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E. A. Ivanov¹, A. O. Sutulin²

DIVERSE $\mathcal{N} = (4, 4)$ TWISTED MULTIPLETS
IN $\mathcal{N} = (2, 2)$ SUPERSPACE

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¹E-mail: eivanov@thsun1.jinr.ru

²E-mail: sutulin@thsun1.jinr.ru

Иванов Е. А., Сутулин А. О.

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Разные $\mathcal{N}=(4, 4)$ твистованные мультиплеты
в $\mathcal{N}=(2, 2)$ суперпространстве

Дано описание четырех различных типов твистованных $\mathcal{N}=(4, 4)$ супермультиплетов в двумерном $\mathcal{N}=(2, 2)$ суперпространстве $\mathbf{R}^{(1,1|2,2)}$. Рассмотренные мультиплеты представляются парой кирального и твистованного кирального суперполей и отличаются законами преобразований относительно дополнительной скрытой $\mathcal{N}=(2, 2)$ -суперсимметрии. Для каждого типа твистованного $\mathcal{N}=(4, 4)$ -мультиплета рассмотрены сигма-модельные $\mathcal{N}=(2, 2)$ суперполевые лагранжианы, которые являются действительными функциями, удовлетворяющими некоторым дифференциальным связям, налагаемым скрытой суперсимметрией. Показано, что общее сигма-модельное действие, зависящее одновременно от различных твистованных $\mathcal{N}=(4, 4)$ -мультиплетов и инвариантное относительно $\mathcal{N}=(4, 4)$ -суперсимметрии, распадается в сумму сигма-модельных действий для каждого мультиплета. Показано, что только для мультиплетов, принадлежащих самодуальной паре, возможно взаимодействие через подходящие смешанные массовые члены.

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Ivanov E. A., Sutulin A. O.

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Diverse $\mathcal{N}=(4, 4)$ Twisted Multiplets in $\mathcal{N}=(2, 2)$ Superspace

We describe four different types of the $\mathcal{N}=(4, 4)$ twisted supermultiplets in two-dimensional $\mathcal{N}=(2, 2)$ superspace $\mathbf{R}^{(1,1|2,2)}$. All these multiplets are presented by a pair of chiral and twisted chiral superfields and differ in the transformation properties under an extra hidden $\mathcal{N}=(2, 2)$ supersymmetry. The sigma model $\mathcal{N}=(2, 2)$ superfield Lagrangians for each type of the $\mathcal{N}=(4, 4)$ twisted supermultiplets are real functions subjected to some differential constraints implied by the hidden supersymmetry. We prove that the general sigma-model action, with all types of $\mathcal{N}=(4, 4)$ twisted multiplets originally included, is reduced to a sum of sigma-model actions for separate types. An interaction between the multiplets of different sorts is possible only through the appropriate mass terms, and only for those multiplets which belong to the same «self-dual» pair.

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INTRODUCTION

An important class of 2D supersymmetric sigma models is constituted by $\mathcal{N}=(2,2)$ and $\mathcal{N}=(4,4)$ supersymmetric models with torsionful bosonic target manifolds and two independent mutually commuting left and right complex structures [1–6]. These models and, in particular, their group manifold WZNW representatives can provide nontrivial backgrounds for 4D superstrings [7–9] and be relevant to 2D black holes in stringy context [10, 11]. Manifestly supersymmetric formulations of $\mathcal{N}=(2,2)$ models in terms of chiral and twisted chiral $\mathcal{N}=(2,2)$ superfields, as well as in terms of semi-chiral superfields, have been studied in [2, 12–16]. For $\mathcal{N}=(4,4)$ models with commuting structures there also exist manifestly supersymmetric *off-shell* formulations in the projective [8, 10, 17, 18] and ordinary [19, 20] $\mathcal{N}=(4,4)$, 2D superspaces. The basic object of these formulations is the $\mathcal{N}=(4,4)$ twisted multiplet. In $\mathcal{N}=(2,2)$, 2D superspace, the latter amounts to a pair of chiral and twisted-chiral superfields.

On the other hand, an adequate framework for 4D theories with extended supersymmetry, e.g., with $\mathcal{N}=2$ one, is provided by the harmonic superspace (HSS) approach [21, 22]. The $\mathcal{N}=(4,4)$, 2D sigma models which can be obtained via a direct dimensional reduction of $\mathcal{N}=2$, 4D sigma models constitute a subclass in a more general variety of $\mathcal{N}=(4,4)$, 2D sigma models. Indeed, their bosonic target manifolds are hyper-Kähler or quaternionic-Kähler, and so are torsionless and exhibit only one set of complex structures. In order to describe 2D supersymmetric theories with torsion one needs a more general type of the harmonic superspace as compared to that one which is of use in the 4D case. This new type of harmonic superspace, the $SU(2) \times SU(2)$ bi-harmonic superspace, has been constructed in our paper [23]. Its key feature is the presence of two independent sets of harmonic variables which are associated with two mutually commuting automorphism $SU(2)$ groups in the left and right light-cone sectors. In [23] we showed how to describe one type of the $\mathcal{N}=(4,4)$ twisted supermultiplets in the $SU(2) \times SU(2)$ bi-harmonic superspace and wrote the most general *off-shell* action of this multiplet as an integral over an analytic subspace of the full superspace. This action corresponds to a general $\mathcal{N}=(4,4)$ supersymmetric sigma models with torsion and mutually commuting sets of left and right complex structures*. Later on, a more general class of *off-shell* torsionful $\mathcal{N}=(4,4)$ sigma-model actions with non-commuting left and right complex struc-

*Some other off-shell $\mathcal{N}=(4,4)$ multiplets and the conformal $\mathcal{N}=(4,4)$ supergravity were presented in the $SU(2) \times SU(2)$ HSS in Refs. [24] and [25], respectively.

tures was constructed within the same $SU(2) \times SU(2)$ bi-harmonic superspace approach [26,27].

In the most previous cases the general sigma-model actions with $\mathcal{N}=(4,4)$ supersymmetry were written for arbitrary number of twisted multiplets of one fixed type, i.e., for those having the same transformation properties under the $SO(4)_L \times SO(4)_R$ automorphism group of $\mathcal{N}=(4,4)$, 2D Poincaré superalgebra. In $\mathcal{N}=(2,2)$ superspace, this basically means that each pair of chiral and twisted-chiral superfields one deals with has the same transformation properties under the extra $\mathcal{N}=(2,2)$ supersymmetry. On the other hand, as we shall explicitly show in this paper, the extra supersymmetry can be realized in *different* ways on *different* such pairs. As found in [28–30], in fact there are four essentially distinct types of the twisted $\mathcal{N}=(4,4)$ multiplets (up to additional twists related to space-time parities [29,31,32]). The basic distinction between them, in the $\mathcal{N}=(4,4)$ superfield formulations, is the different realization of the above automorphism group [29,30]. One of the purposes of the present paper is to demonstrate that it is a freedom in defining the extra $\mathcal{N}=(2,2)$ supersymmetry transformations on a pair of chiral and twisted chiral $\mathcal{N}=(2,2)$ superfields, which is responsible for this diversity of $\mathcal{N}=(4,4)$ twisted multiplets from the $\mathcal{N}=(2,2)$, 2D superspace perspective.

In our previous paper [30], the description of four basic different types of the $\mathcal{N}=(4,4)$ twisted multiplets in the $SU(2) \times SU(2)$ bi-harmonic superspace has been presented and general sigma-model actions for all of them have been constructed. We also discussed the special case of the $SU(2) \times SU(2)$ group manifold WZNW sigma model for one of these multiplets. We have shown that the general sigma-model action of any pair of different multiplets is reduced to a sum of sigma-model actions of separate multiplets.

In many aspects, the $\mathcal{N}=(2,2)$ superspace formulations of $\mathcal{N}=(4,4)$ supersymmetric models are more transparent than the $\mathcal{N}=(4,4)$ superspace ones. In the present paper we give the description of different types of twisted multiplets in the $\mathcal{N}=(2,2)$, 2D superspace $\mathbf{R}^{(1,1|2,2)}$. This $\mathcal{N}=(2,2)$ superspace approach is used to unravel some proofs and conclusions of Ref. [30].

In Sec. 1 we review how $\mathcal{N}=(4,4)$ twisted multiplets are described in the conventional $\mathcal{N}=(4,4)$, 2D superspace $\mathbf{R}^{(1,1|4,4)}$. Then, in Sec. 2, we reformulate the irreducibility conditions for these multiplets in $\mathcal{N}=(2,2)$, 2D superspace $\mathbf{R}^{(1,1|2,2)}$ and show, keeping only one $\mathcal{N}=(2,2)$ supersymmetry manifest, that in all cases they are equivalent to the chirality and twisted chirality conditions for the pair of constituent $\mathcal{N}=(2,2)$ superfields. The difference between various $\mathcal{N}=(4,4)$ twisted multiplets proves to be attributed to a difference in the transformation properties of these pairs with respect to the extra hidden $\mathcal{N}=(2,2)$ supersymmetry. The actions for the twisted multiplets are constructed as $\mathcal{N}=(2,2)$ superspace integrals of some real functions of the constituent $\mathcal{N}=(2,2)$ superfields. These functions are subjected to the appropriate differential constraints

implied by the hidden supersymmetry. In Sec. 3 we examine the $\mathcal{N}=(2,2)$ superspace actions of two types. The Lagrangians in these actions depend either on the $\mathcal{N}=(4,4)$ twisted multiplets belonging to a «self-dual» pair, or on those from different such pairs*. We find that in both cases the sigma-model-type actions are reduced to a sum of such actions for separate multiplets, and this property extends to the cases when a larger number of non-equivalent twisted multiplets is originally involved. This confirms the analogous result of Ref. [30] obtained within the bi-harmonic $\mathcal{N}=(4,4)$ superspace approach. We also show that the only possibility to gain an interaction between different twisted multiplets is to add $\mathcal{N}=(4,4)$ supersymmetric potential (or mass) terms mixing up multiplets from the same «self-dual» pair. The multiplets from different such pairs cannot interact at all, once again in the full agreement with Ref. [30]. Our results are summarized in Conclusions.

1. TWISTED MULTIPLTS IN $\mathcal{N}=(4,4)$, 2D SUPERSPACE

We begin by recapitulating some necessary facts about $\mathcal{N}=(4,4)$, 2D supersymmetry, basically following our papers [23,30]. In the light-cone parametrization, the standard real $\mathcal{N}=(4,4)$, 2D superspace is defined as the set of the light-cone coordinates

$$\mathbf{R}^{(1,1|4,4)} = (Z_L, Z_R) = (z^{++}, \theta^{+i\bar{k}}, z^{--}, \theta^{-a\bar{b}}). \quad (1.1)$$

Here $+, -$ are light-cone indices and i, \bar{k}, a, \bar{b} are doublet indices of four commuting $SU(2)$ groups which constitute the full automorphism group $SO(4)_L \times SO(4)_R$ of $\mathcal{N}=(4,4)$, 2D Poincaré superalgebra. The anticommutators of the corresponding spinor derivatives read

$$\{D_{i\bar{k}}, D_{j\bar{l}}\} = 2i \varepsilon_{ij} \varepsilon_{\bar{k}\bar{l}} \partial_{++}, \quad \{D_{a\bar{b}}, D_{c\bar{d}}\} = 2i \varepsilon_{ac} \varepsilon_{\bar{b}\bar{d}} \partial_{--}, \quad (1.2)$$

where

$$D_{i\bar{k}} = \frac{\partial}{\partial \theta^{i\bar{k}}} + i\theta_{i\bar{k}} \partial_{++}, \quad D_{a\bar{b}} = \frac{\partial}{\partial \theta^{a\bar{b}}} + i\theta_{a\bar{b}} \partial_{--}. \quad (1.3)$$

We shall use both the quartet notation for spinor derivatives and the Grassmann coordinates and their complex doublet form. To make the presentation more concise, we omit the light-cone indices in the quartet notation. The precise

*The definition of «dual» twisted multiplets is given in [20,29] and [30] (see also the end of Sec. 1 below).

relation between these two notations is as follows:

$$\begin{aligned}
(\theta^{+i}, \bar{\theta}^{+i}) &\equiv \theta^{i\bar{k}}, & (\theta^{i\bar{k}})^\dagger &= \varepsilon_{il} \varepsilon_{\bar{k}\bar{n}} \theta^{l\bar{n}}, \\
(\theta^{-a}, \bar{\theta}^{-a}) &\equiv \theta^{a\bar{b}}, & (\theta^{a\bar{b}})^\dagger &= \varepsilon_{ac} \varepsilon_{\bar{b}\bar{d}} \theta^{c\bar{d}}, \\
(D_{+i}, \bar{D}_{+i}) &\equiv D_{i\bar{k}}, & (D_{i\bar{k}})^\dagger &= -\varepsilon^{il} \varepsilon_{\bar{k}\bar{n}} D_{l\bar{n}}, \\
(D_{-a}, \bar{D}_{-a}) &\equiv D_{a\bar{b}}, & (D_{a\bar{b}})^\dagger &= -\varepsilon^{ac} \varepsilon_{\bar{b}\bar{d}} D_{c\bar{d}}.
\end{aligned} \tag{1.4}$$

Here the symbol \dagger means the complex conjugation.

The torsionful $\mathcal{N}=(4,4)$ supersymmetric sigma models can be formulated either in terms of pairs of $\mathcal{N}=(2,2)$ chiral and twisted chiral superfields, or via the properly constrained $\mathcal{N}=(4,4)$ superfields. Both superfield sets represent off-shell twisted $\mathcal{N}=(4,4)$ multiplets.

In $\mathcal{N}=(2,2)$, 2D superspace these models are described by an action which is an integral of some real potential [2,8,10] depending on several pairs of $\mathcal{N}=(2,2)$ chiral and twisted chiral superfields. This function satisfies a set of differential constraints which ensure the off-shell invariance of the action under an extra $\mathcal{N}=(2,2)$ supersymmetry. The latter mixes up chiral superfields with the twisted chiral ones and, together with the manifest $\mathcal{N}=(2,2)$ supersymmetry, constitute the full $\mathcal{N}=(4,4)$ supersymmetry.

On the other hand, in the superspace $\mathbf{R}^{(1,1|4,4)}$ the same $\mathcal{N}=(4,4)$ twisted multiplet is described by a real quartet superfield subjected to the proper irreducibility constraints [1,2,4,33]. These constraints reduce the full set of the field components of this superfield to the *off-shell* field content $(\mathbf{8} + \mathbf{8})$ of the twisted multiplet.

In the most previous studies, the general sigma-model actions with $\mathcal{N}=(4,4)$ supersymmetry, both in the $\mathbf{R}^{(1,1|2,2)}$ [2] and $\mathbf{R}^{(1,1|4,4)}$ [23] superspace approaches, were constructed in terms of only *one* kind of $\mathcal{N}=(4,4)$ twisted multiplet. However, in fact there are a few types of these multiplets which differ in the transformation properties of their component fields with respect to the full R -symmetry (or automorphism) group $SO(4)_L \times SO(4)_R$ of $\mathcal{N}=(4,4)$, 2D Poincaré superalgebra. This degeneracy of the twisted multiplets was first noticed in [19,28,29].

Following Refs. [28,30], one can consider four types of twisted multiplets in the superspace $\mathbf{R}^{(1,1|4,4)}$, in accord with the four possibilities to pair the doublet indices of various $SU(2)$ factors of the left and right $SO(4)$ subgroups in the full $SO(4)_L \times SO(4)_R$ automorphism group

$$\hat{q}^{ia}, \quad \hat{q}^{i\bar{a}}, \quad \hat{q}^{\dot{i}a}, \quad \hat{q}^{\dot{i}\bar{a}}. \tag{1.5}$$

The reality properties of these multiplets in $\mathbf{R}^{(1,1|4,4)}$ are defined by the rules

$$\begin{aligned}
(\hat{q}^{ia})^\dagger &= \varepsilon_{ik} \varepsilon_{ab} \hat{q}^{kb}, & (\hat{q}^{i\bar{a}})^\dagger &= \varepsilon_{ik} \varepsilon_{\bar{a}\bar{b}} \hat{q}^{k\bar{b}}, \\
(\hat{q}^{\dot{i}a})^\dagger &= \varepsilon_{\dot{i}\dot{k}} \varepsilon_{ab} \hat{q}^{\dot{k}b}, & (\hat{q}^{\dot{i}\bar{a}})^\dagger &= \varepsilon_{\dot{i}\dot{k}} \varepsilon_{\bar{a}\bar{b}} \hat{q}^{\dot{k}\bar{b}}.
\end{aligned} \tag{1.6}$$

The irreducibility constraints leaving just total of $(\mathbf{8}+\mathbf{8})$ independent components in every such a superfield read as

$$D^{(k\underline{k}\hat{q}^i)a} = 0, \quad D^{(b\underline{b}\hat{q}^i a)} = 0, \quad (1.7)$$

$$D^{(k\underline{k}\hat{q}^i)\underline{a}} = 0, \quad D^{b(\underline{b}\hat{q}^i\underline{a})} = 0, \quad (1.8)$$

$$D^k(\underline{k}\hat{q}^{\underline{i}})^a = 0, \quad D^{(b\underline{b}\hat{q}^{\underline{i}a})} = 0, \quad (1.9)$$

$$D^k(\underline{k}\hat{q}^{\underline{i}})\underline{a} = 0, \quad D^{b(\underline{b}\hat{q}^{\underline{i}\underline{a}})} = 0, \quad (1.10)$$

where $()$ means the symmetrization in the appropriate indices. The natural description of these twisted multiplets is achieved [23,30] in $\mathcal{N}=(4,4)$, $SU(2) \times SU(2)$ HSS with the double sets of harmonic variables. The corresponding general *off-shell* sigma-model actions for each kind of these multiplets can be written in the *analytic* bi-harmonic superspace [23,30], which is a subspace of the HSS just mentioned.

As shown in [30], in this analytic superspace one can generalize the sigma-model actions of twisted multiplets of one given type to the cases when the superfield Lagrangian bears a dependence on two or even more different species of such a multiplet. The basic goal of the present paper is to rephrase these results in terms of $\mathcal{N}=(2,2)$ chiral and twisted chiral superfields, making them more tractable and transparent.

To close this Section, let us recall the definition of the «self-dual» and «non-self-dual» pairs of the multiplets from set (1.5). The «self-dual» pairs are comprised by those superfields which have no $SU(2)$ doublet indices in common, i.e., by $(\hat{q}^{i a}, \hat{q}^{\underline{i}\underline{a}})$ and $(\hat{q}^{\underline{i}\underline{a}}, \hat{q}^{i a})$. Their distinguishing feature is that the physical bosonic fields of one superfield within the given pair have the same $SU(2)$ content as the auxiliary fields of the other. Any other pair is by definition «non-self-dual».

2. VARIOUS TWISTED MULTIPLICETS IN $\mathcal{N}=(2,2)$, 2D SUPERSPACE

In this Section we show how different types of $\mathcal{N}=(4,4)$ twisted multiplets can be described in the $\mathcal{N}=(2,2)$, 2D superspace and how the difference between them manifests itself in this $\mathcal{N}=(2,2)$ superspace setting. We find that each type is represented by a pair of chiral and twisted chiral $\mathcal{N}=(2,2)$ superfields having, however, different properties under the extra $\mathcal{N}=(2,2)$ supersymmetry transformations. Then we demonstrate that the general $\mathcal{N}=(4,4)$ supersymmetric sigma-model actions of separate multiplets can be constructed in $\mathcal{N}=(2,2)$ superspace as integrals of real functions subjected to some differential constraints.

2.1. Supersymmetry Transformations and Constraints. Let us pass to the equivalent notation for the $\mathbf{R}^{(1,1|4,4)}$ Grassmann coordinates and spinor derivatives (1.1), (1.4)

$$\theta^{+i} = (\theta^+, \eta^+), \bar{\theta}_i^+ = (\bar{\theta}^+, \bar{\eta}^+), \theta^{-a} = (\theta^-, \xi^-), \bar{\theta}_a^- = (\bar{\theta}^-, \bar{\xi}^-), \quad (2.1)$$

$$D_{+i} = (D_+, d_+), \bar{D}_+^i = (\bar{D}_+, \bar{d}_+), D_{-a} = (D_-, d_-), \bar{D}_-^a = (\bar{D}_-, \bar{d}_-). \quad (2.2)$$

In what follows, the coordinates θ 's and spinor derivatives D refer to the manifest $\mathcal{N}=(2,2)$ supersymmetry, while the coordinates η 's, ξ 's and the derivatives d to the extra hidden $\mathcal{N}=(2,2)$ one (see Appendix A for the precise relation between the $\mathcal{N}=(4,4)$ superspace covariant derivatives (1.4) and these $\mathcal{N}=(2,2)$ ones).

The standard real $\mathcal{N}=(2,2)$, 2D superspace $\mathbf{R}^{(1,1|2,2)}$ is parametrized by the following set of coordinates:

$$\mathbf{R}^{(1,1|2,2)} = (z^{++}, \theta^+, \bar{\theta}^+, z^{--}, \theta^-, \bar{\theta}^-). \quad (2.3)$$

Now we wish to see what the irreducibility conditions (1.7)–(1.10) look like in the superspace $\mathbf{R}^{(1,1|2,2)}$. Keeping in mind the reality conditions (1.6) for twisted multiplets in $\mathbf{R}^{(1,1|4,4)}$, we introduce the following complex $\mathcal{N}=(4,4)$ superfields:

$$\hat{q}^{11} = \mathbf{A}, \quad \hat{q}^{12} = \mathbf{B}, \quad \hat{q}^{21} = -\bar{\mathbf{B}}, \quad \hat{q}^{22} = \bar{\mathbf{A}}, \quad (2.4)$$

$$\hat{q}^{1\bar{1}} = \mathbf{a}, \quad \hat{q}^{1\bar{2}} = \mathbf{b}, \quad \hat{q}^{2\bar{1}} = -\bar{\mathbf{b}}, \quad \hat{q}^{2\bar{2}} = \bar{\mathbf{a}}, \quad (2.5)$$

$$\hat{q}^{11} = \mathcal{A}, \quad \hat{q}^{12} = \mathcal{B}, \quad \hat{q}^{21} = -\bar{\mathcal{B}}, \quad \hat{q}^{22} = \bar{\mathcal{A}}, \quad (2.6)$$

$$\hat{q}^{1\bar{1}} = \mathbf{A}, \quad \hat{q}^{1\bar{2}} = \mathbf{B}, \quad \hat{q}^{2\bar{1}} = -\bar{\mathbf{B}}, \quad \hat{q}^{2\bar{2}} = \bar{\mathbf{A}}. \quad (2.7)$$

Expanding these $\mathcal{N}=(4,4)$ superfields with respect to the extra Grassmann coordinates η^\pm, ξ^\pm (and their conjugates) and using constraints (1.7)–(1.10), one finds that only the first components of each $\mathcal{N}=(4,4)$ superfield are independent $\mathcal{N}=(2,2)$ superfields defined on the superspace $\mathbf{R}^{(1,1|2,2)}$. The $\mathcal{N}=(2,2)$ coefficients of the higher η, ξ monomials are expressed as spinor D derivatives of the coefficients associated with the lower-order monomials. The leading lowest-order components satisfy the chirality and the twisted chirality conditions in $\mathbf{R}^{(1,1|2,2)}$.

Thus for the multiplet \hat{q}^{ia} these conditions read (see Appendix A for the analogous conditions for other types of twisted multiplets)

$$\begin{aligned} \bar{D}_+ A = 0, & \quad \bar{D}_- A = 0, & \quad \bar{D}_+ B = 0, & \quad D_- B = 0, \\ D_+ \bar{A} = 0, & \quad D_- \bar{A} = 0, & \quad D_+ \bar{B} = 0, & \quad \bar{D}_- \bar{B} = 0. \end{aligned} \quad (2.8)$$

Here

$$A = \mathbf{A}|_{\eta=\xi=0}, \quad \bar{A} = \bar{\mathbf{A}}|_{\eta=\xi=0}, \quad B = \mathbf{B}|_{\eta=\xi=0}, \quad \bar{B} = \bar{\mathbf{B}}|_{\eta=\xi=0}. \quad (2.9)$$

To see how the difference between non-equivalent $\mathcal{N}=(4,4)$ twisted multiplets manifests itself in this approach, let us explicitly quote the extra Grassmann coordinate expansions for those $\mathcal{N}=(4,4)$ superfields which have as the first component the $\mathcal{N}=(2,2)$ chiral field

$$\begin{aligned}
(i) \quad & \mathbf{A} = A - \bar{\eta}^+ \bar{D}_+ \bar{B} + \bar{\xi}^- \bar{D}_- B + \dots, \quad \mathbf{A} = \mathbf{a} + \eta^+ \bar{D}_+ \bar{\mathbf{b}} - \xi^- \bar{D}_- \mathbf{b} + \dots, \\
(ii) \quad & \mathbf{a} = a - \bar{\eta}^+ \bar{D}_+ \bar{b} - \xi^- \bar{D}_- b + \dots, \quad \mathbf{A} = \alpha + \eta^+ \bar{D}_+ \bar{\beta} + \bar{\xi}^- \bar{D}_- \beta + \dots, \\
& A = \mathbf{A}|_{\eta=\xi=0}, \quad a = \mathbf{a}|_{\eta=\xi=0}, \quad \alpha = \mathbf{A}|_{\eta=\xi=0}, \quad \mathbf{a} = \mathbf{A}|_{\eta=\xi=0}. \quad (2.10)
\end{aligned}$$

In the superspace $\mathbf{R}^{(1,1|4,4)}$ the supersymmetry transformation of the $\mathcal{N}=(4,4)$ superfield Φ are generated by differential operators Q the explicit form of which is given in (A.9)

$$\delta\Phi = i \left(\varepsilon^{+k} Q_{+k} - \bar{\varepsilon}_k^+ \bar{Q}_+^k + \varepsilon^{-a} Q_{-a} - \bar{\varepsilon}_a^- \bar{Q}_-^a \right) \Phi. \quad (2.11)$$

When we reduce the superspace $\mathbf{R}^{(1,1|4,4)}$ to its subspace $\mathbf{R}^{(1,1|2,2)}$, half of supersymmetries become non-manifest and their transformations explicitly involve spinor derivatives. These extra supersymmetries mix different $\mathcal{N}=(2,2)$ superfields which are the first components of the $\mathcal{N}=(4,4)$ superfields defined in Eqs. (2.9), (A.10)–(A.12).

Substituting the expansions, such as (2.10), for each $\mathcal{N}=(4,4)$ superfield into the transformation law (2.11) and singling out the subset with $k = a = 2$ there (the corresponding generators and infinitesimal parameters are associated just with the hidden $\mathcal{N}=(2,2)$ supersymmetry), one finds the extra $\mathcal{N}=(2,2)$ supersymmetry transformation laws of the first components of these superfields

$$\begin{aligned}
\delta A &= \bar{\varepsilon}^+ \bar{D}_+ \bar{B} - \bar{\varepsilon}^- \bar{D}_- B, & \delta \mathbf{a} &= -\varepsilon^+ \bar{D}_+ \bar{\mathbf{b}} + \varepsilon^- \bar{D}_- \mathbf{b}, \\
\delta \bar{A} &= -\varepsilon^+ D_+ B + \varepsilon^- D_- \bar{B}, & \delta \bar{\mathbf{a}} &= \bar{\varepsilon}^+ D_+ \mathbf{b} - \bar{\varepsilon}^- D_- \bar{\mathbf{b}}, \\
\delta B &= -\bar{\varepsilon}^+ \bar{D}_+ \bar{A} - \varepsilon^- D_- A, & \delta \mathbf{b} &= \varepsilon^+ \bar{D}_+ \bar{\mathbf{a}} + \bar{\varepsilon}^- D_- \mathbf{a}, \\
\delta \bar{B} &= \varepsilon^+ D_+ A + \bar{\varepsilon}^- \bar{D}_- \bar{A}, & \delta \bar{\mathbf{b}} &= -\bar{\varepsilon}^+ D_+ \mathbf{a} - \varepsilon^- \bar{D}_- \bar{\mathbf{a}}, \quad (2.12)
\end{aligned}$$

$$\begin{aligned}
\delta a &= \bar{\varepsilon}^+ \bar{D}_+ \bar{b} + \varepsilon^- \bar{D}_- b, & \delta \alpha &= -\varepsilon^+ \bar{D}_+ \bar{\beta} - \bar{\varepsilon}^- \bar{D}_- \beta, \\
\delta \bar{a} &= -\varepsilon^+ D_+ b - \bar{\varepsilon}^- D_- \bar{b}, & \delta \bar{\alpha} &= \bar{\varepsilon}^+ D_+ \beta + \varepsilon^- D_- \bar{\beta}, \\
\delta b &= -\bar{\varepsilon}^+ \bar{D}_+ \bar{a} + \bar{\varepsilon}^- D_- a, & \delta \beta &= \varepsilon^+ \bar{D}_+ \bar{\alpha} - \varepsilon^- D_- \alpha, \\
\delta \bar{b} &= \varepsilon^+ D_+ a - \varepsilon^- \bar{D}_- \bar{a}, & \delta \bar{\beta} &= -\bar{\varepsilon}^+ D_+ \alpha + \bar{\varepsilon}^- \bar{D}_- \bar{\alpha}. \quad (2.13)
\end{aligned}$$

In (2.12) and (2.13) we collected, respectively, the transformation laws of $\mathcal{N}=(2,2)$ superfields belonging to the «self-dual» pairs of the $\mathcal{N}=(4,4)$ twisted multiplets, i.e., $(\hat{q}^{i a}, \hat{q}^{\bar{i} \bar{a}})$ and $(\hat{q}^{i \bar{a}}, \hat{q}^{\bar{i} a})$.

Looking at the sets of transformation laws (2.12) and (2.13), we come to the conclusion that the only difference in the description of various $\mathcal{N}=(4,4)$ twisted multiplets in the superspace $\mathbf{R}^{(1,1|2,2)}$ lies in the transformation laws of their chiral and twisted chiral $\mathcal{N}=(2,2)$ constituents under the extra $\mathcal{N}=(2,2)$ supersymmetry. These laws are specific for each multiplet.

Thus, in the superspace $\mathbf{R}^{(1,1|2,2)}$, each type of the $\mathcal{N}=(4,4)$ twisted multiplets is represented by a pair of chiral and twisted chiral $\mathcal{N}=(2,2)$ superfields. Despite this common feature, the extra $\mathcal{N}=(2,2)$ supersymmetry is realized differently on the $\mathcal{N}=(2,2)$ superfields from non-equivalent twisted multiplets. For $\mathcal{N}=(2,2)$ superfields of one kind (chiral or twisted chiral) belonging to different $\mathcal{N}=(4,4)$ twisted multiplets one cannot *simultaneously* bring the hidden $\mathcal{N}=(2,2)$ supersymmetry transformation laws into the same form by any redefinition of the transformation parameters and/or involved superfields.

Note that the full automorphism group $SO(4)_L \times SO(4)_R$ of the $\mathcal{N}=(4,4)$, 2D Poincaré superalgebra has different realizations on diverse $\mathcal{N}=(4,4)$ twisted multiplets in the $\mathcal{N}=(2,2)$, 2D superspace formulation, precisely as in the $\mathcal{N}=(4,4)$ superspace one. However, only an $U(1)_L \times U(1)_R$ subgroup of this automorphism group is manifest in the $\mathcal{N}=(2,2)$ setting. So it is the difference in the realizations of hidden $\mathcal{N}=(2,2)$ supersymmetry which is the basic distinguishing feature of non-equivalent twisted multiplets in the $\mathcal{N}=(2,2)$ superfield description.

2.2. Action for $\hat{q}^{i a}$ Multiplet. The general action of k chiral superfields C^k and n twisted chiral superfields T^n can be written in $\mathbf{R}^{(1,1|2,2)}$ as an integral of some real function K

$$S_{(2,2)} = \int \mu K(C^k, \bar{C}^k, T^n, \bar{T}^n), \quad (2.14)$$

where

$$\mu = d^2x d^2\theta^+ d^2\theta^- = d^2x d\theta^+ d\bar{\theta}^+ d\theta^- d\bar{\theta}^- \quad (2.15)$$

is the integration measure. The superpotential K is defined modulo a generalized Kähler transformation

$$\delta K = f(C^k, T^n) + g(C^k, \bar{T}^n) + \bar{f}(\bar{C}^k, \bar{T}^n) + \bar{g}(\bar{C}^k, T^n). \quad (2.16)$$

When this action describes a theory with only $\mathcal{N}=(2,2)$ supersymmetry, the numbers of chiral and twisted chiral superfields are not obliged to coincide, $k \neq n$. One can also add, without introducing any central charges, scalar potential interaction terms to Eq. (2.14), which involve two holomorphic functions $P_1(C^k)$

and $P_2(T^n)$

$$S_{(2,2)}^{\text{pot}} = im \int \mu \left\{ (\bar{\theta}^+ \bar{\theta}^-) P_1(C^k) + (\theta^+ \theta^-) \bar{P}_1(\bar{C}^k) + (\bar{\theta}^+ \theta^-) P_2(T^n) + (\theta^+ \bar{\theta}^-) \bar{P}_2(\bar{T}^n) \right\}. \quad (2.17)$$

Despite the presence of explicit θ 's, these terms are invariant under the manifest $\mathcal{N}=(2,2)$ supertranslations as a consequence of the chirality and twisted-chirality constraints for the corresponding superfields. After elimination of the auxiliary fields in the sum of actions (2.14) and (2.17), the resulting component actions acquire some scalar potentials of physical bosonic fields.

Let us firstly assume that the chiral C (antichiral \bar{C}) and twisted chiral T (twisted antichiral \bar{T}) superfields comprise one kind of $\mathcal{N}=(4,4)$ twisted multiplet. Then, requiring action (2.14) to possess an extra $\mathcal{N}=(2,2)$ supersymmetry implies, first, that the numbers of chiral and twisted chiral superfields must coincide, $k = n$, and, second, gives rise to some differential constraints on the function K . These constraints read [2]

$$\frac{\partial^2 K}{\partial C^l \partial \bar{C}^k} - \frac{\partial^2 K}{\partial C^k \partial \bar{C}^l} = 0, \quad \frac{\partial^2 K}{\partial T^l \partial \bar{T}^k} - \frac{\partial^2 K}{\partial T^k \partial \bar{T}^l} = 0, \quad (2.18)$$

$$\frac{\partial^2 K}{\partial C^{(k} \partial \bar{C}^{l)}} + \frac{\partial^2 K}{\partial T^{(l} \partial \bar{T}^{k)}} = 0. \quad (2.19)$$

In particular, if the potential K involves only one chiral and one twisted chiral multiplet ($m = 1$), the set of constraints (2.18), (2.19) amounts to the single four-dimensional Laplace equation

$$\frac{\partial^2 K}{\partial C \partial \bar{C}} + \frac{\partial^2 K}{\partial T \partial \bar{T}} = 0. \quad (2.20)$$

Thus for any real K obeying (2.18), (2.19) the action

$$S_{(4,4)} = \int \mu K(C^m, \bar{C}^m, T^m, \bar{T}^m) \quad (2.21)$$

is $\mathcal{N} = (4,4)$ supersymmetric.

It can be also checked that, for the potential (or mass) terms (2.17) to preserve $\mathcal{N}=(4,4)$ supersymmetry, the functions P_1, P_2 should be linear in the corresponding superfields

$$S_{(4,4)}^m = im \int \mu \left\{ (\bar{\theta}^+ \bar{\theta}^-) C + (\theta^+ \theta^-) \bar{C} + (\bar{\theta}^+ \theta^-) T + (\theta^+ \bar{\theta}^-) \bar{T} \right\}. \quad (2.22)$$

Although in the component form action (2.22) contains only terms linear in the auxiliary fields, the elimination of the latter in the full action including also a

sigma-model part can give rise to non-trivial scalar potentials of physical bosonic fields (provided that the target space bosonic metric is non-trivial) [1, 4, 19, 23].

This consideration can be extended to any other type of $\mathcal{N}=(4, 4)$ twisted multiplet. If the action bears dependence only on those chiral and twisted chiral $\mathcal{N}=(2, 2)$ superfields which comprise the same type of $\mathcal{N}=(4, 4)$ twisted multiplets, the extra $\mathcal{N}=(2, 2)$ supersymmetry differential constraints for the relevant potential K have the same form (2.18), (2.19), whatever the twisted multiplet is. The structure of the off-shell potential terms in this case is also uniquely fixed by the requirement of extra $\mathcal{N}=(2, 2)$ supersymmetry. They are given by a sum of actions (2.22).

3. ACTIONS FOR A PAIR OF TWISTED MULTIPLIETS

3.1. Preface: the Free Actions. In the previous Section we have found that the chiral and twisted chiral superfields forming one or another type of the $\mathcal{N}=(4, 4)$ twisted multiplet have different transformation properties under the hidden $\mathcal{N}=(2, 2)$ supersymmetry. Now we are going to construct, in the superspace $\mathbf{R}^{(1,1|2,2)}$, the supersymmetric sigma-model actions of two types, with the dependence on either a «non-self-dual» or «self-dual» pair of the $\mathcal{N}=(4, 4)$ twisted multiplets. We will show that in both cases the corresponding sigma-model actions are reduced to a sum of sigma-model actions for the separate twisted multiplets. The results presented below are in full agreement with those obtained in Ref. [30] within the harmonic superspace approach.

Before turning to the general case, let us recall the HSS description of two instructive examples of the action with two twisted multiplets, viz. the actions which are bilinear in the corresponding superfields [30]. The first option is the general quadratic actions depending only on one kind of the twisted multiplet and a priori including some harmonic constants. Requiring it to be $\mathcal{N}=(4, 4)$ supersymmetric leads to the conditions on these constants and, as a result, the corresponding actions are reduced to the relevant free actions. In the second case, the bilinear actions involve different sorts of the twisted multiplets. The inspection of such actions in HSS [30] leads to the conclusion that the requirement of invariance under the $\mathcal{N}=(4, 4)$ supersymmetry implies them to vanish.

These results can be easily reproduced in the $\mathbf{R}^{(1,1|2,2)}$ superspace formalism in terms of chiral and twisted chiral superfields. It is easy to show that the bilinear sigma-model action which contains chiral and twisted chiral superfields comprising one sort of the twisted $\mathcal{N}=(4, 4)$ multiplet,

$$S_{(4,4)}^{\text{free}} = \int \mu (C^m \bar{C}^m - T^m \bar{T}^m), \quad (3.1)$$

is equivalent to the $\mathcal{N}=(4,4)$ supersymmetric free actions of such twisted multiplets, while the $\mathbf{R}^{(1,1|2,2)}$ actions constructed bilinear in chiral and twisted chiral superfields from different types of the $\mathcal{N}=(4,4)$ twisted multiplets are vanishing as a consequence of extra $\mathcal{N}=(2,2)$ supersymmetry. Note, that the relative sign between two terms in (3.1) is uniquely fixed by hidden $\mathcal{N}=(2,2)$ supersymmetry (the component Lagrangian is positive-definite despite this sign minus in (3.1)).

Below we shall repeat in $\mathbf{R}^{(1,1|2,2)}$ our general analysis of sigma-model actions for both the «non-self-dual» and «self-dual» pairs of twisted multiplets [30]. The unique form of free action (3.1) and the property that the actions bilinear in the $\mathcal{N}=(2,2)$ superfields from different kinds of $\mathcal{N}=(4,4)$ twisted multiplets are vanishing will follow from this general analysis.

3.2. Action for Non-Dual Twisted Multiplets. We start from the action for the multiplets $\hat{q}^{i a}$ and $\hat{q}^{\bar{i} \bar{a}}$ belonging to different «self-dual» pairs. It is given as an integral of some real function K over the superspace $\mathbf{R}^{(1,1|2,2)}$

$$S_{(4,4)} = \int \mu K(A, \bar{A}, B, \bar{B}, a, \bar{a}, b, \bar{b}). \quad (3.2)$$

Since K is a function of $\mathcal{N}=(2,2)$ superfields, action (3.2) is evidently invariant under the manifest $\mathcal{N}=(2,2)$ supersymmetry. It is also invariant under generalized Kähler gauge transformations

$$\delta K = f(A, B, a, b) + g(A, \bar{B}, a, \bar{b}) + \bar{f}(\bar{A}, \bar{B}, \bar{a}, \bar{b}) + \bar{g}(\bar{A}, B, \bar{a}, b). \quad (3.3)$$

Using the chirality and twisted chirality conditions for the involved superfields and the definition of the integration measure on the superspace $\mathbf{R}^{(1,1|2,2)}$, it is easy to show that the gauge functions in (3.3) indeed do not contribute into the $\mathcal{N}=(2,2)$ superfield action.

We require this action to admit an extra $\mathcal{N}=(2,2)$ supersymmetry which is realized on superfields according to Eqs. (2.12), (2.13). This requirement amounts to some additional constraints on K . The general condition of the invariance of action (3.2) under these transformations is as follows:

$$\begin{aligned} \delta S_{(4,4)} = & \int \mu \left\{ \varepsilon^+ D_+ F - \bar{\varepsilon}^+ \bar{D}_+ \bar{F} + \varepsilon^+ \bar{D}_+ G - \bar{\varepsilon}^+ D_+ \bar{G} \right. \\ & \left. + \varepsilon^- D_- H - \bar{\varepsilon}^- \bar{D}_- \bar{H} + \varepsilon^- \bar{D}_- P - \bar{\varepsilon}^- D_- \bar{P} \right\}, \quad (3.4) \end{aligned}$$

where, for the moment, the functions in the r.h.s. are arbitrary. Explicitly computing the variation $\delta S_{(4,4)}$ and comparing both parts of (3.4), as the result of equating the coefficients before independent infinitesimal parameters we obtain

$$\begin{aligned} \varepsilon^+ & \Rightarrow \frac{\partial K}{\partial B} = \frac{\partial F}{\partial A}, \quad \frac{\partial K}{\partial A} = -\frac{\partial F}{\partial B}, \quad \frac{\partial K}{\partial b} = \frac{\partial F}{\partial a}, \quad \frac{\partial K}{\partial \bar{a}} = -\frac{\partial F}{\partial b}, \\ \bar{\varepsilon}^+ & \Rightarrow \frac{\partial K}{\partial B} = \frac{\partial \bar{F}}{\partial A}, \quad \frac{\partial K}{\partial A} = -\frac{\partial \bar{F}}{\partial B}, \quad \frac{\partial K}{\partial b} = \frac{\partial \bar{F}}{\partial \bar{a}}, \quad \frac{\partial K}{\partial a} = -\frac{\partial \bar{F}}{\partial b}, \quad (3.5) \end{aligned}$$

$$\begin{aligned}
\varepsilon^- &\Rightarrow \frac{\partial K}{\partial B} = -\frac{\partial H}{\partial A}, \quad \frac{\partial K}{\partial \bar{A}} = \frac{\partial H}{\partial \bar{B}}, \quad \frac{\partial K}{\partial b} = -\frac{\partial P}{\partial \bar{a}}, \quad \frac{\partial K}{\partial a} = \frac{\partial P}{\partial b}, \\
\bar{\varepsilon}^- &\Rightarrow \frac{\partial K}{\partial \bar{B}} = -\frac{\partial \bar{H}}{\partial \bar{A}}, \quad \frac{\partial K}{\partial A} = \frac{\partial \bar{H}}{\partial B}, \quad \frac{\partial K}{\partial b} = -\frac{\partial \bar{P}}{\partial a}, \quad \frac{\partial K}{\partial \bar{a}} = \frac{\partial \bar{P}}{\partial \bar{b}}. \quad (3.6)
\end{aligned}$$

Along with these constraints, we find that the functions in the r.h.s. of (3.4) should obey the additional analyticity-type conditions

$$\varepsilon^+ \Rightarrow G = 0, \quad \bar{\varepsilon}^+ \Rightarrow \bar{G} = 0, \quad (3.7)$$

$$\begin{aligned}
\varepsilon^- &\Rightarrow \frac{\partial H}{\partial a} = 0, \quad \frac{\partial H}{\partial \bar{b}} = 0, \quad \frac{\partial P}{\partial \bar{A}} = 0, \quad \frac{\partial P}{\partial B} = 0, \\
\bar{\varepsilon}^- &\Rightarrow \frac{\partial \bar{H}}{\partial \bar{a}} = 0, \quad \frac{\partial \bar{H}}{\partial b} = 0, \quad \frac{\partial \bar{P}}{\partial A} = 0, \quad \frac{\partial \bar{P}}{\partial \bar{B}} = 0. \quad (3.8)
\end{aligned}$$

The integrability conditions for K following from constraints (3.5) read

$$\begin{aligned}
\frac{\partial^2 K}{\partial A \partial \bar{A}} + \frac{\partial^2 K}{\partial B \partial \bar{B}} &= 0, & \frac{\partial^2 K}{\partial a \partial \bar{a}} + \frac{\partial^2 K}{\partial b \partial \bar{b}} &= 0, \\
\frac{\partial^2 K}{\partial a \partial \bar{B}} - \frac{\partial^2 K}{\partial \bar{b} \partial A} &= 0, & \frac{\partial^2 K}{\partial \bar{a} \partial \bar{B}} - \frac{\partial^2 K}{\partial b \partial A} &= 0, \\
\frac{\partial^2 K}{\partial a \partial A} + \frac{\partial^2 K}{\partial \bar{b} \partial B} &= 0, & \frac{\partial^2 K}{\partial \bar{a} \partial A} + \frac{\partial^2 K}{\partial b \partial B} &= 0. \quad (3.9)
\end{aligned}$$

Analogously, from Eqs. (3.6), (3.8) one finds further constraints on K

$$\begin{aligned}
\frac{\partial^2 K}{\partial a \partial B} = 0, \quad \frac{\partial^2 K}{\partial \bar{b} \partial B} = 0, \quad \frac{\partial^2 K}{\partial a \partial \bar{A}} = 0, \quad \frac{\partial^2 K}{\partial \bar{b} \partial \bar{A}} = 0, \\
\frac{\partial^2 K}{\partial \bar{a} \partial \bar{B}} = 0, \quad \frac{\partial^2 K}{\partial b \partial \bar{B}} = 0, \quad \frac{\partial^2 K}{\partial \bar{a} \partial A} = 0, \quad \frac{\partial^2 K}{\partial b \partial A} = 0 \quad (3.10)
\end{aligned}$$

and, in addition, the same two Laplace equations as in the first line of (3.9).

To find a solution to these constraints, it is convenient to introduce doublets of $\mathcal{N}=(2,2)$ superfields as follows:

$$a^\alpha = (a, \bar{b}), \quad A^\alpha = (\bar{A}, B), \quad \bar{a}^\alpha = (\bar{a}, b), \quad \bar{A}^\alpha = (A, \bar{B}), \quad (3.11)$$

where $\alpha, \beta = 1, 2$. With this new notation the set of Eqs. (3.9), (3.10) takes the more concise form

$$\frac{\partial^2 K}{\partial A \partial \bar{A}} + \frac{\partial^2 K}{\partial B \partial \bar{B}} = 0, \quad \frac{\partial^2 K}{\partial a \partial \bar{a}} + \frac{\partial^2 K}{\partial b \partial \bar{b}} = 0, \quad (3.12)$$

$$\frac{\partial^2 K}{\partial a^\alpha \partial A^\beta} = 0, \quad \frac{\partial^2 K}{\partial \bar{a}^\alpha \partial \bar{A}^\beta} = 0, \quad (3.13)$$

$$\varepsilon^{\alpha\beta} \frac{\partial^2 K}{\partial a^\alpha \partial A^\beta} = 0, \quad \bar{\varepsilon}^{\alpha\beta} \frac{\partial^2 K}{\partial \bar{a}^\alpha \partial \bar{A}^\beta} = 0. \quad (3.14)$$

The solution of Eqs. (3.13), (3.14) is as follows (see Appendix B for details):

$$K(A^\alpha, \bar{A}^\alpha, a^\alpha, \bar{a}^\alpha) = T(A^\alpha, \bar{A}^\alpha) + h(a^\alpha, \bar{a}^\alpha). \quad (3.15)$$

In addition, each term in the r.h.s. of (3.15) obeys its own four-dimensional Laplace equation, so constraints (3.12) are also satisfied.

3.3. Action for Dual Twisted Multiplets. In the general case the «test» action of multiplets $\hat{q}^{i a}$ and $\hat{q}^{i \underline{a}}$ can be written in the superspace $\mathbf{R}^{(1,1|2,2)}$ as

$$S_{(4,4)} = \int \mu K(a, \bar{a}, b, \bar{b}, \alpha, \bar{\alpha}, \beta, \bar{\beta}). \quad (3.16)$$

Action (3.16) is invariant under both manifest $\mathcal{N}=(2,2)$ supersymmetry and generalized Kähler gauge transformations

$$\delta K = f(a, b, \alpha, \beta) + g(a, \bar{b}, \alpha, \bar{\beta}) + \bar{f}(\bar{a}, \bar{b}, \bar{\alpha}, \bar{\beta}) + \bar{g}(\bar{a}, b, \bar{\alpha}, \beta). \quad (3.17)$$

As in the previous case, the requirement that this action possesses an extra $\mathcal{N}=(2,2)$ supersymmetry leads to some differential constraints on the function K . To find them we exploit the general invariance condition (3.4) (denoting the relevant functions by the same letters) and the superfield transformation laws (2.13). The resulting constraints are:

$$\begin{aligned} \varepsilon^+ &\Rightarrow \frac{\partial K}{\partial \bar{b}} = \frac{\partial F}{\partial a}, \quad \frac{\partial K}{\partial \bar{a}} = -\frac{\partial F}{\partial b}, \quad \frac{\partial K}{\partial \beta} = \frac{\partial G}{\partial \bar{\alpha}}, \quad \frac{\partial K}{\partial \alpha} = -\frac{\partial G}{\partial \bar{\beta}}, \\ \bar{\varepsilon}^+ &\Rightarrow \frac{\partial K}{\partial \bar{b}} = \frac{\partial \bar{F}}{\partial \bar{a}}, \quad \frac{\partial K}{\partial a} = -\frac{\partial \bar{F}}{\partial \bar{b}}, \quad \frac{\partial K}{\partial \beta} = \frac{\partial \bar{G}}{\partial \bar{\alpha}}, \quad \frac{\partial K}{\partial \bar{\alpha}} = -\frac{\partial \bar{G}}{\partial \bar{\beta}}, \\ \varepsilon^- &\Rightarrow \frac{\partial K}{\partial \bar{b}} = -\frac{\partial P}{\partial \bar{a}}, \quad \frac{\partial K}{\partial a} = \frac{\partial P}{\partial b}, \quad \frac{\partial K}{\partial \beta} = -\frac{\partial H}{\partial \bar{\alpha}}, \quad \frac{\partial K}{\partial \bar{\alpha}} = \frac{\partial H}{\partial \bar{\beta}}, \\ \bar{\varepsilon}^- &\Rightarrow \frac{\partial K}{\partial \bar{b}} = -\frac{\partial \bar{P}}{\partial \bar{a}}, \quad \frac{\partial K}{\partial a} = \frac{\partial \bar{P}}{\partial \bar{b}}, \quad \frac{\partial K}{\partial \beta} = -\frac{\partial \bar{H}}{\partial \bar{\alpha}}, \quad \frac{\partial K}{\partial \bar{\alpha}} = \frac{\partial \bar{H}}{\partial \bar{\beta}}. \end{aligned} \quad (3.18)$$

In addition to these equations, there arise analyticity-type conditions for the functions in the r.h.s. of Eq. (3.4)

$$\begin{aligned} \varepsilon^+ &\Rightarrow \frac{\partial G}{\partial \bar{a}} = 0, \quad \frac{\partial G}{\partial \bar{b}} = 0, \quad \frac{\partial F}{\partial \alpha} = 0, \quad \frac{\partial F}{\partial \beta} = 0, \\ \bar{\varepsilon}^+ &\Rightarrow \frac{\partial \bar{G}}{\partial a} = 0, \quad \frac{\partial \bar{G}}{\partial b} = 0, \quad \frac{\partial \bar{F}}{\partial \bar{\alpha}} = 0, \quad \frac{\partial \bar{F}}{\partial \bar{\beta}} = 0, \\ \varepsilon^- &\Rightarrow \frac{\partial H}{\partial a} = 0, \quad \frac{\partial H}{\partial \bar{b}} = 0, \quad \frac{\partial P}{\partial \bar{\alpha}} = 0, \quad \frac{\partial P}{\partial \beta} = 0, \\ \bar{\varepsilon}^- &\Rightarrow \frac{\partial \bar{H}}{\partial \bar{a}} = 0, \quad \frac{\partial \bar{H}}{\partial \bar{b}} = 0, \quad \frac{\partial \bar{P}}{\partial \alpha} = 0, \quad \frac{\partial \bar{P}}{\partial \bar{\beta}} = 0. \end{aligned} \quad (3.19)$$

From Eqs. (3.18), (3.19) one finds that the potential K satisfies the following integrability conditions:

$$\begin{aligned}
\frac{\partial^2 K}{\partial \bar{a} \partial \beta} = 0, & \quad \frac{\partial^2 K}{\partial \bar{b} \partial \beta} = 0, & \quad \frac{\partial^2 K}{\partial \bar{a} \partial \alpha} = 0, & \quad \frac{\partial^2 K}{\partial \bar{b} \partial \alpha} = 0, \\
\frac{\partial^2 K}{\partial a \partial \beta} = 0, & \quad \frac{\partial^2 K}{\partial b \partial \beta} = 0, & \quad \frac{\partial^2 K}{\partial a \partial \bar{\alpha}} = 0, & \quad \frac{\partial^2 K}{\partial b \partial \bar{\alpha}} = 0, \\
\frac{\partial^2 K}{\partial a \partial \beta} = 0, & \quad \frac{\partial^2 K}{\partial \bar{b} \partial \bar{\alpha}} = 0, & \quad \frac{\partial^2 K}{\partial \bar{a} \partial \beta} = 0, & \quad \frac{\partial^2 K}{\partial b \partial \alpha} = 0.
\end{aligned} \tag{3.20}$$

Besides this, the potential K should also obey two independent Laplace equations

$$\frac{\partial^2 K}{\partial a \partial \bar{a}} + \frac{\partial^2 K}{\partial b \partial \bar{b}} = 0, \quad \frac{\partial^2 K}{\partial \alpha \partial \bar{\alpha}} + \frac{\partial^2 K}{\partial \beta \partial \bar{\beta}} = 0. \tag{3.21}$$

Introducing the doublet notation for superfields like in the previous case

$$a^\alpha = (a, \bar{b}), \quad \Gamma^\alpha = (\bar{\alpha}, \beta), \quad \bar{a}^\alpha = (\bar{a}, b), \quad \bar{\Gamma}^\alpha = (\alpha, \bar{\beta}), \tag{3.22}$$

we cast set (3.20) in the following compact form:

$$\frac{\partial^2 K}{\partial a^\alpha \partial \bar{\Gamma}^\beta} = 0, \quad \frac{\partial^2 K}{\partial \bar{a}^\alpha \partial \bar{\Gamma}^\beta} = 0, \tag{3.23}$$

$$\varepsilon^{\alpha\beta} \frac{\partial^2 K}{\partial a^\alpha \partial \bar{\Gamma}^\beta} = 0, \quad \varepsilon^{\alpha\beta} \frac{\partial^2 K}{\partial \bar{a}^\alpha \partial \bar{\Gamma}^\beta} = 0 \tag{3.24}$$

(no summation over repeating indices). Equations (3.21), (3.23), (3.24) form the full set of differential constraints on the function K . The solution of Eqs. (3.23), (3.24) shows up the same separation as in the previous case (i.e., for a pair of non-dual $\mathcal{N}=(4,4)$ twisted multiplets)

$$K(a^\alpha, \bar{a}^\alpha, \Gamma^\alpha, \bar{\Gamma}^\alpha) = T(\Gamma^\alpha, \bar{\Gamma}^\alpha) + h(a^\alpha, \bar{a}^\alpha). \tag{3.25}$$

Then Eqs. (3.21) imply that T and h should satisfy their own Laplace equations.

Thus we have demonstrated that the general sigma-model actions of both «self-dual» and «non-self-dual» pairs of the twisted $\mathcal{N}=(4,4)$ multiplets are reduced to the proper sums of $\mathcal{N}=(4,4)$ supersymmetric actions of single multiplets. In other words, each action in a sum involves only those $\mathcal{N}=(2,2)$ chiral and twisted chiral superfields which belong to one $\mathcal{N}=(4,4)$ twisted multiplet. From these results it follows that chiral and twisted chiral $\mathcal{N}=(2,2)$ superfields belonging to different $\mathcal{N}=(4,4)$ twisted multiplets and hence having different transformation properties under the hidden $\mathcal{N}=(2,2)$ supersymmetry cannot interact with each other via the sigma-model-type actions.

Our $\mathcal{N}=(2,2)$ superspace analysis confirms the results obtained for different types of $\mathcal{N}=(4,4)$ twisted multiplets in the bi-harmonic superspace approach [30]. In that paper the separation property was proved for the actions depending on arbitrary number of $\mathcal{N}=(4,4)$ twisted multiplets of various kind. The above analysis can be also extended to this general case, with the same ultimate conclusions.

3.4. Potential Terms. In Sec. 2, on the example of the twisted multiplets $\hat{q}^{i a}$, we showed how to construct $\mathcal{N}=(4,4)$ supersymmetric potential (or mass) terms for such multiplets in the superspace $\mathbf{R}^{(1,1|2,2)}$. In our previous paper [30] we found that for the $\mathcal{N}=(4,4)$ twisted multiplets belonging to a «self-dual» pair one can write invariant mixed mass terms. These terms are in fact of the same form as those given in [1, 4, 19, 20].

The results of Ref. [30] can be easily reformulated in terms of the $\mathcal{N}=(2,2)$ chiral and twisted chiral superfields. Indeed, for a «self-dual» pair of the twisted multiplets, e.g., $\hat{q}^{i a}$ and $\hat{q}^{\dot{i} a}$, the corresponding «test» potential term bilinear in superfields can be written as

$$S_{(4,4)}^M = i M \int \mu \left\{ l (\theta^+ \theta^-) \bar{\alpha} \bar{a} + k (\bar{\theta}^+ \bar{\theta}^-) \alpha a + n (\theta^+ \bar{\theta}^-) \bar{\beta} \bar{b} + p (\bar{\theta}^+ \theta^-) \beta b \right\}, \quad (3.26)$$

while for the case of «non-self-dual» pair, e.g., $\hat{q}^{i a}$ and $\hat{q}^{\dot{i} a}$, it has the following form:

$$S_{(4,4)}^{\tilde{M}} = i \tilde{M} \int \mu \left\{ l' (\theta^+ \theta^-) \bar{a} \bar{A} + k' (\bar{\theta}^+ \bar{\theta}^-) a A + n' (\theta^+ \bar{\theta}^-) \bar{b} \bar{B} + p' (\bar{\theta}^+ \theta^-) b B \right\}. \quad (3.27)$$

In Eqs. (3.26), (3.27), l, l' etc. are some numerical coefficients unspecified for the moment. These coefficients have the same meaning as the harmonic constants $C^{p,q}$ with the $U(1) \times U(1)$ charges p and q introduced in [30]. Computing the variation of action (3.26) with respect to the extra $\mathcal{N}=(2,2)$ supersymmetry, it is easy to find that (3.26) is invariant provided that these constants are equal

$$l = k = n = p. \quad (3.28)$$

At the same time, requiring potential action (3.27) of non-dual multiplets to be invariant under this extra $\mathcal{N}=(2,2)$ supersymmetry implies the relevant constants to identically vanish

$$l' = k' = n' = p' = 0, \quad (3.29)$$

which forbids mixed mass terms for such a pair. Analogous results were obtained in [30], where the harmonic constants in the «probe» potential terms for a pair of non-self-dual twisted multiplets were found to vanish as the result of imposing the requirement of $\mathcal{N}=(4,4)$ supersymmetry, whereas in the mass terms for a «self-dual» pair of the twisted multiplets these constants proved to be non-vanishing, with properly constrained dependence on harmonics.

Note that the most general $\mathcal{N}=(4, 4)$ supersymmetric off-shell potential term is a sum of mixed terms (3.26) with condition (3.28) and linear terms (2.22) (for each twisted multiplet involved). A net effect of eliminating the auxiliary fields in the full component action is the generation of some potential and mass terms for the physical bosonic fields [30] (plus some Yukawa-type couplings of physical fermions).

CONCLUSIONS

In this paper we presented the description of four different types of $\mathcal{N}=(4, 4)$ twisted multiplets in $\mathcal{N}=(2, 2)$, 2D superspace $\mathbf{R}^{(1,1|2,2)}$. We showed that each type amounts off-shell to a pair of chiral and twisted chiral $\mathcal{N}=(2, 2)$ superfields, with essentially different transformation properties under the extra $\mathcal{N}=(2, 2)$ supersymmetry which completes the manifest one to the entire $\mathcal{N}=(4, 4)$ supersymmetry. The general off-shell sigma-model action for the $\mathcal{N}=(4, 4)$ twisted multiplet of any fixed kind can be written as an $\mathbf{R}^{(1,1|2,2)}$ integral of real functions K which depend on the relevant pairs of the $\mathcal{N}=(2, 2)$ superfields and are subjected to some differential constraints. These constraints have the same form for every type of the twisted multiplet and ensure the corresponding sigma-model actions to exhibit $\mathcal{N}=(4, 4)$ supersymmetry. We also showed how the requirement of extra $\mathcal{N}=(2, 2)$ supersymmetry constrains the potential (or mass) terms P_1, P_2 .

We demonstrated that in more general cases, when the superpotential K depends on $\mathcal{N}=(2, 2)$ chiral and twisted chiral superfields belonging to different $\mathcal{N}=(4, 4)$ twisted multiplets, the extra $\mathcal{N}=(2, 2)$ supersymmetry requires the general sigma-model action to split into to a sum of sigma-model actions for separate multiplets. The only possibility to arrange mutual interactions of the twisted multiplets of different types is via the appropriate invariant mixed mass terms. The latter are bilinear in the chiral and twisted chiral superfields belonging to a «self-dual» pair of the $\mathcal{N}=(4, 4)$ twisted multiplets. The multiplets from different such pairs can interact with each other neither via sigma-model actions nor via mass terms.

To summarize, the analysis performed in the present paper in the standard $\mathcal{N}=(2, 2)$, 2D superfield formalism revealed a full agreement with that one given in Ref. [30] for different types of $\mathcal{N}=(4, 4)$ twisted multiplets within the bi-harmonic $SU(2) \times SU(2)$ superspace approach. The $\mathcal{N}=(2, 2)$, 2D superfield description of all types of $\mathcal{N}=(4, 4)$ twisted multiplets developed here can find applications in many physical and geometric problems to which these multiplets are relevant.

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APPENDIX A

$\mathcal{N}=(2, 2)$, 2D SPINOR DERIVATIVES AND CONSTRAINTS

Here we give some details of our notations for spinor derivatives. Starting from the quartet notation, one can define two different types of covariant derivatives in the left and right light-cone coordinate sectors, such that they are doublets with respect to different automorphism groups $SU(2)$ (these groups form, respectively, $SO(4)_L$ and $SO(4)_R$)

$$\begin{aligned} D_{i\underline{k}} &= (D_{i\underline{1}}, D_{i\underline{2}}) \equiv (D_{+i}, \bar{D}_{+i}) \\ &= (D_{1\underline{k}}, D_{2\underline{k}}) \equiv (D_{+\underline{k}}, \bar{D}_{+\underline{k}}), \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} D_{a\underline{b}} &= (D_{a\underline{1}}, D_{a\underline{2}}) \equiv (D_{-a}, \bar{D}_{-a}) \\ &= (D_{1\underline{b}}, D_{2\underline{b}}) \equiv (D_{-\underline{b}}, \bar{D}_{-\underline{b}}). \end{aligned} \quad (\text{A.2})$$

The $\mathcal{N}=(2, 2)$ spinor derivatives D and d which correspond to the manifest and hidden supersymmetry, respectively, are defined by

$$D_{+i} = (D_+, d_+) = D_{i\underline{1}}, \quad \bar{D}_+^i = (\bar{D}_+, \bar{d}_+) = \varepsilon^{ik} \bar{D}_{+k} = \varepsilon^{ik} D_{k\underline{2}}, \quad (\text{A.3})$$

$$D_{-a} = (D_-, d_-) = D_{a\underline{1}}, \quad \bar{D}_-^a = (\bar{D}_-, \bar{d}_-) = \varepsilon^{ab} \bar{D}_{-b} = \varepsilon^{ab} D_{b\underline{2}}. \quad (\text{A.4})$$

The relations between the $\mathcal{N}=(2, 2)$ spinor derivatives D , d and the $SU(2)$ -doublet ones with the underlined indices defined in the second lines of (A.1), (A.2) are as follows:

$$D_{+\underline{k}} = D_{1\underline{k}} = (D_+, \bar{d}_+), \quad \bar{D}_+^i = \varepsilon^{ik} \bar{D}_{+\underline{k}} = (\bar{D}_+, d_+), \quad (\text{A.5})$$

$$D_{-\underline{b}} = D_{1\underline{b}} = (D_-, \bar{d}_-), \quad \bar{D}_-^a = \varepsilon^{ab} \bar{D}_{-\underline{b}} = (\bar{D}_-, d_-). \quad (\text{A.6})$$

The explicit form of the covariant spinor derivatives as differential operators in the left sector of $\mathbf{R}^{(1,1|4,4)}$ and $\mathbf{R}^{(1,1|2,2)}$ is

$$\begin{aligned} D_{+i} &= \frac{\partial}{\partial \theta^{+i}} + i\bar{\theta}_i^+ \partial_{+++}, & \bar{D}_+^i &= -\frac{\partial}{\partial \bar{\theta}_i^+} - i\theta^{+i} \partial_{+++}, \\ D_+ &= \frac{\partial}{\partial \theta^+} + i\bar{\theta}^+ \partial_{+++}, & \bar{D}_+ &= -\frac{\partial}{\partial \bar{\theta}^+} - i\theta^+ \partial_{+++}, \\ d_+ &= \frac{\partial}{\partial \eta^+} + i\bar{\eta}^+ \partial_{+++}, & \bar{d}_+ &= -\frac{\partial}{\partial \bar{\eta}^+} - i\eta^+ \partial_{+++}. \end{aligned} \quad (\text{A.7})$$

The analogous expressions in the right sector are

$$\begin{aligned}
D_{-a} &= \frac{\partial}{\partial \theta^{-a}} + i\bar{\theta}_a^- \partial_{--}, & \bar{D}_-^a &= -\frac{\partial}{\partial \bar{\theta}_a^-} - i\theta^{-a} \partial_{--}, \\
D_- &= \frac{\partial}{\partial \theta^-} + i\bar{\theta}^- \partial_{--}, & \bar{D}_- &= -\frac{\partial}{\partial \bar{\theta}^-} - i\theta^- \partial_{--}, \\
d_- &= \frac{\partial}{\partial \xi^-} + i\bar{\xi}^- \partial_{--}, & \bar{d}_- &= -\frac{\partial}{\partial \bar{\xi}^-} - i\xi^- \partial_{--}.
\end{aligned} \tag{A.8}$$

The $\mathcal{N}=(4,4)$ supersymmetry generators in $\mathbf{R}^{(1,1|4,4)}$ read

$$\begin{aligned}
Q_{+i} &= i \frac{\partial}{\partial \theta^{+i}} + \bar{\theta}_i^+ \partial_{++}, & \bar{Q}_+^i &= -i \frac{\partial}{\partial \bar{\theta}_i^+} - \theta^{+i} \partial_{++}, \\
Q_{-a} &= i \frac{\partial}{\partial \theta^{-a}} + \bar{\theta}_a^- \partial_{--}, & \bar{Q}_-^a &= -i \frac{\partial}{\partial \bar{\theta}_a^-} - \theta^{-a} \partial_{--}.
\end{aligned} \tag{A.9}$$

We denote the first components of the expansions of $\mathcal{N}=(4,4)$ superfields in (2.5)–(2.7) with respect to the extra Grassmann coordinates η 's and ξ 's as

$$\mathbf{a}|_{\eta=\xi=0} = a, \quad \bar{\mathbf{a}}|_{\eta=\xi=0} = \bar{a}, \quad \mathbf{b}|_{\eta=\xi=0} = b, \quad \bar{\mathbf{b}}|_{\eta=\xi=0} = \bar{b}, \tag{A.10}$$

$$\mathcal{A}|_{\eta=\xi=0} = \alpha, \quad \bar{\mathcal{A}}|_{\eta=\xi=0} = \bar{\alpha}, \quad \mathcal{B}|_{\eta=\xi=0} = \beta, \quad \bar{\mathcal{B}}|_{\eta=\xi=0} = \bar{\beta}, \tag{A.11}$$

$$\mathbf{A}|_{\eta=\xi=0} = \mathbf{a}, \quad \bar{\mathbf{A}}|_{\eta=\xi=0} = \bar{\mathbf{a}}, \quad \mathbf{B}|_{\eta=\xi=0} = \mathbf{b}, \quad \bar{\mathbf{B}}|_{\eta=\xi=0} = \bar{\mathbf{b}}. \tag{A.12}$$

In the superspace $\mathbf{R}^{(1,1|2,2)}$, these $\mathcal{N}=(2,2)$ superfields are subjected to the following chirality and twisted chirality conditions:

$$\begin{aligned}
\bar{D}_+ a &= 0, & \bar{D}_- a &= 0, & \bar{D}_+ b &= 0, & D_- b &= 0, \\
D_+ \bar{a} &= 0, & D_- \bar{a} &= 0, & D_+ \bar{b} &= 0, & \bar{D}_- \bar{b} &= 0,
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
\bar{D}_+ \alpha &= 0, & \bar{D}_- \alpha &= 0, & \bar{D}_+ \beta &= 0, & D_- \beta &= 0, \\
D_+ \bar{\alpha} &= 0, & D_- \bar{\alpha} &= 0, & D_+ \bar{\beta} &= 0, & \bar{D}_- \bar{\beta} &= 0,
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
\bar{D}_+ \mathbf{a} &= 0, & \bar{D}_- \mathbf{a} &= 0, & \bar{D}_+ \mathbf{b} &= 0, & D_- \mathbf{b} &= 0, \\
D_+ \bar{\mathbf{a}} &= 0, & D_- \bar{\mathbf{a}} &= 0, & D_+ \bar{\mathbf{b}} &= 0, & \bar{D}_- \bar{\mathbf{b}} &= 0.
\end{aligned} \tag{A.15}$$

These conditions directly follow from the defining $\mathcal{N}=(4,4)$ constraints (1.7)–(1.10).

APPENDIX B
SOLVING CONSTRAINTS FOR K

Here we deduce the explicit solution of the constraints on the superpotential K which involves $\mathcal{N}=(4,4)$ twisted multiplets of two different types belonging to a «non-self-dual» pair. These constraints are given by Eqs. (3.13), (3.14):

$$\frac{\partial^2 K}{\partial a^\alpha \partial A^\beta} = 0, \quad \frac{\partial^2 K}{\partial \bar{a}^\alpha \partial \bar{A}^\beta} = 0. \quad (\text{B.1})$$

The doublet notation was explained in (3.11).

As a first step, we partly solve (B.1) by introducing the complex quantity

$$F_\beta(\bar{a}^\alpha, A^\alpha, \bar{A}^\alpha) \equiv \frac{\partial K}{\partial A^\beta}, \quad \bar{F}_\beta(a^\alpha, A^\alpha, \bar{A}^\alpha) \equiv \frac{\partial K}{\partial \bar{A}^\beta}. \quad (\text{B.2})$$

From the definition of F_α and \bar{F}_α one derives the integrability conditions

$$\frac{\partial F_\alpha}{\partial \bar{A}^\beta} - \frac{\partial \bar{F}_\beta}{\partial A^\alpha} = 0. \quad (\text{B.3})$$

Acting on this equation by the operator $\frac{\partial}{\partial a^\rho}$ and again using (B.2), one obtains

$$\frac{\partial^2 \bar{F}_\beta}{\partial A^\alpha \partial a^\rho} = 0, \quad (\text{B.4})$$

which implies

$$\frac{\partial \bar{F}_\alpha}{\partial A^\beta} = G_{\alpha\beta}(A^\rho, \bar{A}^\rho). \quad (\text{B.5})$$

Analogously, for F_α we find

$$\frac{\partial F_\alpha}{\partial \bar{A}^\beta} = \bar{G}_{\alpha\beta}(A^\rho, \bar{A}^\rho). \quad (\text{B.6})$$

Integrating Eqs. (B.5) and (B.6), we find the following general solution for F_α and \bar{F}_α :

$$\begin{aligned} F_\alpha(\bar{a}^\beta, A^\beta, \bar{A}^\beta) &= f_\alpha(A^\beta, \bar{A}^\beta) + \bar{G}_\alpha(A^\beta, \bar{a}^\beta), \\ \bar{F}_\alpha(a^\beta, A^\beta, \bar{A}^\beta) &= \bar{f}_\alpha(A^\beta, \bar{A}^\beta) + G_\alpha(\bar{A}^\beta, a^\beta). \end{aligned} \quad (\text{B.7})$$

Substituting this into (B.3), one finds

$$\frac{\partial f_\alpha}{\partial \bar{A}^\beta} - \frac{\partial \bar{f}_\beta}{\partial A^\alpha} = 0. \quad (\text{B.8})$$

The solution of the last equation can be easily found

$$f_\alpha = \frac{\partial}{\partial A^\alpha} T(A^\beta, \bar{A}^\beta), \quad \bar{f}_\alpha = \frac{\partial}{\partial \bar{A}^\alpha} T(A^\beta, \bar{A}^\beta). \quad (\text{B.9})$$

Then from Eqs. (B.2), (B.7), (B.9) one derives

$$\begin{aligned} \frac{\partial K}{\partial A^\alpha} &= F_\alpha(\bar{a}^\beta, A^\beta, \bar{A}^\beta) = \bar{G}_\alpha(A^\beta, \bar{a}^\beta) + \frac{\partial}{\partial A^\alpha} T(A^\beta, \bar{A}^\beta), \\ \frac{\partial K}{\partial \bar{A}^\alpha} &= \bar{F}_\alpha(a^\beta, A^\beta, \bar{A}^\beta) = G_\alpha(\bar{A}^\beta, a^\beta) + \frac{\partial}{\partial \bar{A}^\alpha} T(A^\beta, \bar{A}^\beta), \end{aligned} \quad (\text{B.10})$$

or

$$\bar{G}_\alpha(A^\beta, \bar{a}^\beta) = \frac{\partial}{\partial A^\alpha} \{K - T\}, \quad G_\alpha(\bar{A}^\beta, a^\beta) = \frac{\partial}{\partial \bar{A}^\alpha} \{K - T\}. \quad (\text{B.11})$$

These relations imply the integrability conditions

$$\frac{\partial \bar{G}_\alpha}{\partial A^\beta} - \frac{\partial \bar{G}_\beta}{\partial A^\alpha} = 0, \quad \frac{\partial G_\alpha}{\partial \bar{A}^\beta} - \frac{\partial G_\beta}{\partial \bar{A}^\alpha} = 0, \quad (\text{B.12})$$

which, in turn, give that

$$\bar{G}_\alpha = \frac{\partial}{\partial A^\alpha} G(A^\beta, \bar{a}^\beta), \quad G_\alpha = \frac{\partial}{\partial \bar{A}^\alpha} \bar{G}(\bar{A}^\beta, a^\beta). \quad (\text{B.13})$$

Substituting (B.13) into (B.11) leads to the set of equations

$$\frac{\partial}{\partial A^\alpha} \{K - T - G(A^\beta, \bar{a}^\beta)\} = 0, \quad \frac{\partial}{\partial \bar{A}^\alpha} \{K - T - \bar{G}(\bar{A}^\beta, a^\beta)\} = 0, \quad (\text{B.14})$$

which can be easily solved as

$$\begin{aligned} (i) \quad K - T &= G(A^\alpha, \bar{a}^\alpha) + \Omega(\bar{A}^\alpha, a^\alpha, \bar{a}^\alpha), \\ (ii) \quad K - T &= \bar{G}(\bar{A}^\alpha, a^\alpha) + \bar{\Omega}(A^\alpha, a^\alpha, \bar{a}^\alpha). \end{aligned} \quad (\text{B.15})$$

Expressing $K - T$ from Eq. (ii) in (B.15) and substituting it into the first equation in (B.14), one finds

$$\bar{\Omega} = G(A^\alpha, \bar{a}^\alpha) + h(a^\alpha, \bar{a}^\alpha). \quad (\text{B.16})$$

Analogously, expressing $K - T$ from Eq. (i) and substituting it into the second equation in (B.14), one obtains

$$\Omega = \bar{G}(\bar{A}^\alpha, a^\alpha) + h(a^\alpha, \bar{a}^\alpha). \quad (\text{B.17})$$

Finally, the full solution of the constraints on the superpotential K is a sum of four pieces

$$K(A^\alpha, \bar{A}^\alpha, a^\alpha, \bar{a}^\alpha) = T(A^\alpha, \bar{A}^\alpha) + h(a^\alpha, \bar{a}^\alpha) + G(A^\alpha, \bar{a}^\alpha) + \bar{G}(\bar{A}^\alpha, a^\alpha). \quad (\text{B.18})$$

Taking into account the definition of the doublets A^α , a^α and their complex conjugates, as well as Laplace equations (3.12), we conclude that the first two terms in (B.18) correspond to the potentials of $\mathcal{N}=(4,4)$ supersymmetric sigma-model actions for two independent twisted multiplets $\hat{q}^{i a}$ and $\hat{q}^{i \bar{a}}$. The last two terms can be removed by the generalized Kähler gauge transformations in (3.3) corresponding to the gauge function g . So they do not make contribution into the $\mathbf{R}^{(1,1|2,2)}$ superfield action.

Thus, the final result for the potential K is

$$K(A^\alpha, \bar{A}^\alpha, a^\alpha, \bar{a}^\alpha) = T(A^\alpha, \bar{A}^\alpha) + h(a^\alpha, \bar{a}^\alpha). \quad (\text{B.19})$$

The proof for the case of the chiral and twisted-chiral $\mathcal{N}=(2,2)$ superfields which form a «self-dual» pair of $\mathcal{N}=(4,4)$ twisted multiplets follows the same route. It can be also straightforwardly extended to the case with multiple twisted multiplets of various types.

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Издательский отдел Объединенного института ядерных исследований
141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.

E-mail: publish@pds.jinr.ru

www.jinr.ru/publish/