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SOME SHARP NORM ESTIMATES IN THE SPECTRAL SUBSPACE PERTURBATION PROBLEM

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Некоторые точные оценки в задаче возмущения
спектральных подпространств

Обсуждается задача возмущения спектральных подпространств, отвечающих изолированным частям спектра самосопряженного оператора. Получены точные априорные оценки вариации спектрального подпространства при внедиагональных возмущениях заданного самосопряженного оператора в предположении, что выпуклая оболочка соответствующей части спектра не пересекается с остальным спектром этого оператора. Дано обобщение $\tan 2\Theta$-теоремы Дэвиса–Кагана на случай неограниченных возмущений.

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Some Sharp Norm Estimates in the
Spectral Subspace Perturbation Problem

We discuss the spectral subspace perturbation problem for a self-adjoint operator. Assuming that the convex hull of a part of its spectrum does not intersect the remainder of the spectrum, we establish an *a priori* sharp bound on variation of the corresponding spectral subspace under off-diagonal perturbations. We also extend the Davis–Kahan $\tan 2\Theta$ Theorem in case of some unbounded perturbations.

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1. INTRODUCTION

Assume that the spectrum of a self-adjoint operator A on a Hilbert space $\mathfrak{H}$ consists of two disjoint components σ_- and σ_+, i.e. $\mathrm{spec}(A) = \sigma_- \cup \sigma_+$ and

$$d = \mathrm{dist}(\sigma_-, \sigma_+) > 0. \tag{1.1}$$

Then $\mathfrak{H}$ is decomposed into the orthogonal sum $\mathfrak{H} = \mathfrak{H}_- \oplus \mathfrak{H}_+$ of the spectral subspaces $\mathfrak{H}_\pm = \mathrm{Ran}\mathsf{E}_A(\sigma_\pm)$, where $\mathsf{E}_A(\delta)$ denotes the spectral projection of A associated with a Borel set $\delta \subset \mathbb{R}$. It is well known (see, e.g. [18, §135]) that sufficiently small self-adjoint perturbation V of A does not close the gaps between the sets σ_- and σ_+, which allows one to think of the corresponding disjoint spectral components σ'_- and σ'_+ of the perturbed operator $L = A + V$ as a result of the perturbation of the spectral sets of σ_- and σ_+, respectively. Moreover, the decomposition $\mathfrak{H} = \mathfrak{H}'_- \oplus \mathfrak{H}'_+$ with $\mathfrak{H}'_\pm = \mathrm{Ran}\,\mathsf{E}_L(\sigma'_\pm)$ is continuous in V in the sense that the projections $\mathsf{E}_L(\sigma'_\pm)$ converge to $\mathsf{E}_A(\sigma_\pm)$ in the operator norm topology as $\|V\| \to 0$.

Given a mutual disposition of the spectral components $\sigma_\pm$ of the operator A, the problem of perturbation theory is to study variation of these components and the corresponding spectral subspaces under the perturbation V. In particular, the questions of interest are the following (see [12, 15]):

(i) Under what (sharp) condition imposed on $\|V\|$ do the gaps between the sets σ_- and σ_+ remain open, i.e. $\mathrm{dist}(\sigma'_-, \sigma'_+) > 0$?

(ii) Having established this condition, can one ensure that it implies inequality

$$\|\mathsf{E}_L(\sigma'_-) - \mathsf{E}_A(\sigma_-)\| < 1? \tag{1.2}$$

(Surely, (1.2) holds if and only if inequality $\|\mathsf{E}_L(\sigma'_+) - \mathsf{E}_A(\sigma_+)\| < 1$ does.)

In general, answer to the question (i) is well known: the gaps between σ_- and σ_+ remain open if

$$\|V\| < \frac{d}{2}. \tag{1.3}$$

Among all perturbations of the operator A we distinguish the ones that are off-diagonal with respect to the decomposition $\mathfrak{H} = \mathrm{Ran}\,\mathsf{E}_A(\sigma_-) \oplus \mathrm{Ran}\,\mathsf{E}_A(\sigma_+)$, i.e. the perturbations that anticommute with the difference

$$J = \mathsf{E}_A(\sigma_+) - \mathsf{E}_A(\sigma_-) \tag{1.4}$$

of the spectral projections $\mathsf{E}_A(\sigma_+)$ and $\mathsf{E}_A(\sigma_-)$. If one restricts oneself to perturbations V of this class, then inequality $\operatorname{dist}(\sigma'_-, \sigma'_+) > 0$ is ensured by the weaker condition

$$\|V\| < \frac{\sqrt{3}}{2} d \tag{1.5}$$

proven in [15, Theorem 1]. Similarly to (1.3), condition (1.5) is sharp.

For a review of the known answers to the question (ii), we refer to [12] in case of the general bounded perturbations and to [15] in case of the off-diagonal ones. Notice that complete answers to the question (ii) were found only by certain additional assumptions on the mutual disposition of the sets σ_- and σ_+. It is still an open problem whether the corresponding conditions (1.3) and (1.5) imply (1.2) under the only assumption (1.1) or not.

In the present paper, we are concerned with the off-diagonal perturbations and restrict ourselves to two particular mutual dispositions of the spectral sets σ_- and σ_+. The first one corresponds to the case where the sets σ_- and σ_+ are subordinated, say

$$\sup \sigma_- < \inf \sigma_+. \tag{1.6}$$

The second case under consideration corresponds to a disposition with one of the sets σ_- and σ_+ lying in a gap of the other set, say

$$\sigma_+ \cap \operatorname{conv}(\sigma_-) = \varnothing, \tag{1.7}$$

where $\operatorname{conv}(\sigma)$ denotes the convex hull of a set $\sigma \subset \mathbb{R}$.

In both these cases, the perturbed spectral sets σ'_- and σ'_+ are known to remain disjoint under requirements on $\|V\|$ much weaker than that of (1.5).

In particular, if (1.6) holds then for any bounded off-diagonal perturbation V the interval $(\sup \sigma_-, \inf \sigma_+)$ belongs to the resolvent set of the perturbed operator $L = A + V$, and thus $\sigma'_- \subset (-\infty, \sup \sigma_-]$ and $\sigma'_+ \subset [\inf \sigma_+, +\infty)$ (see [2], [7]; cf. [14]). Moreover, in this case the following norm estimate holds [7]:

$$\|\mathsf{E}_L(\sigma'_-) - \mathsf{E}_A(\sigma_-)\| \leqslant \sin\Big(\frac{1}{2} \arctan \frac{2\|V\|}{d}\Big) < \frac{\sqrt{2}}{2}.$$

This (sharp) bound on the difference of the spectral projection $\mathsf{E}_L(\sigma'_-)$ and $\mathsf{E}_A(\sigma_-)$ is known as the Davis–Kahan $\tan 2\Theta$ Theorem, since it can be written in the equivalent form $\|\tan 2\Theta\| \leqslant \dfrac{\|V\|}{d}$, where Θ is the operator angle between the subspaces $\mathfrak{H}'_-$ and $\mathfrak{H}_-$ (or between the subspaces $\mathfrak{H}'_+$ and $\mathfrak{H}_+$). For definition of the operator angle between two subspaces see, e.g. [13].

Our first principal result is an extension of the $\tan 2\Theta$ Theorem, which holds not only for bounded but also for some unbounded off-diagonal perturbations V.

Theorem 1. *Given a self-adjoint operator A on the Hilbert space $\mathfrak{H}$, assume that*

$$\operatorname{spec}(A) = \sigma_- \cup \sigma_+ \quad \textit{and} \quad \sup \sigma_- < \inf \sigma_+.$$

Suppose that a symmetric operator V on $\mathfrak{H}$ with $\operatorname{Dom}(V) \supset \operatorname{Dom}(A)$ is off-diagonal with respect to the decomposition $\mathfrak{H} = \operatorname{Ran} \mathsf{E}_A(\sigma_-) \oplus \operatorname{Ran} \mathsf{E}_A(\sigma_+)$ and the closure $L = \overline{A+V}$ of the sum $A+V$ with $\operatorname{Dom}(A+V) = \operatorname{Dom}(A)$ is a self-adjoint operator. Then the spectrum of L consists of two subordinate components σ'_- and σ'_+ such that

$$\sigma'_- \subset (-\infty, \sup \sigma_-], \quad \sigma'_+ \subset [\inf \sigma_+, +\infty),$$

and the following inequality holds

$$\|\mathsf{E}_L(\sigma'_-) - \mathsf{E}_A(\sigma_-)\| \leqslant \sin\left(\frac{1}{2} \arctan \varkappa\right), \tag{1.8}$$

where

$$\varkappa = \inf_{\sup \sigma_- < \mu < \inf \sigma_+} \sup_{\substack{x \in \operatorname{Dom}(A) \\ \|x\| = 1}} \frac{|\langle x, JVx \rangle|}{\langle x, |A - \mu| x \rangle}$$

with J given by (1.4).

Notice that throughout the paper we adopt the natural convention that $\arctan(+\infty) = \pi/2$. In particular, under this convention, inequality (1.8) for $\varkappa = +\infty$ reads $\|\mathsf{E}_L(\sigma'_-) - \mathsf{E}_A(\sigma_-)\| \leqslant \frac{\sqrt{2}}{2}$.

By Remark 4.6 (*iii*) below, the estimate (1.8) is sharp.

Theorem 1 is a corollary to a more general statement (Theorem 4.4) that is valid even in the case where $\sup \sigma_- = \inf \sigma_+$. In its turn, the Davis–Kahan $\tan 2\Theta$ Theorem (Theorem 4.7) appears to be a simple corollary to Theorem 1.

We also remark that for a class of unbounded off-diagonal perturbations studied in [1] (cf. [10], [17]), the rough estimate $\|\mathsf{E}_L(\sigma'_-) - \mathsf{E}_A(\sigma_-)\| \leqslant \frac{\sqrt{2}}{2}$ can be proven by combining [1, Theorem 5.3] and [16, Theorem 5.6]. Example 4.5 to Theorem 1 shows that estimate (1.8) may hold (even with finite $\varkappa$) for unbounded perturbations that do not fit the assumptions of [1].

As regards the spectral disposition (1.7), it has been proven in [15] (see also [14]) that the gaps between σ_- and σ_+ remain open and the bound (1.2) holds if the perturbation V satisfies condition

$$\|V\| < \sqrt{2} d.$$

The only known sharp bound [15, Theorem 2.4] for the norm of the difference $\mathsf{E}_{A+V}(\sigma'_-) - \mathsf{E}_A(\sigma_-)$ involves the distance from the initial spectral set σ_+ to the perturbed spectral set σ'_-, and thus this bound is an *a posteriori* estimate.

Our second principal result just adds an *a priori* sharp bound for the norm $\|\mathsf{E}_{A+V}(\sigma'_-)-\mathsf{E}_A(\sigma_-)\|$ in the case where (1.7) holds and $\|V\|<d$.

Theorem 2. *Given a self-adjoint operator A on the Hilbert space $\mathfrak{H}$, assume that*

$$\operatorname{spec}(A)=\sigma_-\cup\sigma_+,\quad \operatorname{dist}(\sigma_+,\sigma_-)=d>0,\quad \textit{and}\quad \sigma_+\cap\operatorname{conv}(\sigma_-)=\varnothing.$$

Let V be a bounded self-adjoint operator on $\mathfrak{H}$ off-diagonal with respect to the decomposition $\mathfrak{H}=\operatorname{Ran}\mathsf{E}_A(\sigma_-)\oplus\operatorname{Ran}\mathsf{E}_A(\sigma_+)$. Assume in addition that

$$\|V\|<d.$$

Then the spectrum of $L=A+V$ consists of two disjoint components σ'_- and σ'_+ such that

$$\sigma'_-\subset\delta,\quad \sigma'_+\subset\mathbb{R}\backslash\delta,\quad \delta=(\inf\sigma_--d,\sup\sigma_-+d), \tag{1.9}$$

and

$$\|\mathsf{E}_L(\sigma'_-)-\mathsf{E}_A(\sigma_-)\|\leqslant\sin\Bigl(\arctan\frac{\|V\|}{d}\Bigr)=\frac{\|V\|}{\sqrt{d^2+\|V\|^2}}. \tag{1.10}$$

We *conjecture* that estimate (1.10) also holds for $d\leqslant\|V\|<\sqrt{2}d$.

The proofs of both Theorems 1 and 2 are performed by constructing the direct rotation [6] from the subspace $\operatorname{Ran}\mathsf{E}_A(\sigma_-)$ to the subspace $\operatorname{Ran}\mathsf{E}_L(\sigma'_-)$.

Recall that the direct rotation U from a closed subspace $\mathfrak{M}$ of a Hilbert space $\mathfrak{H}$ to a closed subspace $\mathfrak{N}\subset\mathfrak{H}$ with $\dim(\mathfrak{M}\cap\mathfrak{N}^\perp)=\dim(\mathfrak{M}^\perp\cap\mathfrak{N})$ is a unitary operator on $\mathfrak{H}$ mapping $\mathfrak{M}$ onto $\mathfrak{N}$ and being such that for any other unitary W on $\mathfrak{H}$ with $\operatorname{Ran}W|_{\mathfrak{M}}=\mathfrak{N}$ the following inequality holds: $\|I-U\|\leqslant\|I-W\|$, where I is the identity operator on $\mathfrak{H}$. That is, the direct rotation is closer (in the operator norm topology) to the identity operator than any other unitary operator on $\mathfrak{H}$ mapping $\mathfrak{M}$ onto $\mathfrak{N}$. The norm of the difference between the corresponding orthogonal projections onto $\mathfrak{M}$ and $\mathfrak{N}$ is completely determined by location of $\operatorname{spec}(U)$ on the unit circumference.

We extract information on the spectrum of the direct rotation from $\operatorname{Ran}\mathsf{E}_A(\sigma_-)$ to $\operatorname{Ran}\mathsf{E}_L(\sigma'_-)$ using the following auxiliary result which, we think, is of independent interest.

Theorem 3. *Let T be a closed densely defined operator on a Hilbert space $\mathfrak{H}$ with the polar decomposition $T=W|T|$. Assume that G is a bounded operator on $\mathfrak{H}$ such that both GT and G^*T^* are accretive (resp. strictly accretive). Then the products GW and WG are also accretive (resp. strictly accretive) operators.*

Notice that in this theorem and below an operator T on the Hilbert space $\mathfrak{H}$ is called *accretive* (resp. *strictly accretive*) if

$$\operatorname{Re}\langle x,Tx\rangle\geqslant 0\ \ (\text{resp.}\ \ \operatorname{Re}\langle x,Tx\rangle>0)\ \ \text{for any}\, x\in\operatorname{Dom}(T),\|x\|=1.$$

We also adopt the convention that the partial isometry W in the polar decomposition $T = W|T|$ is extended to $\mathrm{Ker}(T)$ by

$$W|_{\mathrm{Ker}(T)} = 0. \tag{1.11}$$

In this way, the isometry W is uniquely defined on the whole space $\mathfrak{H}$ (see, e.g. [11, §VI.7.2]).

A convenient way to construct the direct rotation between two closed subspaces of a Hilbert space is rendered by using a pair of self-adjoint involutions associated with these subspaces. Although the relative geometry of two subspaces is studied in great detail (see, e.g. [9, 11, 18]), for convenience of the reader we give in Sec. 2 a short but self-contained exposition of the subject reformulating some results in terms of a pair of involutions.

The remaining part of the article is organized as follows. Section 3 contains a proof of Theorem 3. The principal result of this section is Theorem 3.4, which allows one to compare two involutions, one of which is associated with a self-adjoint operator. Theorem 1 and some others related statements are proven in Sec. 4. Section 5 contains a proof of Theorem 2.

We conclude the introduction with description of some more notations that are used throughout the paper. The identity operator on any Hilbert space $\mathfrak{H}$ is denoted by I. Given a linear operator T on $\mathfrak{H}$, by $\mathscr{W}(T)$ we denote its numerical range,

$$\mathscr{W}(T) = \{\lambda \in \mathbb{C} \,|\, \lambda = \langle x, Tx \rangle \text{ for some } x \in \mathrm{Dom}(T), \|x\| = 1\}.$$

We use the standard concepts of commuting and anticommuting operators dealing only with the case where at least one of the operators involved is bounded (see, e.g. [5, §3.1.1]). Assuming that S and T are operators on $\mathfrak{H}$, suppose that the operator S is bounded. We say that the operators S and T *commute* (resp. *anticommute*) and write $S \smile T$ or $T \smile S$ (resp. $S \frown T$ or $T \frown S$) if $ST \subset TS$ (resp. $ST \subset -TS$).

2. A PAIR OF INVOLUTIONS

2.1. An Involution. We start with recalling the concept of a (self-adjoint) involution on a Hilbert space. This concept is a main tool we use in the present paper. Notice that in the theory of spaces with indefinite metric the involutions are often called canonical symmetries (see, e.g. [4]).

Definition 2.1. A linear operator J on the Hilbert space $\mathfrak{H}$ is called an *involution* if

$$J^* = J \quad \text{and} \quad J^2 = I. \tag{2.1}$$

In particular, if P^- and $P^+ = I - P^-$ are two complementary orthogonal projections on $\mathfrak{H}$, then the differences $P^+ - P^-$ and $P^- - P^+$ are involutions.

By definition, any involution J is a self-adjoint operator. In fact, it is also a unitary operator since (2.1) yields $J^* = J^{-1}$. Hence $\operatorname{spec}(J) = \{-1, 1\}$ and the spectral decomposition of J reads

$$J = \int_{\mathbb{R}} \lambda \, \mathsf{E}_J(d\lambda) = \mathsf{E}_J(\{+1\}) - \mathsf{E}_J(\{-1\}),$$

which implies that any involution on $\mathfrak{H}$ is the difference between two complementary orthogonal projections. Obviously, the projections $\mathsf{E}_J(\{\pm 1\})$ are equal to

$$\mathsf{E}_J(\{+1\}) = \frac{1}{2}(I + J) \quad \text{and} \quad \mathsf{E}_J(\{-1\}) = \frac{1}{2}(I - J). \tag{2.2}$$

Definition 2.2. Let J be an involution on the Hilbert space $\mathfrak{H}$. The subspaces

$$\mathfrak{H}_- = \operatorname{Ran} \mathsf{E}_J(\{-1\}) \quad \text{and} \quad \mathfrak{H}_+ = \operatorname{Ran} \mathsf{E}_J(\{+1\}) \tag{2.3}$$

are called the *negative* and *positive* subspaces of the involution J, respectively. The decomposition

$$\mathfrak{H} = \mathfrak{H}_- \oplus \mathfrak{H}_+ \tag{2.4}$$

of $\mathfrak{H}$ into the orthogonal sum of the subspaces (2.3) is said to be *associated* with J.

Recall that a linear operator A on $\mathfrak{H}$ is called diagonal with respect to decomposition (2.4) if the subspace $\mathfrak{H}_-$ (and hence the subspace $\mathfrak{H}_+$) reduces A. A linear operator V on $\mathfrak{H}$ is said to be off-diagonal with respect to decomposition (2.4) if

$$\mathfrak{H}_- \cap \operatorname{Dom}(V) = \operatorname{Ran} P^-|_{\operatorname{Dom}(V)}, \quad \mathfrak{H}_+ \cap \operatorname{Dom}(V) = \operatorname{Ran} P^+|_{\operatorname{Dom}(V)},$$

where P^- and P^+ are orthogonal projections onto $\mathfrak{H}_-$ and $\mathfrak{H}_+$, respectively, and

$$\operatorname{Ran} V|_{\mathfrak{H}_- \cap \operatorname{Dom}(V)} \subset \mathfrak{H}_+, \quad \operatorname{Ran} V|_{\mathfrak{H}_+ \cap \operatorname{Dom}(V)} \subset \mathfrak{H}_-. \tag{2.5}$$

A criterion for an operator on $\mathfrak{H}$ to be diagonal or off-diagonal with respect to the orthogonal decomposition of $\mathfrak{H}$ associated with an involution J can be formulated in terms of a commutation relation between this operator and J.

Lemma 2.3. *A linear operator A on the Hilbert space $\mathfrak{H}$ is diagonal with respect to the orthogonal decomposition of $\mathfrak{H}$ associated with an involution J if and only if $J \smile A$.*

Proof. This assertion is an immediate corollary to [5, Theorem 1 in §3.6]. $\square$

Lemma 2.4. *A linear operator V on the Hilbert space $\mathfrak{H}$ is off-diagonal with respect to the orthogonal decomposition of $\mathfrak{H}$ associated with an involution J if and only if $J \frown V$.*

Proof. "Only if part." Assume that V is off-diagonal with respect to an orthogonal decomposition of $\mathfrak{H}$ associated with J. Let $P^{\pm} = \mathsf{E}_J(\{\pm 1\})$. Then $J = P^+ - P^-$ and $P^+ + P^- = I$. By the hypothesis, one infers that $P^{\pm}x \in \mathrm{Dom}(V)$ for any $x \in \mathrm{Dom}(V)$. Hence $x \in \mathrm{Dom}(V)$ implies $Jx \in \mathrm{Dom}(V)$. Moreover, for any $x \in \mathrm{Dom}(V)$ the following chain of equalities holds

$$\begin{aligned} VJx &= VP^+x - VP^-x \\ &= P^-VP^+x - P^+VP^-x \\ &= P^-V(P^+ + P^-)x - P^+V(P^+ + P^-)x \\ &= (P^- - P^+)Vx \\ &= -JVx, \end{aligned}$$

since $P^+VP^+x = P^-VP^-x = 0$ (cf. (2.5)). Thus $J \frown V$.

"If part." Suppose that $J \frown V$, which means that (i) $x \in \mathrm{Dom}(V)$ implies $Jx \in \mathrm{Dom}(V)$ and (ii) $VJx = -JVx$ for all $x \in \mathrm{Dom}(V)$. Let $\mathfrak{H}_{\pm} = \mathrm{Ran}\,\mathsf{E}_J(\{\pm\})$. Condition (i) and equalities (2.2) imply that $\mathsf{E}_J(\{\pm 1\})x \in \mathrm{Dom}(V)$ whenever $x \in \mathrm{Dom}(V)$. Therefore, it follows from condition (ii) that if $x_- \in \mathfrak{H}_- \cap \mathrm{Dom}(V)$, then $Vx_- = -VJx_- = JVx_-$. Hence $Vx_- \in \mathfrak{H}_+$ for all $x_- \in \mathfrak{H}_- \cap \mathrm{Dom}(V)$. In a similar way one verifies that $Vx_+ \in \mathfrak{H}_-$ for all $x_+ \in \mathfrak{H}_+ \cap \mathrm{Dom}(V)$. Hence V is off-diagonal with respect to the decomposition of $\mathfrak{H}$ associated with J, which completes the proof. ☐

Remark 2.5. Operators that are diagonal or off-diagonal with respect to the decomposition (2.4) are often written in the block operator matrix form,

$$A = \begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix}, \quad V = \begin{pmatrix} 0 & V_+ \\ V_- & 0 \end{pmatrix},$$

where $A_{\pm}$ are the parts of the diagonal operator A in $\mathfrak{H}_{\pm}$, and $V_{\pm}$ are the corresponding restrictions of the off-diagonal operator V to $\mathfrak{H}_{\pm}$,

$$A_{\pm} = A|_{\mathrm{Dom}(A) \cap \mathfrak{H}_{\pm}}, \quad V_{\pm} = V|_{\mathrm{Dom}(V) \cap \mathfrak{H}_{\pm}}.$$

In particular, if both A and V are closed operators and, in addition, V is bounded, then the closed operator $L = A + V$ with $\mathrm{Dom}(L) = \mathrm{Dom}(A)$ admits the block operator matrix representation

$$L = \begin{pmatrix} A_- & V_+ \\ V_- & A_+ \end{pmatrix}. \tag{2.6}$$

In this case

$$A = \frac{1}{2}(L + JLJ), \quad V = \frac{1}{2}\overline{(L - JLJ)},$$

where J is the involution that corresponds to the decomposition (2.4).

Notice that the study of invariant subspaces for block operator matrices of the form (2.6) is closely related to the question concerning existence of solutions to the associated operator Riccati equations (see, e.g. [3] and references therein).

2.2. Involutions in the Acute Case. Recall that two closed subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of a Hilbert space $\mathfrak{H}$ are said to be in the acute case if

$$\mathfrak{M} \cap \mathfrak{N}^{\perp} = \{0\} \quad \text{and} \quad \mathfrak{M}^{\perp} \cap \mathfrak{N} = \{0\}.$$

To formulate the notion of the acute case in terms of the corresponding involutions, we adopt the following definition.

Definition 2.6. Involutions J and J' on the Hilbert space $\mathfrak{H}$ are said to be in the *acute case* if

$$\operatorname{Ker}(I + J'J) = \{0\}.$$

Remark 2.7. By inspection, $\operatorname{Ker}(I + J'J) = \operatorname{Ker}(I + JJ')$, which means that this definition is symmetric with respect to the entries J and J'.

Lemma 2.8. *If involutions J and J' are in the acute case and $J \smile J'$, then $J = J'$.*

Proof. Taking into account the self-adjointness of both J and J', the hypothesis $JJ' = J'J$ implies that the unitary operator $J'J$ is self-adjoint. Hence $\operatorname{spec}(J'J) \subset \{-1, 1\}$. Then from the assumption that J and J' are in the acute case it follows that $-1 \notin \operatorname{spec}(J'J)$. This yields $J'J = I$ and hence $J = J'$. □

Some criteria for a pair of involutions J and J' to be in the acute case are presented in Lemma 2.9 below. In particular, this lemma justifies Definition 2.6 stating that J and J' are in the acute case if and only if their negative (resp. positive) subspaces are in the acute case.

One of the criteria in Lemma 2.9 involves the numerical range $\mathscr{W}(J'J)$ of the product $J'J$. Since $J'J$ is a unitary operator, its numerical range is a subset of the unit disc $\{\lambda \in \mathbb{C} \,|\, |\lambda| \leqslant 1\}$. Equalities $J'J = J(JJ')J = J(JJ')J^{-1}$ imply that the products $J'J$ and JJ' are unitarily equivalent. Hence $\mathscr{W}(J'J) = \mathscr{W}(JJ')$. By $JJ' = (J'J)^*$ this means that the numerical range of $J'J$ is symmetric with respect to the real axis.

Lemma 2.9. *Let J and J' be two involutions on $\mathfrak{H}$. Assume that $\mathfrak{H}_{\pm} = \operatorname{Ran} \mathsf{E}_J(\{\pm 1\})$ and $\mathfrak{H}'_{\pm} = \operatorname{Ran} \mathsf{E}_{J'}(\{\pm 1\})$. The following four statements are equivalent:*

(*i*) $\mathfrak{H}_{-} \cap \mathfrak{H}'_{+} = \{0\}$ *and* $\mathfrak{H}_{+} \cap \mathfrak{H}'_{-} = \{0\}$, (*ii*) $\operatorname{Ker}(I + J'J) = \{0\}$,

(*iii*) $\|(J' - J)x\| < 2\|x\|$ *for all* $x \in \mathfrak{H}$, $x \neq 0$,

(*iv*) $-1 \notin \mathscr{W}(J'J)$.

Proof. We prove the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. We prove this implication by contradiction. Suppose that $\operatorname{Ker}(I + J'J) \neq \{0\}$ and $x \in \operatorname{Ker}(I + J'J)$ is a nonzero vector. Representing this vector as $x = x_- + x_+$ with $x_- \in \mathfrak{H}_-$ and $x_+ \in \mathfrak{H}_+$, one obtains $(I + J'J)x = (I - J')x_- + (I + J')x_+$ and hence

$$(I - J')x_- + (I + J')x_+ = 0 \tag{2.7}$$

since $(I + J'J)x = 0$. Applying $(I - J')$ to both parts of (2.7) gives $(I - J')^2 x_- = 0$ and thus $J'x_- = x_-$. Therefore, x_- is an eigenvector of the operator J' corresponding to the eigenvalue $+1$, which means $x_- \in \mathfrak{H}_- \cap \mathfrak{H}'_+$. In a similar way, by applying $(I + J')$ to both parts of (2.7), one concludes that $J'x_+ = -x_+$ and hence $x_+ \in \mathfrak{H}_+ \cap \mathfrak{H}'_-$. Then it follows from condition (*i*) that $x_- = x_+ = 0$ and thus $x = 0$, which contradicts the assumption.

$(ii) \Rightarrow (iii)$. It follows from condition (*ii*) that $\|(I + J'J)x\| > 0$ for any nonzero $x \in \mathfrak{H}$. Then by taking into account the identities

$$\|(J - J')x\|^2 + \|(J + J')x\|^2 = 4\|x\|^2$$

and

$$\|(J + J')x\| = \|J'(J' + J)x\| = \|(I + J'J)x\|$$

one easily concludes that (*ii*) implies (*iii*).

$(iii) \Rightarrow (iv)$. By inspection,

$$\|x\|^2 + \operatorname{Re}\langle x, J'Jx\rangle = \frac{1}{2}\Big\{4\|x\|^2 - \|(J - J')x\|^2\Big\}.$$

Hence (*iii*) implies

$$\|x\|^2 + \operatorname{Re}\langle x, J'Jx\rangle > 0 \quad \text{for any nonzero } x \in \mathfrak{H}.$$

In particular, this means that $\operatorname{Re}\langle x, J'Jx\rangle > -1$ for any $x \in \mathfrak{H}$ such that $\|x\| = 1$ and therefore $-1 \notin \mathscr{W}(J'J)$.

$(iv) \Rightarrow (i)$. Suppose that at least one of the subspaces $\mathfrak{H}_- \cap \mathfrak{H}'_+$ and $\in \mathfrak{H}_+ \cap \mathfrak{H}'_-$ is nontrivial. Pick up vectors $x_- \in \mathfrak{H}_- \cap \mathfrak{H}'_+$ and $x_+ \in \mathfrak{H}_+ \cap \mathfrak{H}'_-$ in such a way that at least one of them is nonzero. Clearly, $J'J(x_- + x_+) = J'(-x_- + x_+) = -(x_- + x_+)$, which means that -1 is an eigenvalue of the operator $J'J$, and thus $-1 \in \mathscr{W}(J'J)$. This contradicts the assumption (*iv*) and thus proves the implication. $\square$

Remark 2.10. Making use of relationship (2.2) between an involution and its spectral projections yields

$$P'^{+} - P^{+} = P^{-} - P'^{-} = \frac{J' - J}{2},$$

where $P^{\pm} = \mathsf{E}_J(\{\pm 1\})$ and $P'^{\pm} = \mathsf{E}_{J'}(\{\pm 1\})$.

Corollary 2.11. *If*

$$\|P'^{-} - P^{-}\| < 1 \quad (or \quad \|P'^{+} - P^{+}\| < 1)$$

holds, then the involutions J and J' are in the acute case. Hence, the negative (resp. positive) subspaces of J and J' are also in the acute case.

2.3. The Direct Rotation. Let J and J' be involutions on $\mathfrak{H}$. Assume that $\mathfrak{H}_-$ and $\mathfrak{H}_+$ are the negative and positive subspaces of J, respectively. Similarly, assume that $\mathfrak{H}'_-$ and $\mathfrak{H}'_+$ are the negative and positive subspaces of J'. It is well known (see, e.g. [6, Theorem 3.1]) that if

$$\dim(\mathfrak{H}_- \cap \mathfrak{H}'_+) = \dim(\mathfrak{H}_+ \cap \mathfrak{H}'_-), \tag{2.8}$$

then there exists a unitary operator W on $\mathfrak{H}$ mapping $\mathfrak{H}_-$ onto $\mathfrak{H}'_-$ and $\mathfrak{H}_+$ onto $\mathfrak{H}'_+$. Clearly, this W satisfies the commutation relation

$$J'W = WJ. \tag{2.9}$$

In particular, by Lemma 2.9 such a unitary W exists if J and J' are in the acute case. The canonical choice of the unitary mapping of one subspace in the Hilbert space onto another, the so-called direct rotation, was suggested by C. Davis in [6] and T. Kato in [11, Sections I.4.6 and I.6.8]. The idea of this choice goes back yet to B. Sz.-Nagy (see [18, §105]). We adopt the following definition of the direct rotation.

Definition 2.12. Let J and J' be involutions on the Hilbert space $\mathfrak{H}$. A unitary operator U on $\mathfrak{H}$ is called *the direct rotation* from J to J' if

$$(i) \quad J'U = UJ, \quad (ii) \quad U^2 = J'J, \quad (iii) \quad \mathrm{Re}\, U \geqslant 0. \tag{2.10}$$

Remark 2.13. The spectrum of any direct rotation is a subset of the unit circumference lying in the closed right half-plane symmetrically with respect to the real axis. To see this, observe that equalities (i) and (ii) imply $U^* = JUJ$ by taking into account that U is a unitary operator. Hence the operator U is unitary equivalent to its adjoint and thus the spectrum of U is symmetric with respect to the real axis. From (iii) it follows that this spectrum is a subset of the half-plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \geqslant 0\}$. To complete the proof of the statement, it only remains to recall that the spectrum of any unitary operator lies on the unit circumference.

We give a short proof of the existence and uniqueness of the direct rotation for the instance where the corresponding involutions are in the acute case. For a different proof of this fact see [7, Propositions 3.1 and 3.3].

Theorem 2.14. *If involutions J and J' are in the acute case, then there is a unique direct rotation from J to J'.*

Proof. We divide the proof into two parts. In the first part, we prove the existence of a direct rotation from J to J'. The uniqueness of the direct rotation is proven in the second part.

(Existence.) Set $T = I + J'J$. One easily verifies that T is a normal operator. By hypothesis,

$$\mathrm{Ker}(T) = \mathrm{Ker}(T^*) = \{0\} \tag{2.11}$$

taking into account Remark 2.7. Hence the isometry U in the polar decomposition

$$T = U|T| = |T|U, \tag{2.12}$$

is a unitary operator (see [18, §110]).

By inspection,

$$J'T = TJ \tag{2.13}$$

and thus

$$J|T|^2 = JT^*T = T^*J'T = T^*TJ = |T|^2J,$$
$$J'|T|^2 = J'TT^* = TJT^* = TT^*J' = |T|^2J'.$$

Hence $J \smile |T|$ and $J' \smile |T|$. Then (2.12) and (2.13) yield $|T|(J'U - UJ) = 0$, which implies that

$$J'U = UJ \tag{2.14}$$

since $\mathrm{Ker}(|T|) = \mathrm{Ker}(T) = \{0\}$. Observing that $J'JT^* = T$, by the same reasoning one obtains $|T|(U - J'JU^*) = 0$. Hence $U = J'JU^*$ and thus

$$U^2 = J'J. \tag{2.15}$$

Finally, $T + T^* = |T|^2$ and $T + T^* = |T|(U + U^*)$ imply $|T|(U + U^* - |T|) = 0$. Therefore,

$$\mathrm{Re}\,U = \frac{1}{2}|T| \geqslant 0. \tag{2.16}$$

Comparing (2.14), (2.15), and (2.16) with (2.10), one concludes that U is the direct rotation from J to J'.

(Uniqueness.) Suppose that U' is another unitary operator such that $U'^2 = U^2$ and $\operatorname{Re} U' \geqslant 0$. By inspection,

$$(\operatorname{Re} U')^2 = \frac{1}{2}\Big(I + \operatorname{Re}(U'^2)\Big) = \frac{1}{2}\Big(I + \operatorname{Re}(U^2)\Big) = (\operatorname{Re} U)^2.$$

Then it follows from the uniqueness of the positive square root of a positive operator that $\operatorname{Re} U = \operatorname{Re} U'$. In addition, the requirement $\operatorname{Im}(U^2) = \operatorname{Im}(U'^2)$ implies $\operatorname{Re} U(\operatorname{Im} U - \operatorname{Im} U') = 0$, which means that $\operatorname{Im} U = \operatorname{Im} U'$ since $\operatorname{Ker}(\operatorname{Re} U) = \operatorname{Ker}(|T|) = \{0\}$ by combining (2.11) and (2.16). Thus $U' = \operatorname{Re} U + i \operatorname{Im} U = U$, completing the proof. ☐

Remark 2.15. In the nonacute case, the direct rotation exists if and only if (2.8) holds (see [7, Proposition 3.2]). If it exists, it is not unique.

To specify location of the spectrum of a unitary operator on the unit circumference, we introduce the notion of the spectral angle.

Definition 2.16. Let W be a unitary operator. The number

$$\vartheta(W) = \sup_{z \in \operatorname{spec}(W)} |\arg z|, \quad \arg z \in (-\pi, \pi],$$

is called the *spectral angle* of W.

Remark 2.17. $\vartheta(W^*) = \vartheta(W)$.

Remark 2.18. The (self-adjoint) operator angle between two closed subspaces in a Hilbert space is expressed through the direct rotation U from one of these subspaces to the other one by $\Theta = \arccos(\operatorname{Re} U)$ (see [7, Eq. (1.18)]). This implies that $\vartheta(U)$ is nothing but the spectral radius of the corresponding operator angle Θ.

The next statement shows that the spectral angle $\vartheta(W)$ is a quantity that characterizes the distinction of the unitary operator W from the identity operator.

Lemma 2.19. *Let W be a unitary operator. Then*

$$\|I - W\| = 2\sin\left(\frac{\vartheta(W)}{2}\right). \tag{2.17}$$

Proof. Observe that $I - W$ is a normal operator. Then using the spectral mapping theorem one concludes that the following chain of equalities holds:

$$\begin{aligned}
\|I-W\| &= \sup_{\lambda\in \mathrm{spec}(I-W)} |\lambda| \\
&= \sup_{z\in \mathrm{spec}(W)} |1-z| \\
&= \sup_{z\in \mathrm{spec}(W)} 2\sin\left(\frac{|\arg z|}{2}\right) \\
&= 2\sin\Bigl(\frac{1}{2}\sup_{z\in \mathrm{spec}(W)} |\arg z|\Bigr) \\
&= 2\sin\left(\frac{\vartheta(W)}{2}\right),
\end{aligned}$$

where $\arg z \in (-\pi,\pi]$. □

Remark 2.20. If U is the direct rotation from an involution J to an involution J' then it possesses the extremal property

$$\vartheta(U) \leqslant \vartheta(W),$$

where W is any other unitary operator satisfying (2.9). This can be easily seen from (2.17) by using [6, Theorem 7.1], which states that $\|I-U\| \leqslant \|I-W\|$.

Remark 2.21. Again assume that U is the direct rotation from an involution J to an involution J'. Then by (2.10) the spectral mapping theorem implies

$$0 \leqslant \vartheta(U) \leqslant \frac{\pi}{2} \quad \text{and} \quad \vartheta(U) = \frac{1}{2}\vartheta(J'J). \tag{2.18}$$

Since $\|J'-J\| = \|I-J'J\|$, by (2.17) it follows from (2.18) that

$$\|J'-J\| = 2\sin\left(\frac{\vartheta(J'J)}{2}\right) = 2\sin\vartheta(U).$$

Hence by Remark 2.10,

$$\|P'^{+}-P^{+}\| = \|P'^{-}-P^{-}\| = \sin\vartheta(U), \tag{2.19}$$

where $P^{\pm} = \mathsf{E}_J(\{\pm 1\})$ and $P'^{\pm} = \mathsf{E}_{J'}(\{\pm 1\})$.

In the proof of the next lemma, we will use the following notation. Assume that $\mathscr{S}$ is a subset of the complex plane. Then $e^{i\varphi}\mathscr{S}$ denotes the result of rotation of $\mathscr{S}$ by the angle $\varphi \subset (-\pi,\pi]$ around the origin, that is,

$$e^{i\varphi}\mathscr{S} = \{z\in\mathbb{C} \mid z = e^{i\varphi}\zeta \ \text{ for some } \zeta\in\mathscr{S}\}.$$

Lemma 2.22. *Let W_1 and W_2 be two unitary operators on the Hilbert space $\mathfrak{H}$. Then*

$$|\vartheta(W_1)-\vartheta(W_2)| \leqslant \vartheta(W_2W_1) \leqslant \vartheta(W_1)+\vartheta(W_2). \tag{2.20}$$

Proof. First, we prove inequality

$$\vartheta(W_2W_1) \leqslant \vartheta(W_1)+\vartheta(W_2). \tag{2.21}$$

Denote by ϑ_1, ϑ_2 and ϑ_3 the spectral angles of W_1, W_2, and W_2W_1, respectively. The case $\vartheta_1+\vartheta_2 \geqslant \pi$ is trivial since $\vartheta_3 \leqslant \pi$ by Definition 2.16. If $\vartheta_1+\vartheta_2 < \pi$, we prove (2.21) by contradiction. Suppose that the opposite inequality holds, that is,

$$\vartheta_3 > \vartheta_1+\vartheta_2.$$

Then there is a number $\varphi \in (-\pi,\pi]$ such that $e^{i\varphi} \in \operatorname{spec}(W_2W_1)$ and

$$\vartheta_1+\vartheta_2 < |\varphi| \leqslant \pi. \tag{2.22}$$

Since W_2W_1 is a normal (unitary) operator, there exists a sequence of vectors $x_n \in \mathfrak{H}$, $n=1,2,\dots$, such that

$$\|x_n\| = 1 \ \text{ and } \ \|W_2W_1x_n - e^{i\varphi}x_n\| \to 0, \quad n\to\infty. \tag{2.23}$$

Indeed, if $e^{i\varphi}$ is an eigenvalue of W_2W_1, to satisfy (2.23) one simply takes $x_n = x_\varphi$, $n=1,2,\dots$, where x_φ is a normalized eigenvector of W_2W_1 corresponding to the eigenvalue $e^{i\varphi}$, i.e. $W_2W_1x_\varphi = e^{i\varphi}x_\varphi$. Otherwise such a sequence exists by the Weyl criterion for the essential spectrum.

Let $z_{1,n} = \langle x_n, W_1x_n\rangle$ and $z_{2,n} = \langle x_n, W_2^*x_n\rangle$. Clearly, (2.23) yields

$$|z_{1,n} - e^{i\varphi}z_{2,n}| \to 0, \quad n\to\infty, \tag{2.24}$$

since by the Schwartz inequality we have

$$|z_{1,n} - e^{i\varphi}z_{2,n}| = |\langle x_n, W_1x_n - e^{i\varphi}W_2^*x_n\rangle| \leqslant \\ \leqslant \|W_1x_n - e^{i\varphi}W_2^*x_n\| = \|W_2W_1x_n - e^{i\varphi}x_n\|.$$

Taking into account that $z_{1,n} \subset \mathscr{W}(W_1)$ and $z_{2,n} \subset \mathscr{W}(W_2^*)$, from (2.24) one concludes that

$$\operatorname{dist}\big(\mathscr{W}(W_1), e^{i\varphi}\mathscr{W}(W_2^*)\big) = 0. \tag{2.25}$$

Meanwhile, if W is a unitary operator with the spectral angle ϑ, the spectral theorem implies

$$\mathscr{W}(W) \subset \mathscr{S}_\vartheta \quad \text{and} \quad \mathscr{W}(W^*) \subset \mathscr{S}_\vartheta,$$

where

$$\mathscr{S}_\vartheta = \{z\in\mathbb{C} \,|\, \operatorname{Re} z \geqslant \cos\vartheta \ \text{ and } \ |z|\leqslant 1\}$$

is a segment of the closed unit disc centered at the origin. Therefore, $\mathscr{W}(W_1) \subset \mathscr{S}_{\vartheta_1}$ and $\mathscr{W}(W_2^*) \subset \mathscr{S}_{\vartheta_2}$. Obviously, $e^{i\varphi}\mathscr{W}(W_2^*) \subset e^{i\varphi}\mathscr{S}_{\vartheta_2}$ and hence

$$\operatorname{dist}\bigl(\mathscr{W}(W_1), e^{i\varphi}\mathscr{W}(W_2^*)\bigr) \geqslant \operatorname{dist}\bigl(\mathscr{S}_{\vartheta_1}, e^{i\varphi}\mathscr{S}_{\vartheta_2}\bigr). \tag{2.26}$$

One easily verifies by inspection that by the assumption (2.22) we have

$$\operatorname{dist}\bigl(\mathscr{S}_{\vartheta_1}, e^{i\varphi}\mathscr{S}_{\vartheta_2}\bigr) = 2\sin\left(\frac{|\varphi| - \vartheta_1 - \vartheta_2}{2}\right)\sin\left(\frac{|\varphi| + \vartheta_2 - \vartheta_1}{2}\right) > 0$$

and thus by (2.26) we get

$$\operatorname{dist}\bigl(\mathscr{W}(W_1), e^{i\varphi}\mathscr{W}(W_2^*)\bigr) > 0,$$

which contradicts (2.25). This completes the proof of (2.21).

By Remark 2.17, inequality (2.21) implies

$$\vartheta(W_2) = \vartheta(W_2W_1W_1^*) \leqslant \vartheta(W_2W_1) + \vartheta(W_1^*) = \vartheta(W_2W_1) + \vartheta(W_1), \tag{2.27}$$
$$\vartheta(W_1) = \vartheta(W_2^*W_2W_1) \leqslant \vartheta(W_2^*) + \vartheta(W_2W_1) = \vartheta(W_2) + \vartheta(W_2W_1). \tag{2.28}$$

Combining (2.27) and (2.28) yields the left inequality in (2.20). The proof is complete. □

Remark 2.23. Setting $W_1 = e^{i\vartheta_1}I$ and $W_2 = e^{i\vartheta_2}I$ with ϑ_1, ϑ_2 appropriate reals, one verifies that both inequalities of (2.20) are sharp.

3. A PROPERTY OF THE POLAR DECOMPOSITION

In this section, we give a proof of Theorem 3. We also derive corollaries to this theorem for the case where one of the involved operators is self-adjoint and the other one is related to an involution.

We start with an auxiliary result.

Lemma 3.1. *Let A be a positive operator on the Hilbert space $\mathfrak{H}$. Suppose that $x, y \in \mathfrak{H}$ are such that*

$$\operatorname{Re}\langle x, A(A^2+\alpha)^{-1}y\rangle > 0 \quad (\geqslant 0) \quad \textit{for any} \quad \alpha > 0. \tag{3.1}$$

Then

$$\operatorname{Re}\langle x, Qy\rangle > 0 \quad (\geqslant 0), \tag{3.2}$$

where Q is the orthogonal projection onto $\operatorname{Ker}(A)^{\perp}$.

Proof. By the spectral theorem

$$\mathrm{Re}\langle x, A(A^2+\eta^2)^{-1}y\rangle = \int_{\mathbb{R}} \frac{\lambda m(d\lambda)}{\lambda^2+\eta^2} = \int_{(0,+\infty)} \frac{\lambda m(d\lambda)}{\lambda^2+\eta^2}, \quad 0 \neq \eta \in \mathbb{R},$$

where for any Borel set $\delta \subset \mathbb{R}$ the Lebesgue–Stieltjes measure $m(\delta)$ reads

$$m(\delta) = \mathrm{Re}\langle x, \mathsf{E}_A(\delta)y\rangle.$$

Hence for any $\varepsilon > 0$

$$\begin{aligned}\int_{\varepsilon}^{1/\varepsilon} d\eta\, \mathrm{Re}\langle x, A(A^2+\eta^2)^{-1}y\rangle &= \int_{\varepsilon}^{1/\varepsilon} d\eta \int_{(0,+\infty)} \frac{\lambda m(d\lambda)}{\lambda^2+\eta^2} \\ &= \int_{(0,+\infty)} m(d\lambda) \int_{\varepsilon}^{1/\varepsilon} \frac{\lambda\, d\eta}{\lambda^2+\eta^2}\end{aligned}$$

by the Fubini theorem. Therefore,

$$\begin{aligned}&\int_{\varepsilon}^{1/\varepsilon} d\eta\, \mathrm{Re}\langle x, A(A^2+\eta^2)^{-1}y\rangle = \\ &\qquad = \int_{(0,+\infty)} m(d\lambda) \left[\arctan\left(\frac{1}{\lambda\varepsilon}\right) - \arctan\left(\frac{\varepsilon}{\lambda}\right)\right]. \quad (3.3)\end{aligned}$$

From (3.3) one immediately infers that

$$\lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{1/\varepsilon} d\eta\, \mathrm{Re}\langle x, A(A^2+\eta^2)^{-1}y\rangle = \frac{\pi}{2} m\big((0,+\infty)\big). \quad (3.4)$$

Notice that $m\big((0,+\infty)\big) = \mathrm{Re}\langle x, Qy\rangle$ since $Q = \mathsf{E}_A\big((0,+\infty)\big)$. Hence (3.4) yields

$$\mathrm{Re}\langle x, Qy\rangle = \lim_{\varepsilon\downarrow 0} \frac{2}{\pi} \int_{\varepsilon}^{1/\varepsilon} d\eta\, \mathrm{Re}\langle x, A(A^2+\eta^2)^{-1}y\rangle. \quad (3.5)$$

Clearly, by (3.5) inequalities (3.2) follow directly from the corresponding assumptions (3.1). The proof is complete. □

With Lemma 3.1 we are ready to prove Theorem 3.

Proof of Theorem 3. First assume that the operators GT and G^*T^* are both accretive. To prove that GW is also an accretive operator, pick up arbitrary $\alpha > 0$ and $x \in \mathfrak{H}$ and set

$$g = (T^*T + \alpha)^{-1}x. \tag{3.6}$$

Taking into account that $g \in \mathrm{Dom}(T)$, introduce

$$h = Tg = T(T^*T + \alpha)^{-1}x. \tag{3.7}$$

Clearly, $h \in \mathrm{Dom}(T^*)$ and

$$x = \alpha g + T^*h. \tag{3.8}$$

By using (3.6), (3.7), and (3.8), it is easy to verify that the following chain of equalities holds

$$\begin{aligned}
\mathrm{Re}\langle W^*G^*x, |T|(|T|^2+\alpha)^{-1}x\rangle &= \mathrm{Re}\langle G^*x, W|T|g\rangle \\
&= \mathrm{Re}\langle G^*x, Tg\rangle \\
&= \mathrm{Re}\langle x, Gh\rangle \\
&= \mathrm{Re}\langle \alpha g + T^*h, Gh\rangle \\
&= \alpha\,\mathrm{Re}\langle g, Gh\rangle + \mathrm{Re}\langle Gh, T^*h\rangle \\
&= \alpha\,\mathrm{Re}\langle g, GTg\rangle + \mathrm{Re}\langle h, G^*T^*h\rangle.
\end{aligned} \tag{3.9}$$

Since by hypothesis both GT and G^*T^* are accretive, (3.9) implies that

$$\mathrm{Re}\langle W^*G^*x, |T|(|T|^2+\alpha)^{-1}x\rangle \geqslant 0 \quad \text{for any} \quad \alpha > 0 \quad \text{and} \quad x \in \mathfrak{H},$$

and hence by Lemma 3.1

$$\mathrm{Re}\langle W^*G^*x, Qx\rangle = \mathrm{Re}\langle x, GWQx\rangle \geqslant 0,$$

where Q is the orthogonal projection onto $\mathrm{Ker}(|T|)^{\perp}$. According to the convention (1.11) we have $\mathrm{Ker}(|T|) = \mathrm{Ker}(T) = \mathrm{Ker}(W)$. Then one concludes that $WQ = W$ and hence

$$\mathrm{Re}\langle x, GWx\rangle \geqslant 0 \quad \text{for all} \quad x \in \mathfrak{H},$$

which proves that the operator GW is accretive.

Further, assume that GT and G^*T^* are both strictly accretive operators. In particular, this implies that

$$\mathrm{Ker}(T) = \mathrm{Ker}(|T|) = \{0\}. \tag{3.10}$$

In this case, if $x \neq 0$ then neither g nor h defined in (3.6) and (3.7) can be zero vectors. Indeed, the equality $g = 0$ implies $h = Tg = 0$ and hence by (3.8) it contradicts the assumption $x \neq 0$. Independently, the equality $h = 0$ yields

$g \in \operatorname{Ker}(T)$ by taking into account (3.7). Then $x \in \operatorname{Ker}(T)$ since $x = \alpha g$ by (3.8). This is again a contradiction because of (3.10).

Therefore, if $x \neq 0$ and $\alpha > 0$, then necessarily $g \neq 0$, $h \neq 0$. Hence, by (3.9) now we have a strict inequality

$$\operatorname{Re}\langle W^* G^* x, |T|(|T|^2+\alpha)^{-1}x\rangle > 0.$$

Then by taking into account (3.10), Lemma 3.1 proves the strict accretiveness of the operator GW.

The accretiveness (resp., the strict accretiveness) of the operator WG can be proven in a similar way. $\square$

Now assume that T is a self-adjoint operator on the Hilbert space $\mathfrak{H}$ and $\operatorname{Ker}(T) = \{0\}$. Then the isometry J' in the polar decomposition

$$T = J'|T| \tag{3.11}$$

is an involution that reads

$$J' = \mathsf{E}_T\big((0,+\infty)\big) - \mathsf{E}_T\big((-\infty,0)\big).$$

Clearly, the negative and positive subspaces of this involution coincide with the corresponding spectral subspaces of T:

$$\mathfrak{H}'_- = \operatorname{Ran}\mathsf{E}_T((-\infty,0)) \quad \text{and} \quad \mathfrak{H}'_+ = \operatorname{Ran}\mathsf{E}_T((0,+\infty)).$$

Below we will show that in some cases Theorem 3 allows one to determine the spectral angle of the product $J'J$, where J is another involution on $\mathfrak{H}$. The norm of the difference between the orthogonal projections onto the corresponding positive (or negative) subspaces of J' and J is then easily computed by using (2.19).

We study the two following cases.

Hypothesis 3.2. *Let J be an involution on the Hilbert space $\mathfrak{H}$. Assume that T is a self-adjoint operator on $\mathfrak{H}$ such that*

(a) $\operatorname{Ker}(T) = \{0\}$ *and the product JT is accretive*

or

(b) *the product JT is strictly accretive.*

Obviously, if the assumption (b) holds, then the assumption (a) holds, too. Therefore, both (a) and (b) assume that $\operatorname{Ker}(T) = \{0\}$. Hence, any of these two assumptions implies that the isometry J' in the polar decomposition (3.11) of T is an involution.

To describe the accretive operators in some more detail, we introduce the following definition.

Definition 3.3. Let S be an accretive operator on the Hilbert space $\mathfrak{H}$. Then the finite or infinite number

$$k(S)=\sup_{z\in\mathscr{W}(S)\setminus\{0\}}\frac{|\mathrm{Im}\,z|}{\mathrm{Re}\,z}$$

is called the *sector bound* of S.

Clearly, if $k(S)$ is finite, then S is a sectorial operator (see [11, §V.3.10]) with vertex 0 and semiangle $\theta=\arctan k(S)$.

Main result of this section is the following.

Theorem 3.4. *Assume Hypothesis 3.2 (a). Let $T=J'|T|$ be the polar decomposition of T. Then the involutions J' and J are in the acute case, and*

$$\vartheta(U)\leqslant\frac{1}{2}\arctan k(JT)\quad\left(\leqslant\frac{\pi}{4}\right),\tag{3.12}$$

where U is the direct rotation from J to J'.

Proof. Since JT is accretive and $T=J'|T|$, it follows from Theorem 3 that the operator $J'J$ is also accretive. Hence $-1\notin\mathscr{W}(J'J)$, and thus by Lemma 2.9 the involutions J and J' are in the acute case.

If $k(JT)=0$ then $\mathscr{W}(JT)$ is a subset of the real axis, which means that JT is a symmetric operator. This implies $J\smile T$ since T is self-adjoint. Hence $J'\smile J$ (see, e.g. [11, Lemma VI.2.37]) and thus $J=J'$ by Lemma 2.8. In this case, estimate (3.12) is trivial since $\vartheta(U)=0$.

Further, assume that $k(JT)>0$. Set

$$\varphi=\frac{\pi}{2}-\arctan k(JT),\quad \varphi\in[0,\pi/2),$$

and observe that the operators GT and G^*T^* with $G=e^{i\varphi}J$ are both accretive. Then by Theorem 3 one concludes that the products $e^{i\varphi}J'J$ and $e^{-i\varphi}J'J$ are also accretive operators. Hence $\mathscr{W}(J'J)$ is a subset of the closed sector $\left\{z\in\mathbb{C}\,\middle|\,|\arg z|\leqslant\frac{\pi}{2}-\varphi,\arg z\in(-\pi,\pi]\right\}$. Then from the inclusion $\mathrm{spec}(J'J)\subset\overline{\mathscr{W}(J'J)}$ it follows that the spectral angle of the unitary operator $J'J$ satisfies

$$\vartheta(J'J)\leqslant\arctan k(JT).\tag{3.13}$$

Now (3.12) follows immediately from (3.13) and (2.18), completing the proof. □

In the two following statements, we present some uniqueness results concerning the involution J' referred to in Theorem 3.4.

Theorem 3.5. *Assume Hypothesis 3.2* (a). *Let* $\widetilde{J}'$ *be an involution on* $\mathfrak{H}$ *such that*

$$(i)\quad \widetilde{J}' \text{ and } J \text{ are in the acute case,}\quad (ii)\quad \widetilde{J}' \smile T,\quad \text{and}\quad (iii)\quad \widetilde{J}' \neq J',$$

where J' *is the involution in the polar decomposition of* T. *Then*

$$\vartheta(\widetilde{U}) \geqslant \frac{\pi}{2} - \frac{1}{2}\arctan k(JT) \quad \left(\geqslant \frac{\pi}{4}\right), \tag{3.14}$$

where $\widetilde{U}$ *is the direct rotation from* J *to* $\widetilde{J}'$.

Theorem 3.6. *Assume Hypothesis 3.2* (b). *Let* $T = J'|T|$ *be the polar decomposition of* T. *Then* J' *is a unique involution on* $\mathfrak{H}$ *such that*

$$(i)\quad J \text{ and } J' \text{ are in the acute case,}\quad (ii)\quad J' \smile T,\quad \text{and}\quad (iii)\quad \vartheta(U) \leqslant \frac{\pi}{4},$$

where U *is the direct rotation from* J *to* J'.

Proof of Theorem 3.5. For the proof by contradiction suppose that instead of (3.14) the opposite inequality holds. Then by (2.18) in Remark 2.21 we have

$$\vartheta(\widetilde{J}'J) < \pi - \arctan k(JT). \tag{3.15}$$

Similarly, Theorem 3.4 yields

$$\vartheta(JJ') \leqslant \arctan k(JT). \tag{3.16}$$

By (3.15) and (3.16), Lemma 2.22 implies that

$$\vartheta(\widetilde{J}'J') = \vartheta((\widetilde{J}'J)(JJ')) \leqslant \vartheta(\widetilde{J}'J) + \vartheta(JJ') < \pi.$$

In particular, this means that $-1 \notin \operatorname{spec}(\widetilde{J}'J')$, which proves that the involutions J' and $\widetilde{J}'$ are in the acute case.

By hypothesis, $\widetilde{J}'$ commutes with T and J' is the isometry in the polar decomposition of T. Hence [11, Lemma VI.2.37] implies $\widetilde{J}' \smile J'$. Then from Lemma 2.8 it follows that $\widetilde{J}' = J'$, which contradicts the assumption (iii). Therefore, $\vartheta(\widetilde{U})$ satisfies (3.14) completing the proof. □

Proof of Theorem 3.6. Arguing by contradiction, suppose that there is an involution $\widetilde{J}'$ distinct from J' and such that conditions (i)–(iii) are satisfied. In particular, this implies that $\vartheta(\widetilde{J}'J) \leqslant \pi/2$ and hence

$$\operatorname{Re}\langle x, J\widetilde{J}'x\rangle \geqslant 0 \quad \text{for all} \quad x \in \mathfrak{H}.$$

Since JT is strictly accretive and $T = J'|T|$, by Theorem 3 the operator JJ' is also strictly accretive, that is,

$$\operatorname{Re}\langle x, JJ'x\rangle > 0 \quad \text{for all} \quad x \in \mathfrak{H}, \quad x \neq 0.$$

Therefore,

$$\mathrm{Re}\langle x, JJ'x\rangle + \mathrm{Re}\langle x, J\widetilde{J'}x\rangle > 0 \ \text{ for all } \ x \in \mathfrak{H}, \quad x \neq 0. \tag{3.17}$$

Now assume that there is $y \in \mathrm{Ker}\,(I + \widetilde{J'}J')$ such that $y \neq 0$. Then applying $\widetilde{J'}$ to both parts of the equality $y + \widetilde{J'}J'y = 0$ yields $J'y + \widetilde{J'}y = 0$. Hence

$$\mathrm{Re}\langle y, JJ'y\rangle + \mathrm{Re}\langle y, J\widetilde{J'}y\rangle = 0,$$

and it follows from (3.17) that $y = 0$. This proves that $\mathrm{Ker}\,(I + \widetilde{J'}J') = \{0\}$, i.e. the involutions $\widetilde{J'}$ and J' are in the acute case.

Clearly, $\widetilde{J'} \smile J'$ since by hypothesis $\widetilde{J'}$ commutes with T, and J' is the isometry in the polar decomposition of T (see [11, Lemma VI.2.37]). Hence, by Lemma 2.8 $\widetilde{J'} = J'$, which contradicts the assumption that $\widetilde{J'}$ is distinct from J'.

The proof is complete. □

4. AN EXTENSION OF THE DAVIS–KAHAN $\tan 2\Theta$ THEOREM. PROOF OF THEOREM 1

Throughout this section we adopt the following hypothesis.

Hypothesis 4.1. *Given a self-adjoint operator A on the Hilbert space $\mathfrak{H}$, assume that*

$$\mathrm{Ker}\,(A - \mu) = \{0\} \quad \textit{for some} \quad \mu \in \mathbb{R}. \tag{4.1}$$

Let V be a symmetric operator on $\mathfrak{H}$ such that

- (i) $\mathrm{Dom}(A) \subset \mathrm{Dom}(V)$,
- (ii) $V \frown J$*, where* $J = \mathsf{E}_A((\mu, +\infty)) - \mathsf{E}_A((-\infty, \mu))$,

and

- (iii) *the closure $L = \overline{L_0}$ of the operator $L_0 = A + V$ with $\mathrm{Dom}(L_0) = \mathrm{Dom}(A)$ is a self-adjoint operator.*

By this hypothesis, the product $J(L - \mu)$ appears to be a strictly accretive operator. Moreover, the sector bound $k\big(J(L-\mu)\big)$ admits an explicit description in terms of the perturbation V.

Lemma 4.2. *Assume Hypothesis 4.1. Then $J(L - \mu)$ is a strictly accretive operator and*

$$k\big(J(L-\mu)\big) = \sup_{\substack{x \in \mathrm{Dom}(A) \\ \|x\| = 1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A - \mu| x\rangle}. \tag{4.2}$$

Proof. Obviously, by Hypothesis 4.1

$$J(A-\mu) = |A-\mu| > 0.$$

Hence by items (*ii*) and (*iii*) of this hypothesis we have

$$\operatorname{Re}\langle x, J(A+V-\mu)x\rangle = \langle x, |A-\mu|x\rangle \quad \text{for all} \quad x \in \operatorname{Dom}(A). \tag{4.3}$$

Pick up an arbitrary $y \in \operatorname{Dom}(L)$. By the assumption *(iii)*, it follows that there exists a sequence of vectors $y_n \in \operatorname{Dom}(A)$ such that $y_n \to y$ and $L_0 y_n \to Ly$ as $n \to \infty$, and thus

$$\operatorname{Re}\langle y_n, J(L_0-\mu)y_n\rangle \to \operatorname{Re}\langle y, J(L-\mu)y\rangle \quad \text{as} \quad n \to \infty. \tag{4.4}$$

Then (4.3) and (4.4) imply $\operatorname{Re}\langle y, J(L-\mu)y\rangle \geqslant 0$. Moreover, $y \in \operatorname{Ker}(|A-\mu|) \subset \operatorname{Dom}(A)$ whenever $\operatorname{Re}\langle y, J(L-\mu)y\rangle = 0$. Taking into account that $\operatorname{Ker}(|A-\mu|) = \operatorname{Ker}(A-\mu) = \{0\}$, one infers that

$$\operatorname{Re}\langle y, J(L-\mu)y\rangle > 0 \quad \text{for all nonzero} \quad y \in \operatorname{Dom}(L),$$

which means that the operator $J(L-\mu)$ is strictly accretive.

Now observe

$$k\big(J(L-\mu)\big) \geqslant \varkappa, \tag{4.5}$$

where

$$\varkappa = \sup_{\substack{x \in \operatorname{Dom}(A) \\ \|x\|=1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A-\mu|x\rangle}. \tag{4.6}$$

Indeed,

$$\begin{aligned} k\big(J(L-\mu)\big) &= \sup_{\substack{x \in \operatorname{Dom}(L) \\ \|x\|=1}} \frac{|\operatorname{Im}\langle x, J(L-\mu)x\rangle|}{\operatorname{Re}\langle x, J(L-\mu)x\rangle} \\ &\geqslant \sup_{\substack{x \in \operatorname{Dom}(A) \\ \|x\|=1}} \frac{|\operatorname{Im}\langle x, J(A+V-\mu)x\rangle|}{\operatorname{Re}\langle x, J(A+V-\mu)x\rangle} \end{aligned}$$

since by Hypothesis 4.1 (*iii*) $\operatorname{Dom}(A) \subset \operatorname{Dom}(L)$ and $L|_{\operatorname{Dom}(A)} = A+V$. Then (4.5) holds by (4.3), since Hypothesis 4.1 (*ii*) implies

$$\operatorname{Im}\langle x, J(A+V-\mu)x\rangle = \langle x, JVx\rangle \quad \text{for any} \quad x \in \operatorname{Dom}(A). \tag{4.7}$$

Clearly, if $\varkappa = \infty$, then (4.2) follows immediately from inequality (4.5). If $\varkappa$ is finite, then by (4.3) and (4.7) from (4.6) we have

$$|\operatorname{Im}\langle x, J(L_0-\mu)x\rangle| \leqslant \varkappa \operatorname{Re}\langle x, J(L_0-\mu)x\rangle \quad \text{for any} \quad x \in \operatorname{Dom}(L_0) = \operatorname{Dom}(A).$$

Since L is the closure of L_0, by continuity of the inner product the same inequality holds for L, that is,

$$|\operatorname{Im}\langle x, J(L-\mu)x\rangle| \leqslant \varkappa \operatorname{Re}\langle x, J(L-\mu)x\rangle \quad \text{for any} \quad x \in \operatorname{Dom}(L).$$

In particular, this means that

$$\sup_{\substack{x \in \operatorname{Dom}(L) \\ \|x\|=1}} \frac{|\operatorname{Im}\langle x, J(L-\mu)x\rangle|}{\operatorname{Re}\langle x, J(L-\mu)x\rangle} = k\big(J(L-\mu)\big) \leqslant \varkappa. \tag{4.8}$$

Now combining (4.5), (4.6), and (4.8) completes the proof. ☐

Remark 4.3. Since $J(L-\mu)$ is a strictly accretive operator, the isometry J' in the polar decomposition $L-\mu = J'|L-\mu|$ is an involution. Clearly, it reads

$$J' = \mathsf{E}_L((\mu,+\infty)) - \mathsf{E}_L((-\infty,\mu)).$$

Theorem 4.4. *Assume Hypothesis 4.1. Let $L-\mu = J'|L-\mu|$ be the polar decomposition of $L-\mu$. Then the involutions J and J' are in the acute case, and*

$$\vartheta(U) \leqslant \frac{1}{2}\arctan\Big(\sup_{\substack{x \in \operatorname{Dom}(A) \\ \|x\|=1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A-\mu|x\rangle}\Big) \quad \Big(\leqslant \frac{\pi}{4}\Big), \tag{4.9}$$

where U is the direct rotation from J to J'. Moreover, J' is a unique involution on $\mathfrak{H}$ with the properties

$$(i)\ J' \text{ and } J \text{ are in the acute case,} \quad (ii)\ J' \smile L, \text{ and } (iii)\ \vartheta(U) \leqslant \frac{\pi}{4}. \tag{4.10}$$

The spectral angle of the direct rotation $\widetilde{U}$ from J to any other involution $\widetilde{J}'$ distinct from J' and satisfying (i) and (ii) is bounded from below as follows

$$\vartheta(\widetilde{U}) \geqslant \frac{\pi}{2} - \frac{1}{2}\arctan\Big(\sup_{\substack{x \in \operatorname{Dom}(A) \\ \|x\|=1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A-\mu|x\rangle}\Big) \quad \Big(\geqslant \frac{\pi}{4}\Big). \tag{4.11}$$

Proof. The operators J and $T = L-\mu$ satisfy Hypothesis 3.2 (b) (and hence Hypothesis 3.2 (a)). Then the assertion is proven simply by combining Theorems 3.4, 3.5, and 3.6 with Lemma 4.2. ☐

With Theorem 4.4 one can easily prove Theorem 1.

Proof of Theorem 1. Pick up arbitrary $\mu,\nu\in(\sup\sigma_-,\inf\sigma_+)$, $\mu<\nu$. Clearly, Hypothesis 4.1 holds for both μ and ν with the same involution $J=\mathsf{E}_A(\sigma_+)-\mathsf{E}_A(\sigma_-)$. By Remark 4.3, the isometries J'_μ and J'_ν in the polar decompositions $L-\mu=J'_\mu|L-\mu|$ and $L-\nu=J'_\nu|L-\nu|$ are involutions. By Theorem 4.4, the involutions J and J'_μ are in the acute case, $J'_\mu\smile L$, and $\vartheta(U_\mu)\leqslant\pi/4$, where U_μ is the direct rotation from J to J'_μ. The same holds for J'_ν and the corresponding direct rotation U_ν from J to J'_ν. Therefore, (4.10) is satisfied for both $J'=J'_\mu$ and $J'=J'_\nu$. Hence, Theorem 4.4 implies $J'_\mu=J'_\nu$, which by Remark 4.3 yields $\mathsf{E}_L\big((\mu,\nu)\big)=0$. Since $\mu,\nu\in(\sup\sigma_-,\inf\sigma_+)$ are arbitrary, then one concludes that $\mathsf{E}_L\big((\sup\sigma_-,\inf\sigma_+)\big)=0$, and thus the interval $(\sup\sigma_-,\inf\sigma_+)$ belongs to the resolvent set of L. Hence,

$$J'_\mu=\mathsf{E}_L(\sigma'_+)-\mathsf{E}_L(\sigma'_-)\quad\text{for all}\quad\mu\in(\sup\sigma_-,\inf\sigma_+),$$

where σ'_- and σ'_+ are the parts of the spectrum of L in the intervals $(-\infty,\sup\sigma_-]$ and $[\inf\sigma_+,+\infty)$, respectively. Since J'_μ does not depend on $\mu\in(\sup\sigma_-,\inf\sigma_+)$, the direct rotation U_μ does not, either. Then estimate (4.9) of Theorem 4.4 yields

$$\tan 2\vartheta(U)\leqslant\inf_{\sup\sigma_-<\mu<\inf\sigma_+}\ \sup_{\substack{x\in\mathrm{Dom}(A)\\ \|x\|=1}}\frac{|\langle x,JVx\rangle|}{\langle x,|A-\mu|x\rangle},\tag{4.12}$$

where U is the direct rotation from the involution $\mathsf{E}_A(\sigma_+)-\mathsf{E}_A(\sigma_-)$ to the involution $\mathsf{E}_L(\sigma'_+)-\mathsf{E}_L(\sigma'_-)$. Now inequality (4.12) proves the bound (1.8) by taking into account (2.19) in Remark 2.21. The proof is complete. □

Example 4.5. Let $\mathscr{D}_a=\mathbb{R}\setminus(-a,a)$ for some $a\geqslant0$. Given $\varkappa\geqslant0$, assume that A and V are operators on the Hilbert space $\mathfrak{H}=L^2(\mathscr{D}_a)$ defined by

$$\begin{gathered}(Ax)(t)=|t|x(-t),\quad (Vx)(t)=\varkappa t\,x(t),\quad t\in\mathscr{D}_a,\\ \mathrm{Dom}(A)=\mathrm{Dom}(V)=\Big\{x\in\mathfrak{H}\,\big|\,\int_{\mathscr{D}_a}t^2|x(t)|^2dt<+\infty\Big\}.\end{gathered}\tag{4.13}$$

Both A and $L=A+V$ are self-adjoint operators. The spectrum of the operator A is purely absolutely continuous. For $a>0$ it consists of two disjoint components $\sigma_-=(-\infty,-a]$ and $\sigma_+=[a,+\infty)$ and for $a=0$ it covers the whole real axis. Obviously, the isometry J in the polar decomposition $A=J|A|$ is the parity operator, $(Jx)(t)=x(-t)$, $x\in\mathfrak{H}$, and the absolute value of A is given by $(|A|x)(t)=|t|x(t)$, $x\in\mathrm{Dom}(A)$. Clearly, J is an involution on $\mathfrak{H}$ such that $J\smile A$ and $J\frown V$. Therefore, for $a>0$ (resp. for $a=0$) the operators A and V satisfy the hypothesis of Theorem 1 (resp. the hypothesis of Theorem 4.4 for $\mu=0$).

Our analysis of the subspace perturbation problem involving A and V given by (4.13) is divided into three parts below.

(i) For any $x \in \mathrm{Dom}(A)$, $\|x\| = 1$, we have

$$\begin{aligned}
|\langle x, JVx\rangle| &= \Big|\int_{\mathscr{D}_a} \varkappa t\overline{x(t)}x(-t)dt\Big| \\
&\leqslant \varkappa \int_{\mathscr{D}_a} |t||\overline{x(t)}x(-t)|dt \\
&\leqslant \varkappa \int_{\mathscr{D}_a} |t|\frac{|x(-t)|^2+|x(t)|^2}{2}dt \\
&= \varkappa \int_{\mathscr{D}_a} |t||x(t)|^2 dt \\
&= \varkappa\langle x, |A|x\rangle.
\end{aligned} \tag{4.14}$$

Moreover, if $x \in \mathrm{Dom}(A)$ is such that $x(-t) = i\,\mathrm{sign}(t)x(t)$, then inequalities in (4.14) turn into equalities. Hence, by taking this into account, (4.14) implies

$$\sup_{\substack{x \in \mathrm{Dom}(A) \\ \|x\|=1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A|x\rangle} = \varkappa. \tag{4.15}$$

An explicit evaluation of the involution $J' = \mathsf{E}_L((+\infty,0)) - \mathsf{E}_L((-\infty,0))$ by using the polar decomposition $L = J'|L|$ yields

$$(J'Jx)(t) = \frac{1}{\sqrt{1+\varkappa^2}}x(t) + \mathrm{sign}(t)\frac{\varkappa}{\sqrt{1+\varkappa^2}}x(-t). \tag{4.16}$$

From (4.16) it follows by inspection that the spectrum of the unitary operator $J'J$ consists of the two mutually conjugate eigenvalues,

$$\mathrm{spec}(J'J) = \left\{\frac{1-i\varkappa}{\sqrt{1+\varkappa^2}}, \frac{1+i\varkappa}{\sqrt{1+\varkappa^2}}\right\}.$$

This implies that $\vartheta(J'J) = \arctan\varkappa$ and then the spectral angle of the direct rotation U from J to J' is equal to $\vartheta(U) = \frac{1}{2}\arctan\varkappa$. Combining this with (4.15) yields that in the case under consideration

$$\vartheta(U) = \frac{1}{2}\arctan\Big(\sup_{\substack{x \in \mathrm{Dom}(A) \\ \|x\|=1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A|x\rangle}\Big) \quad \text{for any} \quad a \geqslant 0. \tag{4.17}$$

(ii) Now set $\widetilde{J'} = -J'$. Clearly, $\vartheta(\widetilde{J'}J) = \pi - \vartheta(J'J)$ and thus the spectral angle of the direct rotation $\widetilde{U}$ from J to $\widetilde{J'}$ reads

$$\vartheta(\widetilde{U}) = \frac{\pi}{2} - \frac{1}{2}\arctan\Big(\sup_{\substack{x \in \mathrm{Dom}(A) \\ \|x\|=1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A|x\rangle}\Big). \tag{4.18}$$

Notice that the involution $\widetilde{J}'$ commutes with L since J' does. By (4.16), it follows that $\operatorname{Ker}(I-J'J)=\{0\}$ whenever $\varkappa\neq 0$. Hence, $\operatorname{Ker}(I+\widetilde{J}'J)=\{0\}$ whenever $\varkappa\neq 0$, which means that for $\varkappa>0$ the involutions J and $\widetilde{J}'$ are in the acute case.

(*iii*) For $a>0$ we have

$$\inf_{|\mu|<a}\sup_{\substack{x\in\operatorname{Dom}(A)\\ \|x\|=1}}\frac{|\langle x,JVx\rangle|}{\langle x,|A-\mu|x\rangle}\leqslant\sup_{\substack{x\in\operatorname{Dom}(A)\\ \|x\|=1}}\frac{|\langle x,JVx\rangle|}{\langle x,|A|x\rangle}. \tag{4.19}$$

Since $\sin\vartheta(U)=\|\mathsf{E}_L\big((-\infty,-a]\big)-\mathsf{E}_A\big((-\infty,-a]\big)\|$, by Theorem 1 the strict inequality in (4.19) implies

$$\vartheta(U)<\frac{1}{2}\arctan\Big(\sup_{\substack{x\in\operatorname{Dom}(A)\\ \|x\|=1}}\frac{|\langle x,JVx\rangle|}{\langle x,|A|x\rangle}\Big),$$

which contradicts (4.17). Hence only the equality sign in (4.19) is allowed and thus

$$\inf_{|\mu|<a}\sup_{\substack{x\in\operatorname{Dom}(A)\\ \|x\|=1}}\frac{|\langle x,JVx\rangle|}{\langle x,|A-\mu|x\rangle}=\sup_{\substack{x\in\operatorname{Dom}(A)\\ \|x\|=1}}\frac{|\langle x,JVx\rangle|}{\langle x,|A|x\rangle}. \tag{4.20}$$

Remark 4.6. Example 4.5 shows the following:

(*i*) Estimate (4.9) of Theorem 4.4 is sharp. This is proven by equality (4.17).

(*ii*) Estimate (4.11) of the same theorem is sharp. This is proven by equality (4.18).

(*iii*) Estimate (1.8) of Theorem 1 is sharp. This is proven by combining equalities (4.17) and (4.20).

The celebrated sharp estimate for the operator angle between the spectral subspaces $\operatorname{Ran}\mathsf{E}_A(\sigma_-)$ and $\operatorname{Ran}\mathsf{E}_L(\sigma'_-)$ known as the Davis–Kahan $\tan 2\Theta$ Theorem [7] (cf. [16]) appears to be a simple corollary to Theorem 1.

Theorem 4.7 (The Davis–Kahan $\tan 2\Theta$ Theorem). *Given a self-adjoint operator A on the Hilbert space $\mathfrak{H}$, assume that*

$$\operatorname{spec}(A)=\sigma_-\cup\sigma_+,\quad d=\operatorname{dist}(\sigma_-,\sigma_+)>0,\quad \textit{and}\quad \sup\sigma_-<\inf\sigma_+.$$

Suppose that a bounded self-adjoint operator V on $\mathfrak{H}$ is off-diagonal with respect to the decomposition $\mathfrak{H}=\operatorname{Ran}\mathsf{E}_A(\sigma_-)\oplus\operatorname{Ran}\mathsf{E}_A(\sigma_+)$. Then the spectrum of $L=A+V$ consists of two disjoint components σ'_- and σ'_+ such that

$$\sigma'_-\subset(-\infty,\sup\sigma_-]\quad \textit{and}\quad \sigma'_+\subset[\inf\sigma_+,+\infty),$$

and

$$\|\mathsf{E}_L(\sigma'_-) - \mathsf{E}_A(\sigma_-)\| \leqslant \sin\Big(\frac{1}{2}\arctan\frac{2\|V\|}{d}\Big). \tag{4.21}$$

Proof. Hypothesis of Theorem 1 is satisfied and thus we only need to prove the estimate (4.21). Set $\mu_0 = \frac{1}{2}(\sup\sigma_+ + \inf\sigma_-)$. Clearly,

$$\begin{aligned}\inf_{\sup\sigma_-<\mu<\inf\sigma_+} \sup_{\substack{x\in\mathrm{Dom}(A)\\ \|x\|=1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A-\mu|x\rangle} &\leqslant \sup_{\substack{x\in\mathrm{Dom}(A)\\ \|x\|=1}} \frac{|\langle x, JVx\rangle|}{\langle x, |A-\mu_0|x\rangle} \\ &\leqslant \sup_{\substack{x\in\mathrm{Dom}(A)\\ \|x\|=1}} \frac{\|V\|}{\langle x, |A-\mu_0|x\rangle} \\ &\leqslant \frac{2\|V\|}{d},\end{aligned}$$

which immediately implies (4.21) by taking into account (1.8). □

5. PROOF OF THEOREM 2

In the proof of the main result of this section, we will use some auxiliary statements. We start with the following lemma.

Lemma 5.1. *Let T be a densely defined operator on a Hilbert space $\mathfrak{H}$ with* $\dim(\mathfrak{H}) \geqslant n$ *for some $n\in\mathbb{N}$. Assume that $\mathfrak{t}(x,y)$ is a sesquilinear form on $\mathfrak{H}$ such that*

$$\mathrm{Dom}(T)\subset\mathrm{Dom}(\mathfrak{t}) \quad \textit{and} \quad \mathfrak{t}(x,y) = \langle x, Ty\rangle \quad \textit{for any} \quad x,y\in\mathrm{Dom}(T).$$

Suppose that there are orthogonal projections $P_i\neq 0$, $i=1,2,\dots,n$, on $\mathfrak{H}$ with the properties

$$P_iP_j = 0 \;\; \textit{if} \;\; i\neq j, \quad \sum_{i=1}^n P_i = I, \;\; \textit{and} \;\; P_ix\in\mathrm{Dom}(\mathfrak{t}) \;\; \textit{whenever} \;\; x\in\mathrm{Dom}(\mathfrak{t}).$$

Let $\mathscr{E}$ be a set of ordered n-element orthonormal systems in $\mathfrak{H}$ defined by

$$\mathscr{E} = \big\{\{e_i\}_{i=1}^n \subset \mathrm{Dom}(\mathfrak{t}) \,\big|\, e_i\in\mathrm{Ran}\,P_i \;\; \textit{and} \;\; \|e_i\|=1 \;\; \textit{for all} \;\; i=1,2,\dots,n\big\}.$$

Then

$$\mathscr{W}(T) \subset \bigcup_{\mathbf{e}\in\mathscr{E}} \mathscr{W}(\mathfrak{t}^{\mathbf{e}}), \tag{5.1}$$

where for any $\mathbf{e}\in\mathscr{E}$ *the* $n\times n$ *matrix* $\mathfrak{t}^{\mathbf{e}}$ *is given by*

$$(\mathfrak{t}^{\mathbf{e}})_{ij}=\mathfrak{t}(e_i,e_j)\ \textit{ with }\ e_i,e_j\in\mathbf{e},\quad i,j=1,2,\dots,n.$$

If, in addition, $\mathrm{Dom}(\mathfrak{t})=\mathrm{Dom}(T)$*, then*

$$\mathscr{W}(T)=\bigcup_{\mathbf{e}\in\mathscr{E}}\mathscr{W}(\mathfrak{t}^{\mathbf{e}}).\tag{5.2}$$

Proof. By hypothesis, $\overline{\mathrm{Dom}(T)}=\mathfrak{H}$ and hence $\overline{\mathrm{Dom}(\mathfrak{t})}=\mathfrak{H}$, too. Therefore, there exists $y\in\mathrm{Dom}(\mathfrak{t})$ such that $P_iy\neq 0$ for all $i=1,2,\dots,n$. Set $e_i=\dfrac{P_iy}{\|P_iy\|}$. Taking into account that by hypothesis $P_iy\in\mathrm{Dom}(\mathfrak{t})$ and thus $e_i\in\mathrm{Dom}(\mathfrak{t})$, $i=1,2,\dots,n$, one concludes that $\{e_i\}_{i=1}^n\in\mathscr{E}$. Hence, the set $\mathscr{E}$ is nonempty.

Assume that $z\in\mathscr{W}(T)$. Then there exists $x\in\mathrm{Dom}(T)$ such that $\langle x,Tx\rangle=z$ and $\|x\|=1$. Pick up an arbitrary $\mathbf{f}=\{f_i\}_{i=1}^n\in\mathscr{E}$ and define the orthonormal system $\mathbf{g}=\{g_i\}_{i=1}^n$ by

$$g_i=\begin{cases}\dfrac{P_ix}{\|P_ix\|}, & \|P_ix\|\neq 0,\\[2mm] f_i, & \|P_ix\|=0.\end{cases}$$

Obviously, $\mathbf{g}\in\mathscr{E}$ and

$$\sum_{i,j=1}^n\mathfrak{t}(g_i,g_j)\|P_ix\|\|P_jx\|=\langle x,Tx\rangle=z,$$

which implies $z\in\mathscr{W}(\mathfrak{t}^{\mathbf{g}})$ since $\sum_{i=1}^n\|P_ix\|^2=\|x\|^2=1$. This proves the inclusion (5.1).

To prove the converse inclusion in the case where $\mathrm{Dom}(\mathfrak{t})=\mathrm{Dom}(T)$, pick up an arbitrary $\mathbf{h}=\{h_i\}_{i=1}^n\in\mathscr{E}$ and assume that $z\in\mathscr{W}(\mathfrak{t}^{\mathbf{h}})$. Then there are $\alpha_i\in\mathbb{C}$, $i=1,2,\dots,n$, such that

$$z=\sum_{i,j=1}^n\mathfrak{t}(h_i,h_j)\alpha_i\overline{\alpha}_j,\quad\sum_{i=1}^n|\alpha_i|^2=1.$$

Set $x=\sum_{i=1}^n\alpha_ih_i$. Clearly, $\|x\|=1$ and $x\in\mathrm{Dom}(\mathfrak{t})=\mathrm{Dom}(T)$. Hence $z=\mathfrak{t}(x,x)=\langle Tx,x\rangle$. This yields $z\in\mathscr{W}(T)$ and hence $\mathscr{W}(\mathfrak{t}^{\mathbf{h}})\subset\mathscr{W}(T)$. One then concludes that

$$\bigcup_{\mathbf{e}\in\mathscr{E}}\mathscr{W}(\mathfrak{t}^{\mathbf{e}})\subset\mathscr{W}(T),$$

and hence (5.2) holds, completing the proof. □

The next simple result on the numerical range of a 2×2 numerical matrix is well known (see, e.g. [8, Lemma 1.1–1]).

Lemma 5.2. *Given numbers $\alpha > 0$, $\beta > 0$, and $\gamma \in \mathbb{C}$, let M be a 2×2 matrix of the form*

$$M = \begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \beta \end{pmatrix}.$$

The matrix M is strictly accretive and its sector bound reads

$$k(M) = \frac{|\gamma|}{\sqrt{\alpha\beta}}.$$

The numerical range $\mathscr{W}(M)$ is a (possibly degenerate) elliptical disc with foci at the eigenvalues of M.

Now we are in a position to prove the main statement of the section. We only recall that by a gap of a closed set $\sigma \subset \mathbb{R}$ one understands an *open* finite interval on the real axis that does not intersect this set but both its ends belong to σ.

Theorem 5.3. *Given a self-adjoint operator A on the Hilbert space $\mathfrak{H}$, assume that*

$$\operatorname{spec}(A) = \sigma_- \cup \sigma_+, \quad \operatorname{dist}(\sigma_+, \sigma_-) = d > 0, \quad \textit{and} \quad \sigma_+ \cap \operatorname{conv}(\sigma_-) = \varnothing.$$

Denote by Δ the gap of σ_+ that contains σ_-, and by $|\Delta|$ the length of Δ. Suppose that V is a bounded self-adjoint operator on $\mathfrak{H}$ anticommuting with $J = \mathsf{E}_A(\sigma_+) - \mathsf{E}_A(\sigma_-)$ and such that

$$\|V\| < \sqrt{d(|\Delta| - d)}. \tag{5.3}$$

Then the spectrum of $L = A + V$ consists of two disjoint components σ'_- and σ'_+ such that

$$\sigma'_- \subset \Delta, \quad \sigma'_+ \subset \mathbb{R} \backslash \Delta, \tag{5.4}$$

and the involutions J and $J' = \mathsf{E}_L(\sigma'_+) - \mathsf{E}_L(\sigma'_-)$ are in the acute case. The spectral angle of the direct rotation U from J to J' satisfies the bound

$$\vartheta(U) \leqslant \frac{1}{2}\arctan\kappa\big(\|V\|\big) \quad \Big(\leqslant \frac{\pi}{4}\Big), \tag{5.5}$$

where the function $\kappa(v)$ is defined for $0 \leqslant v < \sqrt{d(|\Delta| - d)}$ by

$$\kappa(v) = \begin{cases} \dfrac{2v}{d} & \textit{if } v \leqslant \sqrt{\dfrac{d}{2}\left(\dfrac{|\Delta|}{2} - d\right)}, \\[2ex] \dfrac{v\dfrac{|\Delta|}{2} + \sqrt{d(|\Delta| - d)\left[\left(\dfrac{|\Delta|}{2} - d\right)^2 + v^2\right]}}{d(|\Delta| - d) - v^2} & \textit{if } v > \sqrt{\dfrac{d}{2}\left(\dfrac{|\Delta|}{2} - d\right)}. \end{cases} \tag{5.6}$$

Moreover, J' is a unique involution on $\mathfrak{H}$ with the properties

$$(i) \quad J' \text{ and } J \text{ are in the acute case}, \quad (ii) \quad J' \smile L, \quad \text{and} \quad (iii) \quad \vartheta(U) \leqslant \frac{\pi}{4}.$$

The spectral angle of the direct rotation $\widetilde{U}$ from J to any involution $\widetilde{J}'$ distinct from J' and satisfying (i) and (ii) is bounded from below as follows

$$\vartheta(\widetilde{U}) \geqslant \frac{\pi}{2} - \frac{1}{2}\arctan\kappa(\|V\|). \tag{5.7}$$

Proof. One may assume without loss of generality that the gap Δ is centered at the point zero. Under this assumption we set

$$\Delta = (-b, b) \text{ with } b = \frac{|\Delta|}{2}.$$

Then

$$\sigma_+ \subset \mathbb{R} \setminus (-b, b) \text{ and } \sigma_- \subset [-a, a], \tag{5.8}$$

where

$$a = \frac{|\Delta|}{2} - d, \quad 0 \leqslant a < b.$$

By Remark 2.5, the operator L admits the matrix representation (2.6) with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_- \oplus \mathfrak{H}_+$, where

$$\mathfrak{H}_- = \operatorname{Ran} \mathsf{E}_A(\sigma_-) \text{ and } \mathfrak{H}_+ = \operatorname{Ran} \mathsf{E}_A(\sigma_+)$$

are the negative and positive subspaces of J, respectively. Then [14, Theorems 1 (i) and 3.2] imply that the intervals $(-b, -a')$ and (a', b) with

$$a' = a + \|V\| \tan\left(\frac{1}{2}\arctan\frac{2\|V\|}{a+b}\right) < b$$

are in the resolvent set of L. Hence the interval (a'^2, b^2) lies in the spectral gap of L^2, and also the inclusions (5.4) hold. Taking into account (5.3), one verifies by inspection that $a'^2 \leqslant a^2 + \|V\|^2 < b^2$. Therefore, the interval $(a^2 + \|V\|^2, b^2)$ belongs to the resolvent set of L^2. Thus, the spectral projections $\mathsf{E}_{L^2-\mu}\big((-\infty, 0)\big)$ and $\mathsf{E}_{L^2-\mu}\big((0, \infty)\big)$ do not depend on

$$\mu \in (a^2 + \|V\|^2, b^2). \tag{5.9}$$

Moreover,

$$\mathsf{E}_{L^2-\mu}\big((-\infty, 0)\big) = \mathsf{E}_L(\sigma'_-), \quad \mathsf{E}_{L^2-\mu}\big((0, \infty)\big) = \mathsf{E}_L(\sigma'_+),$$

and hence

$$\mathsf{E}_{L^2-\mu}\big((0,\infty)\big)-\mathsf{E}_{L^2-\mu}\big((-\infty,0)\big)=J'. \tag{5.10}$$

Now for any μ satisfying (5.9) set

$$T_\mu = J(L^2-\mu), \quad \operatorname{Dom}\big(T_\mu\big)=\operatorname{Dom}(L^2), \tag{5.11}$$

and

$$\mathfrak{t}_\mu(x,y)=\langle LJx,Ly\rangle-\mu\langle x,Jy\rangle, \quad x,y\in\operatorname{Dom}(\mathfrak{t}_\mu)=\operatorname{Dom}(L). \tag{5.12}$$

Clearly, $\operatorname{Dom}\big(T_\mu\big)\subset\operatorname{Dom}(\mathfrak{t}_\mu)$ and $\mathfrak{t}_\mu(x,y)=\langle x,T_\mu y\rangle$ for any $x,y\in\operatorname{Dom}\big(T_\mu\big)$. Further, introduce the set $\mathscr{E}$ of ordered orthonormal two-element systems in $\mathfrak{H}$ by

$$\mathscr{E}=\big\{\{e_-,e_+\}\subset\operatorname{Dom}(\mathfrak{t}_\mu)\,\big|\,e_\pm\in\mathfrak{H}_\pm,\ \|e_\pm\|=1\big\}.$$

Then by Lemma 5.1 the following inclusion holds

$$\mathscr{W}\big(T_\mu\big)\subset\bigcup_{\mathbf{e}\in\mathscr{E}}\mathscr{W}\big(\mathfrak{t}_\mu^{\mathbf{e}}\big), \tag{5.13}$$

where $\mathfrak{t}_\mu^{\mathbf{e}}$ are 2×2 matrices given by

$$\mathfrak{t}_\mu^{\mathbf{e}}=\begin{pmatrix}\mathfrak{t}_\mu(e_-,e_-) & \mathfrak{t}_\mu(e_-,e_+)\\ \mathfrak{t}_\mu(e_+,e_-) & \mathfrak{t}_\mu(e_+,e_+)\end{pmatrix}, \quad \mathbf{e}=\{e_-,e_+\}\in\mathscr{E}.$$

By taking into account that $A\smile J$ and $V\frown J$, one observes

$$\mathfrak{t}_\mu^{\mathbf{e}}=\begin{pmatrix}\mu-\|Ae_-\|^2-\|Ve_-\|^2 & -\overline{(\langle Ae_+,Ve_-\rangle+\langle Ve_+,Ae_-\rangle)}\\ \langle Ae_+,Ve_-\rangle+\langle Ve_+,Ae_-\rangle & \|Ae_+\|^2+\|Ve_+\|^2-\mu\end{pmatrix}. \tag{5.14}$$

From (5.8) it follows that for $\{e_-,e_+\}\in\mathscr{E}$

$$\|Ae_-\|\leqslant a \ \text{ and } \ \|Ae_+\|\geqslant b. \tag{5.15}$$

Hence, by the assumption (5.9) by Lemma 5.2 it follows from (5.14) and (5.15) that for all $\mathbf{e}\in\mathscr{E}$ the numerical ranges $\mathscr{W}\big(\mathfrak{t}_\mu^{\mathbf{e}}\big)$ are elliptical discs that lie in the open right half-plane $\{z\in\mathbb{C}\,|\,\operatorname{Re}z>0\}$. Then (5.13) implies that the numerical range $\mathscr{W}\big(T_\mu\big)$ also lies in the open right half-plane, that is, the operator T_μ is strictly accretive. Hence, taking into account (5.10) and (5.11), Theorem 3.4 yields that the involutions J and J' are in the acute case. Moreover, for the direct rotation U from J to J' the following inequality holds

$$\vartheta(U)\leqslant\frac{1}{2}\arctan k(T_\mu), \tag{5.16}$$

where μ is an arbitrary point from the interval (5.9). In its turn, inclusion (5.13) implies

$$k(T_\mu) \leqslant \sup_{\mathbf{e}\in\mathscr{E}} k(\mathfrak{t}_\mu^{\mathbf{e}}). \tag{5.17}$$

Since

$$|\langle Ae_+, Ve_-\rangle + \langle Ve_+, Ae_-\rangle| \leqslant \|Ae_+\|\|Ve_-\| + \|Ae_-\|\|Ve_+\|, \quad \mathbf{e} = \{e_-, e_+\} \in \mathscr{E},$$

by Lemma 5.2 it follows from (5.14) that

$$k\big(\mathfrak{t}_\mu^{\mathbf{e}}\big) \leqslant f_\mu(\alpha_-, \alpha_+, v_-, v_+), \tag{5.18}$$

where

$$f_\mu(\alpha_-, \alpha_+, v_-, v_+) = \frac{\alpha_- v_+ + \alpha_+ v_-}{(\mu - \alpha_-^2 - v_-^2)^{1/2}(\alpha_+^2 + v_+^2 - \mu)^{1/2}}$$

with $\alpha_\pm = \|Ae_\pm\|$ and $v_\pm = \|Ve_\pm\|$.

By (5.15), we have

$$0 \leqslant \alpha_- \leqslant a \ \text{ and } \ \alpha_+ \geqslant b, \tag{5.19}$$

while

$$0 \leqslant v_- \leqslant \|V\| \ \text{ and } \ 0 \leqslant v_+ \leqslant \|V\|. \tag{5.20}$$

A direct computation shows that the supremum of the function f_μ over the set in $\mathbb{R}^4$ constrained by (5.19) and (5.20) equals

$$\varkappa(\mu) = \begin{cases} \dfrac{\|V\|(a+b)}{(\mu - a^2 - \|V\|^2)^{1/2}(b^2 + \|V\|^2 - \mu)^{1/2}} & \text{if } a(b^2-\mu) > b\|V\|^2, \\[2ex] \dfrac{[b^2\|V\|^2 + a^2(b^2-\mu)]^{1/2}}{(\mu - a^2 - \|V\|^2)^{1/2}(b^2 - \mu)^{1/2}} & \text{if } a(b^2-\mu) \leqslant b\|V\|^2. \end{cases}$$

Then by (5.16)–(5.18) one infers that

$$\vartheta(U) \leqslant \frac{1}{2}\arctan\varkappa(\mu) \ \text{ for any } \ \mu \in (a^2 + \|V\|^2, b^2).$$

In particular,

$$\vartheta(U) \leqslant \frac{1}{2}\arctan\varkappa_{\min}, \tag{5.21}$$

where

$$\varkappa_{\min} = \inf_{a^2+\|V\|^2<\mu<b^2} \varkappa(\mu). \tag{5.22}$$

By inspection, the function $\varkappa(\mu)$ is continuously differentiable on the interval $(a^2 + \|V\|^2, b^2)$. The (global) minimum of $\varkappa$ on this interval is just equal to $\kappa\big(\|V\|\big)$. By (5.21), the equality $\varkappa_{\min} = \kappa\big(\|V\|\big)$ proves the bound (5.5).

The uniqueness of an involution J' with the properties (i)–(iii) follows from Theorem 3.6. Estimate (5.7) is an immediate corollary to Theorem 3.5.

The proof is complete. □

Remark 5.4. Notice that in the case where the operator A is bounded, the estimate $\|\mathsf{E}_L(\sigma'_-) - \mathsf{E}_A(\sigma_-)\| < \frac{\sqrt{2}}{2}$ (or equivalently $\vartheta(U) < \pi/4$) may be obtained by combining [14, Theorem 1 (*ii*)] and [16, Theorem 5.6].

Theorem 2 is an immediate corollary to Theorem 5.3.

Proof of Theorem 2. The inclusions (1.9) follow from [15, Theorem 4].

Let Δ be the gap of the set σ_+ that contains σ_-. Obviously, $|\Delta| \geqslant 2d$, where $|\Delta|$ is the length of Δ, and thus $\|V\| < d \leqslant \sqrt{d(|\Delta| - d)}$. By Theorem 5.3, one concludes that

$$\|\mathsf{E}_L(\sigma'_-) - \mathsf{E}_A(\sigma_-)\| \leqslant \sin\Big(\frac{1}{2}\arctan\kappa\big(\|V\|\big)\Big)$$

with $\kappa(v)$ given by (5.6). Observing that for $0 \leqslant v < d$

$$\kappa(v) \leqslant \frac{2vd}{d^2 - v^2} = \tan\Big(2\arctan\frac{v}{d}\Big)$$

completes the proof. □

Example 5.5. Let A be a self-adjoint operator on $\mathfrak{H} = \mathbb{C}^4$ defined by

$$A = \operatorname{diag}\{-b, -a, a, b\}, \quad 0 \leqslant a < b.$$

Divide the spectrum of A into the two disjoint sets $\sigma_- = \{-a, a\}$ and $\sigma_+ = \{-b, b\}$. Clearly, $d = \operatorname{dist}(\sigma_-, \sigma_+) = b - a > 0$. The interval $\Delta = (-b, b)$ appears to be the gap of the set σ_+ containing the set σ_-. The involution $J = \mathsf{E}_A(\sigma_+) - \mathsf{E}_A(\sigma_-)$ reads

$$J = \operatorname{diag}\{+1, -1, -1, +1\}.$$

Assume that V is a 4×4 matrix of the form

$$V = \begin{pmatrix} 0 & v_1 & v_2 & 0 \\ v_1 & 0 & 0 & v_2 \\ v_2 & 0 & 0 & v_1 \\ 0 & v_2 & v_1 & 0 \end{pmatrix}, \tag{5.23}$$

where $v_1, v_2 \geqslant 0$. By inspection, V anticommutes with J and $\|V\| = v_1 + v_2$. The involution $J' = \mathsf{E}_L(\mathbb{R}\setminus\Delta) - \mathsf{E}_L(\Delta)$ is computed explicitly as soon as the eigenvectors of the 4×4 matrix $L = A + V$ are found. By the assumption that (5.3) holds, that is, for $\|V\|^2 < b^2 - a^2$, the explicit evaluation of the spectral angle of the direct rotation U from J to J' results in

$$\vartheta(U) = \frac{1}{2}\arctan\left(\frac{2a(v_1 - v_2) + 2b\|V\|}{b^2 - a^2 - \|V\|^2 + (v_1 - v_2)^2}\right).$$

Taking into account that the value of $v_1 - v_2$ for different matrices (5.23) with the same norm $\|V\|$ runs through the interval $[-\|V\|, \|V\|]$, one easily verifies that the maximal possible value $\vartheta_{\max}$ of $\vartheta(U)$ is equal to

$$\vartheta_{\max} = \frac{1}{2} \arctan \kappa(\|V\|) \tag{5.24}$$

with $\kappa(v)$ given by (5.6). In particular, if $a = 0$ then

$$\vartheta_{\max} = \arctan \left(\frac{\|V\|}{d} \right). \tag{5.25}$$

Remark 5.6. Example 5.5 shows the following:

(i) Estimate (5.5) of Theorem 5.3 is sharp. This is proven by equality (5.24).

(ii) Estimate (1.10) of Theorem 2 is also sharp. This is proven by equality (5.25).

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