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V. K. Ignatovich

ON UNCERTAINTY RELATIONS IN QUANTUM MECHANICS

Игнатович В. К. E4-2004-72

О соотношениях неопределенности в квантовой механике

Показано, что соотношения неопределенности (СН), являясь общим свойством всех функций, не играют никакой особенной роли в квантовой механике (КМ) и не запрещают частице иметь одновременно абсолютно точно определенные положение и импульс. На примере исследования траектории классической заряженной частицы показано, что интерференция на двух щелях существует и в классической механике. Обсуждается нелинейная система классических уравнений для траектории частицы и ее поля, которые могли бы заменить уравнение Шредингера. Показано, что такой подход не имеет ничего общего с механикой Бома.

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Ignatovich V. K. E4-2004-72

On Uncertainty Relations in Quantum Mechanics

Uncertainty relations (UR) are shown to have nothing specific for quantum mechanics (QM), being the general property valid for the arbitrary function. A wave function of a particle having a precisely defined position and momentum in QM simultaneously is demonstrated. Interference on two slits in a screen is shown to exist in classical mechanics. A nonlinear classical system of equations replacing the QM Schrödinger equation is suggested. This approach is shown to have nothing in common with the Bohm mechanics.

The investigation has been performed at the Frank Laboratory of Neutron Physics, JINR.

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INTRODUCTION

A widely spread belief dominates in physical community that uncertainty relations (UR) are the cornerstone of quantum mechanics (QM). We want to show that they have nothing specific for QM. They are valid in QM, as in any field where we meet functions. The real cornerstone of QM is the Schrödinger equation, which was a great guess, like Maxwellian ones. In Sec. 1 we remind to readers how UR are proven for an arbitrary function. It follows from this proof that UR have nothing specific for QM. In Sec. 2 we show that such notions as position and momentum are a matter of definition for an extended object like a wave function, and demonstrate that nonsingular de Broglie wave packet describes a particle, which simultaneously has precisely defined momentum and position. In Sec. 3 we show that interference is not an exclusive property of a wave mechanics. It takes place also in classical mechanics. In Sec. 4 we discuss whether QM equation can be replaced with classical equations. We suppose that it is possible to define a system of equations for trajectory and field of the particle, propose for mathematicians to solve an electrodynamical problem for an electron moving through a slit in a conducting screen, and show that such system of equations is not contained in the so-called «Bohm mechanics». In conclusion we repeat our main points.

1. WHAT ARE UR

UR is a mathematical theorem which relates ranges of a function and its Fourier image. This theorem is valid in all branches of physics and mathematics dealing with extended objects described with functions. Let us remind this well-known theorem.

Let us take an arbitrary function $f(x)$ of finite range, and its Fourier image

$$F(p) = \int_{-\infty}^{+\infty} f(x) \exp(ipx) dx, \tag{1}$$

and define

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(p)|^2 dp \equiv N < \infty, \tag{2}$$

$$x_0 = \frac{1}{N}\int x|f(x)|^2 dx, \qquad p_0 = \frac{1}{N}\int p|F(p)|^2 dp. \tag{3}$$

With this function we can write the nonnegative integral

$$\frac{1}{N}\int |(\alpha(x-x_0) + d/dx - ip_0)f(x)|^2 = \alpha^2 A + \alpha B + C \tag{4}$$

for arbitrary α, where

$$A = \frac{1}{N}\int (x-x_0)^2|f(x)|^2 dx = \frac{1}{N}\int (x^2 - x_0^2)|f(x)|^2 dx \equiv \langle(\Delta x)^2\rangle, \tag{5}$$

$$B=\frac{1}{N}\int x\frac{d}{dx}|f(x)|^2 dx=\int \frac{d}{dx}(x|f(x)|^2)-\int |f(x)|^2=-\frac{1}{N}\int |f(x)|^2=-1, \tag{6}$$

$$C = \frac{1}{N}\int (p-p_0)^2|F(p)|^2 dp = \frac{1}{N}\int (p^2 - p_0^2)|F(p)|^2 dp \equiv \langle(\Delta p)^2\rangle. \tag{7}$$

Since Eq. (4) is nonnegative for all α, we have

$$\alpha^2\langle(\Delta x)^2\rangle - \alpha + \langle(\Delta p)^2\rangle \geqslant 0,$$

which is possible only for

$$\langle(\Delta p)^2\rangle\langle(\Delta x)^2\rangle \geqslant \frac{1}{4}, \tag{8}$$

which is just the uncertainty relation used in QM, however, it is satisfied for arbitrary function $f(x)$, and therefore is not related specifically to QM. Thus it cannot be a cornerstone of QM. The uncertainty relation takes place in all branches of physics. For example, in classical field theory, thermodynamics, hydrodynamics, and plasma physics. It is valid even in classical mechanics, because for functions $x(t)$ we have UR $(\Delta\omega)^2(\Delta t)^2 \geqslant 1/4$.

UR contain nothing specific to QM. QM is only a particular case, which is very alike to classical field theory.

2. POSITION AND MOMENTUM CAN BE DEFINED ABSOLUTELY PRECISELY SIMULTANEOUSLY

Since the wave function in QM defines a particle and it is an extended object, the question arises: what is a position of the extended object?

The answer to this question is: position of the extended object is the matter of definition.

In classical electrodynamics position of the electron is the singularity of its field.

In classical mechanics position of, say, a ball is the matter of definition. You may choose its center or a point, where you touch it.

For a free particle of mass m in QM we can use the nonsingular de Broglie's wave-packet [1–3]

$$\psi = j_0(s|\boldsymbol{r} - \boldsymbol{v}t|)\exp(i\boldsymbol{v}\boldsymbol{r} - i\omega t), \tag{9}$$

in which $j_0(x)$ is the spherical Bessel function, s is a parameter determining the width of the function, and

$$\omega = (v^2 + s^2)/2. \tag{10}$$

Here we use unities $\hbar = m = 1$, so velocity v of the particle is the same as its wave-vector k. Function (10) is a solution of the Schrödinger equation

$$(i\partial_t + \Delta/2)\psi = 0.$$

We can define its position as a position of maximum of $|\psi|^2$ and as a momentum of corresponding velocity v. They are defined absolutely precisely simultaneously in QM.

3. INTERFERENCE IN CLASSICAL MECHANICS

Let us consider an experiment on interference on two slits in a screen, shown in Fig. 1.

It is usually stated that particle goes through both slits in the screen, and transmitted parts of the particle wave function interfere on the screen of observation, which is manifested by the interference pattern. However, the interference can be explained purely classically with particle going through only one exactly specified slit.

Let us consider the same experiment with a classical electron, moving through one specified slit in the target screen, as is shown in Fig. 2.

Because of interaction of the electron field with the screen, the electron trajectory changes after the screen. Interaction of the electron field with the screen

S_t depends on the screen structure. In particular, it is different when there is one or two slits. It means that the direction of propagation of the electron after S_t depends on whether the second slit is opened or closed. Thus the second slit interferes with electron motion, even if the electron goes precisely through the chosen upper slit.

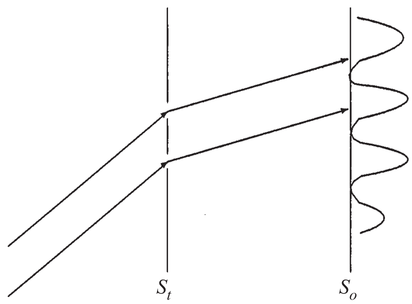

Fig. 1. According to standard QM wave function of a particle transmitted through both slits in target screen S_t interferes after S_t and gives a diffraction pattern on observation screen S_o

Our considerations permit us to predict the change of direction of the electron after the screen S_t, if we perform an experiment shown in Fig. 2, where the second slit can be closed with the shutter. With such simple considerations we cannot predict the diffraction pattern on the screen S_o, shown in Fig. 1, because in classical physics there are no such a parameter as wavelength, however wavelength can enter, if we take into account relativistic retardation of the interaction of electron with its own field reflected from the screen S_t or introduce a quantum of action. Indeed, we can suppose that the shift of the incident electron along distance l can affect the total field of the electron in presence of the screen S_t, and consecutively electron motion only if $pl = h$. Just at this point the quantization can enter into the classical behavior, and give such a parameter as the wavelength.

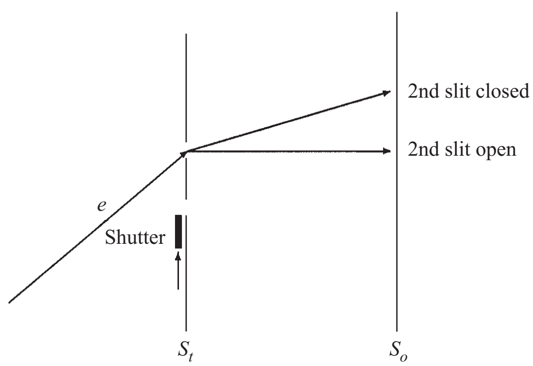

Fig. 2. An experiment with classical electron going through the upper slit in the screen S_t. Because of interaction of the electron field with the S_t its trajectory after the screen depends on whether the other slit is opened or not. This is an interference illustration of two slits in classical physics

4. NONLINEAR CLASSICAL SYSTEM OF EQUATIONS INSTEAD OF QM

All usual equations in mathematical physics can be sorted into two groups:

1. **Field equations** of the type

$$\hat{L}\psi(\boldsymbol{r}) = j(\boldsymbol{r}), \tag{11}$$

where $\hat{L}$ is an operator, which can be linear or nonlinear in field $\psi(\boldsymbol{r})$, and $j(\boldsymbol{r})$ is a source, which can depend on some particle trajectory $\boldsymbol{r}(t)$, and this trajectory is supposed to be fixed. As an example we can mention Maxwell equations with given currents, and with determined boundary conditions.

2. **Trajectory equations** of the type

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}(\boldsymbol{r}(t), t), \tag{12}$$

where the field of force $\boldsymbol{F}(\boldsymbol{r}, t)$ is fixed.

However, above, we had another type of the problem. It differs from (11) and (12). In this problem one has the trajectory equation

$$\frac{d^2\boldsymbol{r}_p}{dt^2} = \boldsymbol{F}(\psi(\boldsymbol{r}_p(t), t)), \tag{13}$$

with the force $\boldsymbol{F}(\psi)$, which depends on unknown field ψ. The field ψ is a solution of the field equation

$$\hat{L}\psi(\boldsymbol{r}, t) = j(\boldsymbol{r}, \boldsymbol{r}_p(t)) \tag{14}$$

with the source which depends on yet unknown solution of Eq. (13).

Formally we can exclude $\psi = \hat{L}^{-1}j(\boldsymbol{r}, \boldsymbol{r}_p(t))$ from Eq. (13); however, then we obtain highly nonlinear equation for trajectory

$$\frac{d^2\boldsymbol{r}_p}{dt^2} = \boldsymbol{F}(\hat{L}^{-1}j(\boldsymbol{r}_p(t), \boldsymbol{r}_p(t'))). \tag{15}$$

Solution of (15) or of the system ((13), (14)) is the **challenge for mathematicians**.

QM avoids solution of such a nonlinear system, however, reduction of nonlinear system to the linear Schrödinger equation costs probabilities instead of determinism.

However, it would be very interesting to try to solve such a nonlinear system, which can be easily formulated in classical electrodynamics.

4.1. The Problem of Classical Electrodynamics. We have the Maxwell equation for 4-tensor $\boldsymbol{F}_{\mu\nu}$:

$$\partial_\mu F_{\mu\nu}(\boldsymbol{r},t) = \frac{4\pi}{c} e u_\nu \delta(\boldsymbol{r}-\boldsymbol{r}(t)), \quad \mu,\nu = 0 \div 3, \tag{16}$$

where u_ν is speed with components $u_0 = c$, $\boldsymbol{u}_k = \boldsymbol{v}_k(t)$ for $k = 1 \div 3$. The functions $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t)$ are not known and are to be determined from the other equation — the trajectory one

$$m\frac{d\boldsymbol{v}(t)}{dt} = e\boldsymbol{E}(\boldsymbol{r},t) + \frac{e}{c}[\boldsymbol{v}(t)\boldsymbol{H}(\boldsymbol{r},t)],$$

where

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}(t)}{dt},$$

and electric and magnetic fields are the components of the 4-tensor $\boldsymbol{F}_{\mu\nu}$

$$\boldsymbol{E}_k(\boldsymbol{r},t) = F_{0k}(\boldsymbol{r},t), \quad \boldsymbol{H}_k = \epsilon_{ijk}F_{ij}(\boldsymbol{r},t),$$

which are formed by field, reflected from the target screen, and the reflection is determined by boundary conditions for the field $F_{\mu\nu}$. The screen can be accepted to be an infinitely thin ideal conductor. Position of slits, their width and the distance between them can be arbitrary.

For the beginning it is sufficient to solve even nonrelativistic, pure Coulomb problem. In the case when there are no slits, solution in nonrelativistic limit is trivial.

We want to remark that this nonlinear system has nothing to do with Bohm mechanics. No quantum potential is introduced, and no Schrödinger equation is presupposed. In the next section we briefly review the Bohm mechanics.

4.2. Bohm Mechanics and Hydrodynamical Interpretation. There are a lot of activity on interpretation of quantum mechanics in terms of classical trajectories and quantum potential (see, for example, [4] and references therein), which are known as Bohm mechanics or hydrodynamical interpretation. However, it is not a classical version, which replaces quantum mechanics, but only an alternative way of solving of the Schrödinger equation. We can find the full wave function ψ by solving the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = \left[-\frac{\hbar^2}{2m}\Delta + V(\boldsymbol{r})\right]\psi, \tag{17}$$

or represent it as $\psi(\boldsymbol{r},t) = R(\boldsymbol{r},t)\exp(iS(\boldsymbol{r})/\hbar)$, where $R(\boldsymbol{r}) = |\psi(\boldsymbol{r})$, substitute into (17), and after separation of real and imaginary parts obtain two other equations for them [4]

$$\frac{\partial R^2}{\partial t} + \boldsymbol{\nabla}\left(R^2\frac{\boldsymbol{\nabla}S}{m}\right) = 0, \tag{18}$$

$$\frac{\partial S}{\partial t} + \frac{(\boldsymbol{\nabla} S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2 R}{R} = 0. \tag{19}$$

Solution of these two equations is equivalent to solution of single Eq. (17). When you find R and S, you can find such things as

$$Q(\boldsymbol{r}, t) = \frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2 R}{R}, \tag{20}$$

which you call «quantum potential», and

$$\boldsymbol{v}(\boldsymbol{r}, t) = \frac{\boldsymbol{\nabla} S(\boldsymbol{r}, t)}{m}, \tag{21}$$

which you call speed. If you apply $\boldsymbol{\nabla}$ to Eq. (19) and use definition (21), you obtain the equation

$$m\frac{\partial \boldsymbol{v}(\boldsymbol{r}, t)}{\partial t} + m(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}(V(\boldsymbol{r}) + Q(\boldsymbol{r}, t)), \tag{22}$$

which is equivalent to

$$\frac{d\boldsymbol{v}(\boldsymbol{r}, t)}{dt} = -\boldsymbol{\nabla}(V(\boldsymbol{r}) + Q(\boldsymbol{r}, t)). \tag{23}$$

However, $\boldsymbol{v}$ is not equal to $\dot{\boldsymbol{r}}(t)$, because it is a field, which depends on both $\boldsymbol{r}$ and t.

Now, if you have already solved Eq. (17), you can consider (23) as the Newton equation and find a family of trajectories. However, in this case you arrive at the problem of finding trajectories for given field (12). It has nothing in common with the proposed classical nonlinear system of equations.

CONCLUSION

We think that the wave function ψ in QM represents some kind of a field, and the force of this field can be proportional to $|\psi|^2$. Then it will explain why in QM probability for a particle to be detected is proportional to $|\psi|^2$. If ψ is a field, then the position and momentum of a particle which is the source of this field can be naturally defined simultaneously, and UR do not forbid it.

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REFERENCES

1. *de Broglie L.* Non-Linear Wave Mechanics: A Causal Interpretation. Amsterdam: Elsevier, 1960.

2. *Ignatovich V. K.* The Physics of Ultracold Neutrons (UCN). Oxford: Clarendon Press, 1990.

3. *Ignatovich V. K., Utsuro M.* Tentative Solution of UCN Problem // Phys. Lett. A. 1997. V. 255. P. 195.

4. *Sanz A. S., Borondo F., Miret-Artes S.* Particle Diffraction Studied Using Quantum Trajectories // J. Phys.: Condens. Matter. 2002. V. 14. P. 6109.

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141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@pds.jinr.ru
www.jinr.ru/publish/