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FINITE-SIZE SCALING IN SYSTEMS WITH STRONG ANISOTROPY: AN ANALYTICAL EXAMPLE

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Конечно-размерный скейлинг для сильно анизотропных систем: аналитическое исследование

Предложен метод исследования систем с сильной анизотропией. Метод основан на использовании аналитических свойств обобщенных функций Миттага–Леффлера, что позволяет преодолеть ряд вычислительных проблем и провести исследование в аналитическом виде. Эффективность метода показана на примере d-мерной $O(\infty)$ спиновой модели с взаимодействием вида $1/r^{(d+\sigma)}$ и геометрией $L^m \times \infty^n, m + n = d$, и где σ зависит от направления.

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Finite-Size Scaling in Systems with Strong Anisotropy: An Analytical Example

The difficulties arising in the investigation of finite-size scaling in d-dimensional $O(N)$ systems with strong anisotropy and/or long-range interactions, decaying with the interparticle distance r as $r^{-d-\sigma}$, are avoided using a technics of calculations based on the analytical properties of the generalized Mittag–Leffler functions. In the case under consideration strong anisotropy means a system with geometry $L^m \times \infty^n; m + n = d$ and the value of the exponent σ depends on the direction.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

In contrast to the theory of finite-size scaling in isotropic systems (see e. g. [1–3]), the theory of finite-size scaling in anisotropic systems is still a field under discussions (see e. g. [4, 5] and Refs. therein). The problems that arise are, not at least, related to the increasing mathematical difficulties due to the lattice direction dependence of the interactions. Notice that, due to the choice of parametrization that has to reduce the many-dimensional problem to an effective one-dimensional one, the pertinent mathematically ensuing integrals can be evaluated only numerically.

Recently [6], a recipe based on some useful analytical properties of the generalized Mittag–Leffler functions is suggested. It permits one to consider isotropic and some anisotropic systems on an equal footing. The generalized Mittag–Leffler functions are defined by the power series [7] (see also [8, 9])

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0. \tag{1.1}$$

Here

$$(\gamma)_0 = 1,\ (\gamma)_k = \gamma(\gamma+1)(\gamma+2)\dots(\gamma+k-1) = \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)}, k = 1, 2, \dots \tag{1.2}$$

These functions are named after Mittag–Leffler who first introduced the particular case with $\beta = \gamma = 1$. The interest in this type of functions has grown up because of their applications to some finite-size scaling problems (see e. g. [1–3]). The present study is an illustration of the possibilities to handle the final expressions of the scaling equations for anisotropic systems analytically.

2. THE MODEL

In this paper we restrict our attention to the N-vector spin model defined at the sites of the lattice. The Hamiltonian of the model reads

$$H = -N \sum_{x,y} J(x-y) \overrightarrow{\sigma_x} \overrightarrow{\sigma_y}, \tag{2.1}$$

where $\overrightarrow{\sigma_x}$ is a classical N-component unit vector defined at site x of the lattice and the spin–spin coupling decays with different power-laws in different lattice directions. We assume a d-dimensional system with geometry $L^m \times \infty^n$ under periodic boundary conditions in the finite dimensions. The interaction between spins enters the expressions of the theory only through its Fourier transform. We will consider the following anisotropic small $\mathbf{q}$ expansion of the Fourier transform of the spin–spin coupling:

$$J(\mathbf{q}) \simeq J(0) + a_{||}|\mathbf{q}_{||}|^{2\sigma} + a_{\perp}|\mathbf{q}_{\perp}|^{2\rho}, \tag{2.2}$$

where the first n directions (called «parallel» and denoted by the subscript $||$) are extended to infinity and the remaining m directions (called «transverse» and denoted by $\perp$) are kept finite with $m+n = d$, and $a_{\perp}$ and $a_{||}$ are metric factors and $\rho, \sigma > 0$. In finite directions the corresponding summations are over the vector $\mathbf{q}_{\perp} = \{q_{\perp 1}, \ldots, k_{\perp m}\}$ that takes values in Λ^m defined by $q_{\perp\nu} = 2\pi n_{\nu}/(aN_0)$ and $-(N_0 - 1)/2 \leq n_{\nu} \leq (N_0 - 1)/2, \nu = 1, \ldots, m$. In infinite directions the sums are substituted with normalized integrals over the corresponding part of the first Brillouin zone $\left[-\frac{\pi}{a}, \frac{\pi}{a}\right]^n$. For our further purposes let us recall that the continuous limit and finite linear dimension $L = N_0 a$ mean that the lattice spacing $a \to 0$ and simultaneously $N_0 \to \infty$. Such type of systems, with $a_{\perp} = a_{||} = -1/2$, is considered in [5], where $0 < \rho, \sigma < 1$. In the large-N limit, the theory is solved in terms of the gap equation for the parameter λ_V related with the finite-volume correlation length of the system. The bulk system is characterized by a vanishing λ_{∞} so that the critical temperature

$$\beta_c = \frac{1}{(2\pi)^d} \int_{[-\pi/a]^d}^{[\pi/a]^d} \frac{d\mathbf{q}}{|\mathbf{q}_{\perp}|^{2\rho} + |\mathbf{q}_{||}|^{2\sigma}} \tag{2.3}$$

is finite whenever the effective dimensionality $D = m/\rho + n/\sigma$ is greater than 2. For more details see Ref. [5]. For the system with layer geometry $\infty^n \times L^m$ the gap equation has the following form:

$$\beta = \frac{1}{(2\pi)^n} \frac{1}{L^m} \int_{[-\pi/a]^d}^{[\pi/a]^d} \sum_{\mathbf{q}_{\perp} \epsilon \Lambda^m} \frac{d^n \mathbf{q}_{||}}{|\mathbf{q}_{\perp}|^{2\rho} + |\mathbf{q}_{||}|^{2\sigma} + \lambda_V}. \tag{2.4}$$

Our analysis will be limited to a system between the upper critical dimension $D_u = 4$ and lower critical one $D_l = 2$.

3. THE GAP EQUATION FOR THE REFERENCE SYSTEM

Let us introduce the notation

$$\Sigma := \frac{1}{(2\pi)^n} \frac{1}{L^m} \sum_{\mathbf{q}_{\perp} \epsilon \Lambda^m} \int_{[-\pi/a]^n}^{[\pi/a]^n} \frac{d^n \mathbf{q}_{||}}{|\mathbf{q}_{\perp}|^{2\rho} + |\mathbf{q}_{||}|^{2\sigma} + \lambda_V}. \tag{3.1}$$

For $n < 2\sigma$ and $\lambda_V \to 0$, due to the convergence of the corresponding integral in (3.1) over $d^n \mathbf{q}_{||}$, one can extend the integration over all R^n, obtaining

$$\Sigma = \frac{1}{(2\pi)^n} \int_{R^n} d^n \mathbf{q}_{||} \left[\frac{1}{L^m} \sum_{\mathbf{q}_\perp \epsilon \Lambda^m} \frac{1}{|\mathbf{q}_\perp|^{2\rho} + |\mathbf{q}_{||}|^{2\sigma} + \lambda_V} \right]. \tag{3.2}$$

The n-dimensional integral in (3.2) can be presented as

$$\frac{1}{(2\pi)^n} \frac{S_n}{L^m} \int_0^\infty \sum_{\mathbf{q}_\perp \epsilon \Lambda^m} \frac{p^{n-1} dp}{|\mathbf{q}_\perp|^{2\rho} + p^{2\sigma} + \lambda_V}, \tag{3.3}$$

where $S_n = 2(\pi)^{n/2}/\Gamma(n/2)$ is the surface of the n-dimensional unit sphere. With the help of the identity

$$\int_0^\infty p^{\alpha-1} dp \frac{1}{t + p^\eta + |\mathbf{q}_\perp|^\tau} = \frac{\Gamma(1-\frac{\alpha}{\eta})\Gamma(\frac{\alpha}{\eta})}{\eta} \frac{1}{(t + |\mathbf{q}_\perp|^\tau)^{1-\frac{\alpha}{\eta}}}, \qquad \eta > \alpha > 0, \tag{3.4}$$

if we choose $\alpha = n$, $\tau = 2\rho$ and $\eta = 2\sigma$ we end up with the result

$$\Sigma = \frac{1}{(2\pi)^n} \frac{S_n \Gamma(1-\frac{n}{2\sigma})\Gamma(\frac{n}{2\sigma})}{2\sigma} \frac{1}{L^m} \sum_{\mathbf{q}_\perp \epsilon \Lambda^m} \frac{1}{(\lambda_V + |\mathbf{q}_\perp|^{2\rho})^{1-\frac{n}{2\sigma}}}, \qquad 2\sigma > n. \tag{3.5}$$

Now the gap equation (2.4) may be presented in the equivalent form

$$\beta = \frac{S_n}{(2\pi)^n} \frac{\Gamma(1-\frac{n}{2\sigma})\Gamma(\frac{n}{2\sigma})}{2\sigma} \frac{1}{L^m} \sum_{\mathbf{q}_\perp \epsilon \Lambda^m} \frac{1}{(\lambda_V + |\mathbf{q}_\perp|^{2\rho})^{1-\frac{n}{2\sigma}}}, \qquad 2\sigma > n. \tag{3.6}$$

Let us emphasize that one can relate Eq. (3.6) with a fictitious *fully finite isotropic* m-dimensional reference system, in which the memory of the anisotropy of the system is retained only in the parameter

$$\gamma := 1 - \frac{n}{2\sigma}, \qquad 0 < \gamma < 1, \tag{3.7}$$

and in the multiplier

$$A_{n,\sigma} = \frac{S_n}{(2\pi)^n} \frac{\Gamma(1-\frac{n}{2\sigma})\Gamma(\frac{n}{2\sigma})}{2\sigma} \tag{3.8}$$

in front of the sum.

The m-dimensional sum in Eq. (3.6)

$$W_{m,2\rho}^{1-\frac{n}{2\sigma}}(\lambda_V, L) := \frac{1}{L^m} \sum_{\mathbf{q}_\perp \epsilon \Lambda^m} \frac{1}{(\lambda_V + |\mathbf{q}_\perp|^{2\rho})^{1-\frac{n}{2\sigma}}}, \qquad 2\sigma > n \tag{3.9}$$

can be evaluated with the help of the identity [6]

$$\frac{1}{(\lambda_V + y^\alpha)^\gamma} = \int_0^\infty dt e^{-yt} t^{\alpha\gamma-1} E^\gamma_{\alpha,\alpha\gamma}(-\lambda_V t^\alpha), \tag{3.10}$$

in terms of the generalized Mittag–Leffler function $E^\gamma_{\alpha,\gamma\alpha}(z)$. If one chooses $\alpha = \rho$, $\gamma = 1 - \dfrac{n}{2\sigma}$ and $y = |\mathbf{q}_\perp|^2$, the needed result is

$$W^\gamma_{m,2\rho}(\lambda_V, L) = \int_0^\infty dx x^{\gamma\rho-1} E^\gamma_{\rho,\gamma\rho}(-\lambda_V x^\rho) \left[\frac{1}{L}\sum_{q\epsilon\Lambda^1} \exp(-q^2 x)\right]^m, \quad \gamma > 0. \tag{3.11}$$

Now let us define

$$Q_{N_0}(x) := \frac{1}{L}\sum_{q\epsilon\Lambda^1} \exp(-q^2 x) = \frac{1}{aN_0}\sum_{l=-N_0/2}^{N_0/2-1} \exp\left(-\frac{4\pi^2 l^2 x}{a^2 N_0^2}\right), \tag{3.12}$$

and using the approximating formula (5.5) of Ref. [11], we have the expression

$$Q_{N_0}(x) \cong \frac{1}{\sqrt{4\pi x}}\left[\mathrm{erf}\left(\frac{\pi x^{1/2}}{a}\right)\right] - \frac{2\pi^2 x}{3}\frac{1}{a}\exp\left[-\left(\frac{\pi}{a}\right)^2 x\right] + \\ + \frac{1}{\sqrt{\pi x}}\left\{\sum_{l=1}^\infty \exp\left[-(laN_0)^2/4x\right]\right\} \tag{3.13}$$

valid in the large-N_0 asymptotic regime. The first and the second terms in the above equation are size-independent and are precisely the infinite volume limit of $Q_{N_0}(x)$. The remainder of the calculation involves the insertion of (3.13) into (3.11). In order to simplify the further calculations, we will consider the important case $m = 1$.

4. FINITE-SIZE SCALING FORM OF THE GAP EQUATION

We can represent the right-hand side of Eq. (3.11) as a sum of three terms. The first term is given by

$$\int_0^\infty dx x^{\gamma\rho-1} E^\gamma_{\rho,\gamma\rho}(-\lambda_V x^\rho) \frac{1}{\sqrt{4\pi x}}\left[\mathrm{erf}\left(\frac{\pi x^{1/2}}{a}\right)\right]. \tag{4.1}$$

Using the definition of erf-function

$$\frac{\mathrm{erf}\,(\Lambda\sqrt{x})}{\sqrt{4\pi x}} = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} \exp(-xk^2) dk \tag{4.2}$$

and the identity (3.10), we get

$$\frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \int_0^\infty dx x^{\gamma\rho-1} E^{\gamma}_{\rho,\gamma\rho}(-\lambda_V x^\rho)\exp\left(-xk^2\right) =$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \frac{1}{(\lambda_V + k^{2\rho})^{1-\frac{n}{2\sigma}}}. \tag{4.3}$$

The second term is

$$-\frac{2\pi^2}{3a}\int_0^\infty dx x^{\gamma\rho} E^{\gamma}_{\rho,\gamma\rho}(-\lambda_V x^\rho)\exp\left[-\left(\frac{\pi}{a}\right)^2 x\right] = -\frac{2\gamma}{3}\frac{(\frac{\pi}{a})^{2\rho(\gamma+1)-1}}{[\lambda_V + (\frac{\pi}{a})^{2\rho}]^{\gamma+1}}. \tag{4.4}$$

The third one equals

$$\int_0^\infty dx x^{\gamma\rho-1} E^{\gamma}_{\rho,\gamma\rho}(-\lambda_V x^\rho)\frac{1}{\sqrt{\pi x}}\left\{\sum_{j=1}^{\infty}\exp\left[-(jL)^2/4x\right]\right\}. \tag{4.5}$$

It is convenient to write Eq. (4.5) in terms of the function (particular case of the Jacobi Θ_3 function)

$$A(x) \equiv \sum_{n=-\infty}^{+\infty} e^{-xn^2} \tag{4.6}$$

and the universal finite-size scaling function

$$F^{\gamma}_{m,2\rho}(y) = \frac{1}{(2\pi)^{\gamma 2\rho}}\int_0^\infty dx x^{\gamma\rho-1} E^{\gamma}_{\rho,\gamma\rho}\left(-\frac{x^\rho}{(2\pi)^{2\rho}}y\right)\left[A^m(x) - 1 - \left(\frac{\pi}{x}\right)^{\frac{m}{2}}\right]. \tag{4.7}$$

This can be done with the help of the Poisson transformation formula

$$A(x) = \sqrt{\frac{\pi}{x}}A\left(\frac{\pi^2}{x}\right) \tag{4.8}$$

and the identity

$$\int_0^\infty dx x^{\gamma\rho-1} E^{\gamma}_{\rho,\gamma\rho}\left(-x^\rho\right) = 1, \quad \rho > 0. \tag{4.9}$$

After some algebra the result for the third term is

$$L^{2\gamma\rho-1}\left[F^{\gamma}_{1,2\rho}(\lambda_V L^{2\rho}) + \frac{1}{(\lambda_V L^{2\rho})^\gamma}\right]. \tag{4.10}$$

Collecting the above results for Eq. (3.11) if $m = 1$, we obtain

$$W^{\gamma}_{1,2\rho}(\lambda_V, L) = \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \frac{1}{(\lambda_V + k^{2\rho})^{1-\frac{n}{2\sigma}}} - \frac{2\gamma}{3}\frac{(\frac{\pi}{a})^{2\rho(\gamma+1)-1}}{[\lambda_V + (\frac{\pi}{a})^{2\rho}]^{\gamma+1}} +$$
$$+L^{2\gamma\rho-1}\left[F^{\gamma}_{1,2\rho}(\lambda_V L^{2\rho}) + \frac{1}{(\lambda_V L^{2\rho})^{\gamma}}\right], \quad \gamma \equiv 1 - \frac{n}{2\sigma} > 0. \quad (4.11)$$

The first term is exactly the bulk limit $W^{\gamma}_{1,2\rho}(\lambda_V, \infty)$. The difference between it and the sum $W^{\gamma}_{1,2\rho}(\lambda_V, L)$ results from the second and third terms. The second term in the considered regime $\lambda_V \to 0$ and $a \to 0$ is of order $O\left(\frac{1}{\pi/a}\right)$ and must be omitted. In the third term apart from the factor $L^{2\gamma\rho-1}$ the intrinsic scaling combination

$$y = \lambda_V L^{2\rho} = (L/\xi_{\perp,L})^{2\rho} \quad (4.12)$$

emerges, where $\xi_{\perp,L}$ is the finite-size transverse correlation length (see [5]). If we introduce the notations

$$K := K(\sigma, n, m) \equiv A^{-1}_{n,\sigma}\beta, \quad (4.13)$$

then (3.6) can be rewritten as $(m = 1)$

$$K - K^{c}_{\infty} = W^{\gamma}_{1,2\rho}(\lambda_V, L) - W^{\gamma}_{1,2\rho}(0, \infty), \quad (4.14)$$

where

$$K^{c}_{\infty} := K^{c}_{\infty}(\sigma, \rho, n, m = 1) \equiv W^{\gamma}_{1,2\rho}(0, \infty) = \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \frac{1}{(k^{2\rho})^{\gamma}} \quad (4.15)$$

is the inverse critical temperature (normalized with $A_{n,\sigma}$) of the «isotropic» bulk system. By substitution of Eq. (4.11) into Eq. (4.14), taking into account the small-argument expansion

$$W^{\gamma}_{1,2\rho}(\lambda_V, \infty) = \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \frac{1}{(\lambda_V + k^{2\rho})^{\gamma}} \simeq K^{c}_{\infty} + \frac{1}{2\pi}\lambda_V^{\frac{1}{2\rho}-\gamma} \times$$
$$\times \int_0^{\infty} dx \frac{x^{\gamma\rho} - (1 + x^{\rho})^{\gamma}}{x^{\gamma\rho+1/2}\,(1 + x^{\rho})^{\gamma}} \quad (4.16)$$

valid for $a \to 0$ and $\lambda_V \to 0$ (the integral over x converges, provided $1 > 2\gamma\rho > 1 - 2\rho$), for the gap equation (2.4) we obtain the scaling form

$$x \approx -a(\gamma, \rho, \sigma)y^{D/2-1} + F^{\gamma}_{1,2\rho}(y) + \frac{1}{y^{\gamma}}, \quad (4.17)$$

where

$$x = L^{2\rho(D/2-1)}(K - K_c) \tag{4.18}$$

and (see also Eq. (3.8))

$$a(\gamma,\rho,\sigma) = D_{\gamma,\rho}/A_{1,\sigma}, \quad D_{\gamma,\rho} = \frac{1}{2\pi}\int_0^\infty dx \frac{x^{\rho\gamma} - (1+x^\rho)^\gamma}{(1+x^\rho)^\gamma x^{\rho\gamma+1/2}}, \;\; 2 > D/2 > 1. \tag{4.19}$$

5. FINITE-SIZE CORRECTIONS

Given the gap equation in scaling form, we are now in a position to explore the various finite-size corrections. Here, we look at different regimes: crossover to the thermodynamic critical behaviour defined by the condition $y \gg 1$ and finite-size scaling regime $y \sim 1$.

A. $y \gg 1$.

The finite-size correction to the bulk critical behaviour can be extracted from the asymptotic form of the functions $F^{\gamma}_{d,\sigma}(y)$ at large argument $y \gg 1$ (see [6]). The result is

$$F^{\gamma}_{1,2\rho}(y) \simeq -y^{-\gamma} + \left[2^{2\rho+1}\gamma\pi^{-\frac{1}{2}}\frac{\Gamma\left(\frac{1+2\rho}{2}\right)}{\Gamma\left(-\rho\right)}\sum_{l=1}^{\infty}\frac{1}{l^{1+2\rho}}\right] y^{-(1+\gamma)}. \tag{5.1a}$$

Further Eq. (5.1a) can be simplified using the definition of Rieman zeta function and the relation

$$\frac{\Gamma(\frac{1+2\rho}{2})}{\Gamma(-\rho)}\zeta(1+2\rho) = \pi^{1/2+2\rho}\zeta(-2\rho). \tag{5.1b}$$

The final result is

$$F^{\gamma}_{1,2\rho}(y) \simeq -y^{-\gamma} + \left[2\gamma(2\pi)^{2\rho}\zeta(-2\rho)\right] y^{-(1+\gamma)}. \tag{5.2}$$

Using Eq. (5.2) for the gap equation (2.4), we obtain

$$x \approx -a(\gamma,\rho,\sigma)y^{D/2-1} + \left[2\gamma(2\pi)^{2\rho}\zeta(-2\rho)\right] y^{-(1+\gamma)}, \qquad y \gg 1. \tag{5.3}$$

As one can see, the finite-size effects governed by the second term in the right-hand side of Eq. (5.2) vary as an algebraic power of the variable y. Since $\zeta(-2\rho) = 0$ for $\rho = k$, k is a natural number, there are no power law dependent finite-size corrections if $\rho = k$. The case $0 < \rho < 1$ corresponds to the long-range interaction. For $\rho = 1$ corresponding to the short-range interaction, the result for the universal finite-size scaling function is

$$F^{\gamma}_{1,2}(y) \simeq -y^{-\gamma} + \left[\frac{1}{2^\gamma\Gamma(\gamma)}\right] y^{-\frac{\gamma}{2}} e^{-\sqrt{y}}, \tag{5.4}$$

which corresponds to exponential fall of finite-size corrections in Eq. (5.3). And as long as $\rho > 1$ is not an integer, the power law corrections take place in the case of the so-called subleading LR interaction [11] but with strong anisotropy.

B. $y \sim 1$.

In order to consider the case of $y \sim 1$, we will derive a new representation for $F^{\gamma}_{1,2\rho}(y)$ and rewrite Eq. (4.17) in a form suitable for obtaining the shift of the bulk critical temperature. First, we represent the integral in Eq. (4.7) as a sum of three terms. The first term is given by

$$(y^{-\frac{1}{\rho}})^{\gamma\rho} \int_0^\infty dt t^{\rho\gamma-1} \left[E^{\gamma}_{\rho,\rho\gamma}(-t^{\rho}) - \frac{1}{\Gamma(\rho\gamma)} \right] \left[A\left(\frac{4\pi^2 t}{y^{\frac{1}{\rho}}}\right) - 1 \right] \equiv S_{\rho,\gamma}(y^{\frac{1}{\rho}}), \tag{5.5}$$

the second term is

$$-\frac{1}{2\sqrt{\pi} y^{\gamma-1/2\rho}} \int_0^\infty dt t^{\rho\gamma-3/2} \left[E^{\gamma}_{\rho,\rho\gamma}(-t^{\rho}) - \frac{1}{\Gamma(\rho\gamma)} \right] \equiv -\frac{1}{y^{\gamma-1/2\rho}} \tilde{D}_{\gamma,\rho}, \tag{5.6}$$

and the third one equals the constant (provided $1 > 2\gamma\rho$)

$$\frac{1}{\Gamma(\rho\gamma)} \frac{1}{(2\pi)^{\gamma 2\rho}} \int_0^\infty dx x^{\gamma\rho-1} \left[A(x) - 1 - \left(\frac{\pi}{x}\right)^{\frac{1}{2}} \right] \equiv C_{\gamma,\rho} = F^{\gamma}_{1,2\rho}(0). \tag{5.7}$$

Let us now calculate the function $S_{\rho,\gamma}(y^{\frac{1}{\rho}})$ and the constants $\tilde{D}_{\gamma,\rho}$ and $C_{\gamma,\rho}$. Making use of the identity (see Eq. (3.10))

$$\int_0^\infty dt e^{-zt} t^{\alpha\gamma-1} \left[E^{\gamma}_{\alpha,\alpha\gamma}(-t^{\alpha}) - \frac{1}{\Gamma(\alpha\gamma)} \right] = \frac{z^{\alpha\gamma} - (1+z^{\alpha})^{\gamma}}{(1+z^{\alpha})^{\gamma} z^{\alpha\gamma}}, \tag{5.8}$$

we represent Eq. (5.5) as

$$S_{\rho,\gamma}(y^{\frac{1}{\rho}}) = 2\sum_{l=1}^{\infty} \frac{(4\pi^2 l^2)^{\rho\gamma} - [y + (4\pi^2 l^2)^{\rho}]^{\gamma}}{(4\pi^2 l^2)^{\rho\gamma} [y + (4\pi^2 l^2)^{\rho}]^{\gamma}}. \tag{5.9}$$

To calculate $\tilde{D}_{\gamma,\rho}$, in Eq. (5.6), we first write

$$t^{-1/2} = \frac{1}{\pi^{1/2}} \int_0^\infty dx x^{-1/2} e^{-tx}, \tag{5.10}$$

then by using the identity (5.8) we take the integral over t, and then

$$\tilde{D}_{\gamma,\rho} = \frac{1}{2\pi} \int_0^\infty dx \frac{x^{\rho\gamma} - (1+x^{\rho})^{\gamma}}{(1+x^{\rho})^{\gamma} x^{\rho\gamma+1/2}}, \tag{5.11}$$

i. e. $\tilde{D}_{\gamma,\rho}$ and $D_{\gamma,\rho}$ coincide. Collecting Eqs. (5.5), (5.6) and (5.7) for (4.7) we get

$$F^{\gamma}_{1,2\rho}(y) = F^{\gamma}_{1,2\rho}(0) - y^{\frac{1-2\gamma\rho}{2\rho}} D_{\gamma,\rho} + 2\sum_{l=1}^{\infty} \frac{(4\pi^2 l^2)^{\rho\gamma} - [y + (4\pi^2 l^2)^{\rho}]^{\gamma}}{(4\pi^2 l^2)^{\rho\gamma}[y + (4\pi^2 l^2)^{\rho}]^{\gamma}},\ 1 > 2\gamma\rho. \tag{5.12}$$

Finally, substituting Eq. (5.12) in Eq. (4.11) we have

$$W^{\gamma}_{1,2\rho}(\lambda_V, L) \simeq \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \frac{1}{(k^{2\rho})^{\gamma}} + D_{\gamma,\rho} L^{-2\rho(D/2-1)} y^{D/2-1} + \\ + L^{-2\rho(D/2-1)}\left[F^{\gamma}_{1,2\rho}(y) + \frac{1}{y^{\gamma}}\right], \quad \gamma > 0, \tag{5.13}$$

and taking into account the expansion (4.16) we obtain the gap equation in the form

$$x \simeq F^{\gamma}_{1,2\rho}(0) + 2\sum_{l=1}^{\infty} \frac{(4\pi^2 l^2)^{\rho\gamma} - [y + (4\pi^2 l^2)^{\rho}]^{\gamma}}{(4\pi^2 l^2)^{\rho\gamma}[y + (4\pi^2 l^2)^{\rho}]^{\gamma}} + \frac{1}{y^{\gamma}}. \tag{5.14}$$

Therefore, when $K \to K^c_{\infty}$, simultaneously with $L \to \infty$, in the way prescribed by the equation

$$K = K^c_{\infty} + \frac{x}{L^{2\rho(D/2-1)}} \tag{5.15}$$

with $x = O(1)$, the leading-order asymptotic form of λ_V is given by

$$\lambda_V \simeq \frac{y(x)}{L^{2\rho}}, \tag{5.16}$$

where $y(x)$ is the positive solution of Eq. (5.14).

Let us remind that when the number of infinite dimensions is less than the lower critical dimension, the singularities of the bulk thermodynamic functions are rounded and no phase transition occurs in the finite-size system. Nevertheless, one can define a pseudocritical temperature, corresponding to the position of the smeared singularities of the finite-size thermodynamic functions, and study its shift with respect to the bulk value of the critical temperature. In the case under consideration the first term in the right-hand side of Eq. (5.14) is identified with the shift of the finite-size pseudocritical temperature. Actually, for the sake of convenience here we study the quantity K. The corresponding result for the pseudocritical K^c_L is

$$K^c_L - K^c_{\infty} = L^{-\lambda} F^{\gamma}_{1,2\rho}(0), \tag{5.17}$$

i. e. the shift critical exponent is $\lambda = 1/\nu_{\perp}; \nu_{\perp} = 1/\rho(D-2)$, in accordance with standard finite-size scaling conjecture, see [1]. The coefficient $F^{\gamma}_{d-n,\sigma}(0)$ can be evaluated analytically as well as numerically for different values of the free parameters d, ρ and γ using the method developed in [10].

6. CONCLUSIONS

We show how the mathematical difficulties that arise in the considered anisotropic model with mixed geometry can be avoided — using the identity (3.4), the problem is effectively reduced to the corresponding isotropic one related to a fully finite reference system, Eq. (3.6). A further step is the recognition that with the help of the identity (3.10) the appearance of $\gamma \neq 1$ in the summand of the gap equation (3.6) is not an obstacle for an analytical treatment. Knowledge of the properties of the generalized Mittag–Leffler function allows one to fulfil all calculations analytically. We show that though the system is strongly anisotropic, the corresponding gap equation (4.17) for the intrinsic scaling variable $y = \lambda_V L^{2\rho}$ has a form similar to the isotropic case with geometry L^D(cf. Eq. (4.115) in [1]). The subsequent calculations verified the finite-size scaling hypothesis in its standard form. The conjecture that finite-size scaling in our system with layer geometry $L^m \times \infty^n$ was the naturally expected one was done in [5]. We are much more interested in the explicit form of the scaling equation in different regimes. As a result, we conclude that the finite-size contributions to the thermodynamic behaviour decay algebraically as a function of L only if $0 < \rho \neq k$, where k is a natural number. In the opposite case, the finite-size contributions decay exponentially as a function of L. The phenomenon that the so-called subleading terms (in our terminology the term with $\rho > 1$) lead to dominant finite-size contributions, being unimportant in the bulk limit, was first discussed in Ref. [11]. This characteristic feature of the long-range interactions is also revealed in our consideration.

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