0$ in (3), we obtain the law for continuing the coordinate $X^{(0)}$ from the physical sheet to the first unphysical sheet:

$$X^{(1)} = \frac{2X^{(0)} + 1}{-X^{(0)} + 4}. \quad (9)$$

Taking the n^{th} power of linear fractional transformation (9) and using crossing-symmetry condition (3)D, we find that

$$X^{(0)} = -2, \; X^{(n)} = \frac{n-2}{n+1}. \quad (10)$$

One of the crossing-symmetry conditions (3)D thus proves unnecessary. This conclusion remains valid for 3×3 crossing-symmetry matrices. The solution of the two-row problem for the line P_1 allows finding only the ratio of the functions S_1 and S_2. The functions themselves can be found from the solution for the projective plane P_2. We write the projective coordinates of the point $(x) = (x_0, x_1, x_2)$ in P_2 in a basis explicitly taking the crossing symmetry into account:

$$\begin{aligned} x_0 &= s - 2a, \\ x_1 &= s + a, \\ x_2 &= c, \end{aligned} \tag{11}$$

where s and c are symmetric functions of z, and a is an antisymmetric function of z.

Considering the transformation $(I_p A_2)^n$ in the basis s, a, c, we can easily see that s, a, and c are related by the equation

$$s^2 - a^2 - sc = 0, \tag{12}$$

which is invariant under the transformations I_p and A_2. In other words, Eq. (12) in P_2 defines an invariant curve C whose points do not leave C under the action of the transformations I_p and A_2. In the basis (x_0, x_1, x_2), the equation of the curve C is given by

$$x^2{}_1 + 2x_0 x_1 - 2x_1 x_2 - x_0 x_2 = 0. \tag{13}$$

Using Eqs. (10) and (13), we can easily find that

$$\frac{x_1}{x_2} = \frac{n}{n-1}, \tag{14}$$

and thus completely define the functions S_1 and S_2. Taking unitarity condition (1)C (which has not been used yet) into account, we can recover formula (4) completely.

We discuss the relation between the descriptions of the two-row problem defined by Conditions 1 for the spaces P_1 and P_2. In the projective plane P_2, the solution is given by the invariant curve (13). It is irreducible and rational as is any algebraic curve of the second order. In the affine coordinates, it becomes

$$x = \frac{x_0}{x_2},\ y = \frac{x_1}{x_2},\ x^2 + 2xy - 2x - y = 0.$$

If we construct a bundle of lines of the form $\lambda_0 g_0 + \lambda_1 g_1$ with the basic point (x_0, y_0) in curve (13), then the coordinates of the second intersection of the lines in the bundle with curve (10) are rational functions of $k = \lambda_1/\lambda_0$:

$$x = \frac{-(x_0 + 2y_0) + 2 + k}{1 + 2k},\ y = y_0 + k(x - x_0).$$

The functions x and y are reduced to formulas (10) and (14) by the specially chosen parametrization

$$k = \frac{(-x_0 - 2y_0 + 1)n + x_0 + 2y_0 - 2}{n+1},$$

which depends on the basic point of the bundle. A bundle of lines behaves as the projective space P_1 under collineations (linear transformations with nonzero determinants) in the space P_2. The projective space P_1 is thus represented by any bundle of lines whose base point lies on the invariant curve (13) of the space P_2. In [7], the invariant manifolds for the problem defined by Conditions 1 with dimensionalities $N \geq 3$ were studied and constructed using series over $1/w$ in a neighborhood of the rest points of dynamic systems (5). Using projective spaces, we can reconsider this problem from a new standpoint. We consider the problem defined by Conditions 1 with the three-row matrix

$$A = \left(\begin{array}{rrr} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{array} \right), \tag{15}$$

which describes the scattering of the particle with angular momentum one on the center with the same momentum. In the space P_3, the matrix A_3 has three eigenvalues equal to $+1$ and one eigenvalue equal to -1. The coordinates of the point (x) in P_3 can be expressed in terms of three symmetric functions s_1, s_2 and s_3 of z and one antisymmetric function a of z by an ordinary collineation (an automorphism of the projective space):

$$x_i = b_{ij}s_j + b_{i4}a.$$

We construct a plane in P_3 that is invariant under the linear transformation of the coordinates of (x) determined by the matrix A_3. It is given by

$$c_0x_0 + c_1x_1 + (2c_0 + c_1)x_2 + c_2x_3 = 0. \tag{16}$$

It is easy to see that the plane $x_1 + x_2 = 0$ is the particular case of a plane (16) and is invariant under the transformation I_p. This plane is the space P_2 in which the problem defined by Conditions 1 with matrix (15) is reduced to the solvable two-row problem [10]. The plane $x_1 + x_2 = 0$ does not contain the rest point $\bar{x} = (1, 1, 1, 1)$ of the dynamic system defined by Conditions 3, i.e., the fixed point of transformation (5). If we require the point $\bar{x}$ to lie in a plane (16), then we obtain the equation

$$c_0x_0 + c_1x_1 + (2c_0 + c_1)x_2 - (3c_0 - 2c_1)x_3 = 0. \tag{17}$$

The transformation I_p maps a plane (17) onto the cubic surface

$$c_0x_1x_2x_3 + c_1x_0x_2x_3 + (2c_0 - c_1)x_0x_1x_3 - (3c_0 + 2x_1)x_0x_1x_2 = 0 \qquad (18)$$

in P_3, which is not invariant under the transformation A_3.

The intersection of a plane (17) and a surface (18) determines a planar spatial curve C, which is not invariant, in general, under the transformation A_3. Indeed, excluding x_3 from Eqs. (17) and (18), we obtain a third-degree homogeneous equation $G(x_0, x_1, x_2) = 0$. In the basis s_1, s_2, a, the function G on the space P_2 contains, in general, odd powers of the antisymmetric function a for any c_0 and c_1. The coefficient of a is a quadratic form with respect to s_1, s_2, and a. The invariance of the planar spatial curve C under the transformation A_3 implies that this quadratic form should vanish. As any second-degree equation, it defines rational functions s_1, s_2 and a of some parameter t. Substituting them in the even part (with respect to a) of the function $G(x_0, x_1, x_2)$, we obtain a third-degree equation with respect to t, which has three solutions in general. An invariant curve exists only if this equation is identically zero, i. e., if G is reducible. The equation determining the coefficients c_0 and c_1 is given by

$$R_{x_0}(G, G'_{x_1}) \equiv 0, \qquad (19)$$

where R_{x_0} is the resultant of G and G'_{x_1} with respect to x_0. From Eq. (19), we obtain $c_0 = -1, c_1 = 3$ and find the function

$$G(x_0, x_1, x_2) = (-3x_1^2 + x_0x_1 + 3x_0x_2 - x_1x_2)(-x_0 + x_2) = 0, \qquad (20)$$

which defines the reducible curve C. The first factor in Eq. (20) is invariant under the transformations I_p and A_2, and together with Eq. (17) defines the well-known solution [11] with a finite number of poles with respect to w. It is represented in P_3 as the intersection of the plane

$$-x_0 + 3x_1 + x_2 - 3x_3 = 0 \qquad (21)$$

and the surface

$$-3{x^2}_1 + x_0x_1 + 3x_0x_2 - x_1x_2 = 0. \qquad (22)$$

Using Eq. (21) and writing Eq. (22) in the form

$$x_1x_3 = x_0x_2,$$

we can easily verify the invariance of (21) under the transformation I_p. Under the action of the transformation A_3, the second factor in (20) becomes $(-x_1 + x_2)$; as a result, we have the degenerate quadratic form

$$(-x_0 + x_2)(-x_1 + x_2) = 0,$$

which is invariant under the transformations I_p and A_3. It determines two bundles of lines that are invariant under the transformation I_p and pass into each other under the transformation A_3:

$$x_0 = x_2, \quad \frac{x_0}{x_1} = \frac{n + 1/6}{n - 7/6}; \quad x_1 = x_2, \quad \frac{x_0}{x_1} = \frac{n - 3/2}{n + 1/2}.$$

CONCLUSION

The nonlinear boundary-value problem of constructing N-dimensional (condition (1)A), elastically unitary (condition (1)C), and crossing-symmetric (condition (1)D) S-matrix is formulated in the projective spaces P_{N-1} and P_N. In the space P_{N-1}, it can be considered as the result of embedding (ignoring one of the unitarity condition (1)C) the initial problem defined by Conditions 1 from the affine space A_N into the projective space P_{N-1}. The condition for the analytic continuation of the S-matrix to unphysical sheets is represented as a nonlinear autonomous system of difference equations, i. e., in the dynamic form. It can also be considered as nonlinear transformation in the spaces A_N, P_{N-1}, and P_N. In particular, among its fixed points, there is a point corresponding to the S-matrix without interaction. In the neighborhood of this point, the S-matrix was studied using power series in $1/w$, which can sometimes be summed [7]. The use of the projective space technique allows analyzing the solutions globally, i. e., constructing the invariant subspaces containing the solutions to be found. The invariant subspaces are determined by functions that are homogeneous in the projective spaces P_{N-1} and P_N, but not in the affine space A_N. This statement disagrees with the conclusion in [12], according to which the invariant subspaces in the affine space A_N are also determined by homogeneous functions. The above geometric interpretation of the boundary-value problem defined by Conditions 1 in the projective spaces P_{N-1} and P_N and the examples considered in [8] and [11] indicate that the homogeneity requirement on the functions defining the invariant subspaces of A_N should be rejected. Concrete applications of the described procedure for solving the nonlinear boundary-value problem are demonstrated in Appendices 1 and 2.

They are follow to the same rule: the Conditions 3 solved in the P_1 for $n \in Z$, then succeed the chain $Z \subset R \subset C$, and at the end the solution one of the unitarity equation in condition (1)C is found.

APPENDIX 1

The two-row crossing-symmetry matrix for the group $SU(2)$ is given by

$$A_2 = \frac{1}{2l+1}\begin{pmatrix} -1 & 2l+2 \\ 2l & 1 \end{pmatrix}, \quad l \in N.$$

The matrix considered in the paper is particular case of it for $l = 1$. We give the calculation scheme for the general case of integer l.

Let us introduce the function $X = S_1/S_2$ and consider it for $z = 0$. Then the continuation of X on to the first unphysical sheet is determined by the rule

$$X^{(1)} = \frac{2lX^{(0)} + 1}{-X^{(0)} + (2l+2)}$$

and together with the crossing-symmetry condition (1)D gives the following expression for $X^{(n)}$

$$X^{(n)} = \frac{n-(l+1)}{n+l}, \quad X^{(0)} = -(1+1/l). \tag{23}$$

Thus, on any unphysical sheet n the ratio S_1/S_2 is defined at $z = 0$, and for construction of S_1 and S_2 it is sufficient to find any of them. Let us denote S_2 by $\varphi = S_2$. This function is determind by the system of functional equations

$$\varphi^{(n)}\varphi^{(1-n)} = 1, \tag{24}$$

$$\frac{\varphi^{(n)}}{\varphi^{(-n)}} = \frac{n+l}{n-l}, \tag{25}$$

which follow from the unitarity and the crossing-symmetry conditions (4) on the unphysical sheets. Here only those equalities are used from (4), which were not used for derivation of Eq. (23). Equation (24) has an obvious solution in the ring of meromorphic functions

$$\varphi^{(n)} = \frac{G(n)}{G(1-n)}, \tag{26}$$

where $G(n)$ is an entire function. Solution (26) can be represented in another form $\ln\varphi^{(n)} = g(n-1/2)$, where $g(n-1/2)$ is any odd function of its argument. That form of $\ln\varphi^{(n)}$ is convenient for the solution to Eq. (25) which is now of the form

$$g(n+1) + g(n) = \ln\frac{n+1/2+l}{n+1/2-l}.$$

A partial solution of this nonhomogeneous difference equation can be found by subsequent substitutions of unknown functions according to the formula

$$g_m(n) = g_{m+1}(n) + \ln\frac{n+(-1)^m\alpha_{m+1}}{n-(-1)^m\alpha_{m+1}},$$

where $\alpha_k = 1/2 + l - k$ and $g_0(n) = g(n)$. The function g_k obeys the equation

$$g_k(n+1) + g_k(n) = \ln \frac{n + 1/2 + (-1)^k(l-k)}{n + 1/2 - (-1)^k(l-k)}.$$

It is clear that

$$g_l(n+1) + g_l(n) = 0, \tag{27}$$

and a general solution to this equation gives a trivial solution of the problem (1), which does not depend on l. Therefore, one gets [8]

$$\varphi^{(n)} = \prod\nolimits_{m=1}^{l} \frac{n - 1/2 - (-1)^m(1/2 + l - m)}{n - 1/2 + (-1)^m(1/2 + l - m)}. \tag{28}$$

One has an infinite product in formula (28) for noninteger $l \in R$. Now Eq. (27) is of the form

$$g_\infty(n+1) + g_\infty(n) = ln(-1). \tag{29}$$

In this case one has, instead of Eq. (28),

$$\varphi^{(n)} = \psi(n) \frac{\Gamma\left[-\dfrac{n+l}{2} + 1\right]\Gamma\left[\dfrac{n-l}{2}\right]}{\Gamma\left[-\dfrac{n-1-l}{2} + 1\right]\Gamma\left[-\dfrac{n-1+l}{2}\right]}, \tag{30}$$

where $\psi(n) = e^{g(n)_\infty}$ is defined by a general solution of Eq. (29) with properties

$$\psi(n+1)\psi(n) = -1, \quad \psi(n)\psi(-n) = 1. \tag{31}$$

Till now one of the unitarity conditions (1)C was not used. It gives the following result

$$n(z) = 1/\pi \arcsin z + i\sqrt{z^2 - 1}\beta(z), \tag{32}$$

where $\beta(z) = -\beta(-z)$ — is a meromorphic function. Equation (32) shows that the Riemann surface of the model has algebraic branch points at $z = \pm 1$ and a logarithmic one at infinify. Now formulae (23), (30), (31), (32) give the general solution to the problem (1) for matrix A_2. The function ψ can be determined from the requirement that Eq. (30) turns into Eq. (28) for integer l. This gives $\psi(n) = -\cot(n)$ for l even and $\psi(n) = -\tan(n)$ for l odd.

Let us remind that in Eq. (30) $l \in R$, but it is clear that this relation can be continued to $l \in C$ and allows explicit determination of the Regge tragectories with definite signature $l_k^{\pm}(z)$. The common part of the Regge tragectories set for $J_\pm = l \pm 1/2$ is of the form $l^{\pm}(z) = \{2 - n(z) + 2k, n(z) + 2k \mid k = 0, 1, 2 \cdots\}$. The Regge trajectories for $J_- = l - 1/2$ contained one additional trajectory $l^{\pm}_{J_-}(z) = -n(z)$. All the Regge trajectories of the model depend on function $\beta(z)$.

APPENDIX 2

We apply the developed method to the problem of scattering of a particle with angular momentum one by a fixed source with the same angular momentum. In this case, the crossing-symmetry matrix is given by expression (15). We decompose the column $S(z)$ into a sum of eigenvectors of the matrix A:

$$S(z) = s_1(z)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4}s_2(z)\begin{pmatrix} 15 \\ -5 \\ 3 \end{pmatrix} + 2\psi(z)\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}. \qquad (33)$$

For $q = 1$, $p = 0$, functional Eq. (5) in the limit $z \to \infty$ determines the fixed (rest) points of the problem. Returning from the basis $s_1(z)$, $s_2(z)$, $\psi(z)$ to the column $S(z)$, we have

$$S_f = \pm i \begin{pmatrix} -(2 \pm \sqrt{5}) \\ -\frac{1}{2}(1 \pm \sqrt{5}) \\ \frac{1}{2}(1 \pm \sqrt{5}) \end{pmatrix}. \qquad (34)$$

From (34) it is clear that $\mathrm{Im}S_i \in Q(\sqrt{5})$. More definitely they are degrees of the roots of the equation

$$x^2 - x - 1 = 0. \qquad (*)$$

These roots are known in the theory Fibonacci numbers and has reflection in the consideration below. Let us come to the linear approximation of the functional Eq. (3) at the vicinity of the point S_f. It can be solved, and the result is of the form

$$S(z) = S_f + c_1\begin{pmatrix} x_-^4 \\ 1 \\ -1 \end{pmatrix}(-1)^n + c_2\begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}x_+^{2n} + c_3\begin{pmatrix} 8x_-^4 \\ -1 \\ 1 \end{pmatrix}x_-^{4n}, \qquad (35)$$

where $x_\pm$ are positive and negative roots of Eq. $(*)$ and c_i is an arbitrary constant. Formula (35) defines three different planes which are linearly invariant under approximate transformation (3). We cosidered below only one of them which is not only linearly but also globally invariant. We can see from (34) that all rest points lie in the plane $S_2 + S_3 = 0$. This plane is invariant under the inversion transformation I and the crossing-symmetry transformation A. In the plane $S_2 + S_3 = 0$, three-row crossing-symmetry matrix (15) passes into the two-row matrix A_2

$$A_2 = \frac{1}{3}\begin{pmatrix} 1 & -8 \\ -1 & -1 \end{pmatrix}, \qquad (36)$$

and the problem is thus reduced to finding two functions $S_1(z)$ and $S_2(z)$. Setting $z=0$ and defining $X^{(n)} = S_1^{(n)}/S_2^{(n)}$, where n is the number of the sheet of the Riemann surface, we see that the transition from the physical sheet to the sheet with the number n is realized by the linear fractional transformation

$$X^{(n)} = \sqrt{5}\frac{\sqrt{5}(-X^{(0)}+2)\mathrm{sh}_y n + (X^{(0)}+4)\mathrm{ch}_y n}{-(X^{(0)}+4)\mathrm{sh}_y n + \sqrt{5}(X^{(0)}-2)\mathrm{ch}_y n}, \tag{37}$$

where we introduce useful notations

$$2\mathrm{sh}_y n = y_+^n - y_-^n, \quad 2\mathrm{ch}_y n = y_+^n + y_-^n,$$

and $y_\pm = (3\pm\sqrt{5})/2$. The unitarity or crossing-symmetry requirements on $X^{(n)}$ gives the condition

$$(X^{(0)}-2)(X^{(0)}+4) = 0 \tag{38}$$

which determines $X(0)$. Consequently, we obtain two different solutions, $X^{(0)} = 2$ and $X^{(0)} = -4$, which are compatible with the unitarity and crossing-symmetry requirements.

The ratio S_1/S_2 is thus determined for $z=0$ on every nonphysical sheet of the Riemann surface defined by Conditions 2 with matrix (36), and to construct S_1 and S_2, it suffices to find any of these functions. We set $S_2(n) = \Phi(n) = -s_2(n) + \psi(n)$, where s_2 and ψ are the functions introduced like in (33). The function Φ satisfies the system of functional equations

$$\Phi(1-n)\Phi(n) = 1, \tag{39}$$

$$\frac{\Phi(n)}{\Phi(-n)} = (-1)\frac{\mathrm{ch}_y(n+1/2)}{\mathrm{ch}_y(n-1/2)}, \ X^{(0)} = 2, \tag{40}$$

$$\frac{\Phi(n)}{\Phi(-n)} = (-1)\frac{\mathrm{sh}_y(n+1/2)}{\mathrm{sh}_y(n-1/2)}, \ X^{(0)} = -4. \tag{41}$$

Relations (37), (38) are used in deriving Eqs. (40), (41). Equation (39) has the solution

$$\Phi(n) = \mathrm{e}^{g(n-1/2)}, \tag{42}$$

where $g(n)$ is an arbitrary odd function, $g(n) = -g(-n)$. Substituting (42) in (40), (41) and changing $n \to n+1/2$, we obtain the difference equations

$$g(n+1) + g(n) = \ln(-1)\frac{\mathrm{ch}_y(n+1)}{\mathrm{ch}_y n}, \quad X^{(0)} = 2, \tag{43}$$

$$g(n+1) + g(n) = \ln(-1)\frac{\mathrm{sh}_y(n+1)}{\mathrm{sh}_y n}, \quad X^{(0)} = -4 \tag{44}$$

for the unknown function $g(n)$.

Solving Eqs. (43), (44) by the method of consecutive functional changes, we obtain

$$g(n) = g_{-1}(n) + g_\infty(n) + \sum_{m=0}^{\infty} G_m(n), \tag{45}$$

where $\mathrm{g}_\infty(n) = n \ln y_+$ and

$$G_m(n) = \ln \frac{\mathrm{ch}_y(n+1+2m)\mathrm{ch}_y(n-2(m+1))}{\mathrm{ch}_y(n-1-2m)\mathrm{ch}_y(n+2(m+1))},\ X^{(0)} = 2, \tag{46}$$

$$G_m(n) = \ln \frac{\mathrm{sh}_y(n+1+2m)\mathrm{sh}_y(n-2(m+1))}{\mathrm{sh}_y(n-1-2m)\mathrm{sh}_y(n+2(m+1))},\ X^{(0)} = -4. \tag{47}$$

The term $g_{-1}(n)$ is introduced to take the factor -1 in Eqs. (43), (44) into account. We set $e^{g_{-1}(n)} = \xi(n)$. The function $\xi(n)$ solves the system of functional equations

$$\xi(n+1)\xi(n) = -1, \quad \xi(n)\xi(-n) = 1. \tag{48}$$

The general solution of this system is expressed in terms of θ-functions. We confine ourselves to the degenerate case here

$$\xi(n) = \mathrm{tg}\frac{\pi}{2}\left(n + \frac{1}{2}\right). \tag{49}$$

Now we use the last unitarity condition (1)C. As a result, the function n considered as a function of the complex variable z is of the same form as in Appendix 1. Formulae (37), (38), (42), (46), (47), (49) now give the solution of the problem defined by Conditions 1 for crossing-symmetry matrix (15) and equation $S_2 + S_3 = 0$.

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