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SCALING REGIMES IN THE MODEL OF PASSIVE SCALAR ADVECTED BY THE TURBULENT VELOCITY FIELD WITH FINITE CORRELATION TIME. INFLUENCE OF HELICITY IN TWO-LOOP APPROXIMATION

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Скейлинговые режимы в модели пассивной скалярной примеси в турбулентном поле скорости с конечными временными корреляциями. Влияние нарушения зеркальной симметрии в двухпетлевом приближении

Исследуется адвекция пассивной скалярной примеси несжимаемым гиротропным турбулентным потоком в рамках расширенной модели Крейчнана. Предполагается, что статистические флуктуации поля скорости имеют гауссово распределение с нулевым средним и шумом с конечными временными корреляциями. Вычисления проведены в двухпетлевом приближении в рамках теоретико-полевой ренормализационной группы. Показано, что нарушение пространственной четности (гиротропия) стохастической среды не влияет на аномальный скейлинг, который является специфическим свойством такого рода моделей без гиротропии. Однако устойчивость асимптотических режимов, в которых имеет место аномальный скейлинг, а также эффективная диффузия сильно зависят от количества гиротропии.

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Scaling Regimes in the Model of Passive Scalar Advected by the Turbulent Velocity Field with Finite Correlation Time. Influence of Helicity in Two-Loop Approximation

The advection of a passive scalar quantity by incompressible helical turbulent flow has been investigated in the framework of an extended Kraichnan model. Statistical fluctuations of the velocity field are assumed to have the Gaussian distribution with zero mean and defined noise with finite-time correlation. Actual calculations have been done up to two-loop approximation in the framework of the field-theoretic renormalization group approach. It turned out that the space parity violation (helicity) of a stochastic environment does not affect anomalous scaling which is the peculiar attribute of a corresponding model without helicity. However, stability of asymptotic regimes, where anomalous scaling takes place, and the effective diffusivity strongly depend on the amount of helicity.

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INTRODUCTION

It has become common practice to say that the theoretical understanding of turbulence remains one of the last unsolved problem of classical physics. Within the one part of the wide concept of turbulence, namely, fully developed turbulence, one of the most interesting open question is the theoretical explanation of the possible deviations from the classical Kolmogorov theory [1–3] which is suggested by both natural as well as numerical experiments. Such a behavior is contained in concepts of intermittency and anomalous scaling. During the last decade this problem was intensively studied within the scope of the models of a passively advected scalar field (concentration of an admixture, or temperature are examples) by a «synthetic» velocity field with prescribed Gaussian statistics. The reason is twofold. First, the deviation from the classical theory is even more strongly noticeable for a passively advected scalar field than for the velocity field itself, see, e. g., [3–6], and second, the problem of passive advection is considerably easier for theoretical investigation [7]. Moreover, it reproduces many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. Thus, the theoretical study of the models of a passive scalar (or also vector) advection can be treated as the first step on the long way of the investigation of intermittency and anomalous scaling in fully developed turbulence. On the other hand, the problem of advection has also its own practical importance.

The central role in the studies of a passive advection was played by a simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space, the so-called Kraichnan rapid-change model [8]. There, for the first time, the anomalous scaling was established on the basis of a microscopic model [9], and corresponding anomalous exponents were calculated within controlled approximations [10] (see also review [11] and references therein).

An effective method for investigation of a self-similar scaling behavior is the renormalization group (RG) technique [12–14]. It was widely used in the theory of critical phenomena to explain the origin of the critical scaling and also to calculate corresponding universal quantities (e.g., critical dimensions). This method can be also directly used in the theory of turbulence [13, 15–17] and related models like a simpler stochastic problem of a passive scalar advected by prescribed stochastic flow. In what follows we use the conventional («quantum field theory» or field theoretic) RG which is based on the standard renormalization procedure, i. e., on an elimination of the ultraviolet (UV) divergences.

In paper [18] the field-theoretic RG and operator-product expansion (OPE) were used in the systematic investigation of the rapid-change model. It was shown that within the field theoretic approach the anomalous scaling is related to the very existence of the so-called «dangerous» composite operators with negative critical dimensions in OPE (see, e. g., [13, 17] for details). In the subsequent papers [19] the anomalous exponents of the model were calculated within the ε expansion to order ε^3 (three-loop approximation). Here ε is a parameter which describes a given equal-time pair correlation function of the velocity field (see subsequent section).

Afterwards, various descendants of the Kraichnan model, namely, models with inclusion of large and small scale anisotropy [20], compressibility [21] and finite correlation time of the velocity field [22–24] were studied by the field theoretic approach. Moreover, advection of a passive vector field by the Gaussian self-similar velocity field (with and without large and small scale anisotropy, pressure, compressibility, and finite correlation time) has been also investigated and all possible asymptotic scaling regimes and cross-over among them have been classified [25]. General conclusion is: the anomalous scaling, which is the most important feature of the Kraichnan rapid change model, remains valid for all generalized models.

Let us describe briefly the solution of the problem in the framework of the field theoretic approach. It can be divided into two main stages. On the first stage the multiplicative renormalizability of the corresponding field theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behavior of the latter on their ultraviolet argument (r/ℓ) for $r \gg \ell$ and any fixed (r/L) is given by infrared stable fixed points of those equations. Here ℓ and L are the inner (ultraviolet) and outer (infrared) scales. It involves some «scaling functions» of the infrared argument (r/L), whose form is not determined by the RG equations. On the second stage, their behavior at $r \ll L$ is found from the OPE within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite operators, which give rise to an infinite family of independent aforementioned scaling exponents (and hence to multiscaling).

In [22] the problem of a passive scalar advected by the Gaussian self-similar velocity field with finite correlation time [26] was studied by the field theoretic RG method. There, the systematic study of the possible scaling regimes and anomalous behavior were presented at one-loop level. The two-loop corrections to the anomalous exponents were obtained in [24]. It was shown that the anomalous exponents are nonuniversal as a result of their dependence on a dimensionless parameter, the ratio of the velocity correlation time, and turnover time of a scalar field.

In what follows, we shall continue with the investigation of this model from the point of view of the influence of helicity on the scaling regimes within two-loop approximation.

Helicity is defined as the scalar product of velocity and vorticity and its nonzero value expresses mirror symmetry breaking of turbulent flow. It plays significant role in the processes of magnetic field generation in electrically conductive fluid [27–33] and represents one of the most important characteristics of large-scale motions as well [34–37]. The presence of helicity is observed in various natural (like large air vortices in atmosphere) and technical flows [35, 38, 39]. Despite of this fact the role of the helicity in hydrodynamical turbulence is not completely clarified up to now.

The Navier–Stokes equations conserve kinetic energy and helicity in inviscid limit. Presence of two quadratic invariants leads to the possibility of appearance of double cascade. It means that cascades of energy and helicity take place in different ranges of wave numbers analogously to the two-dimensional turbulence and/or the helicity cascade appears concurrently to the energy one in the direction of small scales [40, 41]. Particularly, helicity cascade is closely connected with the existence of exact relation between triple and double correlations of velocity known as «2/15» law analogously to the «4/5» Kolmogorov law [42]. Corresponding to [40] aforementioned scenarios of turbulent cascades differ each other by spectral scaling. Theoretical arguments given by Kraichnan [43] and results of numerical calculations of Navier–Stokes equations [44–46] support the scenario of concurrent cascades. The appearance of helicity in turbulent system leads to constraint of non-linear cascade to the small scales. This phenomenon was first demonstrated by Kraichnan [43] within the modelling problem of statistically equilibrium spectra and later in numerical experiments.

Turbulent viscosity and diffusivity, which characterize influence of small-scale motions on heat and momentum transport, are basic quantities investigated in the theoretic and applied models. The constraint of direct energy cascade in helical turbulence has to be accompanied by decrease of turbulent viscosity. However, no influence of helicity on turbulent viscosity was found in some works [47, 48]. Similar situation is observed for turbulent diffusivity in helical turbulence. Although the modelling calculations demonstrate intensification of turbulent transfer in the presence of helicity [49, 50] direct calculation of diffusivity does not confirm this effect [49, 51, 52]. Helicity is the pseudoscalar quantity hence it can be easily understood, that its influence appears only in quadratic and higher terms of perturbation theory or in the combination with other pseudoscalar quantities (e. g. large-scale helicity). Really, simultaneous consideration of memory effects and second-order approximation indicate effective influence of helicity on turbulent viscosity [53,54] and turbulent diffusivity [50,55,56] already in the limit of small or infinite correlation time.

Helicity, as we shall see below, does not affect known results in one-loop approximation and, therefore, it is necessary to turn to the second order (two-loop) approximation to be able to analyze possible consequences. It is also important to say that within the framework of the classical Kraichnan model, i. e., model of passive advection by the Gaussian velocity field with δ-like correlations in time, it is not possible to study the influence of the helicity because all potentially «helical» diagrams are identically equal to zero at all orders in the perturbation theory. In this sense, the investigation of the helicity in the present model can be consider as the first step to analyze the helicity in genuine turbulence.

The paper is organized as follows: In Sec. 1 we present definition of the model and introduce the helicity to the transverse projector of a given pair correlation function of the velocity field. We give the field theoretic formulation of the original stochastic problem and discuss the corresponding diagrammatic technique. In Sec. 2 we analyze the ultraviolet (UV) divergences of the model, establish its multiplicative renormalizability and calculate the renormalization constants in the two-loop approximation. In Sec. 3 we analyze possible scaling regimes of the model, associated with nontrivial and physically acceptable fixed points of the corresponding RG equations. There are five such regimes, anyone of them can be realized in dependence on the values of parameters of the model. We discuss the physical meaning of these regimes (e. g., some of them correspond to zero, finite, or infinite correlation time of the advecting field) and their regions of stability in the space of the model parameters. In Sec. 4 the two-loop corrections to the effective diffusivity are calculated. In Conclusion the discussion of results is presented.

1. FIELD THEORETIC DESCRIPTION OF THE MODEL

The advection of a passive scalar field $\theta(x) \equiv \theta(t, \mathbf{x})$ in helical turbulent environment is described by the stochastic equation

$$\partial_t \theta + v_i \partial_i \theta = \nu_0 \Delta \theta + f, \tag{1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, ν_0 is the molecular diffusivity coefficient (hereafter all parameters with a subscript 0 denote bare parameters of unrenormalized theory; see below), $\triangle \equiv \partial^2$ is the Laplace operator, $v_i \equiv v_i(x)$ is the ith component of the divergence-free (owing to the incompressibility) velocity field $\mathbf{v}(x)$, and $f \equiv f(x)$ is an artificial Gaussian random noise with zero mean and correlation function

$$\langle f(x) f(x') \rangle = \delta(t - t') C(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}', \tag{2}$$

where parentheses $\langle \dots \rangle$ hereafter denote average over corresponding statistical ensemble and L is an integral scale. The random noise maintains the steady-state of the system but the detailed form of the function $C(\mathbf{r}/L)$ in Eq. (2) is inessential in our further consideration. It is only important that the function $C(\mathbf{r}/L)$ is finite and decreases rapidly for $r \equiv |\mathbf{r}| \gg L$. In spite of the fact that in real problems the velocity field $\mathbf{v}(x)$ satisfies the Navier–Stokes equation, in what follows, we suppose that statistics of velocity field is given in the form of the Gaussian distribution with zero mean and pair correlation function

$$\langle v_i(x) v_j(x') \rangle = \int \frac{d\omega d^d\mathbf{k}}{(2\pi)^{d+1}} P^{\rho}_{ij}(\mathbf{k}) D_v(\omega, k) \exp[-i(t-t') + i\mathbf{k}(\mathbf{x}-\mathbf{x}')], \quad (3)$$

with

$$D_v(\omega, k) = \frac{D_0 k^{4-d-2\varepsilon-\eta}}{(i\omega + u_0\nu_0 k^{2-\eta})(-i\omega + u_0\nu_0 k^{2-\eta})}, \quad (4)$$

where $k \equiv |\mathbf{k}|$. The transverse projector $P^{\rho}_{ij}(\mathbf{k})$ reflects vectorial nature of the solenoidal velocity field and will be specified below. Here $D_0 \equiv g_0 \nu_0^3$ is a positive amplitude factor and introduced parameter g_0 plays the role of the coupling constant of the model, the analog of the coupling constant λ_0 in the $\lambda_0 \varphi^4$ model of critical behavior [12, 13]. In addition, g_0 is a formal small parameter of the ordinary perturbation theory. The parameter u_0 gives the ratio of the turnover time of a scalar field and the velocity correlation time. The positive exponents ε and η ($\varepsilon = O(\eta)$) are small RG expansion parameters, the analogs of the parameter $\varepsilon = 4 - d$ in the $\lambda_0\varphi^4$ theory. Thus we have a kind of double expansion model in the $\varepsilon - \eta$ plane around the origin $\varepsilon = \eta = 0$. Correlation function (4) is directly related to the energy spectrum via the frequency integral [22, 58, 59]

$$E(k) \simeq k^{d-1} \int d\omega D_v(\omega, k) \simeq \frac{g_0 \nu_0^2}{u_0} k^{1-2\varepsilon}. \quad (5)$$

Therefore, the coupling constant g_0 and the exponent ε control the behavior of the equal-time pair correlation function of the velocity field (mean square velocity) or, equivalently, energy spectrum. On the other hand, the parameter u_0 and the second exponent η are related to the frequency $\omega \simeq u_0 \nu_0 k^{2-\eta}$ which characterizes the mode k [58–61]. Thus, in our notation, the value $\varepsilon = 4/3$ corresponds to the celebrated Kolmogorov «two-thirds law» for the spatial statistics of the velocity field or, equivalently, «five-thirds law» for the energy spectrum, and $\eta = 4/3$ corresponds to the Kolmogorov frequency. Simple dimensional analysis shows that g_0 and u_0, which we commonly term as charges, are related to the characteristic ultraviolet (UV) momentum scale Λ (or inner length $l \sim \Lambda^{-1}$) by

$$g_0 \simeq \Lambda^{2\varepsilon+\eta}, \;\; u_0 \simeq \Lambda^{\eta}. \quad (6)$$

For completeness, we remain d-dependence in expressions (3) and (4) (d is the dimensionality of the $\mathbf{x}$ space), although, of course, when one investigates system with helicity the dimension of the $\mathbf{x}$ space must be strictly equal to three. It allows one to study d-dependence of the non-helical case of the model. To include helicity aforementioned transverse projector $P^{\rho}_{ij}(\mathbf{k})$ is taken in the form

$$P^{\rho}_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) + H_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2 + i\rho\epsilon_{ijl}\frac{k_l}{k}. \tag{7}$$

Here $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ represents non-helical part of the total transverse projector $P^{\rho}_{ij}(\mathbf{k})$. On the other hand, $H_{ij}(\mathbf{k}) = i\rho\epsilon_{ijl}\frac{k_l}{k}$ mimics the presence of helicity in the flow. Thus, formally, the transition to the helical fluid corresponds to the breaking of spatial parity, and, technically, this is expressed by the fact that the correlation function is specified in the form of mixture of a true tensor and a pseudotensor. The tensor ε_{ijl} is Levi-Civita's completely antisymmetric tensor of rank 3 (it is equal to 1 or -1 according to whether (i,j,l) is an even or odd permutation of $(1,2,3)$ and zero otherwise), and the real parameter of helicity ρ controls the amount of helicity. Due to the requirement of positive definiteness of the correlation function the absolute value of ρ must be in the interval $|\rho| \in \langle 0,1\rangle$ [29,30]. Nonzero helical part proportional to ρ physically expresses existence of nonzero correlations $\langle \mathbf{v}\,\mathrm{rot}\mathbf{v}\rangle$.

The general model (3), (4) contains two important special cases: the rapid-change model limit when $u_0 \to \infty$ and $g_0' \equiv g_0/u_0^2 = \text{const}$,

$$D_v(\omega,k) \to g_0'\nu_0 k^{-d-2\varepsilon+\eta}, \tag{8}$$

and the quenched (time-independent or frozen) velocity field limit which is defined by $u_0 \to 0$ and $g_0'' \equiv g_0/u_0 = \text{const}$,

$$D_v(\omega,k) \to g_0''\nu_0^2 k^{-d+2-2\varepsilon}\pi\delta(\omega), \tag{9}$$

which is similar to the well-known models of the random walks in random environment with long range correlations; see, e. g., Refs. [62,63].

Using the Martin–Siggia–Rose mechanism [64] the stochastic problem (1)–(4) can be treated as a field theory with action functional

$$S(\theta,\theta',\mathbf{v}) = \theta' D_\theta \theta'/2 + \theta'[-\partial_t + \nu_0\triangle - (v_i\partial_i)]\theta - \mathbf{v}D_v^{-1}\mathbf{v}/2, \tag{10}$$

where θ' is an auxiliary scalar field, and D_θ and D_v are correlators (2) and (3), respectively. In the action (10) all the required integrations over $x = (t,\mathbf{x})$ and summations over the vector indices are understood. The first four terms in Eq. (10) represent the Dominicis–Janssen-type action for the stochastic problem (1), (2) at fixed $\mathbf{v}$, and the last term represents the Gaussian averaging over $\mathbf{v}$.

Model (10) corresponds to a standard Feynman diagrammatic technique with the bare propagators $\langle\theta\theta'\rangle_0$ and $\langle v_i v_j\rangle_0$ (in the time-momentum representation)

$$\langle\theta(t,\mathbf{k})\theta'(t',\mathbf{k})\rangle_0 = \theta(t-t')\mathrm{e}^{-\nu_0 k^2(t-t')}, \tag{11}$$

$$\langle v_i(t,\mathbf{k})v_j(t',\mathbf{k})\rangle_0 = \frac{D_0}{2u_0 k^{1+2\epsilon}}\mathrm{e}^{-u_0\nu_0 k^{2-\eta}(t-t')}P^{\rho}_{ij}(\mathbf{k}), \tag{12}$$

where $\theta(t-t')$ is the step function, or (in the momentum-frequency representation)

$$\langle\theta(\omega,\mathbf{k})\theta'(\omega,\mathbf{k})\rangle_0 = \frac{1}{-i\omega+\nu_0 k^2}, \tag{13}$$

$$\langle v_i(\omega,\mathbf{k})v_j(\omega,\mathbf{k})\rangle_0 = P^{\rho}_{ij}(\mathbf{k})D_v(\omega,k), \tag{14}$$

where $D_v(\omega,k)$ is given directly by Eq. (4). In the Feynman diagrams, these propagators are represented by the lines which are shown in Fig. 1 (the end with a slash in the propagator $\langle\theta\theta'\rangle_0$ corresponds to the field θ', and the end without a slash corresponds to the field θ). The triple vertex (or interaction vertex) $-\theta' v_j\partial_j\theta = \theta' v_j V_j\theta$, where $V_j = ik_j$ (in the momentum-frequency representation), is presented in Fig. 1, where momentum $\mathbf{k}$ is flowing into the vertex via the auxiliary field θ'.

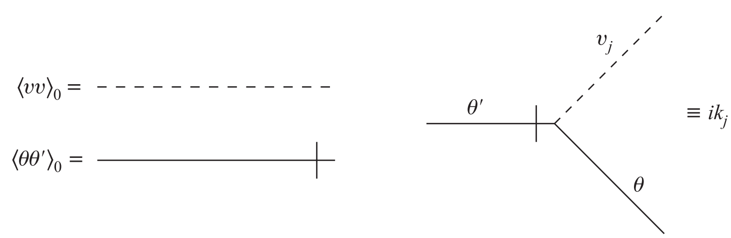

Fig. 1. Left: Graphical representations of the needed propagators of the model. Right: The triple (interaction) vertex of the model. Momentum $\mathbf{k}$ is entering into the vertex via field θ'

2. RENORMALIZATION GROUP ANALYSIS

Model (10) is logarithmic for $\epsilon = \eta = 0$ (the coupling constant g_0 is dimensionless) and, in the framework of known dimensional regularization and minimal substraction (MS) scheme, which we use, possible ultraviolet (UV) divergences have the form of poles in various linear combinations of ϵ and η in the correlation functions. Using the standard analysis of quantum field theory one finds that, in the model under consideration, all divergences can be removed by the only counter-term of the form $\theta' \triangle \theta$ [22]. Thus, the model is multiplicatively renormalizable, which is expressed explicitly in the multiplicative renormalization of the parameters g_0, u_0, and ν_0 in the form

$$\nu_0 = \nu Z_\nu, \;\; g_0 = g\mu^{2\varepsilon+\eta} Z_g, \;\; u_0 = u\mu^\eta Z_u. \tag{15}$$

Here the dimensionless parameters g, u and ν are the renormalized counterparts of the corresponding bare ones, μ is the renormalization mass (a scale setting parameter), an artefact of dimensional regularization. Newly introduced quantities $Z_i = Z_i(g, u; d, \rho; \epsilon, \eta) = Z_i(g, u; d, \rho; \epsilon)$ are renormalization constants (note that if ρ is nonzero then $d = 3$) and, in general, contain poles of linear combinations of ϵ and η. However, as detailed analysis shows, to obtain all important quantities as the γ functions, β functions, coordinates of fixed points, and the critical dimensions, the knowledge of the renormalization constants for the special choice $\eta = 0$ is sufficient up to two-loop approximation (see details in [22]).

The renormalized action functional has the following form:

$$S_R(\theta, \theta', \mathbf{v}) = \theta' D_\theta \theta'/2 + \theta'[-\partial_t + \nu Z_\nu \triangle - (v\partial)]\theta - \mathbf{v} D_v^{-1} \mathbf{v}/2, \tag{16}$$

where the correlator D_v is written in renormalized parameters (in wave-number-frequency representation)

$$D_{v,ij}(\omega, k) = \frac{P_{ij}^\rho(\mathbf{k}) g \nu^3 \mu^{2\varepsilon+\eta} k^{4-d-2\varepsilon-\eta}}{(i\omega + u\nu\mu^\eta k^{2-\eta})(-i\omega + u\nu\mu^\eta k^{2-\eta})}. \tag{17}$$

By comparison of the renormalized action (16) with definitions of the renormalization constants Z_i, $i = g, u, \nu$ (15) one comes to the relations among them

$$Z_\nu = Z_1, \;\; Z_g = Z_\nu^{-3}, \;\; Z_u = Z_\nu^{-1}. \tag{18}$$

The second and third relations are consequences of the absence of the renormalization of the term with D_v

$$g_0 \nu_0^3 = g\mu^{2\epsilon+\eta} \nu^3, u_0 \nu_0 = u\mu^\eta \nu \tag{19}$$

in renormalized action (16).

The issue of interest is, in particular, the behavior of response functions, e. g. $\langle \theta(x)\theta'(x') \rangle$, correlation functions $\langle \theta(x_1)\theta(x_2)\ldots\theta(x_n) \rangle$, and the equal-time structure functions

$$S_n(r) \equiv \langle [\theta(t, \mathbf{x}) - \theta(t, \mathbf{x}')]^n \rangle, \;\; r = |\mathbf{x} - \mathbf{x}'| \tag{20}$$

in the inertial range, specified by the inequalities $l \ll r \ll L$ (l is an internal length). Here parentheses $\langle \ldots \rangle$ mean functional average over fields $\theta, \theta', \mathbf{v}$ with weight $\exp(S_R)$. In the isotropic case, the odd functions S_{2n+1} vanish, while for S_{2n} simple dimensional considerations give

$$S_{2n}(r) = \nu_0^{-n}\, r^{2n}\, R_{2n}(r/l, r/L, g_0, u_0, \rho), \tag{21}$$

where R_{2n} are some functions of dimensionless variables. In principle, they can be calculated within the ordinary perturbation theory (i. e., as series in g_0), but this is not useful for investigation of the inertial-range behavior: the coefficients are singular in the limits $r/l \to \infty$ and/or $r/L \to 0$, which compensate the smallness of g_0, and in order to find correct IR behavior we have to sum the entire series. The desired summation can be accomplished using the field theoretic renormalization group (RG) and operator product expansion (OPE); see [18, 19, 22].

The RG analysis consists of two main stages. On the first stage, the multiplicative renormalizability of the model is demonstrated and the differential RG equations for its correlation (structure) functions are obtained. The asymptotic behavior of the functions like (20) for $r/l \gg 1$ and any fixed r/L is given by IR stable fixed points (see below) of the RG equations and has the form

$$S_{2n}(r) = \nu_0^{-n}\, r^{2n}\, (r/l)^{-\gamma_n} R_{2n}(r/L, \rho), \quad r/l \gg 1 \tag{22}$$

with certain, as yet unknown, «scaling functions» $R_{2n}(r/L, \rho)$. In the theory of critical phenomena [12, 13] the quantity $\Delta[S_{2n}] \equiv -2n + \gamma_n$ is termed the «critical dimension», and the exponent γ_n, the difference between the critical dimension $\Delta[S_{2n}]$ and the «canonical dimension» $-2n$, is called the «anomalous dimension». In the case at hand, the latter has an extremely simple form: $\gamma_n = n\epsilon$. Whatever be the functions $R_n(r/L, \rho)$, the representation (22) implies the existence of a scaling (scale invariance) in the IR region ($r/l \gg 1$, r/L fixed) with definite critical dimensions of all «IR relevant» parameters, $\Delta[S_{2n}] = -2n + n\epsilon$, $\Delta_r = -1$, $\Delta_{L^{-1}} = 1$ and fixed «irrelevant» parameters ν_0 and l.

On the second stage, the small r/L behavior of the functions $R_{2n}(r/L, \rho)$ is studied within the general representation (22) using the OPE. It shows that, in the limit $r/L \to 0$, the functions $R_{2n}(r/L, \rho)$ have the asymptotic form

$$R_{2n}(r/L) = \sum_F C_F(r/L)\, (r/L)^{\Delta_n}, \tag{23}$$

where C_F are coefficients regular in r/L. In general, the summation is implied over certain renormalized composite operators F with critical dimensions Δ_n. In case under consideration the leading operators F have the form $F_n = (\partial_i\theta\partial_i\theta)^n$.

We have performed the complete two-loop calculation of the critical dimensions of the composite operators F_n for arbitrary values of n, d, u and ρ

$$\Delta[F] = \Delta_n^{(1)}\epsilon + \Delta_n^{(2)}\epsilon^2, \tag{24}$$

where

$$\Delta_n^{(1)} = \frac{-n(n-2)(d-1)}{2(d-1)(d+2)} \tag{25}$$

is expression obtained in one-loop approximation.

Two-loop contribution $\Delta_n^{(2)}$ in non-helical case is rather cumbersome and can be found in [24]. The main result of our investigation of the influence of the helicity on the result given in [24] is that although separated two-loop Feynman graphs of operators F_n strongly depend on helicity parameter ρ, such a dependence disappears in their sum, which gives rise to the critical dimensions Δ_n. We can conclude that in two-loop approximation anomalous scaling with *negative* exponents (24) is not affected by the existence of nonzero helical correlations $\langle \mathbf{v}\,\mathrm{rot}\mathbf{v}\rangle$ in turbulent incompressible flow. It turns out, however, that region of stability of possible asymptotic regimes governed by fixed points of RG equations, where anomalous scaling takes place, and effective diffusivity rather strongly depends on ρ.

Let us analyze asymptotic regimes in detail. The structure functions and the other statistical averages of random fields θ, θ' satisfy linear differential RG equations with linear differential operator $\mathcal{D}_{RG}$. For example, RG equation for pair structure function S_2 has the form

$$\mathcal{D}_{RG} S_2(r) = 0, \tag{26}$$

with

$$\mathcal{D}_{RG} = \mathcal{D}_\mu + \beta_g(g,u)\partial_g + \beta_u(g,u)\partial_u - \gamma_\nu(g,u)\mathcal{D}_\nu. \tag{27}$$

Here $\mathcal{D}_x \equiv x\partial_x$ stands for any variable x and the RG functions (the β and γ functions) are given by well-known definitions and in our case, using the relations (18) for the renormalization constants, they acquire the following form:

$$\gamma_\nu \equiv \tilde{\mathcal{D}}_\mu \ln Z_\nu, \tag{28}$$

$$\beta_g \equiv \tilde{\mathcal{D}}_\mu g = g(-2\varepsilon - \eta + 3\gamma_\nu), \tag{29}$$

$$\beta_u \equiv \tilde{\mathcal{D}}_\mu u = u(-\eta + \gamma_\nu). \tag{30}$$

The renormalization constant Z_ν is determined by the requirement that response function $G \equiv \langle \theta\theta' \rangle$ must be UV finite when it is written in renormalized variables. In our case it means that it has no singularities in the limit $\varepsilon, \eta \to 0$. The response function G is related to the self-energy operator $\Sigma_{\theta'\theta}$, which is expressed via the Feynman graphs, by the Dyson equation. In frequency-momentum representation it has the following form:

$$G(\omega, \mathbf{p}) = \frac{1}{-i\omega + \nu_0 p^2 - \Sigma_{\theta'\theta}(\omega, p)}. \tag{31}$$

Thus, Z_ν is found from the requirement that the UV divergences are canceled in Eq. (31) after substitution $\nu_0 = \nu Z_\nu$. This determines Z_ν up to an UV finite contribution, which is fixed by the choice of the renormalization scheme. In the MS scheme all the renormalization constants have the form: 1 + *poles in* ε, η *and their linear combinations*. In contrast to rapid-change model, where only one-loop diagram exists (it is related to the fact that all higher-order loop diagrams contain at least one closed loop which is built on by only retarded propagators, thus are automatically equal to zero), in the case with finite correlations in time of the velocity field, higher-order corrections are nonzero. In two-loop approximation the self-energy operator $\Sigma_{\theta'\theta}$ is defined by diagrams which are shown in Fig. 2.

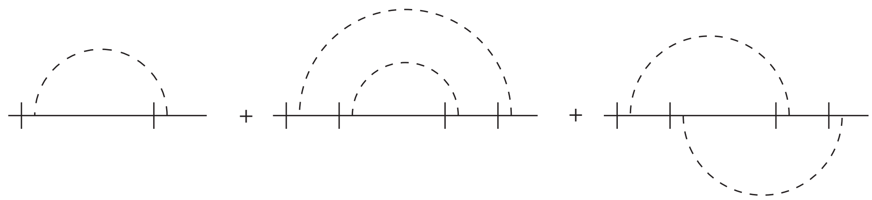

Fig. 2. The one- and two-loops contributions to the self-energy operator $\Sigma_{\theta'\theta}$

As was already mentioned, in our calculations we can put $\eta = 0$. This possibility essentially simplifies the evaluations of all quantities [22–24]. Then the divergent parts of diagrams in Fig. 2 have the following analytical form:

$$A = -\frac{S_d}{(2\pi)^d}\frac{g\nu p^2}{4u(1+u)}\frac{d-1}{d}(\mu L)^{2\varepsilon}\frac{1}{\varepsilon}, \tag{32}$$

$$B_1 = \frac{S_d^2}{(2\pi)^{2d}}\frac{g^2\nu p^2(d-1)^2}{16u^2(1+u)^3d^2}\frac{(\mu L)^{4\varepsilon}}{\varepsilon}\left[\frac{1}{2\varepsilon} + \frac{{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right)}{(d+2)(1+u)^2}\right], \tag{33}$$

$$B_2 = \frac{S_d^2}{(2\pi)^{2d}}\frac{g^2\nu p^2(d-1)}{16u^2(1+u)^3d^2}\frac{(\mu L)^{4\varepsilon}}{\varepsilon}\left[\frac{{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right)}{(d+2)(1+u)} - \frac{(d-2)\pi\rho^2}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1+\frac{d}{2};\frac{1}{(1+u)^2}\right)\right], \tag{34}$$

where A corresponds to one-loop contribution (the first diagram in Fig. 2), B_1 is related to the second diagram in Fig. 2, and B_2 is the result for the third diagram. Here $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the d-dimensional sphere, ${}_2F_1(a,b,c,z) = 1+\frac{a\,b}{c\cdot 1}z+\frac{a(a+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2+\dots$ represents the corresponding hypergeometric function. In further investigations the helical term with ρ^2 in B_2 has to be taken with $d = 3$ but for completeness we remain the d-dependence in this part of B_2 in (34).

In the end, the renormalization constant Z_ν is given as follows:

$$\begin{aligned} Z_\nu = 1 \; - \; & \frac{\bar{g}}{\varepsilon}\frac{d-1}{d}\frac{1}{4u(1+u)} - \frac{\bar{g}^2}{\varepsilon^2}\frac{(d-1)^2}{d^2}\frac{1}{32u^2(1+u)^3} \\ + \; & \frac{\bar{g}^2}{\varepsilon}\frac{(d-1)(d+u)}{d^2(d+2)}\frac{1}{16u^2(1+u)^5}\,{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right) \\ - \; & \rho^2\frac{\bar{g}^2}{\varepsilon}\frac{\pi}{144u^2(1+u)^3}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right), \end{aligned} \tag{35}$$

where in the helical part (the last line) we already substitute $d = 3$, and we have introduced new notation $\bar{g} = gS_d/(2\pi)^d$ for simplicity.

Now using the definition of the anomalous dimension γ_ν in Eq. (28) one comes to the following expression for it:

$$\gamma_\nu = -2(\bar{g}\mathcal{A} + 2\bar{g}^2\mathcal{B}), \tag{36}$$

where

$$\mathcal{A} = -\frac{d-1}{d}\frac{1}{4u(1+u)} \tag{37}$$

is the one-loop contribution to anomalous dimension γ_ν and the two-loop one is

$$\begin{aligned} \mathcal{B} \;\; = \;\; & \frac{(d-1)(d+u)}{16d^2(d+2)u^2(1+u)^5}\,{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right) \\ & -\frac{\pi\rho^2}{144u^2(1+u)^3}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right). \end{aligned} \tag{38}$$

In the following section we shall use these results to determine possible scaling regimes of the model.

3. FIXED POINTS AND SCALING REGIMES

Possible scaling regimes of a renormalized model are directly given by the infrared (IR) stable fixed points of the corresponding system of the RG equations [12, 13]. The fixed point of the RG equations is defined by β functions, namely, by requirement of their vanishing. In our model the coordinates g_*, u_* of the fixed points are found from the system of two equations

$$\beta_g(g_*, u_*) = \beta_u(g_*, u_*) = 0. \tag{39}$$

The beta functions β_g and β_u are defined in Eqs. (29) and (30). To investigate the IR stability of a fixed point it is enough to analyze the eigenvalues of the matrix Ω of first derivatives

$$\Omega_{ij} = \left(\begin{array}{cc} \partial\beta_g/\partial g & \partial\beta_g/\partial u \\ \partial\beta_u/\partial g & \partial\beta_u/\partial u \end{array} \right). \tag{40}$$

The IR asymptotic behavior is governed by the IR stable fixed points, i. e., those for which both eigenvalues are positive.

The possible scaling regimes of the model in one-loop approximation were investigated in [22], and the two-loop approximation without helicity was studied in [24]. Our question is what restrictions on «phase» diagram of scaling regimes are given by helicity (in two-loop approximation).

First of all, we shall study the rapid-change limit: $u \to \infty$. In this regime it is convenient to make transformation to new variables, namely, $w \equiv 1/u$, and $g' \equiv g/u^2$, with the corresponding changes in the β functions

$$\beta_{g'} = g'(\eta - 2\varepsilon + \gamma_\nu), \tag{41}$$
$$\beta_w = w(\eta - \gamma_\nu). \tag{42}$$

In this notation the anomalous dimension γ_ν acquires the following form:

$$\gamma_\nu = -2(\bar{g}'\mathcal{A}' + 2\bar{g}'^2\mathcal{B}'), \tag{43}$$

where again $\bar{g}' = g'S_d/(2\pi)^d$. The one-loop contribution $\mathcal{A}'$ acquires the form

$$\mathcal{A}' = -\frac{d-1}{d}\frac{1}{4(1+w)} \tag{44}$$

and the two-loop correction $\mathcal{B}'$ is

$$\begin{aligned} \mathcal{B}' = & \frac{(d-1)(dw+1)w^2}{16d^2(d+2)(1+w)^5}\,{}_2F_1\left(1,1;2+\frac{d}{2};\frac{w^2}{(1+w)^2}\right) \\ & -\frac{\pi\rho^2 w}{144(1+w)^3}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{w^2}{(1+w)^2}\right). \end{aligned} \tag{45}$$

It is evident that in the rapid-change limit $w \to 0$ ($u \to \infty$) the two-loop contribution $\mathcal{B}'$ is equal to zero. It is not surprising because in the rapid-change model there are no higher-loop corrections to the self-energy operator [18, 19], thus we arrive to the one-loop result of Ref. [22] with the anomalous dimension γ_ν of the form

$$\gamma_\nu = \lim_{w\to 0}\frac{(d-1)\bar{g}'}{2d(1+w)} = \frac{(d-1)\bar{g}'}{2d}. \tag{46}$$

In this limit we have two fixed points denoted as FPI and FPII in [22]. The first fixed point is trivial, namely

$$\mathrm{FPI}: \quad \mathrm{w}_* = \mathrm{g}'_* = 0, \tag{47}$$

with $\gamma_\nu^* = 0$, and diagonal matrix Ω with eigenvalues (diagonal elements)

$$\lambda_1 = \eta, \quad \lambda_2 = \eta - 2\varepsilon. \tag{48}$$

The region of stability is shown in Fig. 3. The second point is defined as

$$\mathrm{FPII}: \quad \mathrm{w}_* = 0, \ \bar{\mathrm{g}}'_* = \frac{2\mathrm{d}}{\mathrm{d}-1}(2\varepsilon - \eta), \tag{49}$$

with $\gamma_\nu^* = 2\varepsilon - \eta$. These are exact one-loop expressions as a result of non-existence of the higher-loop corrections. That means that they have no corrections of order $O(\varepsilon^2)$ and higher (we work with assumption that $\varepsilon \simeq \eta$, therefore it also includes corrections of the types $O(\eta^2)$ and $O(\eta\varepsilon)$). The corresponding «stability matrix» is triangular with diagonal elements (eigenvalues):

$$\lambda_1 = 2(\eta - \varepsilon), \quad \lambda_2 = 2\varepsilon - \eta. \tag{50}$$

The region of stability of this fixed point is shown in Fig. 3.

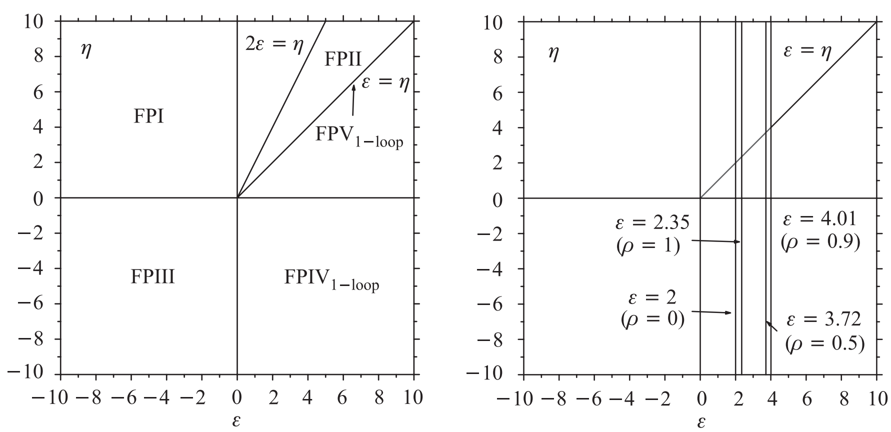

Fig. 3. Left: The «phase» diagram of scaling regimes in one-loop approximation. The fixed points FPI-FPIII are not changed in the two-loop approximation (there are no higher-loops corrections as well). Right: The influence of the helicity on the stability of the fixed point FPIV in two-loop approximation (details see in text)

Now let us analyze the «frozen regime» with frozen velocity field, which is mathematically obtained from the model under consideration in the limit $u \to 0$. To study this transition it is appropriate to change the variable g to the new variable $g'' \equiv g/u$ [22]. Then the β functions are transformed to the following ones:

$$\beta_{g''} = g''(-2\varepsilon + 2\gamma_\nu), \tag{51}$$

$$\beta_u = u(-\eta + \gamma_\nu), \tag{52}$$

where β_u function is not changed, i. e., it is the same as the initial one (30). In this notation the anomalous dimension γ_ν has the form

$$\gamma_\nu = -2(\bar{g}''\mathcal{A}'' + 2\bar{g}''^2\mathcal{B}''), \tag{53}$$

where, as always, $\bar{g}'' = g''S_d/(2\pi)^d$. The one-loop part $\mathcal{A}''$ is now defined as

$$\mathcal{A}'' = -\frac{d-1}{d}\frac{1}{4(1+u)} \tag{54}$$

and the two-loop one, $\mathcal{B}''$, is given as

$$\begin{aligned} \mathcal{B}'' = & \frac{(d-1)(d+u)}{16d^2(d+2)(1+u)^5}\,{}_2F_1\left(1,1;2+\frac{d}{2};\frac{1}{(1+u)^2}\right) \\ & -\frac{\pi\rho^2}{144(1+u)^3}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u)^2}\right). \end{aligned} \tag{55}$$

In the limit $u \to 0$ the functions $\mathcal{A}''$ and $\mathcal{B}''$ aquire the following form:

$$\mathcal{A}''_0 = -\frac{d-1}{4d}, \tag{56}$$

and

$$\mathcal{B}''_0 = \frac{(d-1){}_2F_1\left(1,1;2+\frac{d}{2};1\right)}{16d(d+2)} - \frac{\pi\rho^2{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};1\right)}{144}. \tag{57}$$

The system of β functions (51) and (52) exhibits two fixed points, denoted as FPIII and FPIV in [22], related to the corresponding two scaling regimes. One of them is again trivial, namely,

$$\mathrm{FPIII:} \quad \mathrm{u}_* = \mathrm{g}''_* = 0, \tag{58}$$

with $\gamma_\nu^* = 0$. The eigenvalues of the corresponding matrix Ω, which is diagonal in this case, are

$$\lambda_1 = -2\varepsilon, \quad \lambda_2 = -\eta. \tag{59}$$

Thus this regime is IR stable only if both parameters ε and η are negative simultaneously as can be seen in Fig. 3. The second, non-trivial, point is

$$\text{FPIV}: \quad \mathrm{u}_* = 0, \quad \bar{\mathrm{g}}''_* = -\frac{\varepsilon}{2\mathcal{A}''_0} - \frac{\mathcal{B}''_0}{2\mathcal{A}''^2_0}\varepsilon^2, \tag{60}$$

where $\mathcal{A}''_0$ and $\mathcal{B}''_0$ are defined in Eqs. (56) and (57), respectively.

When we study system with the helicity then the dimension of the space is fixed for $d = 3$. The fixed point FPIV is given as

$$u_* = 0 \quad \bar{g}''_* = 3\varepsilon + \frac{3}{2}\left(1 - \frac{3\pi^2\rho^2}{16}\right)\varepsilon^2. \tag{61}$$

Therefore, in the helical case, the situation is a little bit more complicated as a result of a competition between the non-helical and helical terms within the two-loop corrections. The matrix Ω is triangular with the diagonal elements (they are taken already at the fixed point)

$$\lambda_1 = 2\varepsilon + \left(-1 + \frac{3\pi^2\rho^2}{16}\right)\varepsilon^2, \tag{62}$$

$$\lambda_2 = \varepsilon - \eta, \tag{63}$$

where the explicit dependence of the eigenvalue λ_1 on the parameter ρ takes place. The requirement to have positive values for the parameter $\bar{g}''_*$, and at the same time for the eigenvalues λ_1, λ_2 leads to the region of stable fixed point. The results are shown in Fig. 3. The picture is rather complicated due to the very existence of the «critical» absolute value of ρ

$$\rho_c = \frac{4}{\sqrt{3}\pi}, \tag{64}$$

which is defined from the condition of vanishing of the two-loop corrections in Eqs. (61) and (62):

$$\left(-1 + \frac{3\pi^2\rho^2}{16}\right) = 0. \tag{65}$$

When the helicity is not present, the system exhibits this type of fixed point (and, of course, the corresponding scaling behavior) in the region restricted by inequalities: $\varepsilon > 0$, $\varepsilon = \eta$, and $\varepsilon < 2$ ($d = 3$). The last condition is changing when helicity is switched on. The important feature here is that the two-loop contributions to $\bar{g}''_*$ and λ_1 have the same structure but opposite sign. This leads to the different sources of conditions in the case when $|\rho| < \rho_c$ and $|\rho| > \rho_c$, respectively. In the situation with $|\rho| < \rho_c$ the positiveness of λ_1 plays crucial role and one has the following region of stability of the IR fixed point FPIV:

$$\varepsilon > 0, \quad \varepsilon > \eta, \quad \varepsilon < \frac{32}{16 - 3\pi^2\rho^2}. \tag{66}$$

On the other hand, in the case with $|\rho| > \rho_c$, the principal restriction on the IR stable regime is yielded by the condition $\bar{g}''_* > 0$ with the final IR stable region defined as

$$\varepsilon > 0, \ \ \varepsilon > \eta, \ \ \varepsilon < \frac{32}{-16 + 3\pi^2\rho^2}. \tag{67}$$

Therefore, if absolute value of the helicity parameter ρ continuously increases, the region of stability of the fixed point defined by the last inequality in Eq. (66) is increasing too. This restriction vanishes completely when $|\rho|$ reaches the «critical» value ρ_c, and the picture becomes the same as in the one-loop approximation [22]. In this rather specific situation the two-loop influence on the region of stability of the fixed point is exactly zero: the helical and non-helical two-loop contributions are canceled by each other. Then if the absolute value of parameter ρ increases further, the last condition appears again, namely the third condition in Eq. (67), and restriction becomes stronger when $|\rho|$ tends to it maximal value, $|\rho| = 1$. In the case of the maximal breaking of mirror symmetry (maximal helicity), $|\rho| = 1$, the region of the IR stability of the fixed point is defined by inequalities: $\varepsilon > 0, \varepsilon = \eta$, and $\varepsilon < 2.351$ (see Fig. 3). It is interesting enough that the presence of helicity in the system leads to the enlargement of the stability region.

Now let us turn to the most interesting scaling regime with finite value of the fixed point for the variable u. By short analysis one immediately concludes that the system of equations

$$\beta_g = g(-2\varepsilon - \eta + 3\gamma_\nu) = 0, \tag{68}$$

$$\beta_u = u(-\eta + \gamma_\nu) = 0 \tag{69}$$

can be fulfilled simultaneously for finite values of g, u only in the case when the parameter ε is equal to η: $\varepsilon = \eta$. In this case the function β_g is proportional to the function β_u. As a result we have not one fixed point but a line of fixed points in the $g - u$ plane. The value of the fixed point for the variable g in two-loop approximation is given as follows (we denote it as FPV):

$$\text{FPV}: \ \ \bar{\text{g}}_* = -\frac{1}{2\mathcal{A}_*}\,\varepsilon - \frac{1}{2}\frac{\mathcal{B}_*}{\mathcal{A}_*^3}\,\varepsilon^2, \tag{70}$$

with exact one-loop result for $\gamma_\nu^* = \varepsilon = \eta$ (this is, of course, already directly given by Eq. (69)). Here $\mathcal{A}_*$ and $\mathcal{B}_*$ are expressions $\mathcal{A}$ and $\mathcal{B}$ from Eqs. (37), (38) which are taken in the fixed point value u_* of the variable u. The possible values of the fixed point for variable u_* can be restricted (and will be restricted) as we shall discussed below. The «stability matrix» Ω has the following eigenvalues:

$$\lambda_1 = 0, \ \ \lambda_2 = 3\bar{g}^* \left(\frac{\partial\gamma_\nu}{\partial g}\right)_* + u^* \left(\frac{\partial\gamma_\nu}{\partial u}\right)_*. \tag{71}$$

The vanishing of the eigenvalue λ_1 is an exact result which is related to the degeneracy of the system of Eqs. (68) and (69) when nonzero solutions in respect to g and u are assumed. Equivalently, it reflects the existence of a borderline direction in the $g-u$ plane along the line of the fixed points.

In the helical case the coordinates of the fixed point are defined by the following equation:

$$\begin{aligned}\bar{g}_* &= 3u_*(1+u_*)\varepsilon + \frac{3u_*\varepsilon^2}{20(1+u_*)^2}\Bigg(2(3+u_*)_2F_1\left(1,1;\frac{7}{2};\frac{1}{(1+u_*)^2}\right) \\ &\quad -5\pi(1+u_*)^2\rho^2{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u_*)^2}\right)\Bigg). \end{aligned} \tag{72}$$

The competition between the helical and non-helical terms appears again which yields a nontrivial restriction for the fixed point values of the variable u to have positive fixed values for the variable g. The eigenvalue λ_2 of the matrix Ω is now

$$\begin{aligned}\lambda_2 &= \frac{2+u_*}{1+u_*}\varepsilon + \frac{\varepsilon^2}{140(1+u_*)^6}\Bigg[8u(3+u)_2F_1\left(2,2;\frac{9}{2};\frac{1}{(1+u_*)^2}\right) \\ &+ 14(1+u_*)^2(u_*(3+u_*)-6)\,{}_2F_1\left(1,1;\frac{7}{2};\frac{1}{(1+u_*)^2}\right) \\ &+ 7\pi\rho^2(1+u_*)^2 \\ &\times \left(10(1+u_*)^2\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u_*)^2}\right) - u_*\left(\frac{3}{2},\frac{3}{2};\frac{7}{2};\frac{1}{(1+u_*)^2}\right)\right)\Bigg]\end{aligned} \tag{73}$$

with nontrivial helical part which plays an important role in determination of the region of the IR stability of the fixed point.

It cannot be seen immediately from Eqs. (72) and (73) but numerical analysis shows that again important role is played by $\rho_c = 4/(\sqrt{3}\pi)$. First, let us study the case when $|\rho| < \rho_c = 4/(\sqrt{3}\pi)$. The corresponding regions of stable IR fixed points with $g_* > 0$ are shown in Fig. 4. In the case when helicity is not present ($\rho = 0$, see the corresponding curve in Fig. 4), the only restriction is given by condition that $\lambda_2 > 0$, on the other hand, the condition $g_* > 0$ is fulfilled without any restriction on the parameter space. When arbitrary small helicity is present, i. e., $\rho > 0$, the restriction related to the positiveness of g_* arises and is stronger when $|\rho|$ is increasing (the right curve for the concrete value of ρ in Fig. 4) and starts to play the dominant role. At the same time, with increasing of $|\rho|$ the importance of the positiveness of the eigenvalue λ_2 decreases (the left curve for the concrete value of ρ in Fig. 4). For a given $|\rho| < \rho_c$ there exists an interval of values of the variable u_* for which there is no restriction on the value of the parameter ε. For example, for $|\rho| = 0.1$ it is $1.128 < u_* < 13.502$, for $|\rho| = 0.5$,

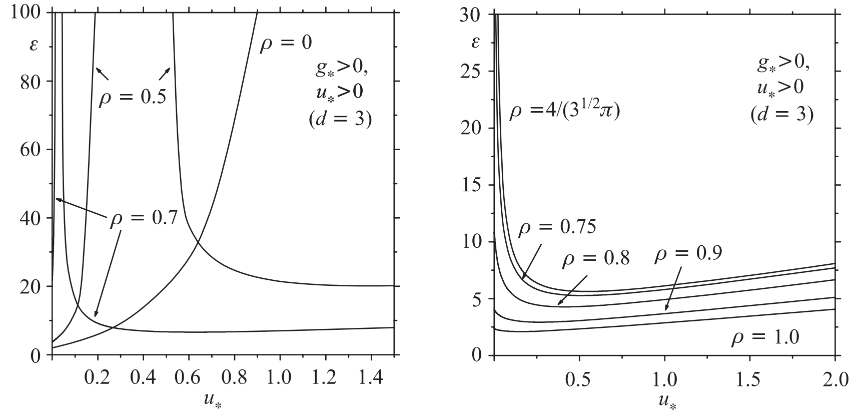

Fig. 4. The region of stability of the fixed point FPV as a function of the helicity parameter ρ for $|\rho| < \rho_c$ (left) and for $|\rho| > \rho_c$ (right). Details see in text

$0.217 < u_* < 0.394$, and for $|\rho| = 0.7$, $0.019 < u_* < 0.029$. Now let us turn to the case $|\rho| \geq \rho_c$. When $|\rho|$ obtains its «critical» value ρ_c, the IR fixed point is stable for all values of $u_* > 0$ and $\varepsilon > 0$, i. e., the condition $\lambda_2 > 0$ becomes fulfilled without any restrictions on the parameter space. On the other hand, the condition $g_* > 0$ yields strong enough restriction and it becomes stronger when $|\rho|$ tends to its maximal value $|\rho| = 1$ as can be seen in Fig. 4.

The most important conclusion of our two-loop investigation of the scaling regimes is the fact that the possible restrictions on the regions of stability of the IR fixed points are «pressed» to the region with rather large values of ϵ, namely, $\varepsilon \geq 2$, and do not disturb the regions with relatively small ε. For example, the Kolmogorov point ($\varepsilon = \eta = 4/3$) is not influenced.

4. EFFECTIVE DIFFUSIVITY

One of the interesting objects from the theoretical as well as experimental point of view is so-called effective diffusivity $\bar{\nu}$. In this section let us briefly investigate the effective diffusivity $\bar{\nu}$, which replaces initial molecular diffusivity ν_0 in equation (1) due to the interaction of a scalar field θ with random velocity field $\mathbf{v}$. Molecular diffusivity ν_0 governs exponential dumping in time all fluctuations in the system in the lowest approximation, which is given by the propagator (response function) (11) $G(t-t', \mathbf{k}) = \langle \theta(t, \mathbf{k}) \theta'(t', \mathbf{k}) \rangle_0 = \theta(t-t') \exp(-\nu_0 k^2 (t-t'))$. Analogously, the effective diffusivity $\bar{\nu}$ governs exponential dumping of all fluctuations described by full response function, which is defined by Dyson equation (31). Its explicit expression can be obtained by the

RG approach. In accordance with general rules of the RG (see, e. g., [13]) all principal parameters of the model g_0, u_0 and ν_0 are replaced by their effective (running) counterparts, which satisfy the Gell-Mann–Low RG equations

$$s\frac{d\bar{g}}{ds} = \beta_g(\bar{g},\bar{u}),\; s\frac{d\bar{u}}{ds} = \beta_u(\bar{g},\bar{u}), \tag{74}$$

$$s\frac{d\bar{\nu}}{ds} = -\bar{\nu}\gamma_\nu(\bar{g},\bar{u}), \tag{75}$$

with initial conditions $\bar{g}|_{s=1} = g, \bar{u}|_{s=1} = u, \bar{\nu}|_{s=1} = \nu$. Here $s = k/\mu$, β and γ functions are defined in (28)–(30) and all running parameters clearly depend on variable s. Straightforward integration (at least numerical) of equations (74) gives way how to find their fixed points. Instead one very often solves the set of equations $\beta_g(g_*, u_*) = \beta_u(g_*, u_*) = 0$ which defines all fixed points g_*, u_*. Just last approach we used above when we classified all fixed points. Due to special form of β functions (29), (30) we are able to solve equation (75) analytically. Using (74) and (29) one immediately rewrites (75) in the form

$$\frac{d\bar{\nu}}{\bar{\nu}} = \frac{\gamma_\nu}{2\varepsilon + \eta - 3\gamma_\nu}\frac{d\bar{g}}{\bar{g}} \tag{76}$$

which can be easily integrated. Using initial conditions the solution acquires the form

$$\bar{\nu} = (\frac{g\nu^3}{\bar{g}s^{2\epsilon+\eta}})^{1/3} = (\frac{D_0}{\bar{g}k^{2\epsilon+\eta}})^{1/3}, \tag{77}$$

where to obtain the last expression we used the equations $g\mu^{2\epsilon+\eta}\nu^3 = g_0\nu_0^3 = D_0$ (see (19)). We emphasize that above solution is exact, i. e., the exponent $2\epsilon + \eta$ is exact too. However, in infrared region $k \ll \mu \sim l^{-1}$, $\bar{g} \to g_*$, which can be calculated only pertubatively. In the two-loop approximation $g_* = g_*^{(1)}\varepsilon + g_*^{(2)}\varepsilon^2$ and after the Taylor expansion of $g_*^{1/3}$ in (77) we obtain

$$\bar{\nu} \approx \nu_* \left(\frac{D_0}{g_*^{(1)}\varepsilon}\right)^{1/3} k^{-\frac{2\epsilon+\eta}{3}}, \qquad \nu_* \equiv 1 - \frac{g_*^{(2)}\varepsilon}{3g_*^{(1)}}. \tag{78}$$

Remind that for the Kolmogorov values $\varepsilon = \eta = 4/3$ the exponent in (78) becomes equal to $-4/3$. Let us estimate the contribution of helicity to the effective diffusivity in nontrivial point above denoted as FPV (72). In this point $\varepsilon = \eta$ $((2\varepsilon + \eta)/3 = 2\varepsilon)$ and

$$\begin{aligned}\nu_* &= 1 - \frac{\varepsilon}{12(1+u_*)} \\ &\times\left(\frac{2(3+u_*)}{5(1+u_*)^2}{}_2F_1\left(1,1;\frac{7}{2};\frac{1}{(1+u_*)^2}\right) - \pi\rho^2{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u_*)^2}\right)\right).\end{aligned} \tag{79}$$

In Fig. 5 is shown dependence of the ν_* on the helicity parameter ρ and the IR fixed point u_* of the parameter u. As one can see from these figures when $u_* \to \infty$ (the rapid change model limit) the two-loop corrections to $\nu_* = 1$ are vanishing. Such a behavior is related to the fact, which was already stressed in the text when the IR fixed points were analyzed, that within the rapid change model there are no two- and higher-loop corrections at all. On the other hand, the largest two-loop corrections to the ν_* are given in the frozen velocity field limit ($\nu_* \to 0$) (especially for the non-helical case, see Fig. 5). It is interesting

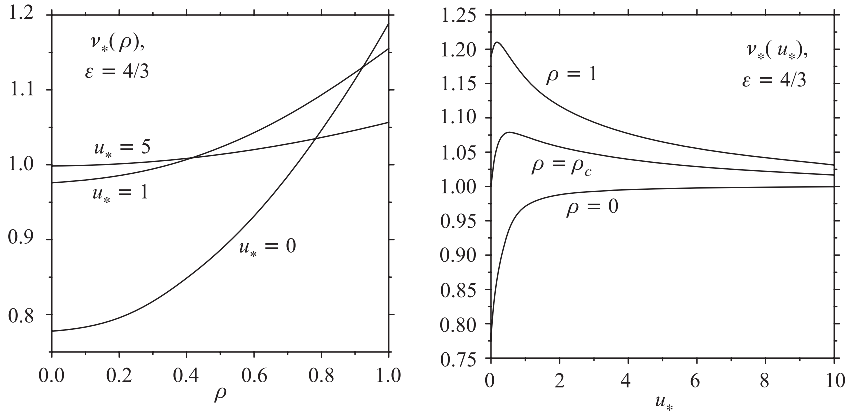

Fig. 5. Left: The dependence of ν_* on the helicity parameter ρ for definite IR fixed point values u_* of the parameter u. Right: The dependence of ν_* on the IR fixed point u_* for the concrete values of the helicity parameter ρ

that for all finite values of the parameter u_* there exists a value of the helicity parameter ρ for which the two-loop contribution to ν_* are canceled. For example, for the frozen velocity field limit ($u_* = 0$) such a situation arises when the helicity parameter ρ is equal to its «critical» value $\rho_c = 4/(\sqrt{3}\pi)$ (this situation can be seen in Fig. 5 (right)). It is again the result of the competition between the non-helical and helical parts of the the two-loop corrections as is shown in Eq. (79). Further important feature of expression (79) is that it is linear in the parameter ε. Thus, when one varies the value of ε the picture is the same as in Fig. 5 and only the scale of corrections is changed. In Fig. 5 we have shown the situation for the most interesting case when ε is equal to its «Kolmogorov» value, namely, $\varepsilon = 4/3$.

CONCLUSION

We have studied the advection of a scalar field by turbulent flow in the framework of the extended Kraichnan model and investigated the influence of helicity on anomalous scaling, stability of asymptotic regimes and effective diffusivity. Such an investigation is useful for understanding of efficiency of toy models (like Kraichnan model) to study the real turbulent motions by means of modern theoretical methods including the renormalization group approach. Actually, we performed two-loop calculations of the divergent parts of the Feynman graphs, which are necessary to achieve multiplicative renormalization of equivalent field theoretic model. In this way we have shown that anomalous scaling, which is typical for the Kraichnan model and its numerous extensions [10, 24], is not violated by inclusion of helicity to the incompressible fluid. On the other hand, stability of the asymptotic regimes, values of the fixed RG points, and the turbulent diffusivity strongly depend on amount of helicity. It is shown that the presence of helicity in the system leads to the restrictions of the possible values of the parameters of the model. The most interesting fact is existence of a «critical» value ρ_c of the helicity parameter ρ which divides the interval of possible absolute values of ρ into two parts with completely different behavior. Such a situation is given by the fact that in two-loop approximation there is a competition between the non-helical and helical contributions. Within of the so-called frozen limit the presence of helicity enlarges the region of parameter space with stable scaling regime, and, as a result of above-mentioned competition, if $|\rho| = \rho_c$ the corresponding two-loop restriction is vanished completely and one is coming to the one-loop results [22]. Similar splitting, although more complicated, into two nontrivial behavior of the fixed point was also obtained in the general case with finite correlations in time of the velocity field. The influence of helicity on the effective diffusivity was also discussed. The main role in its determination is played by the fixed point value ν_*. It was shown that ν_* strongly depends on the helicity parameter ρ mainly in the so-called frozen limit case when $u_* \to 0$.

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REFERENCES

1. *Monin A. S., Yaglom A. M.* Statistical Fluid Mechanics. V. 2. Cambridge, MA: MIT Press, 1975.

2. *Frisch U.* Turbulence: The Legacy of A. N. Kolmogorov. Cambridge: Cambridge University Press, 1995.

3. *Sreenivasan K. R., Antonia R. A.* // Ann. Rev. Fluid Mech. 1997. V. 29. P. 435.

4. *Antonia R. A., Hopfinger E., Gagne Y., Anselmet F.* // Phys. Rev. A. 1984. V. 30. P. 2704.

5. *Warhaft Z.* // Ann. Rev. Fluid Mech. 2000. V. 32. P. 203.

6. *Moisy F., Willaime H., Anderson J. S., Tabeling P.* // Phys. Rev. Lett. 2001. V. 86. P. 4827.

7. *Shraiman B. I., Siggia E.* // Nature. 2000. V. 405. P. 639.

8. *Kraichnan R. H.* // Phys. Fluids. 1968. V. 11. P. 945.

9. *Kraichnan R. H.* // Phys. Rev. Lett. 1994. V. 72. P. 1016.

10. *Gawedzki K., Kupiainen A.* // Phys. Rev. Lett. 1995. V. 75. P. 3834;
Bernard D., Gawedzki K., Kupiainen A. // Phys. Rev. E. 1996. V. 54. P. 2564;
Chertkov M., Falkovich G., Kolokov I., Lebedev V. // Phys. Rev. E. 1995. V. 52. P. 4924;
Chertkov M., Falkovich G. // Phys. Rev. Lett. 1996. V. 76. P. 2706;
Pumir A. // Europhys. Lett. 1996. V. 34. P. 25; 1997. V. 37. P. 529; Phys. Rev. E. 1998. V. 57. P. 2914;
Shraiman B. I., Siggia E. D. // Phys. Rev. Lett. 1996. V. 77. P. 2463;
Pumir A., Shraiman B. I., Siggia E. D. // Phys. Rev. E. 1997. V. 55. P. R1263.

11. *Falkovich G., Gawedzki K., Vergassola M.* // Rev. Mod. Phys. 2001. V. 73. P. 913.

12. *Zinn-Justin J.* Quantum Field Theory and Critical Phenomena. Oxford: Clarendon, 1989.

13. *Vasil'ev A. N.* The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics. N. Y.: Chapman & Hall/ CRS Press Company, 2004.

14. *Collins J.* Renormalization. Cambridge: Cambridge University Press, 1984.

15. *de Dominicis C., Martin P. C.* // Phys. Rev. A. 1979. V. 19. P. 419.

16. *Adzhemyan L. Ts., Antonov N. V., Vasil'ev A. N.* // Usp. Fiz. Nauk. 1996. V. 166. P. 1257 (in Russian); [Phys. Usp. 1996. V. 39. P. 1193].

17. *Adzhemyan L. Ts., Antonov N. V., Vasil'ev A. N.* The Field Theoretic Renormalization Group in Fully Developed Turbulence. London: Gordon & Breach, 1999.

18. *Adzhemyan L. Ts., Antonov N. V., Vasil'ev A. N.* // Phys. Rev. E. 1998. V. 58. P. 1823.

19. *Adzhemyan L. Ts., Antonov N. V., Barinov V. A., Kabrits Yu. S., Vasil'ev A. N.* // Phys. Rev. E. 2001. V. 63. P. 025303(R); 2001. V. 64. P. 056306.

20. *Adzhemyan L. Ts., Antonov N. V., Hnatich M., Novikov S. V.* // Phys. Rev. E. 2000. V. 6. P. 016309.

21. *Adzhemyan L. Ts., Antonov N. V.* // Phys. Rev. E. 1998. V. 58. P. 7381;
Antonov N. V., Honkonen J. Ibid. 2001. V. 63. P. 036302.

22. *Antonov N. V.* // Phys. Rev. E. 1999. V. 60. P. 6691.

23. *Antonov N. V.* // Physica. D. 2000. V. 144. P. 370.

24. *Adzhemyan L. Ts., Antonov N. V., Honkonen J.* // Phys. Rev. E. 2002. V. 66. P. 036313.

25. *Antonov N. V., Lanotte A., Mazzino A.* // Phys. Rev. E. 2000. V. 61. P. 6586;
Antonov N. V., Honkonen J., Mazzino A., Muratore-Ginanneschi P. Ibid. 2000. V. 62. P. R5891;
Adzhemyan L. Ts., Antonov N. V., Runov A. V. Ibid. 2001. V. 64. P. 046310;
Antonov N. V., Hnatic M., Honkonen J., Jurcisin M. Ibid. 2004. V. 68. P. 046306;
Hnatic M., Jurcisin M., Mazzino A., Sprinc S. // Acta Phys. Slov. 2002. V. 52. P. 559.

26. *Shraiman B. I., Siggia E. D.* // Phys. Rev. E. 1994. V. 49. P. 2912; C. R. Acad. Sci., Ser. IIb: Mec., Phys., Chim., Astron. 1995. V. 321. P. 279; Phys. Rev. Lett. 1996. V. 77. P. 2463.

27. *Vainstein S. I., Zel'dovich Ya. B., Ruzmaykin A. A.* Turbulent Dynamo in Astrophysics. M.: Nauka, 1980 (in Russian).

28. *Moffatt H. K.* Magnetic Field Generation in Electrically Conducting Fluids. Cambridge: Cambridge Univ. Press, 1978.

29. *Adzhemyan L. Ts., Vasil'ev A. N., Hnatich M.* // Teor. Mat. Fiz. 1987. V. 72. P. 369 (in Russian).

30. *Hnatich M., Jurcisin M., Stehlik M.* // Magnit. Gidrodin. 2001. V. 37. P. 80.

31. *Fedorov Yu., Katz M., Kichatinov L. L., Stehlik M.* // Astron. Astroph. 1992. V. 260. P. 499.

32. *Katz M., Fedorov Yu., Stehlik M.* // Astroph. Space Sci. 1990. V. 166. P. 49.

33. *Dorman L. I., Katz M., Stehlik M.* // BAC. 1990. V. 41. P. 312.

34. *Etling D.* // Beitr. Phys. Atmosph. 1985. V. 58. P. 88.

35. *Moffat H. K., Tsinober A.* // Ann. Rev. Fluid Mech. 1992. V. 24. P. 281.

36. *Kurgansky S.* Adiabatic Invariants in Large-Scale Atmospheric Dynamics. London/Philadelphia: Taylor & Francis, 2002.

37. *Ponomarev V. M., Khapaev A. A., Chkhetiani O. G.* // Izvestya: Atmospheric and Oceanic Physics. 2003. V. 39, No. 4. P. 391.

38. *Kholmyansky M., Shapiro-Orot M., Tsinober A.* // Proc. R. Soc. Lond. A. 2001. V. 457. P. 2699.

39. *Koprov B. M., Koprov V. M., Ponomarev V. M., Chkhetiani O. G.* // Doklady RAN. 2005. V. 50, No. 11 (in Russian).

40. *Brissaud A., Frisch U., Leorat J., Lesieur M., Mazure A.* // Phys. Fluids. 1973. V. 16. P. 1363.

41. *Moiseev S. S., Chkhetiani O. G.* // JETP. 1996. V. 83. P. 192.

42. *Chkhetiani O. G.* // JETP Lett. 1996. V. 63. P. 808.

43. *Kraichnan R. H.* // J. Fluid Mech. 1973. V. 59. P. 745.

44. *Andre J. C., Lesieur M.* // J. Fluid Mech. 1977. V. 81. P. 187.

45. *Borue V., Orszag S. A.* // Phys. Rev. E. 1997. V. 55. P. 7005.

46. *Chen Q., Chen S., Eyink G. L.* // Phys. Fluids. 2003. V. 15. P. 361.

47. *Pouquet A., Fournier J. D., Sulem P. L.* // J. De Physique Lett. 1978. V. 39, No. 13. P. 199.

48. *Ye Zhou* // Phys. Rev. A. 1991. V. 41. P. 5683.

49. *Kraichnan R. H.* // J. Fluid Mech. 1976. V. 77. P. 753.

50. *Drummond S. T., Duane S., Horgan R. R.* // J. Fluid Mech. 1984. V. 138. P. 75.

51. *Knobloch E.* // J. Fluid Mech. 1977. V. 83. P. 129.

52. *Lipscombe T. C., Frencel A. L., ter Haar D.* // J. Stat. Phys. 1991. V. 63. P. 305.

53. *Belyan A. V., Moiseev S. S., Chkhetiani O. G.* // Doklady RAN. 1994. V. 39, No. 1. P. 13 (in Russian).

54. *Belyan A. V., Moiseev S. S., Golbraih E. I., Chkhetiani O. G.* // Physica. A. 1998. V. 258. P. 55.

55. *Dolginov A. Z., Silantiev N. A.* // JETP. 1987. V. 93. P. 159.

56. *Dean D. S., Drummond I. T., Horgan R. P.* // Phys. Rev. E. 2002. V. 63. P. 61205.

57. *Chketiani O. G.* // JETP. 2004. V. 99, No. 2. P. 328.

58. *Avellaneda M., Majda A.* // Commun. Math. Phys. 1990. V. 131. P. 381; 1992. V. 146. P. 139;
Majda A. // J. Stat. Phys. 1993. V. 73. P. 515;
Hontrop D., Majda A. // J. Math. Sci. Univ. Tokyo. 1994. V. 1. P. 23.

59. *Zhang Q., Glimm J.* // Commun. Math. Phys. 1992. V. 146. P. 217.

60. *Chertkov M., Falkovich G., Lebedev V.* // Phys. Rev. E. 1996. V. 76. P. 3707.

61. *Eyink G.* // Phys. Rev. E. 1996. V. 54. P. 1497.

62. *Bouchaud J. P., Comtet A., Georges A., Le Doussal P.* // J. Phys. (Paris) 1987. V. 48. P. 1445; 1988. V. 49. P. 369;
Bouchaud J. P., Georges A. // Phys. Rep. 1990. V. 195. P. 127.

63. *Honkonen J., Karjalainen E.* // J. Phys. A. 1988. V. 21. P. 4217;
Honkonen J., Pis'mak Yu. M., Vasil'ev A. N. Ibid. 1989. V. 21. P. L835;
Honkonen J., Pis'mak Yu. M. Ibid. 1989. V. 22. P. L899.

64. *Martin P. C., Siggia E. D., Rose H. A.* // Phys. Rev. A. 1973. V. 8. P. 423.

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