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UNSTABLE EVEN-PARITY EIGENMODES
OF THE REGULAR STATIC $SU(2)$
YANG–MILLS-DILATON SOLUTIONS

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Четные собственные неустойчивые моды регулярных стационарных решений системы взаимодействующих $SU(2)$ Янга–Миллса и дилатонного полей

Получены четные собственные неустойчивые моды регулярных стационарных решений системы уравнений, описывающей систему взаимодействующих сферически-симметричных полей Янга–Миллса калибровочной группы $SU(2)$ и дилатонного поля в пространстве Минковского размерности $3 + 1$. Соответствующая матричная задача Штурма–Лиувилля решалась численно на основе непрерывного аналога метода Ньютона. Этот метод также был применен для решения краевой задачи и детально представлен в работе.

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Unstable Even-Parity Eigenmodes of the Regular Static $SU(2)$ Yang–Mills–Dilaton Solutions

In this paper, we obtain unstable even-parity eigenmodes of the regular static spherically symmetric solutions of the $SU(2)$ Yang–Mills–dilaton coupled system of equations in the $3 + 1$ Minkowski space-time. The corresponding matrix Sturm–Liouville problem is solved numerically by means of the continuous analogue of Newton’s method. The method, being the powerful tool for solving both boundary-value and Sturm–Liouville problems, is described in detail.

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INTRODUCTION

Regular stationary/static solutions of nonlinear field equations play a very important role in understanding of nonperturbative aspects of field dynamics in various modern field models, including gravity. During the last decade since Bartnik and McKinnon's discovery [1] of the regular particle-like solutions to the coupled system of the Einstein–Yang–Mills (EYM) equations there was a growing interest in revealing their possible physical and mathematical significance [2–5]. At the same time, the qualitatively similar (essentially non-Abelian ones) regular solutions were found in other various field models, i. e. in the coupled system of the Yang–Mills–dilaton (YMD) equations [6], in the coupled system of the Einstein–Yang–Mills–dilaton (EYMD) equations [7–10], in more sophisticated stringy inspired field models [11], etc (see Ref. [12] for a review).

The main physical and mathematical properties of the mentioned solutions are determined by the nonlinear nature of the $SU(2)$ Yang–Mills (YM) field, which is a substantial part of all these models. All the mentioned coupled systems of equations admit spherically symmetric particle-like solutions in which the Yang–Mills field configuration is bounded by gravity (EYM), by the scalar dilaton field (YMD), or by both of them (EYMD) in unstable equilibrium. (Note, pure YM field in the $3+1$ Minkowski space-time is self-repulsive and no regular pure YM configurations exist.) In terms of the linear perturbation analysis these solutions have even-parity unstable modes [6, 9, 13], which are called gravitating (EYM) or dilatonic (YMD) unstable modes.

On the other hand, in the case of a spherical symmetry the $SU(2)$ Yang–Mills self-interaction potential has a double well-shaped form and its two distinct minima correspond to the topologically distinct YM vacua. The relevant YM magnetic function must tend to the vacuum values at the origin and at the spatial infinity for all regular solutions. It was shown that some of those regular solutions could play a role at ultramicroscopic distance analogous to that of electroweak sphalerons [2] due to existence of the odd-parity Yang–Mills negative modes [3] (which are called sphaleronic unstable modes) and the corresponding fermion zero modes, which are responsible for the anomalous fermion production [5].

However, in this paper we concentrate on study of the even-parity unstable modes in the YMD system, since their significance is inspired by an interesting story, related to the singularity formation in nonlinear evolution equations. Indeed,

it has been discovered in the numerical studies of the massless fields collapse [14], there are two types of the collapse behavior. Type-II behavior is characterized by the mass gap absence in the black hole spectrum, hence the black holes with an arbitrary small mass can be obtained in this way. Later on, the Type-I behavior was observed as well [15–16]. It takes place if the considered system of Einstein-matter equations admits regular static finite energy asymptotically flat solutions. In this case the smallest black hole has a finite mass, which is equal to the mass of the lowest static solution. In the EYM system, the lowest static solution is the $N = 1$ Bartnick–McKinnon (BK) solution and its mass determines the smallest mass of the formed black hole in this type of a collapse. $N = 1$ BK solution is occurred to be an intermediate attractor, which the collapsing solution should attain in order to turn to the Type-I black hole formation scenario [15, 17, 18]. Moreover, $N = 1$ BK solution is occurred to be a threshold configuration, which separates the black hole formation and dispersion scenarios. The revealing of the threshold nature remains an unresolved problem and the studies of the static configurations decay via their proper unstable eigenmodes seem to be a very natural step towards understanding the threshold nature.

Thus, the goal of our present paper is to obtain unstable eigenmodes of the regular static solutions in the YMD system. It has been realized recently that the singularity (black hole) formation in gravity and blow-up in nonlinear wave equations share many common features [19–22], since one deals with the same class of the supercritical evolution PDEs [23]. As shown in Ref. [24], a coupled system of Yang–Mills-dilaton equations in $3 + 1$ Minkowski space is also a supercritical system in PDE terminology and it does exhibit blow-up (singularity formation) behavior, which is, in principle, equivalent to the black hole formation. So, we study the unstable modes of the static solutions in the YMD system, which is representative for a wide class of a supercritical evolution PDEs, including self-gravitating systems.

Our main tool to attack the problem numerically is a continuous analogue of the Newton’s method (CANM), which is a really powerful tool for solving both boundary-value and Sturm–Liouville problems. The key point of the method is an introduction of formal evolution parameter t and a corresponding reformulation of the problem in terms of a new system of PDE evolution equations. If necessary conditions are satisfied, the solution of the evolution problem approaches the solution of the initial (boundary-value or Sturm–Liouville) ODE problem as $t \rightarrow +\infty$.

The paper is organized as follows. In Sec. 1, we briefly remind the main features of the regular static spherically symmetric solutions to the coupled system of the $SU(2)$ Yang–Mills-dilaton equations. In Sec. 2, the iterative scheme, based on the continuous analogue of Newton’s method is introduced. The CANM method is used for numerical solution of the matrix Sturm–Liouville problem, which describes (unstable) eigenmodes of the regular static YMD solutions. Section

3 contains the results of the numerical experiments — the eigenvalues and the eigenfunctions of the mentioned Sturm–Liouville problem. We conclude with some remarks. A more detailed description of the CANM method is presented in Appendix.

1. BASIC EQUATIONS

Static spherically symmetric regular solutions of a coupled system of Yang–Mills-dilaton (YMD) equations in 3 + 1 Minkowski space-time were obtained and analyzed in Ref. [6]. In this section, we briefly remind their results.

A coupled system of YMD fields is given by the action

$$S = \frac{1}{4\pi} \int \left(\frac{1}{2} (\partial\Phi)^2 - \frac{\exp\{k\Phi\}}{4g^2} F^{a\mu\nu} F_{\mu\nu}^a \right) d^3x dt, \quad (1)$$

where Φ is the dilaton field, $F^{a\mu\nu}$ is the Yang–Mills field, k and g are dilaton and gauge coupling constants, respectively. Purely magnetic ansatz for the static $SU(2)$ spherically symmetric YM potential has a form

$$A_t^a = 0 \quad A_i^a = \epsilon_{aik} \frac{x^k}{r^2} [f(r) - 1], \quad (2)$$

hence the desired static solutions are described in terms of two independent functions: $f(r)$, the YM function, and $\Phi(r)$, the dilaton function. After substitution (2) into (1) and rescaling $\Phi \rightarrow \Phi/k$, $r \rightarrow (k/g)r$, $t \rightarrow (k/g)t$, and $S \rightarrow g * kS$, the dependence on two parameters k and g vanishes effectively. The integration over the angular variables gives the reduced action:

$$S = - \int_0^\infty \left\{ \frac{1}{2} r^2 \Phi'^2 + \exp\{\Phi\} \left[f'^2 + \frac{(f^2 - 1)^2}{2r^2} \right] \right\} dr, \quad (3)$$

where prime stands for derivatives with respect to the radial variable r . The corresponding field equations

$$f'' + f'\Phi' = -\frac{f(1-f^2)}{r^2} \quad \text{and} \quad (4a)$$

$$\Phi'' + \frac{2\Phi'}{r} = \frac{e^\Phi}{r^2} \left[f'^2 + \frac{(f^2 - 1)^2}{2r^2} \right] \quad (4b)$$

are obtained by a variation of the action (3) in respect to the functions $f(r)$ and $\Phi(r)$ under the following boundary conditions:

$$f(0) = \pm 1, \quad f(\infty) = \pm 1, \quad (5a)$$

$$\Phi'(0) = 0, \quad \Phi'(\infty) = 0. \quad (5b)$$

The invariance of the equations (4) in respect to the replacement $f \rightarrow -f$ allows one to put $f(0) = 1$ without loss of generality. The transformation $\Phi \rightarrow \Phi + \lambda$, $r \rightarrow r \exp\{-\lambda/2\}$ with $\lambda = \text{const}$ does not change the equations either, hence one can put $\Phi_0 = 0$.

The system of nonlinear equations (4) has two singular points $r = 0$ and $r = \infty$ and the solutions are supposed to be bounded at these points.

Regular solutions at the point $r = 0$ admit the following simple series expansions:

$$f(r)_{r \rightarrow 0} = 1 - b r^2 + O(r^4), \quad (6a)$$

$$\Phi(r)_{r \rightarrow 0} = \Phi_0 + b^2 r^2 + O(r^4). \quad (6b)$$

Here b is a free parameter, which finally provides the boundary conditions at the origin as

$$f(0) = 1, \quad f'(0) = 0, \quad (7a)$$

$$\Phi(0) = 0, \quad \Phi'(0) = 0. \quad (7b)$$

The asymptotical behavior of regular solutions at $r \rightarrow \infty$ has the following form:

$$f(r)_{r \rightarrow \infty} = (-1)^N \left(1 - \frac{c}{r} + \frac{3c^2 - cd}{4r^2} - \frac{11c^3 - 6c^2d + cd^2}{20r^3} + O(r^{-4}) \right),$$

$$\Phi(r)_{r \rightarrow \infty} = \Phi_\infty - \frac{d}{r} + O(r^{-4}), \quad (8)$$

where c , d , and Φ_∞ are parameters. As a result, the corresponding boundary conditions at the infinity are

$$\lim_{r \rightarrow \infty} f(r) = (-1)^N, \quad \lim_{r \rightarrow \infty} f'(r) = 0,$$

$$\lim_{r \rightarrow \infty} \Phi(r) = \Phi_\infty, \quad \lim_{r \rightarrow \infty} \Phi'(r) = 0, \quad (9)$$

for $N = 1, 2, \dots$

It is a well-known fact that the system of equations (4) has an infinite set of regular solutions, labelled by the number N of the nodes of YM function $f(r)$ [6, 25]. In Ref. [6], the solutions have been found numerically by use of the shooting strategy in respect to the free parameter b . Indeed, in order to get solutions it is possible to transform the boundary value problem (4), (7a), (9) into the initial value problem with the initial conditions (6) so that the desired solutions meet the asymptotics (9) at infinity. For the selection of the parameter b , the following key property [4] of the YM function was used: YM function $f(r)$ is strictly

bounded within the interval $-1 \leq f(r) \leq 1$ for all globally regular solutions of the system (4).

For our present purposes, the solutions of the system (4) have been found using the continuous analogue of Newton's method [30] (see Sec. 2). In this approach the problem is considered as a boundary value problem within some finite interval $r \in [0, R_\infty]$, where $0 < R_\infty < \infty$ is an «actual infinity». The boundary conditions can be written as follows:

$$\begin{aligned} f(0) = 1, f(R_\infty) = (-1)^N, \\ \Phi'(0) = 0, \Phi'(R_\infty) = 0. \end{aligned} \quad (10)$$

For numerical realization it is effective to use a boundary condition of a mixed type

$$\frac{1}{R_\infty} [f(R_\infty) - (-1)^N] + f'(R_\infty) = 0, \quad (11)$$

which directly follows from the asymptotics (8).

The static solutions corresponding to $N = 1, 2, 3, 4$ are shown in Fig. 1.

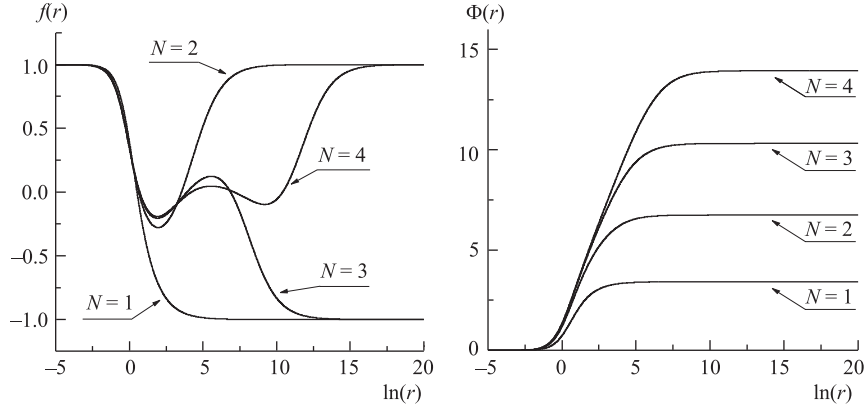


Fig. 1. Static solutions with $N = 1, 2, 3, 4$: YM function $f(r)$ (left), dilaton function $\Phi(r)$ (right)

Following the lines of Ref. [6] we shall consider small spherically symmetric perturbations of the obtained regular static solutions $f_N(r), \Phi_N(r)$ of the following form:

$$f(r) = f_N(r) + \epsilon \exp\{-\Phi_N/2\} v(r) \exp\{i\omega t\}, \quad (12a)$$

$$\Phi(r) = \Phi_N(r) + \epsilon \frac{\sqrt{2}}{r} u(r) \exp\{i\omega t\}. \quad (12b)$$

Here ϵ is a small parameter. Note that parity transformation $\theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi$, where θ, ϕ are spherical angles, is a symmetry operation for the chosen

perturbations; and that is why the corresponding eigenmodes are called even-parity eigenmodes.

After introducing the column of the perturbation functions $\chi \equiv (u, v)^T$ (upper index T everywhere signs the transposition) the effective action for the perturbation $\{u(r), v(r)\}$ in the leading non-vanishing order (ϵ^2) has the form

$$\tilde{S} = \int_0^\infty \left((\chi^+)' (\chi)' + p(r) (\chi^+)' (i\sigma_2) \chi + \chi^+ V(r) \chi - \omega^2 \chi^+ \chi \right) dr + \omega^2, \quad (13)$$

where σ_2 is the corresponding Pauli matrix. Note, we have added the term ω^2 into the effective action (13) in order to get the normalization condition (see below) by a formal variation in respect to the $\lambda = -\omega^2$. The elements of the matrix $V(r)$ and the function $p(r)$ are expressed in terms of the background static solution $\{f_N(r), \Phi_N(r)\}$ as follows:

$$\begin{aligned} V_{11}(r) &= \frac{\exp\{\Phi_N\}}{r^2} \left[f_N'^2 + \frac{(f_N^2 - 1)^2}{2r^2} \right], \\ V_{12}(r) &= V_{21}(r) = \frac{1}{\sqrt{2} r^2} [r \exp\{\Phi_N/2\} f_N']', \\ V_{22}(r) &= \frac{1}{2} \Phi_N'' + \frac{1}{4} \Phi_N'^2 + \frac{3f_N^2 - 1}{r^2}; \\ p(r) &= -2 \frac{\exp\{\Phi_N/2\} f_N'}{\sqrt{2} r}. \end{aligned}$$

In order to bring the eigenvalue problem to a self-adjoint form, the extended derivative D is introduced:

$$\begin{aligned} D\chi &= \chi' - i A' \sigma_2 \chi = \begin{pmatrix} u' - A'v \\ v' + A'u \end{pmatrix} \\ A(r) &= \int_0^r \frac{\exp[\phi_N(\xi)/2] f_N(\xi), \xi}{\sqrt{2} \xi} d\xi. \end{aligned} \quad (14)$$

Then the effective action (13) can be rewritten in the form

$$\tilde{S} = \int_0^\infty \left(D\chi^+ D\chi + \chi^+ \tilde{U}(r) \chi - \omega^2 \chi^+ \chi \right) dr + \omega^2. \quad (15)$$

The elements of the matrix $\tilde{U}(r)$ are expressed as

$$\begin{aligned}\tilde{U}_{11}(r) &= \frac{\exp\{\Phi_N\}}{2r^2} \left[f_N'^2 + \frac{(f_N^2 - 1)^2}{r^2} \right], \\ \tilde{U}_{12}(r) &= \tilde{U}_{21}(r) = \frac{1}{\sqrt{2}r^2} [r \exp\{\Phi_N/2\} f_N']', \\ \tilde{U}_{22}(r) &= \frac{1}{2} \Phi_N'' + \frac{1}{4} \Phi_N'^2 + \frac{3f_N^2 - 1}{r^2} - \frac{\exp\{\Phi_N\} f_N'^2}{2r^2}.\end{aligned}\quad (16)$$

The gauge transformation

$$\chi = \exp\{i A(r)\sigma_2\} \Psi(r), \quad (17)$$

where $\Psi = (\Psi_1, \Psi_2)^T$, allows one to bring the action to the final self-adjoint form

$$\tilde{S} = \int_0^\infty ((\Psi^+)')(\Psi)' + \Psi^+ U(r)\Psi - \omega^2 \Psi^+ \Psi \, dr + \omega^2, \quad (18)$$

where the elements of the matrix potential $U(r)$ are

$$U(r) = \exp\{-iA(r)\sigma_2\} \tilde{U}(r) \exp\{iA(r)\sigma_2\}. \quad (19)$$

The behavior of the matrix elements U_{ij} (19), corresponding to the static solutions with $N = 1, 2, 3, 4$ nodes, is demonstrated in Fig. 2.

Let us put $\omega^2 = -\lambda$, then we get a matrix Sturm–Liouville problem as a result of the unconditional extremum conditions of the functional (18) on the set of the variables $(\Psi_1, \Psi_2, \lambda)$:

$$-\Psi'' + U(r)\Psi + \lambda\Psi = 0, \quad (20)$$

on the semiaxes $0 \leq r < \infty$ with the boundary conditions

$$\Psi_1(0) = 0, \quad \Psi_2'(0) = 0, \quad \Psi_1(\infty) = 0, \quad \Psi_2(\infty) = 0 \quad (21)$$

and the norm condition

$$I(\Psi) = \int_0^\infty \Psi^+ \Psi \, dr - 1 \equiv \int_0^\infty (\Psi_1^2 + \Psi_2^2) \, dr - 1 = 0. \quad (22)$$

The linear stability of the static solutions is related to the spectrum of the eigenvalue problem (20)–(22): the negative eigenvalues $\omega^2 = -\lambda < 0$ correspond to the unstable modes. In Ref. [6], the Calogero phase functions method [26] has been applied in order to prove the unstable eigenmodes existence: it was shown

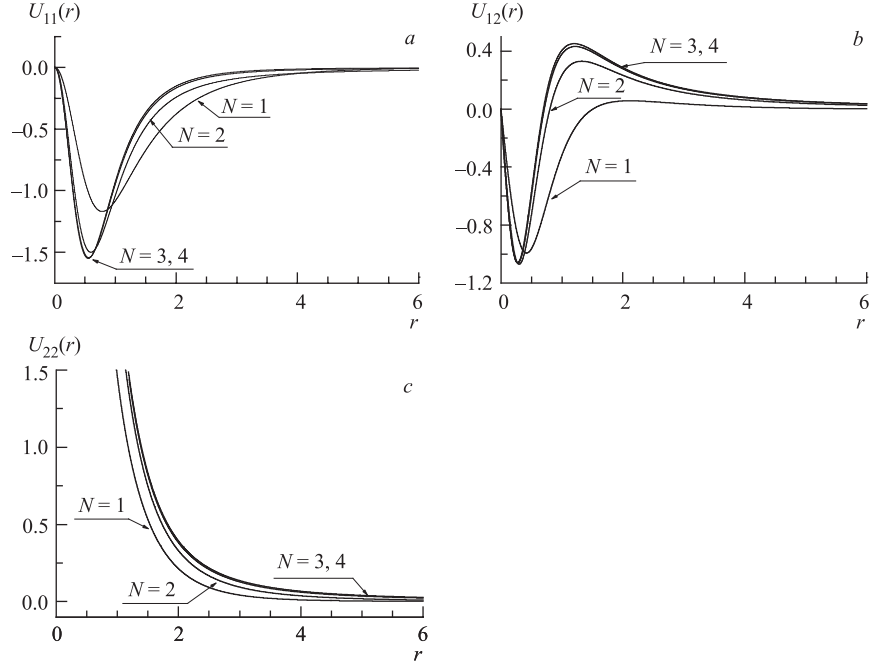


Fig. 2. The matrix elements U_{ij} (19): $U_{11}(r)$ (a), $U_{12}(r)$ (b), $U_{22}(r)$ (c), which correspond to the static solutions with $N = 1, 2, 3, 4$ nodes

that each static regular solution with N nodes of the YM function had exactly N unstable eigenmodes in the spectrum of the Sturm–Liouville problem (20)–(22).

The goal of our present paper is to get numerically the unstable eigenvalues and the corresponding eigenfunctions of the Sturm–Liouville problem (20)–(22) with an accuracy that should be sufficient for further simulation of the regular static solution decay via their unstable eigenmodes.

2. ITERATIVE SCHEME FOR THE STURM–LIOUVILLE PROBLEM

In what follows, the elaborated continuous analogue of Newton’s method (CANM) [27–30] was applied to solve our Sturm–Liouville problem.

The first step is to replace the semi-infinite interval $r \in [0, \infty)$ by the interval $r \in [0, R_\infty]$ for the numerical solution of the problem. Here $R_\infty \gg 0$ is the «actual infinity». At the same time, the problem (20)–(22) can be rewritten in the

form

$$E(\Psi, \lambda) = -\Psi'' + U\Psi + \lambda\Psi = 0, \quad (23a)$$

$$G_L(\Psi(0), \Psi'(0), \lambda) = 0, \quad G_R(\Psi(R_\infty), \Psi'(R_\infty), \lambda) = 0, \quad (23b)$$

$$I(\Psi) = \int_0^{R_\infty} \Psi^+ \Psi dr - 1 = \int_0^{R_\infty} [\Psi_1^2(r) + \Psi_2^2(r)] dr - 1 = 0, \quad (23c)$$

where G_L and G_R are boundary conditions (of mixed type, in general) at the left ($r = 0$) and at the right ($r = R_\infty$) ends of the interval.

The boundary conditions G_L on the left side are already imposed in (21), so we just rewrite it as

$$\begin{aligned} G_{1L}(\Psi(0), \Psi'(0), \lambda) &\equiv \Psi_1(0) = 0, \\ G_{2L}(\Psi(0), \Psi'(0), \lambda) &\equiv \Psi_2'(0) = 0. \end{aligned} \quad (24)$$

The consistency of the imposed boundary condition $G_{1L} = G_{2L} = 0$ is confirmed by the local solutions behavior, obtained from the series expansions of the system (23a) at the vicinity of the origin:

$$\begin{aligned} U_{11}(r)_{r \rightarrow 0} &= 2b^3(b-4)r^2 + O(r^4), \\ U_{12}(r)_{r \rightarrow 0} &= U_{21}(r)_{r \rightarrow 0} = \frac{4}{3}\sqrt{2}b^2(b-3)r + O(r^3), \\ U_{22}(r)_{r \rightarrow 0} &= \frac{2}{r^2} + 3b(b-2) + \frac{6}{5}b^2(4+4b-b^2)r^2 + O(r^4); \\ \Psi_1(r)_{r \rightarrow 0} &= \tilde{C}_1 \left(r + \frac{1}{6}\lambda r^3 \right) + O(r^5), \\ \Psi_2(r)_{r \rightarrow 0} &= \tilde{C}_2 r^2 + O(r^4). \end{aligned}$$

Here \tilde{C}_1 and \tilde{C}_2 are free parameters. The static solutions behavior (6) was used to obtain the matrix elements $U_{ij}(r)$ asymptotics at $r \rightarrow 0$.

The boundary conditions G_R at the right end ($r = R_\infty$) can be defined in a similar way, taking into account the local solutions behavior at $r \rightarrow \infty$. Indeed, using the asymptotic behavior of the static solutions (8) and the matrix elements $U_{ij}(r)$:

$$\begin{aligned} U_{11}(r)_{r \rightarrow \infty} &= \left(\frac{2}{r^2} - \frac{6c+d}{r^3} \right) \sin^2(A_\infty) + O(r^{-4}), \\ U_{12}(r)_{r \rightarrow \infty} &= U_{21}(r)_{r \rightarrow \infty} = \left(-\frac{1}{r^2} + \frac{6c+d}{2r^3} \right) \sin(2A_\infty) + O(r^{-4}), \quad (25) \\ U_{22}(r)_{r \rightarrow \infty} &= \left(\frac{2}{r^2} - \frac{6c+d}{r^3} \right) \cos^2(A_\infty) + O(r^{-4}), \end{aligned}$$

where the value $A_\infty \equiv A(r = \infty)$ (according to (14)), we obtain the local solutions for the functions Ψ given by the following series expansion:

$$\begin{aligned}\Psi_1(r)_{r \rightarrow \infty} &= \left[C_1 \left(1 + \frac{\sin^2(A_\infty)}{\sqrt{\lambda} r} \right) - C_2 \frac{\sin(2A_\infty)}{2\sqrt{\lambda} r} + O(r^{-2}) \right] e^{-\sqrt{\lambda} r}, \\ \Psi_2(r)_{r \rightarrow \infty} &= \left[C_2 \left(1 + \frac{\cos^2(A_\infty)}{\sqrt{\lambda} r} \right) - C_1 \frac{\sin(2A_\infty)}{2\sqrt{\lambda} r} + O(r^{-2}) \right] e^{-\sqrt{\lambda} r}.\end{aligned}\quad (26)$$

If we restrict our considerations within the terms of order $O(r^{-1})e^{-\sqrt{\lambda} r}$, the following couple (G_{1R}, G_{2R}) of the boundary conditions at the right end $r = R_\infty$ is given by

$$G_{1R}(\Psi(R_\infty), \Psi'(R_\infty), \lambda) \equiv \Psi_1'(R_\infty) + \sqrt{\lambda} \Psi_1(R_\infty) = 0, \quad (27a)$$

$$G_{2R}(\Psi(R_\infty), \Psi'(R_\infty), \lambda) \equiv \Psi_2'(R_\infty) + \sqrt{\lambda} \Psi_2(R_\infty) = 0. \quad (27b)$$

The problem (23) is equivalent to the nonlinear functional equation [29]

$$\varphi(z) \equiv \begin{pmatrix} E(z) \\ G_L(z) \\ G_R(z) \\ I(z) \end{pmatrix} = 0, \quad (28)$$

where the elements $z \equiv (\lambda, \Psi)$, $\lambda \in \mathbb{R}$, and $\Psi_i(r)$ belong to the set of twice differentiable functions in the finite interval $[0, R_\infty]$, which satisfy the boundary conditions (23b).

It is supposed that an isolated solution $z^* \equiv (\lambda^*, \Psi^*)$ of the equation (28) does exist. Then, according to the CANM [27–30], a formal evolution parameter t is introduced. Thus, the following evolution equation

$$\frac{d}{dt} \varphi(z(t), t) = -\varphi(z(t), t), \quad 0 < t < \infty, \quad (29)$$

is considered with the initial condition $z(0) = z_0$.

It is also supposed that the given initial data $z_0 \equiv (\lambda_0, \Psi_0)$ is chosen in some vicinity of the exact desired solution z^* in a corresponding functional space. In Ref. [30], it was shown that if additionally:

1. $\varphi(z)$ is a smooth function;
2. the operator $[\varphi'(z)]^{-1}$ exists and is bounded in some vicinity of the point $z(0) = z_0$;

then the equation (29) has a unique local solution $z(t)$ in the vicinity of z^* , and

$$\lim_{t \rightarrow \infty} z(t) = z^*.$$

We assume the above two conditions are fulfilled.

Let us introduce the functions $w(r, t)$ and $\mu(t)$ in the following way:

$$w(r, t) = \frac{\partial \Psi(r, t)}{\partial t} = \begin{pmatrix} w_1(r, t) \\ w_2(r, t) \end{pmatrix}, \quad (30a)$$

$$\mu(t) = \lambda(t) + \frac{d\lambda(t)}{dt}. \quad (30b)$$

Then the equation (29) in our particular example is rewritten in terms of a couple of equations, which allow one to determine the introduced functions $w(r, t)$ and $\mu(t)$ as follows:

$$-w'' + (U + \lambda I)w = -(-\Psi'' + (U + \mu I)\Psi), \quad (31a)$$

$$\frac{d}{dt}G_L(\Psi(R_\infty), \Psi'(R_\infty), \lambda) = -G_L(\Psi(R), \Psi'(R_\infty), \lambda), \quad (31b)$$

$$\frac{d}{dt}G_R(\Psi(R_\infty), \Psi'(R_\infty), \lambda) = -G_R(\Psi(R), \Psi'(R_\infty), \lambda), \quad (31c)$$

$$\int_0^{R_\infty} [\Psi_1(r, t)w_1(r, t) + \Psi_2(r, t)w_2(r, t)] dr - \frac{1}{2} \left[\int_0^{R_\infty} (\Psi_1^2(r, t) + \Psi_2^2(r, t)) dr - 1 \right] = 0, \quad (31d)$$

where I is a 2×2 unit matrix.

In order to solve the PDE evolution problem (30), (31) numerically, we have used a standard finite-difference technique. Let $t^0, t^1, \dots, t^k, \dots, t^0 = 0, t^{k+1} - t^k = \tau^k$ be a given discretization of the «time» t . We assume the step $0 < \tau^k \leq 1$ is properly specified. Then we use the Euler scheme in order to obtain the next $(k+1)$ th approximation to the exact solutions for the eigenfunction $\Psi^{k+1} \equiv \Psi(r, t^{k+1})$ and the eigenvalues $\lambda^{k+1} \equiv \lambda(t^{k+1})$:

$$\Psi^{k+1}(r) = \Psi^k(r) + \tau^k w^k(r), \quad (32a)$$

$$\lambda^{k+1} = \lambda^k + \tau^k (\mu^k - \lambda^k). \quad (32b)$$

It is supposed that $\Psi^k(r)$ and λ^k are known at the k th iteration step, whereas the $\Psi^0(r)$ and λ^0 are some properly chosen initial data.

At each time step k , the iterative corrections $w^k(r)$ are calculated in the following way:

$$w^k = \zeta^k + \mu^k \eta^k, \quad (33)$$

where the functions $\zeta^k \equiv (\zeta_1^k, \zeta_2^k)^T$ are solutions of the problem (for sake of simplicity we will henceforth omit the number of iterations k):

$$-\zeta_1'' + (U_{11} + \lambda) \zeta_1 + U_{12} \zeta_2 = \Psi_1'' - U_{11} \Psi_1 - U_{12} \Psi_2, \quad (34a)$$

$$-\zeta_2'' + (U_{22} + \lambda) \zeta_2 + U_{12} \zeta_1 = \Psi_2'' - U_{11} \Psi_2 - U_{12} \Psi_1, \quad (34b)$$

$$\zeta_1(0) = -\Psi_1(0), \quad \zeta_2'(0) = -\Psi_2'(0), \quad (34c)$$

$$\zeta_1'(R_\infty) + \sqrt{\lambda} \zeta_1(R_\infty) = -\Psi_1'(R_\infty) - \frac{\sqrt{\lambda}}{2} \Psi_1(R_\infty), \quad (34d)$$

$$\zeta_2'(R_\infty) + \sqrt{\lambda} \zeta_2(R_\infty) = -\Psi_2'(R_\infty) - \frac{\sqrt{\lambda}}{2} \Psi_2(R_\infty), \quad (34e)$$

whereas the functions $\eta^k \equiv (\eta_1^k, \eta_2^k)^T$ are solutions of the problem:

$$-\eta_1'' + (U_{11} + \lambda) \eta_1 + U_{12} \eta_2 = -\Psi_1, \quad (35a)$$

$$-\eta_2'' + (U_{22} + \lambda) \eta_2 + U_{12} \eta_1 = -\Psi_2, \quad (35b)$$

$$\eta_1(0) = 0, \quad \eta_2(0) = 0, \quad (35c)$$

$$\eta_1'(R_\infty) + \sqrt{\lambda} \eta_1(R_\infty) = -\frac{1}{2\sqrt{\lambda}} \Psi_1(R_\infty), \quad (35d)$$

$$\eta_2'(R_\infty) + \sqrt{\lambda} \eta_2(R_\infty) = -\frac{1}{2\sqrt{\lambda}} \Psi_2(R_\infty). \quad (35e)$$

We used the finite difference method for the discretization in respect to the spatial variable r . The corresponding discrete linear problem (34), (35) has been solved by a matrix sweat method. After that the value of the parameter μ^k was calculated by means of the expression (31):

$$\mu = \frac{1 - \int_0^{R_\infty} (\Psi_1^2 + \Psi_2^2) dr - 2 \int_0^{R_\infty} (\Psi_1 \zeta_1 + \Psi_2 \zeta_2) dr}{2 \int_0^{R_\infty} (\Psi_1 \eta_1 + \Psi_2 \eta_2) dr}. \quad (36)$$

At the k th iteration step, Ψ^k and λ^k are known, then at the $(k+1)$ th one, Ψ^{k+1} and λ^{k+1} are calculated according to (32). The iteration process goes on until the fulfilment of the inequality $\delta^k \leq \epsilon$, where $\epsilon > 0$ is a given small value, and the discrepancy δ^k is defined by the following relation:

$$\delta^k = \|\varphi(\lambda^k, \Psi^k)\| \equiv \max_{r \in [0, R_\infty]} |\varphi(\lambda^k, \Psi^k)|.$$

The best convergence of the method was achieved for the following choice [30] of the step τ^k , $k > 0$:

$$\tau^k = \begin{cases} \min(1, \tau^{k-1} \delta^{k-1} / \delta^k), & \delta^k \leq \delta^{k-1}, \\ \max(\tau^0, \tau^{k-1} \delta^{k-1} / \delta^k), & \delta^k > \delta^{k-1}, \end{cases} \quad (37)$$

for some given τ_0 , which depends on the initial approximation z_0 .

In order to check and compare our results, we have also solved the following non-self-adjoint Sturm–Liouville problem:

$$-\chi'' + P(r) \chi' + Q(r) \chi = -\lambda \chi, \quad 0 < r < \infty, \quad (38)$$

which follows from the perturbation functional (13). Here $P(r)$ and $Q(r)$ are 2×2 matrices, and $P(r)$ is an antisymmetrical one. The boundary conditions are analogous to the conditions (21).

We used the collocation method for the discretization of appropriate linearized boundary problems at each iteration step. The accuracy of appropriate difference scheme on the analytical grid with the step h is $O(h^2)$. The choice of the iteration step τ_k is realized by the Ermakov–Kalitkin formula [30].

3. NUMERICAL RESULTS

To obtain the eigenmodes of the regular static YMD solutions, the following problems have been solved:

a) the static background solutions, labelled by the number N of the nodes of YM function, which are solutions to the boundary-value problem (4), (5);

b) the eigenmodes, which are solutions to the self-adjoint Sturm–Liouville problem (20)–(22) or (equivalently) to the non-self-adjoint Sturm–Liouville problem (38).

Both problems were solved using the continuous analogue of Newton’s method.

The semi-infinity interval $[0, \infty)$ was changed to the interval $[0, R_\infty]$ for the numerical reasons. The influence of the «actual infinity» R_∞ was studied by the establishment method. The results for the case $N = 1$ on the mesh exponentially condensed to the origin $r = 0$ are shown in Table 1.

Here NM is the number of elements of the mesh. It is evident that each doubled value of R_∞ leads to the establishment of at least one significant digit after decimal point.

One of the specific features of the problem is the rapid increase of R_∞ as the number N (which labels the corresponding background solution) grows.

Table 1. Influence of the parameter R_∞ on the eigenvalue λ_1^1

R_∞	NM	$\lambda_1^1 \equiv -\omega^2$
50	101	0.0905620823346630
100	201	0.0905653977396295
200	401	0.0905657469757358
400	801	0.0905657826471273
800	1601	0.0905657861599309
1600	3201	0.0905657864933593

Table 2 shows sufficient values of R_∞ needed for calculation of the unstable eigenmodes for the background solutions with $N = 1, 2, \dots, 6$.

In Table 2, the corresponding first eigenvalues $\{\lambda_N^i\}_{i=1}^N$ are shown, which correspond to the background solutions with $N = 1, 2, \dots, 6$.

Table 2. First eigenvalues λ_N^1 and appropriate parameters R_∞ and b

N	λ_N^1	R_∞	b
1	$9.0566 \cdot 10^{-2}$	$2 \cdot 10^3$	1.043320582
2	$7.5382 \cdot 10^{-2}$	$2 \cdot 10^5$	1.414072399
3	$4.9346 \cdot 10^{-2}$	$2 \cdot 10^7$	1.500007215
4	$4.3455 \cdot 10^{-2}$	$1 \cdot 10^8$	1.515017863
5	$4.2434 \cdot 10^{-2}$	$1 \cdot 10^9$	1.517493316
6	$4.2266 \cdot 10^{-2}$	$3 \cdot 10^{10}$	1.517897653
∞	$\approx 4.22 \cdot 10^{-2}$		≈ 1.518

All the values of b are obtained numerically by means of the comparison of the background solutions to the boundary-value problem (4), (5) and the local expansions (6). In Ref. [6], it is shown that the main background solution parameters tend to some limit values as $N \rightarrow \infty$. This limit solution can be characterized by the parameter $b_\infty \approx 1.518$. This allows us to determine the first limit eigenvalue using an extrapolation of λ_N^1 as a function of N : $\lambda_\infty^1 \approx 4.22 \cdot 10^{-2}$ (See Table 2).

In Table 3, all the eigenvalues $\{\lambda_N^i\}_{i=1}^N$, which correspond to the background solutions with $N = 1, 2, 3, 4$, are presented. The first three eigenvalues for the

Table 3. Eigenvalues $\{\lambda_N^i\}_{i=1}^N$

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
1	$9.0566 \cdot 10^{-2}$			
2	$7.5382 \cdot 10^{-2}$	$2.0742 \cdot 10^{-4}$		
3	$4.9346 \cdot 10^{-2}$	$1.4957 \cdot 10^{-4}$	$1.9622 \cdot 10^{-7}$	
4	$4.3455 \cdot 10^{-2}$	$5.9905 \cdot 10^{-5}$	$1.3278 \cdot 10^{-7}$	$\sim 10^{-9}$

background $N = 4$ solutions are also presented. For a given background solution, labelled by the number N , the fast decrease of the value λ_N^i as i increases is obvious. Hence, only the main (minimal) eigenmodes are presented below.

The eigenfunctions $\Psi_1^1(r)$ and $\Psi_2^1(r)$ for $N = 1, 2, 3, 4$ are shown in Fig. 3. The eigenfunctions $\Psi_1^2(r)$ and $\Psi_2^2(r)$ for $N = 2, 3, 4$ are shown in Fig. 4. The eigenfunctions $\Psi_1^3(r)$ and $\Psi_2^3(r)$ for $N = 3, 4$ are shown in Fig. 5.

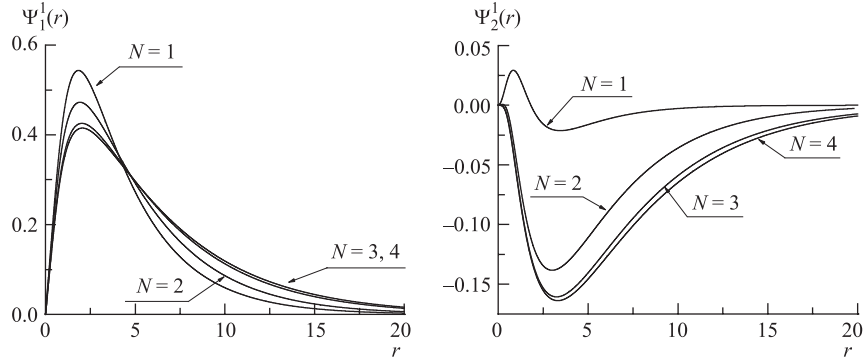


Fig. 3. Eigenfunctions Ψ_1^1 and Ψ_2^1 for $N = 1, 2, 3, 4$

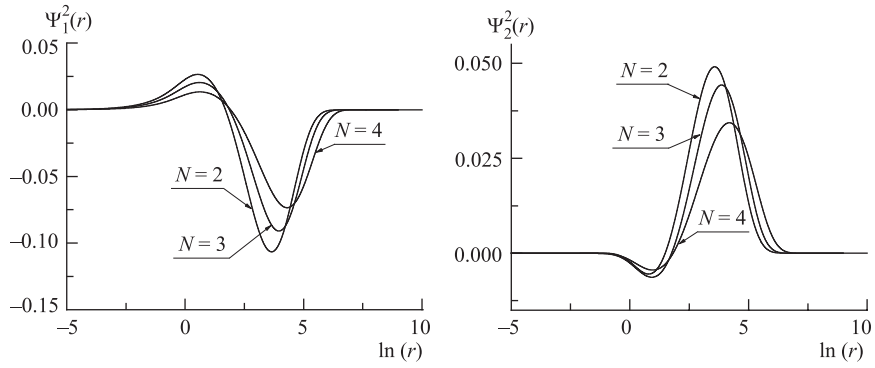


Fig. 4. Eigenfunctions Ψ_1^2 and Ψ_2^2 for $N = 2, 3, 4$

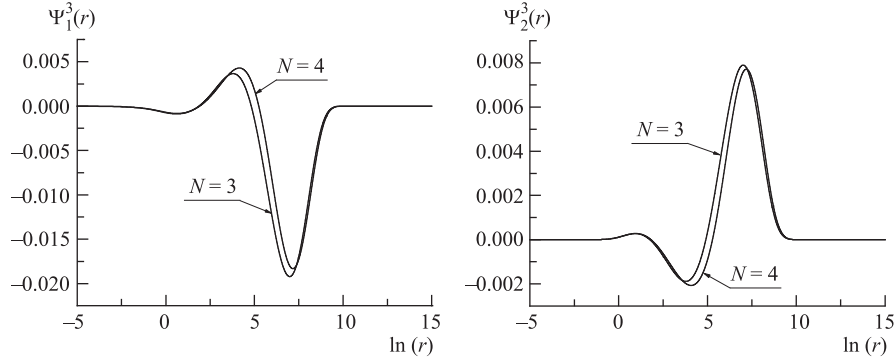


Fig. 5. Eigenfunctions Ψ_1^3 and Ψ_2^3 for $N = 3, 4$

CONCLUSIONS AND OUTLOOK

We have found the eigenvalues and the eigenfunctions for the matrix Sturm–Liouville problem, which are proper unstable modes of the regular static spherically symmetric solutions in the coupled system of Yang–Mills–dilaton equations. The efficiency of the continuous analogue of Newton’s method, which was used to solve the corresponding numerical problem, is demonstrated. The CANM, being a universal tool for numerical solution of both boundary value and Sturm–Liouville problems, is described in detail.

The obtained unstable modes are used as initial data in the static solutions decay problem in a nonlinear regime. This task is in progress now and will be reported soon [31]. The mentioned nonlinear decay problem arises in a wide class of mathematical physics problems, inspired by the recent progress in understanding the singularity formation in nonlinear wave equations and in massless fields collapse problem.

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APPENDIX

To facilitate the readers, the CANM is presented briefly below. Let us consider the nonlinear equation

$$\chi(x) = 0, \quad (39)$$

where the nonlinear operator $y = \chi(x)$ maps the Banach space X (the element $x \in X$) into the Banach space Y ($y \in Y$). Solution to Eq. (39) means finding such elements $x^* \in X$ that are mapped by the operator χ into the trivial element of space Y .

If we suppose that x_k is a known approximation to the sought solution at the k th iteration stage, then the increment Δx_k is computed by the formula $\Delta x_k = \psi(x_k)$, $x_{k+1} = x_k + \Delta x_k$, $k = 0, 1, 2, \dots$, where x_0 is a given element.

The used method of iteration determines the manner of constructing the function $\psi(x)$. For example, in the case of Newton's method $\psi(x) = -[\chi'(x)]^{-1}\chi(x)$, where $\chi'(x)$ is a linear operator, it is the Frechét derivative of function $\chi(x)$. For each iteration process of the kind mentioned above, one can build a continuous analogue introducing a continuous parameter t , $0 \leq t < \infty$, instead of the discrete variable k , ($k = 0, 1, 2, \dots$). Further we suppose the smooth dependence $x = x(t)$ and introduce the derivative $\frac{d}{dt}x(t)$ instead of the increment Δx_k . In this way, we obtain the differential equation

$$\frac{d}{dt}x(t) = -\psi[x(t)], \quad x(0) = x_0. \quad (40)$$

Thus, the solving of the original Eq. (39) is realized by the solving of the above Cauchy problem (40) on the positive half-axis $0 \leq t < \infty$.

There have been proved numerous theorems (see [27, 28, 30]) related to the convergence to the isolated solution x^* of the continuous analogues of various iteration methods.

As a particular case of the continuous analogue of Newton's method (CANM) we can present Eq. (40) as

$$\frac{d}{dt}\chi[x(t)] = -\chi[x(t)], \quad x(0) = x_0. \quad (41)$$

From here we obtain the first integral

$$\chi[x(t)] = e^{-t}\chi(x_0).$$

If the function $\chi(x)$ is smooth and the operator $[\chi'(x)]^{-1}$ is bounded in the vicinity of initial approximation x_0 , then in the same vicinity there exists an isolated root x^* of Eq. (39) and $\lim_{t \rightarrow \infty} x(t) \rightarrow x^*$.

For example, let us consider the convergence conditions of CANM for the following simple nonlinear boundary value problem:

$$\chi(y) \equiv \{y'' + f(x, y), y(0), y(1)\} = 0, \quad x \in (0, 1). \quad (42)$$

Theorem. *Let the solution of BVP (42) exist and be localizable. Furthermore (i) the function $f(x, y)$ is smooth in some domain D ;*

(ii) the boundary-value problem

$$v'' + f'_y(x, y)v = 0, \quad v(0) = v(1) = 0$$

has only a trivial solution for each smooth function $y(x) \in D$;

(iii) $\|y'_0 + f(x, y_0)\| \leq \epsilon$, where $\epsilon > 0$ is little enough, and function $y_0(x)$, smooth in domain D , is an initial approximation of the sought solution $y^*(x)$.

Then the system with respect to functions $y(x, t)$ and $v(x, t)$

$$v''_{xx} + f'_y(x, y)v = -[y''_{xx} + f(x, y)], \quad y'_t = v$$

with boundary conditions

$$v(0, t) = v(1, t) = 0$$

and initial condition

$$y(x, 0) = y_0(x)$$

has on half-strip $s = \{(x, t): 0 \leq x \leq 1, 0 \leq t < \infty\}$ a unique solution, subjected to the condition

$$\lim_{t \rightarrow \infty} \|y(x, t) - y^*(x)\|_{C^2[0,1]} = 0.$$

The most simple method for approximated integration of the problem (40) is the Euler method. Let us build the set t_k , $k = 0, 1, 2, \dots$ and $\tau_k = t_{k+1} - t_k$. Then the following sequence of linear problems is reached:

$$\chi'(x_k)v_k = -\chi(x_k), \quad x_{k+1} = x_k + \tau_k v_k, \quad k = 0, 1, 2, \dots,$$

where x_0 is a predetermined element. When the parameter $\tau_k \equiv 1$, we obtain the classical Newton's method.

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