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THE NUCLEAR SCISSORS MODE BY TWO APPROACHES (Wigner Function Moments Versus RPA)

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Ядерная ножничная мода в двух подходах
(метод моментов функции Вигнера и приближение случайных фаз)

На примере простой модели — гармонического осциллятора с квадруполь-квадрупольным остаточным взаимодействием — сравниваются два взаимодополняющих метода описания коллективного движения, приближение случайных фаз и метод моментов функции Вигнера. Показано, что они дают одинаковые формулы для собственных частот и вероятностей переходов всех коллективных возбуждений, включая ножничную моду, которой уделяется особое внимание. Получены аналитические выражения для нормировочного фактора «синтетической» ножничной моды и ее перекрытия с физическими состояниями. Дано строгое доказательство ортогональности духового состояния всем физическим состояниям модели.

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Two complementary methods to describe the collective motion, RPA and Wigner Function Moments (WFM) method, are compared on an example of a simple model — harmonic oscillator with quadrupole-quadrupole residual interaction. It is shown that they give identical formulae for eigenfrequencies and transition probabilities of all collective excitations of the model including the scissors mode, which is a subject of our especial attention. The normalization factor of the «synthetic» scissors state and its overlap with physical states are calculated analytically. The orthogonality of the spurious state to all physical states is proved rigorously.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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INTRODUCTION

The full analysis of the scissors mode in the framework of a solvable model (harmonic oscillator with quadrupole-quadrupole residual interaction (HO+QQ)) was given in [1]. Many obscure points in the understanding of this mode nature were clarified: for example, its coexistence with the isovector giant quadrupole resonance (IVGQR), the decisive role of the Fermi surface deformation in its creation, and so on.

The Wigner Function Moments (WFM) method was applied to derive analytical expressions for currents of both coexisting modes (it was done for the first time), their excitation energies, magnetic and electric transition probabilities. Unexpectedly, our formulae for energies turned out to be identical with those derived by Hamamoto and Nazarewicz [2] in the framework of the RPA. This fact generated the natural motivation for this work: to check the relation between formulas for transition probabilities derived by two methods. The obvious development of this investigation is the systematic comparison of two approaches with the aim to establish the connection between them. The HO+QQ model is a very convenient proving ground for this kind of researches, because all results can be obtained analytically. There is no need to describe the merits and demerits of the RPA — they are known very well [3]. It is necessary, however, to say a few words about the WFM. Its idea is based on the virial theorems of Chandrasekhar and Lebovitz [4]. Instead of writing the equations of motion for microscopic amplitudes of particle-hole excitations (RPA), one writes the dynamical equations for various multipole phase space moments of a nucleus. This allows one to achieve better physical interpretation of the studied phenomenon without going into its detailed microscopic structure. The WFM method was successfully applied to study isoscalar and isovector giant multipole resonances and low-lying collective modes of rotating and nonrotating nuclei with various realistic forces [5]. The results of calculations were always very close to similar results obtained with the help of RPA. In principle, it should be expected, because the basis of the both methods is the same: Time Dependent Hartree–Fock (TDHF) theory and a small amplitude approximation. On the other hand, it is evident that they are not equivalent, because one deals with equations of motion for different objects. The detailed analysis of the interplay of two methods turns out useful also from a «practical» point of view: firstly, it allows one to obtain additional insight into the nature of the scissors mode; secondly, we find new exact mathematical results for the considered model.

1. THE WFM METHOD

The detailed description of the method of Wigner function moments can be found in [1, 5, 6]. Here we remind briefly only its main points. The basis of the method is the TDHF equation for the one-body density matrix: $i\hbar\dfrac{\partial\hat{\rho}^\tau}{\partial t} = \left[\hat{H}^\tau, \hat{\rho}^\tau\right]$, where $\hat{H}^\tau$ is the one-body self-consistent Hamiltonian depending implicitly on the density matrix $\rho^\tau(\mathbf{r}_1, \mathbf{r}_2, t) = \langle \mathbf{r}_1 | \hat{\rho}^\tau(t) | \mathbf{r}_2 \rangle$ and τ is an isotopic index. It is convenient to modify this equation introducing the Wigner transform of the density matrix [3] known as the Wigner function $f^\tau(\mathbf{r}, \mathbf{p}, t)$:

$$\frac{\partial f^\tau}{\partial t} = \frac{2}{\hbar} \sin\left(\frac{\hbar}{2}(\nabla^H_{\mathbf{r}} \cdot \nabla^f_{\mathbf{p}} - \nabla^H_{\mathbf{p}} \cdot \nabla^f_{\mathbf{r}})\right) H^\tau_{\mathrm{W}} f^\tau, \tag{1}$$

where the upper index on the nabla operator stands for the function on which this operator acts and H_{W} is the Wigner transform of the Hamiltonian H.

It is shown in [5,6] that by integrating Eq. (1) over the phase space $\{\mathbf{p}, \mathbf{r}\}$ with the weights $x_{i_1} x_{i_2} \ldots x_{i_k} p_{i_{k+1}} \ldots p_{i_{n-1}} p_{i_n}$, where k runs from 0 to n, one can obtain a closed finite set of dynamical equations for Cartesian tensors of the rank n. Taking linear combinations of these equations one is able to represent them through various multipole moments which play roles of collective variables of the problem. Here we consider the case $n = 2$.

1.1. Model Hamiltonian, Equations of Motion. The microscopic Hamiltonian of the model is

$$\begin{aligned} H = &\sum_{i=1}^{A} \left(\frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \mathbf{r}_i^2\right) + \bar{\kappa} \sum_{\mu=-2}^{2} (-1)^\mu \sum_{i}^{Z} \sum_{j}^{N} q_{2\mu}(\mathbf{r}_i) q_{2-\mu}(\mathbf{r}_j) \\ &+ \frac{1}{2}\kappa \sum_{\mu=-2}^{2} (-1)^\mu \Big\{\sum_{i \neq j}^{Z} q_{2\mu}(\mathbf{r}_i) q_{2-\mu}(\mathbf{r}_j) + \sum_{i \neq j}^{N} q_{2\mu}(\mathbf{r}_i) q_{2-\mu}(\mathbf{r}_j)\Big\}, \end{aligned} \tag{2}$$

where the quadrupole operator $q_{2\mu} = \sqrt{16\pi/5}\, r^2 Y_{2\mu}$ and N, Z are the numbers of neutrons and protons, respectively. The mean field potential for protons (or neutrons) is

$$V^\tau(\mathbf{r}, t) = \frac{1}{2} m\,\omega^2 r^2 + \sum_{\mu=-2}^{2} (-1)^\mu Z^\tau_{2\mu}(t) q_{2-\mu}(\mathbf{r}), \tag{3}$$

where $Z^{\mathrm{n}}_{2\mu} = \kappa Q^{\mathrm{n}}_{2\mu} + \bar{\kappa} Q^{\mathrm{p}}_{2\mu}$, $\quad Z^{\mathrm{p}}_{2\mu} = \kappa Q^{\mathrm{p}}_{2\mu} + \bar{\kappa} Q^{\mathrm{n}}_{2\mu}$ and the quadrupole moments $Q^\tau_{2\mu}(t)$ are defined as

$$Q^\tau_{2\mu}(t) = \int d\{\mathbf{p}, \mathbf{r}\} q_{2\mu}(\mathbf{r}) f^\tau(\mathbf{r}, \mathbf{p}, t)$$

with $\int d\{\mathbf{p}, \mathbf{r}\} \equiv 2(2\pi\hbar)^{-3} \int d^3p \int d^3r$.

Integration of Eq. (1) with the weights $r^2_{\lambda\mu}, (rp)_{\lambda\mu} \equiv \{r\otimes p\}_{\lambda\mu}$ and $p^2_{\lambda\mu}$ yields the following set of equations [1]:

$$\begin{aligned}
&\frac{d}{dt}R^\tau_{\lambda\mu} - \frac{2}{m}L^\tau_{\lambda\mu} = 0, \qquad \lambda = 0,2,\\
&\frac{d}{dt}L^\tau_{\lambda\mu} - \frac{1}{m}P^\tau_{\lambda\mu} + m\,\omega^2 R^\tau_{\lambda\mu} - 2\sqrt{30}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda1}\}(Z^\tau_2 R^\tau_j)_{\lambda\mu} = 0,\\
&\quad \lambda = 0,1,2,\\
&\frac{d}{dt}P^\tau_{\lambda\mu} + 2m\,\omega^2 L^\tau_{\lambda\mu} - 4\sqrt{30}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda1}\}(Z^\tau_2 L^\tau_j)_{\lambda\mu} = 0,\\
&\quad \lambda = 0,2,
\end{aligned} \tag{4}$$

where $\{^{11j}_{2\lambda1}\}$ is the Wigner $6j$-symbol,

$$r^2_{\lambda\mu} \equiv \{r\otimes r\}_{\lambda\mu} = \sum_{\sigma,\nu} C^{\lambda\mu}_{1\sigma,1\nu} r_\sigma r_\nu$$

is a tensor product [7], and r_ν are cyclic variables

$$r_{+1} = -(x_1 + ix_2)/\sqrt{2}, \quad r_0 = x_3\ , \quad r_{-1} = (x_1 - ix_2)/\sqrt{2}.$$

In terms of these variables $q_{2\mu} = \sqrt{6}r^2_{2\mu}$, $Q^\tau_{2\mu} = \sqrt{6}R^\tau_{2\mu}$. Further the following notation is introduced:

$$P^\tau_{\lambda\mu}(t) = \int d\{\mathbf{p},\mathbf{r}\} p^2_{\lambda\mu} f^\tau(\mathbf{r},\mathbf{p},t), \quad L^\tau_{\lambda\mu}(t) = \int d\{\mathbf{p},\mathbf{r}\} (rp)_{\lambda\mu} f^\tau(\mathbf{r},\mathbf{p},t). \tag{5}$$

By definition $R^\tau_{00} = -Q^\tau_{00}/\sqrt{3}$ with $Q^\tau_{00} = N^\tau < r^2 >$ being the mean square radius. The tensor $L^\tau_{1\nu}$ is connected with the angular momentum by the following relations: $L^\tau_{10} = \frac{i}{\sqrt{2}}I^\tau_3, \quad L^\tau_{1\pm1} = \frac{1}{2}(I^\tau_2 \mp iI^\tau_1)$.

We rewrite Eqs. (4) in terms of the isoscalar and isovector variables $R_{\lambda\mu} = R^{\rm n}_{\lambda\mu} + R^{\rm p}_{\lambda\mu}$, $\bar{R}_{\lambda\mu} = R^{\rm n}_{\lambda\mu} - R^{\rm p}_{\lambda\mu}$ (and so on) with the isoscalar $\kappa_0 = (\kappa + \bar{\kappa})/2$ and isovector $\kappa_1 = (\kappa - \bar{\kappa})/2$ strength constants. There is no problem to solve these equations numerically. However, we want to simplify the situation as much as possible to get the results in analytical form that gives us a maximum of insight into the nature of the modes.

1. The problem is considered in a small amplitude approximation. Writing all variables as a sum of their equilibrium value plus a small deviation

$$R_{\lambda\mu}(t) = R^{\rm eq}_{\lambda\mu} + \mathcal{R}_{\lambda\mu}(t), \quad P_{\lambda\mu}(t) = P^{\rm eq}_{\lambda\mu} + \mathcal{P}_{\lambda\mu}(t), \quad L_{\lambda\mu}(t) = L^{\rm eq}_{\lambda\mu} + \mathcal{L}_{\lambda\mu}(t),$$

$$\bar{R}_{\lambda\mu}(t) = \bar{R}^{\rm eq}_{\lambda\mu} + \bar{\mathcal{R}}_{\lambda\mu}(t), \quad \bar{P}_{\lambda\mu}(t) = \bar{P}^{\rm eq}_{\lambda\mu} + \bar{\mathcal{P}}_{\lambda\mu}(t), \quad \bar{L}_{\lambda\mu}(t) = \bar{L}^{\rm eq}_{\lambda\mu} + \bar{\mathcal{L}}_{\lambda\mu}(t),$$

we linearize the equations of motion in $\mathcal{R}_{\lambda\mu}$, $\mathcal{P}_{\lambda\mu}$, $\mathcal{L}_{\lambda\mu}$ and $\bar{\mathcal{R}}_{\lambda\mu}$, $\bar{\mathcal{P}}_{\lambda\mu}$, $\bar{\mathcal{L}}_{\lambda\mu}$.

2. We study nonrotating nuclei, i.e. nuclei with $L^{\rm eq}_{1\nu} = \bar{L}^{\rm eq}_{1\nu} = 0$.

3. Only axially symmetric nuclei with $R^{\rm eq}_{2\pm 2} = R^{\rm eq}_{2\pm 1} = \bar{R}^{\rm eq}_{2\pm 2} = \bar{R}^{\rm eq}_{2\pm 1} = 0$ are considered.

4. Finally, we take $\bar{R}^{\rm eq}_{20} = \bar{R}^{\rm eq}_{00} = 0$. This means that equilibrium deformation and mean square radius of neutrons are supposed to be equal to that of protons.

Due to approximation No. 4. the equations for isoscalar and isovector systems are decoupled. Further, due to the axial symmetry the angular momentum projection is a good quantum number. As a result, every set of equations splits into five independent subsets with $\mu = 0, \pm 1, \pm 2$. The detailed derivation of formulae for eigenfrequencies and transition probabilities together with all necessary explanations are given in [1]. Here we write out only final results required for the comparison with respective results obtained in the framework of the RPA.

1.2. Isoscalar Eigenfrequencies. Let us analyze the isoscalar set of equations with $\mu = 1$

$$
\begin{aligned}
&\dot{\mathcal{R}}_{21} - 2\mathcal{L}_{21}/m = 0,\\
&\dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + \left[m\,\omega^2 + 2\kappa_0(Q^{\rm eq}_{20} + Q^{\rm eq}_{00})\right]\mathcal{R}_{21} = 0,\\
&\dot{\mathcal{P}}_{21} + 2[m\omega^2 + \kappa_0 Q^{\rm eq}_{20}]\mathcal{L}_{21} = 0,\\
&\dot{\mathcal{L}}_{11} = 0.
\end{aligned}
\tag{6}
$$

Imposing the time evolution via $\mathrm{e}^{-\mathrm{i}\Omega t}$ for all variables one transforms (6) into a set of algebraic equations. The eigenfrequencies are found from its characteristic equation which reads

$$
\Omega^2[\Omega^2 - 4\omega^2 - \frac{6\kappa_0}{m}(Q^{\rm eq}_{20} + \frac{4}{3}Q^{\rm eq}_{00})] = 0. \tag{7}
$$

For κ_0 we take the self-consistent value $\kappa_0 = -\dfrac{m\bar{\omega}^2}{4Q_{00}}$, where $\bar{\omega}^2 = \dfrac{\omega^2}{1+\frac{2}{3}\delta}$ (see Appendix A) with the standard definition of the deformation parameter $Q_{20} = Q_{00}\dfrac{4}{3}\,\delta$. Then

$$
\Omega^2[\Omega^2 - 2\bar{\omega}^2(1+\delta/3)] = 0. \tag{8}
$$

The nontrivial solution of this equation gives the frequency of the $\mu = 1$ branch of the isoscalar GQR

$$
\Omega^2 = \Omega^2_{\rm is} = 2\bar{\omega}^2(1+\delta/3). \tag{9}
$$

Taking into account the relation (63) we find that this result coincides with that of [8]. The trivial solution $\Omega = \Omega_0 = 0$ is characteristic of nonvibrational mode corresponding to the obvious integral of motion $\mathcal{L}_{11} = \mathrm{const}$ responsible for the rotational degree of freedom. This is usually called the «spurious» mode.

1.3. Isovector Eigenfrequencies. The information about the scissors mode is contained in the set of isovector equations with $\mu = 1$

$$
\begin{aligned}
&\dot{\bar{\mathcal{R}}}_{21} - 2\bar{\mathcal{L}}_{21}/m = 0,\\
&\dot{\bar{\mathcal{L}}}_{21} - \bar{\mathcal{P}}_{21}/m + \left[m\,\omega^2 + \kappa Q_{20}^{\mathrm{eq}} + 4\kappa_1 Q_{00}^{\mathrm{eq}}\right]\bar{\mathcal{R}}_{21} = 0,\\
&\dot{\bar{\mathcal{P}}}_{21} + 2[m\omega^2 + \kappa_0 Q_{20}^{\mathrm{eq}}]\bar{\mathcal{L}}_{21} - 6\kappa_0 Q_{20}^{\mathrm{eq}}\,\bar{\mathcal{L}}_{11} = 0,\\
&\dot{\bar{\mathcal{L}}}_{11} + 3\bar{\kappa} Q_{20}^{\mathrm{eq}}\bar{\mathcal{R}}_{21} = 0.
\end{aligned}
\tag{10}
$$

Imposing the time evolution via $\mathrm{e}^{-i\Omega t}$ one transforms (10) into a set of algebraic equations. Again the eigenfrequencies are found from the characteristic equation which reads

$$
\Omega^4 - \Omega^2[4\omega^2 + \frac{8}{m}\kappa_1 Q_{00}^{\mathrm{eq}} + \frac{2}{m}(\kappa_1 + 2\kappa_0)Q_{20}^{\mathrm{eq}}] + \frac{36}{m^2}(\kappa_0 - \kappa_1)\kappa_0 (Q_{20}^{\mathrm{eq}})^2 = 0. \tag{11}
$$

Supposing, as usual, the isovector constant κ_1 to be proportional to the isoscalar one, $\kappa_1 = \alpha\kappa_0$, and taking the self-consistent value for κ_0, we finally obtain

$$
\Omega^4 - 2\Omega^2\bar{\omega}^2(2-\alpha)(1+\delta/3) + 4\bar{\omega}^4(1-\alpha)\delta^2 = 0. \tag{12}
$$

The solutions of this equation are

$$
\Omega_{\pm}^2 = \bar{\omega}^2(2-\alpha)(1+\delta/3) \pm \sqrt{\bar{\omega}^4(2-\alpha)^2(1+\delta/3)^2 - 4\bar{\omega}^4(1-\alpha)\delta^2}. \tag{13}
$$

The solution Ω_+ gives the frequency Ω_{iv} of the $\mu = 1$ branch of the isovector GQR. The solution Ω_- gives the frequency Ω_{sc} of the scissors mode.

We adjust α from the fact that the IVGQR is experimentally known to lie practically at twice the energy of the isoscalar GQR. In our model the experimental situation is satisfied by $\alpha = -2$. Then

$$
\begin{aligned}
\Omega_{\mathrm{iv}}^2 &= 4\bar{\omega}^2(1 + \frac{\delta}{3} + \sqrt{(1+\frac{\delta}{3})^2 - \frac{3}{4}\delta^2}\,),\\
\Omega_{\mathrm{sc}}^2 &= 4\bar{\omega}^2(1 + \frac{\delta}{3} - \sqrt{(1+\frac{\delta}{3})^2 - \frac{3}{4}\delta^2}\,).
\end{aligned}
\tag{14}
$$

1.4. Linear Response and Transition Probabilities. A direct way of calculating the reduced transition probabilities is provided by the theory of linear response of a system to a weak external field

$$
\hat{O}(t) = \hat{O}\exp(-i\Omega t) + \hat{O}^{\dagger}\exp(i\Omega t).
$$

For *magnetic excitations*

$$
\hat{O} = \hat{O}_{1\mu} = -i\sum_{s=1}^{z}\nabla_s(r_s Y_{1\mu})\cdot[\mathbf{r}_s \times \nabla_s]\mu_N, \quad \mu_N = \frac{e\hbar}{2mc}, \tag{15}
$$

$$B(M1)_{\rm sc} = 2|<{\rm sc}|\hat{O}_{11}|0>|^2 = \frac{1-\alpha}{4\pi}\frac{m\bar{\omega}^2}{\hbar}Q_{00}\delta^2\frac{\Omega_{\rm sc}^2-2(1+\delta/3)\bar{\omega}^2}{\Omega_{\rm sc}(\Omega_{\rm sc}^2-\Omega_{\rm iv}^2)}\,\mu_N^2, \tag{16}$$

$$B(M1)_{\rm iv} = 2|<{\rm iv}|\hat{O}_{11}|0>|^2 = \frac{1-\alpha}{4\pi}\frac{m\bar{\omega}^2}{\hbar}Q_{00}\delta^2\frac{\Omega_{\rm iv}^2-2(1+\delta/3)\bar{\omega}^2}{\Omega_{\rm iv}(\Omega_{\rm iv}^2-\Omega_{\rm sc}^2)}\,\mu_N^2. \tag{17}$$

These two formulae can be joined into one expression by the simple transformation of the denominators. Really, we have from (13)

$$\begin{aligned}\pm(\Omega_{\rm iv}^2-\Omega_{\rm sc}^2) &= \pm(\Omega_+^2-\Omega_-^2) = \pm 2\sqrt{\bar{\omega}^4(2-\alpha)^2(1+\delta/3)^2-4\bar{\omega}^4(1-\alpha)\delta^2}\\ &= 2\Omega_\pm^2-2\bar{\omega}^2(2-\alpha)(1+\delta/3) = 2\Omega_\pm^2-(2-\alpha)(\omega_x^2+\omega_z^2).\end{aligned} \tag{18}$$

Using these relations in formulae (16) and (17), we obtain the expression for *B(M*1) value valid for both excitations

$$B(M1)_\nu = \frac{1-\alpha}{8\pi}\frac{m\bar{\omega}^2}{\hbar}Q_{00}\delta^2\frac{\Omega_\nu^2-2(1+\delta/3)\bar{\omega}^2}{\Omega_\nu[\Omega_\nu^2-\bar{\omega}^2(2-\alpha)(1+\delta/3)]}\,\mu_N^2. \tag{19}$$

For *electric excitations* $\hat{O}=\hat{O}_{2\mu}=\sum\limits_{s=1}^{z} er_s^2 Y_{2\mu}$.

$$B(E2)_{\rm sc} = 2|<{\rm sc}|\hat{O}_{21}|0>|^2 = \frac{e^2\hbar}{m}\frac{5}{8\pi}Q_{00}\frac{(1+\delta/3)\Omega_{\rm sc}^2-2(\bar{\omega}\delta)^2}{\Omega_{\rm sc}(\Omega_{\rm sc}^2-\Omega_{\rm iv}^2)}. \tag{20}$$

$$B(E2)_{\rm iv} = 2|<{\rm iv}|\hat{O}_{21}|0>|^2 = \frac{e^2\hbar}{m}\frac{5}{8\pi}Q_{00}\frac{(1+\delta/3)\Omega_{\rm iv}^2-2(\bar{\omega}\delta)^2}{\Omega_{\rm iv}(\Omega_{\rm iv}^2-\Omega_{\rm sc}^2)}. \tag{21}$$

$$B(E2)_{\rm is} = 2|<{\rm is}|\hat{O}_{21}|0>|^2 = \frac{e^2\hbar}{m}\frac{5}{8\pi}Q_{00}[(1+\delta/3)\Omega_{\rm is}^2-2(\bar{\omega}\delta)^2]/[\Omega_{\rm is}]^3. \tag{22}$$

Using relations (18) in formulae (20) and (21) we obtain the expression for *B*(*E*2) value valid for all three excitations

$$B(E2)_\nu = 2|<\nu|\hat{O}_{21}|0>|^2 = \frac{e^2\hbar}{m}\frac{5}{16\pi}Q_{00}\frac{(1+\delta/3)\Omega_\nu^2-2(\bar{\omega}\delta)^2}{\Omega_\nu[\Omega_\nu^2-\bar{\omega}^2(2-\alpha)(1+\delta/3)]}. \tag{23}$$

The isoscalar value (22) is obtained by assuming $\alpha = 1$.

2. RPA

Standard RPA equations in the notation of [3] are

$$\sum_{n,j}\left\{\left[\delta_{ij}\delta_{mn}(\epsilon_m-\epsilon_i)+\bar{v}_{mjin}\right]X_{nj}+\bar{v}_{mnij}Y_{nj}\right\}=\hbar\Omega X_{mi},$$
$$\sum_{n,j}\left\{\bar{v}_{ijmn}X_{nj}+\left[\delta_{ij}\delta_{mn}(\epsilon_m-\epsilon_i)+\bar{v}_{inmj}\right]Y_{nj}\right\}=-\hbar\Omega Y_{mi}. \tag{24}$$

According to the schematic model (2), the matrix element of the residual interaction is

$$\bar{v}_{mjin}=\kappa_{\tau\tau'}D^{\tau *}_{im}D^{\tau'}_{jn}$$

with $D=q_{21}=\sqrt{16\pi/5}\,r^2Y_{21}$ and $\kappa_{\rm nn}=\kappa_{\rm pp}=\kappa, \quad \kappa_{\rm np}=\bar{\kappa}$. This interaction distinguishes between protons and neutrons, so we have to introduce the isospin projection indices τ, τ' into the set of RPA equations (24):

$$(\epsilon^\tau_m-\epsilon^\tau_i)X^\tau_{mi}+\sum_{n,j,\tau'}\kappa_{\tau\tau'}D^{\tau *}_{im}D^{\tau'}_{jn}X^{\tau'}_{nj}+\sum_{n,j,\tau'}\kappa_{\tau\tau'}D^{\tau *}_{im}D^{\tau'}_{nj}Y^{\tau'}_{nj}=\hbar\Omega X^\tau_{mi},$$
$$\sum_{n,j,\tau'}\kappa_{\tau\tau'}D^{\tau *}_{mi}D^{\tau'}_{jn}X^{\tau'}_{nj}+(\epsilon^\tau_m-\epsilon^\tau_i)Y^\tau_{mi}+\sum_{n,j,\tau'}\kappa_{\tau\tau'}D^{\tau *}_{mi}D^{\tau'}_{nj}Y^{\tau'}_{nj}=-\hbar\Omega Y^\tau_{mi}. \tag{25}$$

Its solution is

$$X^\tau_{mi}=\frac{D^{\tau *}_{im}}{\hbar\Omega-\epsilon^\tau_{mi}}K^\tau, \quad Y^\tau_{mi}=-\frac{D^{\tau *}_{mi}}{\hbar\Omega+\epsilon^\tau_{mi}}K^\tau \tag{26}$$

with $\epsilon^\tau_{mi}=\epsilon^\tau_m-\epsilon^\tau_i$ and $K^\tau=\sum_{\tau'}\kappa_{\tau\tau'}C^{\tau'}$.

The constant C^τ is defined as $C^\tau=\sum_{n,j}(D^\tau_{jn}X^\tau_{nj}+D^\tau_{nj}Y^\tau_{nj})$. Using here the above written expressions for X^τ_{nj} and Y^τ_{nj}, one derives the useful relation

$$C^\tau=2S^\tau K^\tau=2S^\tau\sum_{\tau'}\kappa_{\tau\tau'}C^{\tau'}, \tag{27}$$

where the following notation is introduced:

$$S^\tau=\sum_{mi}|D^\tau_{mi}|^2\frac{\epsilon^\tau_{mi}}{E^2-(\epsilon^\tau_{mi})^2} \tag{28}$$

with $E=\hbar\Omega$. Let us write this relation in detail

$$C^{\rm n}-2S^{\rm n}(\kappa C^{\rm n}+\bar{\kappa}C^{\rm p})=0,$$
$$C^{\rm p}-2S^{\rm p}(\bar{\kappa}C^{\rm n}+\kappa C^{\rm p})=0. \tag{29}$$

The condition for existence of a nontrivial solution of this set of equations gives the secular equation

$$(1-2S^{\rm n}\kappa)(1-2S^{\rm p}\kappa)-4S^{\rm n}S^{\rm p}\bar{\kappa}^2=0. \tag{30}$$

Making obvious linear combinations of two equations in (29), we write them in terms of isoscalar and isovector variables $C=C^{\rm n}+C^{\rm p}$, $\bar{C}=C^{\rm n}-C^{\rm p}$

$$\begin{aligned} &C-2(S^{\rm n}+S^{\rm p})\kappa_0 C-2(S^{\rm n}-S^{\rm p})\kappa_1\bar{C}=0,\\ &\bar{C}-2(S^{\rm n}-S^{\rm p})\kappa_0 C-2(S^{\rm n}+S^{\rm p})\kappa_1\bar{C}=0. \end{aligned} \tag{31}$$

Approximation No. 4 allows us to decouple equations for isoscalar and isovector variables. Really, in this case $S^{\rm n}=S^{\rm p}\equiv S/2$; hence, we obtain two secular equations

$$1-2S\kappa_0=0, \quad \text{or} \quad 1-S\kappa=S\bar{\kappa} \tag{32}$$

in the isoscalar case and

$$1-2S\kappa_1=0, \quad \text{or} \quad 1-S\kappa=-S\bar{\kappa} \tag{33}$$

in the isovector one, the difference between them being in the strength constants only. Having in mind the relation $\kappa_1=\alpha\kappa_0$, we come to the conclusion that it is sufficient to analyze the isovector case only — the results for isoscalar one are obtained by assuming $\alpha=1$.

2.1. Eigenfrequencies. The detailed expression for the isovector secular equation is

$$\frac{1}{2\kappa_1}=\sum_{mi}|D_{mi}|^2\frac{\epsilon_{mi}}{E^2-\epsilon_{mi}^2}. \tag{34}$$

The operator D has only two types of nonzero matrix elements D_{mi} in the deformed oscillator basis. Matrix elements of the first type couple the states of the same major shell. All corresponding transition energies are degenerate: $\epsilon_m-\epsilon_i=\hbar(\omega_x-\omega_z)\equiv\epsilon_0$. Matrix elements of the second type couple the states of the different major shells with $\Delta N=2$. All corresponding transition energies are degenerate too: $\epsilon_m-\epsilon_i=\hbar(\omega_x+\omega_z)\equiv\epsilon_2$. Therefore, the secular equation can be rewritten as

$$\frac{1}{2\kappa_1}=\frac{\epsilon_0 D_0}{E^2-\epsilon_0^2}+\frac{\epsilon_2 D_2}{E^2-\epsilon_2^2}. \tag{35}$$

The sums $D_0=\sum\limits_{mi(\Delta N=0)}|D_{mi}|^2$ and $D_2=\sum\limits_{mi(\Delta N=2)}|D_{mi}|^2$ can be calculated analytically (see Appendix B):

$$D_0=\frac{Q_{00}}{m\bar{\omega}^2}\epsilon_0, \quad D_2=\frac{Q_{00}}{m\bar{\omega}^2}\epsilon_2. \tag{36}$$

Let us transform the secular Eq. (35) in the polynomial

$$E^4 - E^2[(\epsilon_0^2 + \epsilon_2^2) + 2\kappa_1(\epsilon_0 D_0 + \epsilon_2 D_2)] + [\epsilon_0^2\epsilon_2^2 + 2\kappa_1\epsilon_0\epsilon_2(\epsilon_0 D_2 + \epsilon_2 D_0)] = 0.$$

Using here the expressions (36) for D_0, D_2 and the self-consistent value of the strength constant (62), we find

$$E^4 - E^2(1-\alpha/2)(\epsilon_0^2 + \epsilon_2^2) + (1-\alpha)\epsilon_0^2\epsilon_2^2 = 0,$$

or

$$\Omega^4 - \Omega^2(2-\alpha)\omega_+^2 + (1-\alpha)\omega_-^4 = 0, \tag{37}$$

where the notation $\omega_+^2 = \omega_x^2 + \omega_z^2$ and $\omega_-^4 = (\omega_x^2 - \omega_z^2)^2$ is introduced. This result coincides with that of [2]. By a trivial rearrangement of the terms in (37) one obtains the useful relation

$$\Omega^2(\Omega^2 - \omega_+^2) = (1-\alpha)(\Omega^2\omega_+^2 - \omega_-^4). \tag{38}$$

Substituting expressions (62) for ω_x^2, ω_z^2 into (38), we reproduce formula (12) for the isovector case

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(2-\alpha)(1+\delta/3) + 4\bar{\omega}^4(1-\alpha)\delta^2 = 0.$$

Taking here $\alpha = 1$ we reproduce formula (8) for the isoscalar case

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(1+\delta/3) = 0.$$

2.2. $B(E2)$-Factors. According to [3], the transition probability for a one-body operator $\hat{F} = \sum_{i=1}^{A}\hat{f}_i$ is calculated with the help of the formula

$$< 0|\hat{F}^\tau|\nu >= \sum_{mi}(\hat{f}_{im}^\tau X_{mi}^{\tau,\nu} + \hat{f}_{mi}^\tau Y_{mi}^{\tau,\nu}). \tag{39}$$

To calculate quadrupole excitations, one has to take $\hat{f}^{\mathrm{p}} = er^2Y_{2\mu} = \tilde{e}D^{\mathrm{p}}$ with $\tilde{e} = e\sqrt{\frac{5}{16\pi}}$. The expressions for X_{mi}^τ, Y_{mi}^τ are given by formulae (26). Combining these results we have

$$< 0|\hat{O}_{21}^{\mathrm{p}}\mathrm{p}|\nu >= 2\tilde{e}K_\nu^{\mathrm{p}}\sum_{mi}|D_{mi}^{\mathrm{p}}|^2\frac{\epsilon_{mi}^{\mathrm{p}}}{E_\nu^2 - (\epsilon_{mi}^{\mathrm{p}})^2} = 2\tilde{e}K_\nu^{\mathrm{p}}S_\nu^{\mathrm{p}} = \tilde{e}C_\nu^{\mathrm{p}}. \tag{40}$$

The constant C_ν^{p} is determined by the normalization condition

$$\delta_{\nu,\nu'} = \sum_{mi,\tau}(X_{mi}^{\tau,\nu*}X_{mi}^{\tau,\nu'} - Y_{mi}^{\tau,\nu*}Y_{mi}^{\tau,\nu'}),$$

that gives

$$\frac{1}{(C^{\rm p}_\nu)^2} = E_\nu \sum_{mi} \left[\frac{|D^{\rm p}_{mi}|^2}{(S^{\rm p}_\nu)^2} \frac{\epsilon^{\rm p}_{mi}}{[E_\nu^2 - (\epsilon^{\rm p}_{mi})^2]^2} + \frac{(C^{\rm n}_\nu)^2}{(C^{\rm p}_\nu)^2} \frac{|D^{\rm n}_{mi}|^2}{(S^{\rm n}_\nu)^2} \frac{\epsilon^{\rm n}_{mi}}{[E_\nu^2 - (\epsilon^{\rm n}_{mi})^2]^2} \right]. \tag{41}$$

The ratio $C^{\rm n}/C^{\rm p}$ is determined by any of equations (29):

$$\frac{C^{\rm n}}{C^{\rm p}} = \frac{1 - 2S^{\rm p}\kappa}{2S^{\rm p}\bar{\kappa}} = \frac{2S^{\rm n}\bar{\kappa}}{1 - 2S^{\rm n}\kappa}. \tag{42}$$

Formula (41) is simplified by approximation No. 4, when $S^{\rm p} = S^{\rm n}, \epsilon^{\rm p}_{mi} = \epsilon^{\rm n}_{mi}$, $D^{\rm p}_{mi} = D^{\rm n}_{mi}$. Applying the second parts of formulae (32), (33) it is easy to find that in this case $C^{\rm n}/C^{\rm p} = \pm 1$. As a result, the final expression for $B(E2)$ value is

$$B(E2)_\nu = 2| < 0|\hat{O}^p_{21}|\nu > |^2 = 2\tilde{e}^2 \left(16 E_\nu \kappa_1^2 \sum_{mi} |D_{mi}|^2 \frac{\epsilon_{mi}}{(E_\nu^2 - \epsilon_{mi}^2)^2} \right)^{-1}. \tag{43}$$

With the help of formulae (36) this expression can be transformed into

$$\begin{aligned} B(E2)_\nu &= \frac{5}{8\pi} \frac{e^2 Q_{00}}{m\bar{\omega}^2 \alpha^2 E_\nu} \left[\frac{\epsilon_0^2}{(E_\nu^2 - \epsilon_0^2)^2} + \frac{\epsilon_2^2}{(E_\nu^2 - \epsilon_2^2)^2} \right]^{-1} \\ &= \frac{5}{8\pi} \frac{e^2 Q_{00}}{m\bar{\omega}^2 \alpha^2 E_\nu} \frac{(E_\nu^2 - \epsilon_0^2)^2 (E_\nu^2 - \epsilon_2^2)^2}{(E_\nu^2 - \epsilon_2^2)^2 \epsilon_0^2 + (E^2 - \epsilon_0^2)^2 \epsilon_2^2} \\ &= \frac{5}{16\pi} \frac{e^2 \hbar Q_{00}}{m\bar{\omega}^2 \Omega_\nu} \frac{(\Omega_\nu^2 \omega_+^2 - \omega_-^4)^2}{\Omega_\nu^4 \omega_+^2 - 2\Omega_\nu^2 \omega_-^4 + \omega_+^2 \omega_-^4}. \end{aligned} \tag{44}$$

At a glance, this expression has nothing common with (23). Nevertheless, it can be shown that they are identical. To this end, we analyze carefully the denominator of the last expression in (44). Summing it with the secular equation (37) (multiplied by ω_+^2), which obviously does not change its value, we find after elementary combinations

$$\begin{aligned} \text{Denom} &= \Omega_\nu^4 \omega_+^2 - 2\Omega_\nu^2 \omega_-^4 + \omega_+^2 \omega_-^4 + \omega_+^2 [\Omega_\nu^4 - \Omega_\nu^2 (2-\alpha)\omega_+^2 + (1-\alpha)\omega_-^4] \\ &= \omega_+^2 \Omega_\nu^2 [2\Omega_\nu^2 - (2-\alpha)\omega_+^2] - \omega_-^4 [2\Omega_\nu^2 - (2-\alpha)\omega_+^2] \\ &= (\Omega_\nu^2 \omega_+^2 - \omega_-^4)[2\Omega_\nu^2 - (2-\alpha)\omega_+^2]. \end{aligned} \tag{45}$$

This result allows us to write the final expression

$$B(E2)_\nu = \frac{5}{16\pi} \frac{e^2 \hbar}{m\bar{\omega}^2} Q_{00} \frac{\Omega_\nu^2 \omega_+^2 - \omega_-^4}{\Omega_\nu [2\Omega_\nu^2 - (2-\alpha)\omega_+^2]}, \tag{46}$$

which coincides with (23) (we remind that $\omega_+^2 = 2\bar{\omega}^2(1+\delta/3)$, $\omega_-^4 = 4\delta^2\bar{\omega}^4$). By the simple transformations this formula is reduced to the result of Hamamoto and Nazarewicz published in [2] without the constant factor $\frac{5}{32\pi}\frac{e^2\hbar}{m\omega_0}Q_{00}^0$.

2.3. ***B*(*M*1)-Factors.** In accordance with formulae (15), (39) and (26), the magnetic transition matrix element is

$$< 0|\hat{O}_{11}^{\rm p}|\nu >= K_\nu^{\rm p}\sum_{mi}\left[\frac{(\hat{O}_{11}^{\rm p})_{im}D_{im}^{\rm p*}}{E_\nu-\epsilon_{mi}^{\rm p}} - \frac{(\hat{O}_{11}^{\rm p})_{mi}D_{mi}^{\rm p*}}{E_\nu+\epsilon_{mi}^{\rm p}}\right]. \qquad (47)$$

As is shown in Appendix B, the matrix element $(O_{11}^{\rm p})_{im}$ is proportional to $D_{im}^{\rm p}$ (formula (74)). So, expression (47) is reduced to

$$\begin{aligned}< 0|\hat{O}_{11}^{\rm p}|\nu > &= -K_\nu^{\rm p}\frac{\tilde{e}\hbar}{2c\sqrt{5}}(\omega_x^2-\omega_z^2)^{\rm p}\sum_{mi}\left[\frac{D_{im}^{\rm p}D_{im}^{\rm p*}}{\epsilon_{im}^{\rm p}(E_\nu-\epsilon_{mi}^{\rm p})} - \frac{D_{mi}^{\rm p}D_{mi}^{\rm p*}}{\epsilon_{mi}^{\rm p}(E_\nu+\epsilon_{mi}^{\rm p})}\right]\\ &= K_\nu^{\rm p}\frac{\tilde{e}\hbar}{c\sqrt{5}}(\omega_x^2-\omega_z^2)^{\rm p}E_\nu\sum_{mi}\frac{|D_{mi}^{\rm p}|^2}{\epsilon_{mi}^{\rm p}[E_\nu^2-(\epsilon_{mi}^{\rm p})^2]}. \qquad (48)\end{aligned}$$

With the help of approximation No. 4 and expressions (36) for D_0, D_2 we find

$$\begin{aligned}< 0|\hat{O}_{11}^{\rm p}|\nu > &= \frac{C_\nu^{\rm p}}{2S_\nu^{\rm p}}\frac{\tilde{e}\hbar}{c\sqrt{5}}(\omega_x^2-\omega_z^2)\frac{Q_{00}}{2m\bar{\omega}^2}(\frac{E_\nu}{E_\nu^2-\epsilon_0^2}+\frac{E_\nu}{E_\nu^2-\epsilon_2^2})\\ &= -2\kappa_1 C_\nu^{\rm p}\frac{\tilde{e}}{c\sqrt{5}}(\omega_x^2-\omega_z^2)\frac{Q_{00}}{m\bar{\omega}^2}\frac{\Omega_\nu(\Omega_\nu^2-\omega_+^2)}{\alpha(\Omega_\nu^2\omega_+^2-\omega_-^4)}\\ &= \frac{C_\nu^{\rm p}}{2}\frac{\tilde{e}}{c\sqrt{5}}(\omega_x^2-\omega_z^2)\frac{1-\alpha}{\Omega_\nu}. \qquad (49)\end{aligned}$$

Relation (38) and the self-consistent value of the strength constant $\kappa_1 = \alpha\kappa_0$ were used at the last step. For the magnetic transition probability we have

$$\begin{aligned}B(M1)_\nu &= 2|< 0|\hat{O}_{11}^{\rm p}|\nu > |^2 =\\ &= 2\frac{(C_\nu^{\rm p})^2}{4}\frac{\tilde{e}^2}{5c^2}\omega_-^4\frac{(1-\alpha)^2}{\Omega_\nu^2} = \frac{\omega_-^4}{20c^2}\frac{(1-\alpha)^2}{\Omega_\nu^2}B(E2). \qquad (50)\end{aligned}$$

This relation between *B*(*M*1) and *B*(*E*2) was also found (to the factor $1/(20c^2)$) by Hamamoto and Nazarewicz [2]. Substituting expression (46) for *B*(*E*2) into (50) we reproduce (with the help of relation (38)) formula (19).

2.4. «Synthetic» Scissors and Spurious State. The nature of collective excitations calculated by the method of Wigner function moments is ascertained quite easily by analyzing the roles of collective variables describing the phenomenon.

The solution of this problem in the RPA approach is not so obvious. That is why the nature of the low-lying states has often been established by considering overlaps of these states with the «pure scissors state» [10,11] or «synthetic state» [2] produced by the action of the scissors operator

$$S_x = \mathcal{N}^{-1}(< I_x^{\mathrm{n}2} > I_x^{\mathrm{p}} - < I_x^{\mathrm{p}2} > I_x^{\mathrm{n}})$$

on the ground state

$$|\mathrm{Syn}>= S_x|0> .$$

Due to axial symmetry one can use the I_y^τ component instead of I_x^τ, or any their linear combination, for example, the variable L_{11}^τ, which is much more convenient for us. The terms $< I_x^{\tau 2} >$ are introduced to ensure the orthogonality of the synthetic scissors to the spurious state $|\mathrm{Sp}>= (I^{\mathrm{n}} + I^{\mathrm{p}})|0>$. However, we do not need these terms because the collective states $|\nu>$ of our model are already orthogonal to $|\mathrm{Sp}>$ (see below); hence, the overlaps $<\mathrm{Syn}|\nu>$ will be free from any admixtures of $|\mathrm{Sp}>$. So, we use the following definitions of the synthetic and spurious states:

$$|\mathrm{Syn}>= \gamma\mathcal{N}^{-1}(L_{11}^{\mathrm{p}} - L_{11}^{\mathrm{n}})|0>= \mathcal{N}^{-1}(\hat{O}_{11}^{\mathrm{p}} - \hat{O}_{11}^{\mathrm{n}})|0>,$$

$$|\mathrm{Sp}>= (\hat{O}_{11}^{\mathrm{p}} + \hat{O}_{11}^{\mathrm{n}})|0>,$$

where $\gamma = -i\dfrac{e}{2mc}\sqrt{\dfrac{3}{2\pi}}$.

Let us demonstrate the orthogonality of the spurious state to all the rest states $|\nu>$. As the first step it is necessary to show that the secular Eq. (30) has the solution $E = 0$. We need the expression for $S^\tau(E = 0) \equiv S^\tau(0)$. In accordance with (28), we have

$$S^\tau(E) = \left[\frac{\epsilon_0 D_0}{E^2 - \epsilon_0^2} + \frac{\epsilon_2 D_2}{E^2 - \epsilon_2^2}\right]^\tau, \quad S^\tau(0) = -\left[\frac{D_0}{\epsilon_0} + \frac{D_2}{\epsilon_2}\right]^\tau .$$

The expressions for D_0^τ, D_2^τ are easily extracted from formulae (71), (72):

$$D_0^\tau = \frac{\hbar}{m}Q_{00}^\tau\left[\frac{1+\frac{4}{3}\delta}{\omega_x} - \frac{1-\frac{2}{3}\delta}{\omega_z}\right]^\tau, \quad D_2^\tau = \frac{\hbar}{m}Q_{00}^\tau\left[\frac{1+\frac{4}{3}\delta}{\omega_x} + \frac{1-\frac{2}{3}\delta}{\omega_z}\right]^\tau . \tag{51}$$

So we find

$$\begin{aligned} S^\tau(0) &= -\frac{\hbar}{m}Q_{00}^\tau\left[\frac{1+\frac{4}{3}\delta}{\omega_x}(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_0}) + \frac{1-\frac{2}{3}\delta}{\omega_z}(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_0})\right]^\tau \\ &= -\frac{\hbar^2}{m}\frac{4\delta^\tau Q_{00}^\tau}{\epsilon_2^\tau\epsilon_0^\tau} = -\frac{1}{m}\frac{3Q_{20}^\tau}{(\omega_x^2 - \omega_z^2)^\tau}, \end{aligned} \tag{52}$$

where, in accordance with (73),

$$(\omega_x^2-\omega_z^2)^{\rm p}=-\frac{6}{m}(\kappa Q_{20}^{\rm p}+\bar\kappa Q_{20}^{\rm n}),\quad (\omega_x^2-\omega_z^2)^{\rm n}=-\frac{6}{m}(\kappa Q_{20}^{\rm n}+\bar\kappa Q_{20}^{\rm p}).\quad (53)$$

Finally, we get

$$2S^{\rm p}(0)=\frac{Q_{20}^{\rm p}}{\kappa Q_{20}^{\rm p}+\bar\kappa Q_{20}^{\rm n}},\quad 1-2S^{\rm p}(0)\kappa=\frac{\bar\kappa Q_{20}^{\rm n}}{\kappa Q_{20}^{\rm p}+\bar\kappa Q_{20}^{\rm n}},$$

$$2S^{\rm n}(0)=\frac{Q_{20}^{\rm n}}{\kappa Q_{20}^{\rm n}+\bar\kappa Q_{20}^{\rm p}},\quad 1-2S^{\rm n}(0)\kappa=\frac{\bar\kappa Q_{20}^{\rm p}}{\kappa Q_{20}^{\rm n}+\bar\kappa Q_{20}^{\rm p}}.$$

It is easy to see that substituting these expressions into (30) we obtain the identity; therefore, the secular equation has the zero solution.

At the second step it is necessary to calculate the overlap $<{\rm Sp}|\nu>$. Summing (47) with an analogous expression for neutrons, we get

$$\begin{aligned}<{\rm Sp}|\nu> &= \frac{\tilde e\hbar}{c\sqrt5}E_\nu\sum_\tau K_\nu^\tau(\omega_x^2-\omega_z^2)^\tau\sum_{mi}\frac{|D_{mi}^\tau|^2}{\epsilon_{mi}^\tau(E_\nu^2-\epsilon_{mi}^2)^\tau}\\ &= \frac{\tilde e\hbar}{c\sqrt5}E_\nu\sum_\tau K_\nu^\tau(\omega_x^2-\omega_z^2)^\tau\sum_{mi}\frac{|D_{mi}^\tau|^2\epsilon_{mi}^\tau}{(\epsilon_{mi}^2)^\tau(E_\nu^2-\epsilon_{mi}^2)^\tau}.\end{aligned}\quad (54)$$

Applying the algebraical identity

$$\frac{1}{\epsilon^2(E^2-\epsilon^2)}=\frac{1}{E^2}(\frac{1}{\epsilon^2}+\frac{1}{E^2-\epsilon^2})$$

and remembering the definition (28) of S^τ, we can rewrite (54) as

$$<{\rm Sp}|\nu>=\frac{\tilde e\hbar}{c\sqrt5E_\nu}\sum_\tau K_\nu^\tau(\omega_x^2-\omega_z^2)^\tau(S^\tau-S^\tau(0))$$

$$=\frac{\tilde e\hbar}{c\sqrt5}\frac{K_\nu^{\rm p}}{E_\nu}\left[(\omega_x^2-\omega_z^2)^{\rm p}(S^{\rm p}-S^{\rm p}(0))+(\omega_x^2-\omega_z^2)^{\rm n}(S^{\rm n}-S^{\rm n}(0))\frac{K_\nu^{\rm n}}{K_\nu^{\rm p}}\right].$$

In accordance with (27) and (42),

$$\frac{K_\nu^{\rm n}}{K_\nu^{\rm p}}=\frac{1-2S^{\rm p}\kappa}{2S^{\rm n}\bar\kappa}.\quad (55)$$

Noting now (see formula (52)) that $(\omega_x^2-\omega_z^2)^\tau S^\tau(0)=-\frac{3}{m}Q_{20}^\tau$ and taking into account relations (53), we find

$$\begin{aligned}
< \mathrm{Sp}|\nu > &= \beta\,\{[(\kappa Q_2^{\mathrm{p}} + \bar\kappa Q_2^{\mathrm{n}})2S^{\mathrm{p}} - Q_2^{\mathrm{p}}] \\
&\quad +[(\kappa Q_2^{\mathrm{n}} + \bar\kappa Q_2^{\mathrm{p}})2S^{\mathrm{n}} - Q_2^{\mathrm{n}}]\frac{1-2S^{\mathrm{p}}\kappa}{2S^{\mathrm{n}}\bar\kappa}\Big\} \\
&= \beta\,\{[(2S^{\mathrm{p}}\kappa - 1)Q_2^{\mathrm{p}} + 2S^{\mathrm{p}}\bar\kappa Q_2^{\mathrm{n}}] \\
&\quad +[(2S^{\mathrm{n}}\kappa - 1)Q_2^{\mathrm{n}} + 2S^{\mathrm{n}}\bar\kappa Q_2^{\mathrm{p}})]\frac{1-2S^{\mathrm{p}}\kappa}{2S^{\mathrm{n}}\bar\kappa}\Big\} \\
&= \beta\left\{2S^{\mathrm{p}}\bar\kappa Q_2^{\mathrm{n}} + (2S^{\mathrm{n}}\kappa - 1)Q_2^{\mathrm{n}}\frac{1-2S^{\mathrm{p}}\kappa}{2S^{\mathrm{n}}\bar\kappa}\right\} \\
&= \beta\frac{Q_2^{\mathrm{n}}}{2S^{\mathrm{n}}\bar\kappa}\,\{2S^{\mathrm{n}}\bar\kappa 2S^{\mathrm{p}}\bar\kappa - (1-2S^{\mathrm{n}}\kappa)(1-2S^{\mathrm{p}}\kappa)\} = 0, \quad (56)
\end{aligned}$$

where $\beta = -\frac{3}{m}\frac{\tilde e\hbar}{c\sqrt{5}}\frac{K_\nu^{\mathrm{p}}}{E_\nu}$ and $Q_2 \equiv Q_{20}$. The expression in the last curly brackets coincides obviously with the secular Eq. (30) that proves the orthogonality of the spurious state to all physical states of the considered model. So we can conclude that strictly speaking this is not a spurious state, but one of the exact eigenstates of the model corresponding to the integral of motion $I^{\mathrm{n}} + I^{\mathrm{p}}$. In other words [3]: «In fact these excitations are not really spurious, but they represent a different type of motion which has to be treated separately». The same conclusion was made by N. Lo Iudice [12] who solved this problem approximately with the help of several assumptions (a small deformation limit, for example).

The problem of the «spurious» state being solved, the calculation of the overlaps $< \mathrm{Syn}|\nu >$ becomes trivial. Really, we have shown that $< 0|\hat O_{11}^{\mathrm{n}} + \hat O_{11}^{\mathrm{p}}|\nu >= 0$. That means that $< 0|\hat O_{11}^{\mathrm{n}}|\nu >= - < 0|\hat O_{11}^{\mathrm{p}}|\nu >$; hence, $< \mathrm{Syn}|\nu >= \mathcal{N}^{-1} < 0|\hat O_{11}^{\mathrm{p}} - \hat O_{11}^{\mathrm{n}}|\nu >= 2\mathcal{N}^{-1} < 0|\hat O_{11}^{\mathrm{p}}|\nu >$ and

$$U^2 \equiv |< \mathrm{Syn}|\nu >|^2 = 2\mathcal{N}^{-2}B(M1)_\nu.$$

The nontrivial part of the problem is the calculation of the normalization factor $\mathcal{N}$. It is important not to forget about the time dependence of the synthetic state which should be determined by the external field

$$|\mathrm{Syn}(t) >= \mathcal{N}^{-1}[(\hat O_{11}^{\mathrm{p}} - \hat O_{11}^{\mathrm{n}})\,\exp^{-i\Omega t} + (\hat O_{11}^{\mathrm{p}} - \hat O_{11}^{\mathrm{n}})^\dagger \exp^{i\Omega t}]|0 > .$$

Then we have

$$\begin{aligned}
\mathcal{N}^2 &= \; < \mathrm{Syn}(t)|\mathrm{Syn}(t) > \\
&= 2 < 0|(\hat{O}^{\mathrm{p}}_{11} - \hat{O}^{\mathrm{n}}_{11})^\dagger (\hat{O}^{\mathrm{p}}_{11} - \hat{O}^{\mathrm{n}}_{11})|0 > \\
&= 2\sum_{\mathrm{ph}} < 0|(\hat{O}^{\mathrm{p}}_{11} - \hat{O}^{\mathrm{n}}_{11})^\dagger |\mathrm{ph} >< \mathrm{ph}|(\hat{O}^{\mathrm{p}}_{11} - \hat{O}^{\mathrm{n}}_{11})|0 > \\
&= 2\sum_{\mathrm{ph}} | < \mathrm{ph}|(\hat{O}^{\mathrm{p}}_{11} - \hat{O}^{\mathrm{n}}_{11})|0 > |^2 = 2\sum_{\tau,\mathrm{ph}} | < \mathrm{ph}|\hat{O}^{\tau}_{11}|0 > |^2. \quad (57)
\end{aligned}$$

With the help of relation (74) we find

$$\begin{aligned}
\mathcal{N}^2 &= \frac{2}{5}(\frac{e\hbar}{2c})^2 \sum_{\tau,\mathrm{ph}} \left(\omega_-^4 \frac{| < \mathrm{ph}|r^2\, Y_{21}|0 > |^2}{\epsilon^2_{\mathrm{ph}}}\right)^\tau \\
&= \frac{1}{8\pi}(\frac{e\hbar}{2c})^2 \sum_{\tau} (\omega_-^4)^\tau \left(\frac{D_0}{\epsilon_0^2} + \frac{D_2}{\epsilon_2^2}\right)^\tau . \quad (58)
\end{aligned}$$

Expressions for D_0^τ, D_2^τ, ω_x^τ, ω_z^τ are given by formulae (51), (73). To get a definite number, it is necessary to make some assumption concerning the relation between neutron and proton equilibrium characteristics. As usual, we apply approximation No. 4, i.e., suppose $Q_{00}^{\mathrm{n}} = Q_{00}^{\mathrm{p}}$, $Q_{20}^{\mathrm{n}} = Q_{20}^{\mathrm{p}}$. It is easy to check that in this case formulae for $\omega_{x,z}^\tau$ are reduced to the ones for the isoscalar case, namely (62), and $D_0^\tau = D_0/2$, $D_2^\tau = D_2/2$, where D_0 and D_2 are given by (36). So we get

$$\mathcal{N}^2 = \frac{\omega_-^4}{8\pi}(\frac{e\hbar}{2c})^2 \frac{Q_{00}}{m\bar{\omega}^2}\left(\frac{1}{\epsilon_0} + \frac{1}{\epsilon_2}\right) = \mu_N^2 \frac{\delta}{2\pi}\frac{m\omega_x}{\hbar} Q_{00}. \quad (59)$$

The estimation of the overlap for ^{156}Gd with $\delta = 0.27$ gives $\mathcal{N}^2 = 34.72\mu_N^2$ and $U^2 = 0.53$, that is two times larger than the result of [10] obtained in QRPA calculations with the Skyrme forces. The disagreement can naturally be attributed to the difference in forces and especially to the lack of pair correlations in our approach (see the next section, nevertheless). In a small deformation limit $U^2 = \frac{1}{2}\sqrt{\frac{3}{2}} \approx 0.6$.

3. SUPERDEFORMATION

A certain drawback of our approach is that, so far, we have not included the superfluidity into our description. Nevertheless, our formulae (14), (19) can be successfully used for the description of superdeformed nuclei where the pairing is very weak [2, 9]. For example, applying them to the superdeformed nucleus ^{152}Dy ($\delta \simeq 0.6$, $\hbar\omega_0 = 41/A^{1/3}$MeV), we get

$$E_{\rm iv} = 20.8\,{\rm MeV}, \qquad B(M1)_{\rm iv} = 15.9\,\mu_N^2$$

for the isovector GQR and

$$E_{\rm sc} = 4.7\,{\rm MeV}, \qquad B(M1)_{\rm sc} = 20.0\,\mu_N^2$$

for the scissors mode. There are not so many results of other calculations to compare with. As a matter of fact, there are only two papers considering this problem.

The phenomenological TRM model [9] predicts

$$E_{\rm iv} \simeq 26\,{\rm MeV}, \quad B(M1)_{\rm iv} \simeq 26\,\mu_N^2, \quad E_{\rm sc} \simeq 6.1\,{\rm MeV}, \quad B(M1)_{\rm sc} \simeq 22\,\mu_N^2.$$

The only existing microscopic calculation [2] in the framework of QRPA with separable forces gives

$$E_{\rm iv} \simeq 28\,{\rm MeV}, \quad B(M1)_{\rm iv} \simeq 37\,\mu_N^2, \quad E_{\rm sc} \simeq 5{-}6\,{\rm MeV}, \quad B(M1)_{1^+} \simeq 23\,\mu_N^2.$$

Here $B(M1)_{1^+}$ denotes the total $M1$ orbital strength carried by the calculated $K^\pi = 1^+$ QRPA excitations modes in the energy region below 20 MeV.

It is easy to see that in the case of IVGQR one can speak, at least, about qualitative agreement. Our results for $E_{\rm sc}$ and $B(M1)_{\rm sc}$ are in good agreement with that of phenomenological model and with $E_{\rm sc}$ and $B(M1)_{1^+}$ of Hamamoto and Nazarewicz.

It is possible to extract from the histogram of [2] the value of the overlap of calculated low-lying 1^+ excitations with the synthetic scissors state: $| < {\rm Syn}|1^+ > |^2 \approx 0.4$. The result of our calculation $U^2 = 0.43$ agrees with it very well. So the natural conclusion of this section is that the correct treatment of pair correlations is more important for a reasonable description of the scissors mode than the thorough choice of an interaction.

CONCLUSION

The properties of collective excitations (the scissors mode, isovector and isoscalar giant quadrupole resonances) of the harmonic oscillator Hamiltonian with the quadrupole-quadrupole residual interaction (HO+QQ) were studied by two methods: WFM and RPA. We have found that both methods give the same analytical expressions for energies and transition probabilities of all considered excitations. Does it mean that WFM and RPA are identical approaches? Certainly, not. First of all, we have the experience of previous WFM calculations [5] with realistic forces which show that, for example, we reproduce only centroids of giant resonances whereas RPA describes their fine structure. Secondly, we suppose that one can find such nuclear characteristics that will be described differently by two approaches even in this simple model. Thirdly, to establish completely (and finally) the relation between the two approaches, it is necessary to analyze the equations of motion for multipole moments from the point of view of RPA. It will be done in the subsequent publication.

There is no sense to speak about advantages or disadvantages of one of the two discussed methods — they are complementary. Of course, RPA gives complete, exhaustive information concerning the microscopic (particle-hole) structure of collective excitations. However, sometimes considerable additional efforts are required to understand their physical nature. On the contrary, WFM method gives information only on the physical nature of excitations and does not touch their microscopic structure. Our results serve as a very good illustration of this situation. Really, what do we know about the scissors mode and IVGQR from each method? RPA says that the scissors mode is mostly created by $\Delta N = 0$ particle-hole excitations with a small admixture of $\Delta N = 2$ ph excitations and vice versa for IVGQR. And that is all! One can even not suspect about the key role of the relative angular momentum in the creation of the scissors mode. On the other hand, the WFM method says that the scissors mode appears due to oscillations of the relative angular momentum with a small admixture of the quadrupole moment oscillations and vice versa for IVGQR. Further, it informs us about the extremely important role of the Fermi surface deformation in the formation of the scissors mode.

Two new mathematical results are obtained for the HO+QQ model. We have proved exactly, without any approximations, the orthogonality of the «spurious» state to all physical states. In this sense, we have generalized the result of Lo Iudice [12] derived in a small deformation approximation. The analytical expressions are derived for the normalization factor of the synthetic scissors state and overlaps of this state with eigenstates of the model.

APPENDIX A

It is known that the deformed harmonic oscillator Hamiltonian can be obtained in a Hartree approximation «by making the assumption that the isoscalar part of the Q-Q force builds the one-body container well» [13]. In our case it is obtained quite easily by summing the expressions for V^{p} and V^{n} (formula (3)):

$$V(\mathbf{r},t) = \frac{1}{2}(V^{\mathrm{p}}(\mathbf{r},t) + V^{\mathrm{n}}(\mathbf{r},t)) = \frac{1}{2}m\,\omega^2 r^2 + \kappa_0 \sum_{\mu=-2}^{2} (-1)^{\mu} Q_{2\mu}(t) q_{2-\mu}(\mathbf{r}). \tag{60}$$

In the state of equilibrium (i.e., in the absence of an external field) $Q_{2\pm1} = Q_{2\pm2} = 0$. Using the definition [14] $Q_{20} = Q_{00}\frac{4}{3}\delta$ and the formula $q_{20} = 2z^2 - x^2 - y^2$, we obtain the potential of the anisotropic harmonic oscillator

$$V(\mathbf{r}) = \frac{m}{2}[\omega_x^2(x^2+y^2) + \omega_z^2 z^2]$$

with oscillator frequencies

$$\omega_x^2 = \omega_y^2 = \omega^2(1+\sigma\delta), \quad \omega_z^2 = \omega^2(1-2\sigma\delta),$$

where $\sigma = -\kappa_0\dfrac{8Q_{00}}{3m\omega^2}$. The definition of the deformation parameter δ must be reproduced by the harmonic oscillator wave functions, which allows one to fix the value of σ. We have

$$Q_{00} = \frac{\hbar}{m}(\frac{\Sigma_x}{\omega_x} + \frac{\Sigma_y}{\omega_y} + \frac{\Sigma_z}{\omega_z}), \quad Q_{20} = 2\frac{\hbar}{m}(\frac{\Sigma_z}{\omega_z} - \frac{\Sigma_x}{\omega_x}),$$

where $\Sigma_x = \Sigma_{i=1}^{A}(n_x + \dfrac{1}{2})_i$, and n_x is the oscillator quantum number. Using the self-consistency condition [14]

$$\Sigma_x\omega_x = \Sigma_y\omega_y = \Sigma_z\omega_z = \Sigma_0\omega_0,$$

where Σ_0 and ω_0 are defined in the spherical case, we get

$$\frac{Q_{20}}{Q_{00}} = 2\frac{\omega_x^2 - \omega_z^2}{\omega_x^2 + 2\omega_z^2} = \frac{2\sigma\delta}{1-\sigma\delta} = \frac{4}{3}\delta.$$

Solving the last equation with respect to σ, we find

$$\sigma = \frac{2}{3+2\delta}. \tag{61}$$

Therefore, the oscillator frequences and the strength constant can be written as

$$\omega_x^2 = \omega_y^2 = \bar{\omega}^2(1+\frac{4}{3}\delta), \quad \omega_z^2 = \bar{\omega}^2(1-\frac{2}{3}\delta), \quad \kappa_0 = -\frac{m\bar{\omega}^2}{4Q_{00}} \tag{62}$$

with $\bar{\omega}^2 = \omega^2/(1+\frac{2}{3}\delta)$. The condition for volume conservation $\omega_x\omega_y\omega_z = \mathrm{const} = \omega_0^3$ makes ω δ-dependent

$$\omega^2 = \omega_0^2\frac{1+\frac{2}{3}\delta}{(1+\frac{4}{3}\delta)^{2/3}(1-\frac{2}{3}\delta)^{1/3}}.$$

So the final expressions for oscillator frequences are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1+\frac{4}{3}\delta}{1-\frac{2}{3}\delta}\right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1-\frac{2}{3}\delta}{1+\frac{4}{3}\delta}\right)^{2/3}.$$

It is easy to see that they correspond to the case when the deformed density $\rho(\mathbf{r})$ is obtained from the spherical density $\rho_0(r)$ by the scale transformation [8]

$$(x, y, z) \to (xe^{\alpha/2}, ye^{\alpha/2}, ze^{-\alpha})$$

with

$$e^{\alpha} = \left(\frac{1+\frac{4}{3}\delta}{1-\frac{2}{3}\delta}\right)^{1/3}, \quad \delta = \frac{3}{2}\frac{e^{3\alpha}-1}{e^{3\alpha}+2}, \tag{63}$$

which conserves the volume and does not destroy the self-consistency, because the density and potential are transformed in the same way.

It is necessary to note that Q_{00} also depends on δ

$$Q_{00} = \frac{\hbar}{m}(\frac{\Sigma_x}{\omega_x} + \frac{\Sigma_y}{\omega_y} + \frac{\Sigma_z}{\omega_z}) = \frac{\hbar}{m}\Sigma_0\omega_0(\frac{2}{\omega_x^2} + \frac{1}{\omega_z^2}) = Q_{00}^0 \frac{1}{(1+\frac{4}{3}\delta)^{1/3}(1-\frac{2}{3}\delta)^{2/3}},$$

where $Q_{00}^0 = A\frac{3}{5}R^2$, $R = r_0 A^{1/3}$. As a result, the final expression for the strength constant becomes

$$\kappa_0 = -\frac{m\omega_0^2}{4Q_{00}^0}\left(\frac{1-\frac{2}{3}\delta}{1+\frac{4}{3}\delta}\right)^{1/3} = -\frac{m\omega_0^2}{4Q_{00}^0}e^{-\alpha},$$

that coincides with the respective result of [8].

APPENDIX B

To calculate the sums $D_0 = \sum\limits_{mi(\Delta N=0)} |D_{mi}|^2$ and $D_2 = \sum\limits_{mi(\Delta N=2)} |D_{mi}|^2$ we employ the sum-rule techniques of Suzuki and Rowe [8]. The well-known harmonic oscillator relations

$$x\psi_{n_x} = \sqrt{\frac{\hbar}{2m\omega_x}}(\sqrt{n_x}\psi_{n_x-1} + \sqrt{n_x+1}\psi_{n_x+1}),$$
$$\hat{p}_x\psi_{n_x} = -i\sqrt{\frac{m\hbar\omega_x}{2}}(\sqrt{n_x}\psi_{n_x-1} - \sqrt{n_x+1}\psi_{n_x+1}) \tag{64}$$

allow us to write

$$
\begin{aligned}
&xz\psi_{n_x}\psi_{n_z} = \\
&\quad = \frac{\hbar}{2m\sqrt{\omega_x\omega_z}}(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} + \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1} \\
&\qquad\qquad + \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} + \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\hat{p}_x\hat{p}_z}{m^2\omega_x\omega_z}\psi_{n_x}\psi_{n_z} = \\
&\quad = -\frac{\hbar}{2m\sqrt{\omega_x\omega_z}}(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} + \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1} \\
&\qquad\qquad - \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}).
\end{aligned}
$$

These formulae demonstrate in an obvious way that the operators

$$
P_0 = \frac{1}{2}(zx + \frac{1}{m^2\omega_x\omega_z}\hat{p}_x\hat{p}_z) \quad \text{and} \quad P_2 = \frac{1}{2}(zx - \frac{1}{m^2\omega_x\omega_z}\hat{p}_x\hat{p}_z)
$$

contribute only to the excitation of the $\Delta N = 0$ and $\Delta N = 2$ states, respectively. Following [8], we express the zx component of $r^2Y_{21} = \sqrt{\frac{5}{16\pi}}D$ $= -\sqrt{\frac{15}{8\pi}}z(x+iy)$ as

$$
zx = P_0 + P_2.
$$

Hence, we have

$$
\begin{aligned}
\epsilon_0 \sum_{mi(\Delta N=0)} |<0|\sum_{s=1}^{A} z_s x_s|mi>|^2 &= \epsilon_0 \sum_{mi} |<0|\sum_{s=1}^{A} P_0(s)|mi>|^2 \\
= \frac{1}{2} <0|[\sum_{s=1}^{A} P_0(s), [H, \sum_{s=1}^{A} P_0(s)]]|0>, &
\end{aligned} \tag{65}
$$

where $\epsilon_0 = \hbar(\omega_x - \omega_z)$. The above commutator is easily evaluated for the Hamiltonian (60), as

$$
\begin{aligned}
&<0|[\sum_{s=1}^{A} P_0(s), [H, \sum_{s=1}^{A} P_0(s)]]|0> = \\
&\qquad = \frac{\hbar}{2m}\epsilon_0 \left(\frac{<0|\sum_{s=1}^{A} z_s^2|0>}{\omega_x} - \frac{<0|\sum_{s=1}^{A} x_s^2|0>}{\omega_z} \right).
\end{aligned}
$$

Taking into account the axial symmetry and using the definitions

$$Q_{00} =< 0|\sum_{s=1}^{A}(2x_s^2 + z_s^2)|0 >,\ Q_{20} = 2 < 0|\sum_{s=1}^{A}(z_s^2 - x_s^2)|0 >,\ Q_{20} = Q_{00}\frac{4}{3}\delta,$$

we transform this expression to

$$< 0|[\sum_{s=1}^{A} P_0(s), [H, \sum_{s=1}^{A} P_0(s)]]|0 >= \frac{\hbar}{6m}\epsilon_0 Q_{00}\left(\frac{1+\frac{4}{3}\delta}{\omega_x} - \frac{1-\frac{2}{3}\delta}{\omega_z}\right). \quad (66)$$

With the help of the self-consistent expressions for ω_x, ω_z (62) one comes to the following result:

$$< 0|[\sum_{s=1}^{A} P_0(s), [H, \sum_{s=1}^{A} P_0(s)]]|0 >= \frac{Q_{00}}{6m}\frac{\epsilon_0^2}{\bar{\omega}^2} = \frac{\hbar^2}{6m}Q_{00}^0\left(\frac{\omega_0}{\omega_z} - \frac{\omega_0}{\omega_x}\right)^2. \quad (67)$$

By using the fact that the matrix elements for the zy component of $r^2 Y_{21}$ are identical to those for the zx component, because of axial symmetry, we finally obtain

$$\epsilon_0 \sum_{mi(\Delta N=0)} |< 0|\sum_{s=1}^{A} r_s^2 Y_{21}|mi > |^2 = \frac{5}{16\pi}\frac{Q_{00}}{m\bar{\omega}^2}\epsilon_0^2 =$$

$$= \frac{5}{16\pi}\frac{Q_{00}^0}{m}\frac{\epsilon_0^2}{\omega_0^2}\left(\frac{1+\frac{4}{3}\delta}{1-\frac{2}{3}\delta}\right)^{1/3}. \quad (68)$$

By calculating a double commutator for the P_2 operator, we find

$$\epsilon_2 \sum_{mi(\Delta N=2)} |< 0|\sum_{s=1}^{A} r_s^2 Y_{21}|mi > |^2 = \frac{5}{16\pi}\frac{Q_{00}}{m\bar{\omega}^2}\epsilon_2^2 =$$

$$= \frac{5}{16\pi}\frac{Q_{00}^0}{m}\frac{\epsilon_2^2}{\omega_0^2}\left(\frac{1+\frac{4}{3}\delta}{1-\frac{2}{3}\delta}\right)^{1/3}, \quad (69)$$

where $\epsilon_2 = \hbar(\omega_x + \omega_z)$.

We need also the sums D_0^τ and D_2^τ calculated separately for neutron and proton systems with the mean fields $V^{\rm n}$ and $V^{\rm p}$, respectively. The necessary formulae are easily derivable from the already obtained results. There are no any reasons to require the fulfillment of the self-consistency conditions for neutrons and protons separately, so one has to use formula (66). The trivial change of notation gives

$$< 0|[\sum_{s=1}^{Z} P_0(s), [H^{\rm p}, \sum_{s=1}^{Z} P_0(s)]]|0 >= \frac{\hbar}{6m}\epsilon_0^{\rm p} Q_{00}^{\rm p}\left(\frac{1+\frac{4}{3}\delta^{\rm p}}{\omega_x^{\rm p}} - \frac{1-\frac{2}{3}\delta^{\rm p}}{\omega_z^{\rm p}}\right), \quad (70)$$

$$\epsilon_0^{\rm p}\sum_{mi(\Delta N=0)}|<0|\sum_{s=1}^{Z}r_s^2Y_{21}|mi>|^2=\frac{5}{16\pi}\frac{\hbar}{m}\epsilon_0^{\rm p}Q_{00}^{\rm p}\left(\frac{1+\frac{4}{3}\delta^{\rm p}}{\omega_x^{\rm p}}-\frac{1-\frac{2}{3}\delta^{\rm p}}{\omega_z^{\rm p}}\right), \tag{71}$$

$$\epsilon_2^{\rm p}\sum_{mi(\Delta N=2)}|<0|\sum_{s=1}^{Z}r_s^2Y_{21}|mi>|^2=\frac{5}{16\pi}\frac{\hbar}{m}\epsilon_2^{\rm p}Q_{00}^{\rm p}\left(\frac{1+\frac{4}{3}\delta^{\rm p}}{\omega_x^{\rm p}}+\frac{1-\frac{2}{3}\delta^{\rm p}}{\omega_z^{\rm p}}\right). \tag{72}$$

The nontrivial information is contained in oscillator frequences of the mean fields $V^{\rm p}$ and $V^{\rm n}$ (formula (3))

$$\begin{aligned}(\omega_x^{\rm p})^2&=\omega^2[1-\frac{2}{m\omega^2}(\kappa Q_{20}^{\rm p}+\bar\kappa Q_{20}^{\rm n})], \quad (\omega_z^{\rm p})^2=\omega^2[1+\frac{4}{m\omega^2}(\kappa Q_{20}^{\rm p}+\bar\kappa Q_{20}^{\rm n})],\\ (\omega_x^{\rm n})^2&=\omega^2[1-\frac{2}{m\omega^2}(\kappa Q_{20}^{\rm n}+\bar\kappa Q_{20}^{\rm p})], \quad (\omega_z^{\rm n})^2=\omega^2[1+\frac{4}{m\omega^2}(\kappa Q_{20}^{\rm n}+\bar\kappa Q_{20}^{\rm p})].\end{aligned} \tag{73}$$

The above-written formulae can be used also to calculate the analogous sums for various components of the angular momentum. Really, by definition $\hat I_1=y\hat p_z-z\hat p_y, \quad \hat I_2=z\hat p_x-x\hat p_z$. In accordance with (64), we have

$$\begin{aligned}&x\hat p_z\psi_{n_x}\psi_{n_z}=\\ &=-i\frac{\hbar}{2}\sqrt{\frac{\omega_z}{\omega_x}}(\sqrt{n_xn_z}\psi_{n_x-1}\psi_{n_z-1}-\sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1}+\\ &\qquad+\sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1}-\sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}).\end{aligned}$$

Therefore,

$$\begin{aligned}&\hat I_2\psi_{n_x}\psi_{n_z}=\\ &=i\frac{\hbar}{2}(\sqrt{\frac{\omega_z}{\omega_x}}-\sqrt{\frac{\omega_x}{\omega_z}})(\sqrt{n_xn_z}\psi_{n_x-1}\psi_{n_z-1}-\sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1})+\\ &+i\frac{\hbar}{2}(\sqrt{\frac{\omega_z}{\omega_x}}+\sqrt{\frac{\omega_x}{\omega_z}})(\sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1}-\sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}).\end{aligned}$$

Having these formulae, one derives the following expressions for matrix elements coupling the ground state with $\Delta N=2$ and $\Delta N=0$ excitations:

$$<n_x+1,n_z+1|\hat I_2|0>=i\frac{\hbar}{2}\frac{(\omega_x^2-\omega_z^2)}{\omega_x+\omega_z}\sqrt{\frac{(n_x+1)(n_z+1)}{\omega_x\omega_z}},$$

$$< n_x+1, n_z+1|xz|0 >= \frac{\hbar}{2m}\sqrt{\frac{(n_x+1)(n_z+1)}{\omega_x\omega_z}},$$

$$< n_x+1, n_z-1|\hat{I}_2|0 >= i\frac{\hbar}{2}\frac{(\omega_x^2-\omega_z^2)}{\omega_x-\omega_z}\sqrt{\frac{(n_x+1)n_z}{\omega_x\omega_z}},$$

$$< n_x+1, n_z-1|xz|0 >= \frac{\hbar}{2m}\sqrt{\frac{(n_x+1)n_z}{\omega_x\omega_z}}.$$

It is easy to see that

$$< n_x+1, n_z+1|\hat{I}_2|0 >= im\frac{(\omega_x^2-\omega_z^2)}{\omega_x+\omega_z} < n_x+1, n_z+1|xz|0 >,$$

$$< n_x+1, n_z-1|\hat{I}_2|0 >= im\frac{(\omega_x^2-\omega_z^2)}{\omega_x-\omega_z} < n_x+1, n_z-1|xz|0 > .$$

Due to the degeneracy of the model all particle-hole excitations with $\Delta N = 2$ have the same energy ϵ_2, and all particle-hole excitations with $\Delta N = 0$ have the energy ϵ_0. This fact allows one to join the last two formulae into one general expression

$$< \mathrm{ph}|\hat{I}_2|0 >= i\hbar m\frac{(\omega_x^2-\omega_z^2)}{\epsilon_{\mathrm{ph}}} < \mathrm{ph}|xz|0 > .$$

Taking into account the axial symmetry we can write the analogous formula for $\hat{I}_1$:

$$< \mathrm{ph}|\hat{I}_1|0 >= -i\hbar m\frac{(\omega_x^2-\omega_z^2)}{\epsilon_{\mathrm{ph}}} < \mathrm{ph}|yz|0 > .$$

The magnetic transition operator $\hat{O}_{1\pm1}$ is proportional (15) to the angular momentum: $\hat{O}_{1\pm1} = -\frac{ie}{4mc}\sqrt{\frac{3}{2\pi}}\sum_{s=1}^{z}(\hat{I}_2 \mp i\hat{I}_1)_s$. Therefore, we can write

$$< \mathrm{ph}|\hat{O}_{1\pm1}|0 >= -\frac{e\hbar}{2c\sqrt{5}}\frac{(\omega_x^2-\omega_z^2)}{\epsilon_{\mathrm{ph}}} < \mathrm{ph}|r^2\, Y_{2\pm1}|0 > . \quad (74)$$

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