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E. A. Hayryan[1], E. Jurcisinova[1,2], M. Jurcisin[2,3],
I. Pokorny[4], M. Stehlik[2]

STABILITY OF KOLMOGOROV SCALING IN THE THEORY OF ANISOTROPICALLY DRIVEN DEVELOPED TURBULENCE

[1] LIT, Joint Institute for Nuclear Research, 141 980 Dubna, Russia
[2] Institute of Experimental Physics, SAS, 043 53 Košice, Slovakia
[3] BLTP, Joint Institute for Nuclear Research, 141 980 Dubna, Russia
[4] Department of Metal Forming, Technical University, 042 00 Košice, Slovakia

Айрян Э. А. и др. E17-2005-208
Устойчивость скейлинга Колмогорова в теории анизотропной развитой турбулентности

Методами ренормализационной группы исследована развитая турбулентность с аксиальной анизотропией в пространствах с размерностями $d > 2$. Проанализировано влияние анизотропии на устойчивость режима Колмогорова. Показано, что нарушение анизотропией режима скейлинга в трехмерном пространстве происходит только при достаточно специфических значениях параметров анизотропии. Граничное значение размерности пространства между устойчивым и неустойчивым режимами найдено как функция параметров анизотропии.

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Stability of Kolmogorov Scaling in the Theory of Anisotropically Driven Developed Turbulence

The fully developed turbulence with axial anisotropy for dimensions $d > 2$ was investigated by means of renormalization group approach. The influence of anisotropy on the stability of the Kolmogorov scaling regime was analyzed. It was shown that there are only rather specific values of the anisotropy parameters in which the three-dimensional scaling regime is destroyed by the influence of axial anisotropy. The borderline dimension between stable scaling regime and unstable one was calculated as a function of the anisotropy parameters.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

It is believed that traditional approach to the description of fully developed turbulence based on the stochastic Navier–Stokes equation is the most realistic one [1]. The complexity of this equation leads to great difficulties which defend to solve it even in the simplest case when one assumes the isotropy of the system under consideration. On the other hand, almost all real hydrodynamic turbulent systems are more or less anisotropic, and strictly isotropic situations are rather rare. Therefore, if one wants to model realistic developed turbulence, one is pushed to consider anisotropically forced turbulence rather than isotropic turbulence. Without doubt, this, of course, rapidly increases complexity of the corresponding differential equation which itself has to involve a part responsible for a description of the anisotropy. However, even in the isotropic case we have no exact solution of the Navier–Stokes equation. In this situation, one is forced to find some convenient methods to treat the problem at least step by step.

Among other approaches which were applied in the theory of fully developed turbulence during the last decades, one of the most suitable and also powerful tool is the so-called renormalization group (RG) method* which was widely applied as an effective method of studying self-similar scaling behavior, e. g., it was successfully used in the theory of critical phenomena to explain the origin of critical scaling and to calculate universal quantities (critical dimensions and scaling functions). During the last two decades the RG technique was widely used in the fully developed turbulence, and gives answers to some principal questions (e. g., the fundamental description of the infrared (IR) scale invariance) and is also useful for the calculations of many universal parameters (e. g., critical dimensions of the fields and their gradients, etc.). A detailed survey of these questions can be found in Refs. [7, 8, 9], and references therein.

In early papers, the RG approach was applied only to isotropic models of developed turbulence. However, the method can also be used (with corresponding modifications) in the theory of anisotropically developed turbulence. A crucial

*Here we consider and discuss quantum-field renormalization group (or also known as field-theoretic renormalization group) approach [2] rather than Wilson renormalization group technique [3] (see also [4–6]). It is this version of the RG that is the simplest and the most convenient in practical calculations, especially in higher orders of the perturbation expansion.

question immediately arises here: whether the principal properties of the isotropic case and the anisotropic one are the same, at least at the qualitative level. If they are, then it is possible to consider the isotropic case as a first step in the investigation of the real turbulent systems. Maybe the most interesting question is the following one: does the scaling regime remain stable under transition from isotropically developed turbulence into the anisotropically developed turbulence? In the framework of the RG approach the stable regimes correspond to the very existence of the stable fixed points of the corresponding RG equations. Thus, the aforementioned question can be reformulated in other words, namely, do the stable fixed points of the RG equations remain stable under the influence of anisotropy?

During the last decade a few papers have appeared in which the above question was considered in the framework of the RG approach in fully developed turbulence and related problems (magnetohydrodynamic developed turbulence, advection of passive and vector fields by a given turbulent environment, etc.). In some cases, it was found that stability actually takes place (see, e. g., [10, 11]). On the other hand, the existence of systems without such a stability has also been proven. As was shown in Ref. [12] in the anisotropic* magnetohydrodynamic developed turbulence a stable regime generally does not exist. In Refs. [11, 13], d-dimensional models with $d > 2$ were investigated for two cases: weak anisotropy [11] and strong anisotropy [13], and it was shown that the stability of the isotropic fixed point is lost for dimensions $d < d_c \simeq 2.685$. In Ref. [13], where strong anisotropy was investigated, it was also stated that stability of the fixed point, even for dimension $d = 3$, takes place only for sufficiently weak anisotropy. In the present paper, we would like to return to the problem of the influence of strong anisotropy on the stability of the scaling regime in fully developed turbulence which was studied in Ref. [13]. The reason is the suspicion that their results are, at least, not precise. Our conclusion will be the following: the numerical results and conclusions of Ref. [13] are not exact and must be specified although the conceptual framework of their approach is accurate. It will be discussed in detail in the subsequent sections in the present paper.

Another problem in these investigations was that it is impossible to use them in the physically important case $d = 2$, because new ultraviolet (UV) divergences appear in the Green functions, when one considers $d = 2$, and they were not taken into account in the papers [11, 13]. Let us analyzed this problem a little bit more, even though a solution of this problem is not the aim of our investigations below. In Ref. [14], a correct treatment of the two-dimensional isotropic turbulence was given. The correctness in the renormalization procedure was reached by

*Now and in what follows we always have in mind the uniaxial anisotropy, i. e., the anisotropy defined by one specific direction (see next section).

introducing a new local term (with a new coupling constant) into the model, which allows one to remove additional UV divergences. From this point of view, the results obtained earlier for anisotropically developed turbulence, presented in [15] and based on [16] (the results of the last paper are in conflict with Ref. [14]) cannot be correct because they are inconsistent with the basic requirement of the UV renormalization, namely, with the requirement of the localness of the counterterms [5, 6].

The authors of the recent paper [17] used the double-expansion procedure introduced in [14] together with minimal substraction (MS) scheme [18] for an investigation of developed turbulence with weak anisotropy for $d = 2$. The double-expansion procedure is a combination of the well-known Wilson dimensional regularization procedure and an analytical one. In such a perturbation approach the deviation of the spatial dimension from $d = 2$, $\delta = (d-2)/2$, and that of the exponent of the powerlike correlation function of random forcing from their critical value ϵ, play the role of expansion parameters. The main result of the paper was the conclusion that the two-dimensional (2D) fixed point is not stable under weak anisotropy. It means that 2D turbulence is very sensitive to the anisotropy, and nonstable scaling regimes exist in this case. In the case $d = 3$, for both isotropic and anisotropic turbulences, the existence of a stable fixed point, which governs the Kolmogorov asymptotic regime, was established by means of the RG approach and by using the analytical regularization procedure [10, 11, 13]. Using the analytical continuation from $d = 2$ to the three-dimensional (3D) turbulence (in the same sense as in the theory of critical phenomena) one can also verify whether the stability of the fixed point (or, equivalently, stability of the Kolmogorov scaling regime) is restored. From the analysis made in Ref. [17], it follows that it is impossible to restore the stable regime by transition from dimension $d = 2$ to $d = 3$. In Ref. [19], it was supposed that main reason for above-described discrepancy is related to the straightforward application of the standard MS scheme. In the standard MS scheme one works with a purely divergent part of the Green functions only, and its tensor part is neglected. In the case of isotropic models, the stability of the fixed points is independent of dimension d. However, in anisotropic models the stability of fixed points depends on the dimension d, and the tensor structure of the Feynman diagrams becomes to be important.

In Ref. [19], it was suggested to apply a modified MS scheme. The modification is based on the keeping of the d dependence of the UV divergences of diagrams. After such a modification d dependence is correctly taken into account, and can be used to investigate whether it is possible to restore the stability of the anisotropically developed turbulence for some dimension d_c when going from two-dimensional system to three-dimensional one. Thus, after renormalization which is made for the value $d = 2$, the d dependence of the tensor parts of counterterms is remained. In Ref. [19] the influence of weak anisotropy on the stability of the fixed point and the corresponding dependence of the borderline dimension d_c on weak anisotropy were studied. It was shown that in the limit of infinitesimally weak anisotropy for the physically most reasonable value of $\epsilon = 2$, the value of the borderline dimension is $d_c \simeq 2.44$, which is lower than the value $d_c \simeq 2.68$ obtained within the traditional ϵ expansion [11, 13]. Below the borderline dimension, the stable regime of the fixed point of the isotropically developed turbulence is lost by influence of weak anisotropy.

Further step is the inclusion of a nonrestricted axial anisotropy or the so-called strong axial anisotropy. Within ϵ expansion it was investigated in Ref. [13], where the influence of anisotropy on Kolmogorov constant was also studied. In Ref. [20], the influence of the strong anisotropy on the scaling regime was studied within double-expansion scheme for some special situations. During the calculations, the results obtained in Ref. [13] were also recalculated (part of results were published in Ref. [20]), and numerical inconsistences with earlier results of Ref. [13] were obtained. In this situation, it is necessary to return and verify conclusions of the above-mentioned paper. This is the aim of the present paper to specify these results in detail.

The paper is organized as follows: In Sec. 2 we discuss the field theoretic functional formulation of the stochastic problem of fully developed turbulence with strong anisotropy. In Sec. 3 the RG analysis of the problem is given. In Sec. 4 we discuss the stability of the fixed point under influence of strong anisotropy. Our results are compared to the results of Ref. [13]. In Sec. 5 we discuss in detail the numerical method which was used in calculations. In Conclusion discussion of the results is presented. Appendix I contains explicit expressions for the divergent parts of the important Feynman diagram. In Appendix II the necessary and sufficient conditions for convergence of some integrals are proven.

2. DESCRIPTION OF MODEL. FIELD THEORETIC FORMULATION

We are going to study anisotropically driven fully developed turbulence. The anisotropy is characterized by one specific direction, i. e., we shall work with uniaxial anisotropy. The value of the anisotropy parameters will not be restricted.

In the statistical theory of anisotropically developed turbulence, the turbulent flow is characterized by the random velocity field $\mathbf{v}(\mathbf{x}, t)$, where $\mathbf{v}$ and $\mathbf{x}$ are supposed to be d-dimensional vectors. Its evolution is governed by the randomly forced Navier–Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu_0 \Delta \mathbf{v} - \mathbf{f}^A = \mathbf{f}, \qquad (1)$$

where the incompressibility of the fluid is assumed, which is given mathematically by the well-known conditions $\nabla \cdot \mathbf{v} = 0$ and $\nabla \cdot \mathbf{f} = 0$. The parameter ν_0 is the kinematic viscosity (hereafter all parameters with a subscript 0 denote bare parameters of unrenormalized theory; see below). The term $\mathbf{f}^A$ is related to anisotropy and will be specified later. The large-scale random force per unit mass $\mathbf{f}$ is assumed to have Gaussian statistics defined by the averages

$$\langle f_i \rangle = 0,\ \langle f_i(\mathbf{x}_1, t) f_j(\mathbf{x}_2, t) \rangle = D_{ij}(\mathbf{x}_1 - \mathbf{x}_2, t_1 - t_2). \tag{2}$$

The two-point correlation matrix

$$D_{ij}(\mathbf{x}, t) = \delta(t) \int \frac{d^d\mathbf{k}}{(2\pi)^d} \tilde{D}_{ij}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \tag{3}$$

is convenient to parametrize in the following way [10, 12]:

$$\tilde{D}_{ij}(\mathbf{k}) = g_0 \nu_0^3 k^{4-d-2\epsilon} [(1 + \alpha_1 \xi_k^2) P_{ij}(\mathbf{k}) + \alpha_2 R_{ij}(\mathbf{k})] , \tag{4}$$

where a vector $\mathbf{k}$ is the wave vector, d is the dimension of the space (in our case: $2 < d$), $\epsilon \geqslant 0$ is dimensionless parameter of the model. The physical value of this parameter is $\epsilon = 2$ (so-called energy pumping regime). We shall not discuss here more complicated case $d = 2$. The value $\epsilon = 0$ corresponds to a logarithmic perturbation theory for a calculation of Green functions when g_0, which plays the role of a bare coupling constant of the model, becomes dimensionless. The problem of the continuation from $\epsilon = 0$ to the physical values was discussed in Ref. [22]. The $(d \times d)$-matrices P_{ij} and R_{ij} are the transverse projection operators. Their explicit forms are defined by the relations (in the wave-number space)

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2},\ \ R_{ij}(\mathbf{k}) = \left(n_i - \xi_k \frac{k_i}{k} \right) \left(n_j - \xi_k \frac{k_j}{k} \right), \tag{5}$$

where ξ_k is given by the equation

$$\xi_k = \mathbf{k} \cdot \mathbf{n}/k. \tag{6}$$

In Eq. (5), the unit vector $\mathbf{n}$ specifies the direction of the anisotropy axis. The tensor $\tilde{D}_{ij}$, given by Eq. (4), is the most general form with respect to the condition of incompressibility of the system under consideration and contains two dimensionless parameters α_1 and α_2. The positiveness of the correlator tensor D_{ij} leads to restrictions on the above parameters, namely, $\alpha_1 \geqslant -1$ and $\alpha_2 \geqslant -1$. In what follows, we assume no further restrictions on these parameters.

Using the well-known Martin–Siggia–Rose formalism [23–26], the stochastic problem (1) with correlator (3) can be transformed into the field theoretic model of fields $\mathbf{v}$ and $\mathbf{v}'$, where $\mathbf{v}'$ is independent of the velocity field $\mathbf{v}$ auxiliary incompressible field, which we have to introduce when transforming the stochastic problem into a functional form. After this transformation the action of the fields $\mathbf{v}$ and $\mathbf{v}'$ is given in the form

$$
\begin{aligned}
S &= \frac{1}{2}\int d^d\mathbf{x}_1 dt_1 d^d\mathbf{x}_2 dt_2 \left[v_i'(\mathbf{x}_1,t_1)D_{ij}(\mathbf{x}_1-\mathbf{x}_2,t_1-t_2)v_j'(\mathbf{x}_2,t_2)\right] \\
&+ \int d^d\mathbf{x}dt \left\{\mathbf{v}'(\mathbf{x},t)\left[-\partial_t\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{v}+\nu_0\nabla^2\mathbf{v}+\mathbf{f}^A\right](\mathbf{x},t)\right\}. \qquad (7)
\end{aligned}
$$

The functional formulation gives the possibility to use the quantum field theory methods, including the RG technique, to solve the problem. Standardly, the formulation through action functional (7) replaces the statistical averages of random quantities in the stochastic problem (1)–(4) with equivalent functional averages with weight $\exp S(\mathbf{v},\mathbf{v}')$. Generating functionals of total Green functions $G(A)$ and connected Green functions $W(A)$ are then defined by the functional integral

$$
G(A) = \mathrm{e}^{W(A)} = \int \mathcal{D}\Phi \; \mathrm{e}^{S(\Phi)+A\Phi}, \qquad (8)
$$

where $\Phi = \{\mathbf{v},\mathbf{v}'\}$, $A(x) = \{\mathbf{A}^{\mathbf{v}},\mathbf{A}^{\mathbf{v}'}\}$ represents a set of arbitrary sources for the set of fields Φ, $\mathcal{D}\Phi \equiv \mathcal{D}\mathbf{v}\mathcal{D}\mathbf{v}'$ denotes the measure of functional integration, and linear form $A\Phi$ is defined as

$$
A\Phi = \int d\,x[A_i^v(x)v_i(x) + A_i^{v'}(x)v_i'(x)]. \qquad (9)
$$

By means of the RG approach it is possible to extract large-scale asymptotic behavior of the correlation functions after an appropriate renormalization procedure which is needed to remove UV divergences.

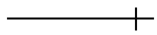
Now we can return to give an explicit form of the anisotropic dissipative term $\mathbf{f}^A$ (see, e. g., Ref. [13]). In our case we suppose that $d > 2$. In this situation, the UV divergences are only present in the one-particle-irreducible Green function $\langle\mathbf{v}'\mathbf{v}\rangle$. To remove them, one needs to introduce into the action, in addition to the counterterm $\mathbf{v}'\nabla^2\mathbf{v}$ (the only counterterm needed in the isotropic model), the following ones: $\mathbf{v}'(\mathbf{n}\cdot\nabla)^2\mathbf{v}$, $(\mathbf{n}\cdot\mathbf{v}')\nabla^2(\mathbf{n}\cdot\mathbf{v})$, and $(\mathbf{n}\cdot\mathbf{v}')(\mathbf{n}\cdot\nabla)^2(\mathbf{n}\cdot\mathbf{v})$. These additional terms are needed to remove divergences related to anisotropic structures. Therefore, in order to arrive at a multiplicatively renormalizable model, we have to take the term $\mathbf{f}^A$ in the form

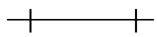
$$
\mathbf{f}^A = \nu_0\left[\chi_{10}(\mathbf{n}\cdot\nabla)^2\mathbf{v} + \chi_{20}\mathbf{n}\nabla^2(\mathbf{n}\cdot\mathbf{v}) + \chi_{30}\mathbf{n}(\mathbf{n}\cdot\nabla)^2(\mathbf{n}\cdot\mathbf{v})\right], \qquad (10)
$$

where bare parameters χ_{10}, χ_{20} and χ_{30} characterize the weight of the individual structures in Eq. (10) on the viscous dissipation.

Action (7) with kernel (4) is given in the form convenient for a realization of the field theoretic perturbation analysis with the standard Feynman diagram technique. From the quadratic part of the action, one obtains the matrix of bare propagators (in the wave-number — frequency representation):

$$= \langle v_i v_j \rangle_0 \equiv \Delta_{ij}^{vv}(\mathbf{k}, \omega_k),$$


$$= \langle v_i v'_j \rangle_0 \equiv \Delta_{ij}^{vv'}(\mathbf{k}, \omega_k),$$


$$= \langle v'_i v'_j \rangle_0 \equiv \Delta_{ij}^{v'v'}(\mathbf{k}, \omega_k) = 0,$$

where

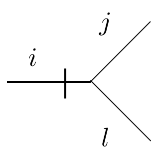
$$\Delta_{ij}^{vv}(\mathbf{k}, \omega_k) = -\frac{K_3}{K_1 K_2} P_{ij} + \frac{1}{K_1(K_2 + \tilde{K}(1-\xi_k^2))} \left[\frac{\tilde{K} K_3}{K_2} + \frac{\tilde{K}(K_3 + K_4(1-\xi_k^2))}{(K_1 + \tilde{K}(1-\xi_k^2))} - K_4 \right] R_{ij}$$

$$\Delta_{ij}^{vv'}(\mathbf{k}, \omega_k) = \frac{1}{K_2} P_{ij} - \frac{\tilde{K}}{K_2(K_2 + \tilde{K}(1-\xi_k^2))} R_{ij}, \qquad (11)$$

with

$$\begin{aligned} K_1 &= i\omega_k + \nu_0 k^2 + \nu_0 \chi_{10} (\mathbf{n} \cdot \mathbf{k})^2 , \\ K_2 &= -i\omega_k + \nu_0 k^2 + \nu_0 \chi_{10} (\mathbf{n} \cdot \mathbf{k})^2 , \\ K_3 &= -g_0 \nu_0^3 k^{4-d-2\epsilon} (1 + \alpha_{10} \xi_k^2), \\ K_4 &= -g_0 \nu_0^3 k^{4-d-2\epsilon} \alpha_{20}, \\ \tilde{K} &= \nu_0 \chi_{20} k^2 + \nu_0 \chi_{30} (\mathbf{n} \cdot \mathbf{k})^2. \end{aligned} \qquad (12)$$

In the case of weak anisotropy one can make the expansion and work only with linear terms with respect to all parameters which characterize anisotropy [19]. The interaction vertex in our model is given by the expression



$$\equiv V_{ijl} = i(k_j \delta_{il} + k_l \delta_{ij}),$$

where wave vector $\mathbf{k}$ corresponds to the field $\mathbf{v}'$. Now one can use the above introduced Feynman rules for computation of the needed diagram.

3. RENORMALIZATION, RG FUNCTIONS, AND RG EQUATIONS

The analysis of the UV divergences is based standardly on the analysis of canonical dimensions (see, e. g., [5, 6, 8]). Our model has two scales [2, 7, 8, 9], i. e., the canonical dimension of some quantity F is defined by two numbers, the momentum dimension d_F^k and the frequency dimension d_F^ω. The total canonical dimension is introduced as $d_F = d_F^k + 2d_F^\omega$ (because, in the free theory, $\partial_t \propto \triangle$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems. To find the dimensions of all quantities it is appropriate to use the standard normalization conditions $d_k^k = -d_x^k = 1, d_\omega^\omega = -d_t^\omega = 1, d_k^\omega = d_x^\omega = d_\omega^k = d_t^k = 0$, and the requirement that each term of the action functional must be dimensionless separately with respect to the momentum and frequency dimensions. The dimensions for our model (7) are given in Table 1, including the parameters which will be introduced later on. The model is logarithmic (the coupling constant g_0 is dimensionless) at $\epsilon = 0$. It means that the UV divergences have the form of the poles in ϵ in the Green functions.

Table 1. Canonical dimensions of the fields and parameters of the model under consideration

F	$\mathbf{v}$	$\mathbf{v}'$	m, Λ, μ	ν_0, ν	g_0	$g, \chi_{i0}, \chi_i, \alpha_1, \alpha_2$
d_F^k	−1	d + 1	1	−2	2ε	0
d_F^ω	1	−1	0	1	0	0
d_F	1	d − 1	1	0	2ε	0

The total canonical dimension of an arbitrary one-irreducible Green function $\Gamma = \langle \Phi \ldots \Phi \rangle_{1-\mathrm{ir}}$ is given by the relation

$$d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - d_v N_v - d_{v'} N_{v'}, \tag{13}$$

where N_v and $N_{v'}$ are the numbers of corresponding fields entering into the function Γ. The total dimension d_Γ is the formal index of the UV divergence. It is well known that superficial UV divergences, whose removal requires counterterms, can be present only in those Green functions Γ for which the total canonical index d_Γ is nonnegative integer.

Detail analysis of divergences in our anisotropic stochastic model is the same as in the case of isotropic fully developed turbulence model, and it was presented in Ref. [2] (see also, e. g., [8, 9]), therefore we shall not repeat it here. The conclusion of this analysis is that the UV divergences can be present only in the 1-particle-irreducible Green function $\langle \mathbf{v}\mathbf{v}' \rangle_{1-\mathrm{ir}}$.

It can be shown (for example by direct calculations) that the field theoretic model (7) with anisotropic terms (10) is multiplicatively renormalizable, i. e., all terms needed for removing of the divergences are included in action (7). Thus one can immediately write down the renormalized action (in the wave-number-frequency representation, where $\nabla \to i\mathbf{k}, \partial_t \to -i\omega_k$, and all needed integrations and summations are assumed)

$$S^R(\mathbf{v},\mathbf{v}') = \frac{1}{2} v_i' \left[g\nu^3 \mu^{2\epsilon} k^{4-d-2\epsilon} \left(\left(1 + \alpha_1 \xi_k^2\right) P_{ij} + \alpha_2 R_{ij} \right) \right] v_j' +$$
$$+ v_i' \left[(i\omega_k - Z_1 \nu k^2) P_{ij} - \nu k^2 \left(Z_2 \chi_1 \xi_k^2 P_{ij} + \left(Z_3 \chi_2 + Z_4 \chi_3 \xi_k^2 \right) R_{ij} \right) \right] v_j +$$
$$+ \frac{1}{2} v_i' v_j v_l V_{ijl}, \quad (14)$$

where μ is a scale setting parameter with the same canonical dimension as the wave number. Quantities g, $\chi_i, i = 1,2,3$, and ν are the renormalized counterparts of bare ones, and Z_i are renormalization constants which are expressed via the UV divergent parts of the function $\langle \mathbf{v}\mathbf{v}' \rangle_{1-\mathrm{ir}}$. In the one-loop approximation their general form is

$$Z_i = 1 - \frac{1}{\epsilon} F_i. \quad (15)$$

In standard MS scheme the amplitudes F_i are only some functions of g, $\chi_i, i = 1,2,3$, α_1, α_2, and d and are independent of ϵ.

The transition from the action (7) to the renormalized one (14) is given by the introduction of the following renormalization constants Z:

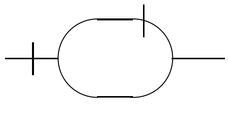
$$\nu_0 = \nu Z_\nu\,, \;\; g_0 = g\mu^{2\epsilon} Z_g\,, \;\; \chi_{i0} = \chi_i Z_{\chi_i}\,, \quad (16)$$

where $i = 1,2$, and 3. By comparison of the corresponding terms in the action (14) with definitions of the renormalization constants Z for parameters (16), one can immediately write the following relations between them:

$$Z_\nu = Z_1\,, \;\; Z_g = Z_1^{-3}\,, \;\; Z_{\chi_i} = Z_{1+i} Z_1^{-1}\,, \quad (17)$$

where again $i = 1,2$, and 3.

In one-loop approximation divergent one-irreducible Green function $\langle \mathbf{v}\mathbf{v}' \rangle_{1-\mathrm{ir}}$ is represented by the Feynman diagram:

$\langle \mathbf{v}'\mathbf{v} \rangle_{1-\mathrm{ir}} =$  (18)

The divergent part of this diagram has the following structure:

$$\Gamma_{ij}^{v'v} = -\frac{\nu g A}{2\epsilon} \left[a_1 \delta_{ij} k^2 + a_2 \delta_{ij} (\mathbf{n}\cdot\mathbf{k})^2 + a_3 n_i n_j k^2 + a_4 n_i n_j (\mathbf{n}\cdot\mathbf{k})^2 \right], \quad (19)$$

where parameter A is defined as $A = S_{d-1}/((2\pi)^d(d^2-1))$, S_d is d-dimensional sphere given as $S_d = 2\pi^{(d/2)}/\Gamma(d/2)$, and functions a_i ($i = 1, \ldots, 4$) are given in Appendix I. They are expressed in an integral form. The counterterms are built up from these divergent parts, which lead to the following equations for renormalization constants:

$$Z_1 = 1 - A\frac{g}{2\epsilon}a_1\,,\; Z_{1+i} = 1 - \frac{A}{\chi_i}\frac{g}{2\epsilon}a_{1+i},\; i = 1, 2, 3. \tag{20}$$

From these expressions one can define corresponding anomalous dimensions $\gamma_i = \mu\partial_\mu \ln Z_i$ for all renormalization constants Z_i (the logarithmic derivative $\mu\partial_\mu$ is taken at fixed values of all bare parameters). The β functions for all invariant charges (running coupling constant g, and parameters χ_i) are given by the following relations: $\beta_g = \mu\partial_\mu g$, and $\beta_{\chi_i} = \mu\partial_\mu \chi_i$ ($i = 1, 2, 3$). Now using Eqs. (17) and definitions given above, one can immediately write the β functions in the forms:

$$\begin{aligned} \beta_g &= -g(2\epsilon + \gamma_g) = g(-2\epsilon + 3\gamma_1), \\ \beta_{\chi_i} &= -\chi_i\gamma_{\chi_i} = -\chi_i(\gamma_{i+1} - \gamma_1), \end{aligned} \tag{21}$$

where

$$\gamma_1 = Aga_1\,,\;\; \gamma_{i+1} = \frac{Ag}{\chi_i}a_{i+1}\,,\;\; i = 1, 2, 3\,. \tag{22}$$

By substitution of the functions γ_i (22) into the expressions for the β functions one obtains:

$$\beta_g = g(-2\epsilon + 3Aga_1)\,,\;\; \beta_{\chi_i} = -Ag\left(a_{i+1} - \chi_i a_1\right)\,,\;\; i = 1, 2, 3. \tag{23}$$

4. STABILITY OF THE KOLMOGOROV SCALING REGIME

Fully developed turbulence is characterized by the large Reynolds number Re. On the other hand, the large Re corresponds to the existence of a large inertial interval, which is defined by the inequalities $1/\Lambda = l \ll r \ll L = 1/m$, where l corresponds to an inner scale (the scale where dissipation forces are dominated, or the scale of the smallest eddies), and L is an outer scale of the system (the scale of the energy pumping into the system, or the scale of the largest eddies). In fully developed turbulence we are interested in the behavior of the correlation functions of velocity field, $\langle v_{i_1}(\mathbf{x}_1, t), \ldots, v_{i_N}(\mathbf{x}_N, t)\rangle$, deep inside of the inertial interval, i. e., far away from the dissipation effects as well as far away from energy pumping scale. Within the field theoretic approach they are given by the following functional integral (see also Sec. 2):

$$\langle v_{i_1}(\mathbf{x}_1, t), \ldots, v_{i_N}(\mathbf{x}_N, t)\rangle = \int \mathcal{D}\Phi \; v_{i_1}(\mathbf{x}_1, t), \ldots, v_{i_N}(\mathbf{x}_N, t) e^{S(\Phi)}, \qquad (24)$$

where $\Phi = \{\mathbf{v}, \mathbf{v}'\}$, $1 \leqslant i_j \leqslant d, j = 1, \ldots, N$, and $S(\Phi)$ is given by Eq. (7).

The behavior of the correlation functions inside the inertial interval is the main issue of the famous Kolmogorov–Obukhov phenomenological theory [27,28] (see also Ref. [29]). It was formulated in the form of two hypotheses which lead to the scaling behavior of the correlation functions within the inertial interval. In what follows we will discussed only the second Kolmogorov hypothesis related to the IR scaling and our aim is to investigate the influence of the axial anisotropy on this scaling behavior.

As was mentioned in Introduction the appropriate method to investigate self-similar systems is the RG method. Within the RG technique the correlation functions are obtained directly in the scaling form (with correct critical dimensions) and their large-scale limit (i. e., IR limit) is described by the stable fixed points of the renormalization theory, i. e., the scaling regime is stable if the corresponding fixed point is IR stable. The IR fixed point is obtained using the system of differential equations (also called the flow equations) which drive the effective variables $\bar{C} = \{\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3\}$ which are the functions of the dimensionless scale parameter (wave number) $t = k/\Lambda$. Their explicit forms are the following:

$$t\frac{d\bar{g}}{dt} = \beta_g(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d)\,, \qquad (25)$$

$$t\frac{d\bar{\chi}_i}{dt} = \beta_{\chi_i}(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d)\,, \quad i = 1, 2, 3. \qquad (26)$$

The dimensionless wave number t belongs to the interval $0 \leqslant t \leqslant 1$, and the initial conditions for the above differential equations are taken at $t = 1$. The IR stable fixed point corresponds to the values in the limit $t \to 0$, i. e., $(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3)|_{t\to 0} = (g^*, \chi_1^*, \chi_2^*, \chi_3^*)$.

In principle, one has two possible ways how to find the IR fixed point of the model. First of all, one can solve the system of four equations

$$\beta_C(C^*, \alpha_1, \alpha_2, d) = 0\,, \qquad (27)$$

where we denote $C^* = \{g^*, \chi_1^*, \chi_2^*, \chi_3^*\}$. In this case, the IR stability of the fixed point is determined by the positive real parts of the eigenvalues of the matrix

$$\omega_{lm} = \left(\frac{\partial \beta_{C_l}}{\partial C_m}\right)_{C=C^*}, \quad l, m = 1, \ldots, 4. \qquad (28)$$

This is a comfortable way for the determination of the fixed point but in our case it cannot be used. The reason is the presence of the integrals in the β functions (see Appendix I) which makes this way rather complicated.

The second possibility is to solve directly the system of the differential equations (25) and (26). It is the way which can be used, and will be used, in our case. This method was also applied in Ref. [13]. More about numerical methods will be said in the next section.

Now we have all necessary tools at hand to investigate the fixed point and its stability. Because the aim of the present paper is to specify the results obtained in the Ref. [13], in this sense, first of all, we have to discuss their results, and we shall try to argue why they are not precise. After that, we shall present our recalculated results.

As for results of Ref. [13], we shall concentrate on Fig. 1 therein. It shows the dimensional borderline (in the spatial region $2 < d \leqslant 3$) between the IR stable regime (Kolmogorov scaling regime) and unstable regime as a function of the anisotropy parameters α_1, α_2 (see Fig. 1 in Ref. [13]). The main conclusions in [13] are the following:

- They confirm the existence of a universal kinetic scaling regime. It corresponds to a stable fixed point of the renormalization group.

- The Kolmogorov scaling regime becomes unstable when anisotropy parameters are even not too large. This situation concerns mainly the parameter α_2, when even very weak anisotropy represented by the parameter α_2 leads to the destruction of the Kolmogorov scaling regime. When value of the parameter $\alpha_2 > 0.0235$ then for all values of parameter α_1 the Kolmogorov regime is destroyed.

- According to their investigation the authors declare that in the limit of the weak anisotropy, the nonzero χ_3 parameter is irrelevant for the stability of the scaling regime at $d = 3$.

Let us analyze what is inaccurate in these conclusions. The first one is a general statement which is correct but the other two are a little bit problematic.

First problem in Fig. 1 of Ref. [13] is related to the isotropic limit of the model. It is well known (see Ref. [11]) that in the isotropic limit ($\alpha_{1,2} \to 0$) the borderline dimension d_c between stable and unstable regimes is $d_c = (3\sqrt{17} - 7)/2 \simeq 2.6846$. This result is also confirmed in Ref. [13] (Eq. (3.4)). On the other hand, in Fig. 1 of Ref. [13], the borderline dimension in this limit is closed to the value $d_c = 2.72$, and the contour which corresponds to the borderline value $d_c = 2.68$ is rather far away from the point $(\alpha_1, \alpha_2) = (0, 0)$ especially in the α_1 direction. This is the first and the most important discrepancy.

The second problem is the rather strong dependence of the d_c on the parameter α_2 (the region of the stability of the scaling regime is very narrow in α_2 direction). Why is it a problem? The answer is the following: the values of α_2 from the interval $-0.025 < \alpha_2 < 0.025$ belong to the weak anisotropy limit, i. e., one

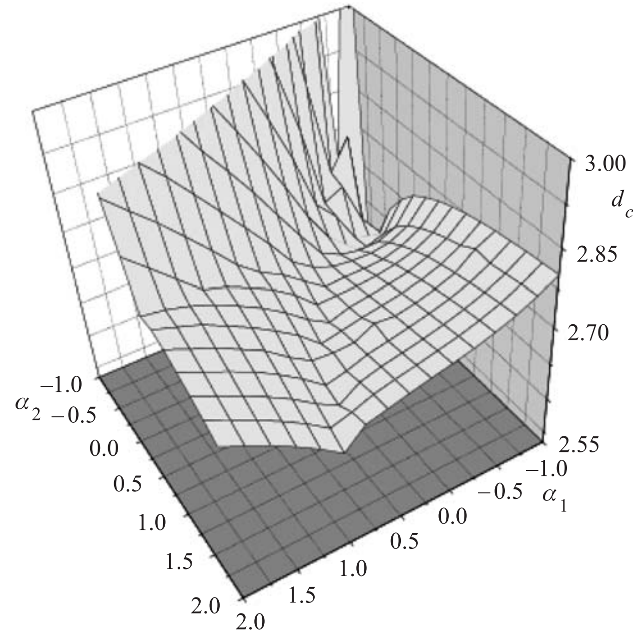


Fig. 1. The three-dimensional view on the dependence of the borderline dimension d_c on the parameters α_1 and α_2

can expand all β functions into the series in anisotropy parameters and keep only the linear parts. This is what we have in mind when we are talking about the weak anisotropy approximation. The region close enough around the point $(\alpha_1, \alpha_2) = (0, 0)$ fulfils this condition. In Ref. [19] the influence of the weak anisotropy was studied in the similar problem but in the double expansion scheme to be able to study also two-dimensional turbulence. The dependence of the borderline dimension d_c on the small parameters α_1 and α_2 was calculated, and it was shown that this dependence is very simple and without radical increase or decrease of d_c near $(\alpha_1, \alpha_2) = (0, 0)$. It can be shown that the same situation takes place also in the weak anisotropy limit of the present model. These linear parts of the β functions must also play the principal role in the case of unrestricted uniaxial anisotropy if the parameters of the anisotropy α_1 and α_2 are small enough. It is fulfil comfortably in the square area $\alpha_1 \times \alpha_2 = \langle -0.025, 0.025\rangle \times \langle -0.025, 0.025\rangle$. From this point of view the drastic dependence of d_c on the parameter α_2 in Fig. 1 of Ref. [13] is a little bit strange.

The third conclusion is not exact too. As was shown in Ref. [19], exactly the parameters χ_3 and α_5 (the last one in related to the double expansion model) play the crucial role in the determination of the IR stability of the fixed point.

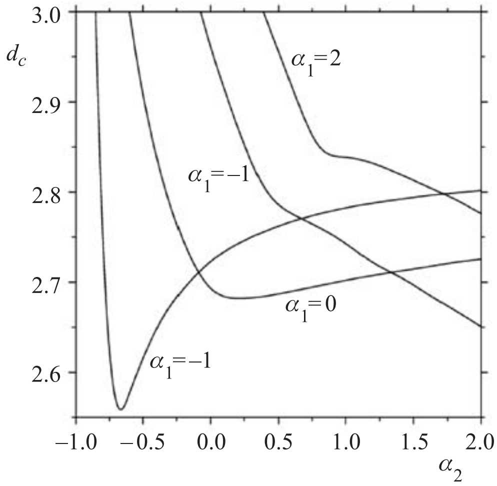


Fig. 2. The dependence of the borderline dimension d_c on the parameter α_2 for concrete values of the parameter α_1

Again, it can be shown that in the weak anisotropy limit of the model under consideration the parameter χ_3 alone plays the same role as the parameters χ_3 and α_5 in double expansion case. Even though that the value of the fixed point for parameter χ_3 is $\chi_3^* = 0$, namely the eigenvalue of the matrix of the first derivatives related to this parameter is responsible for the very existence of the borderline dimension $d_c \in (2, 3\rangle$.

In Figs. 1 and 2 our results for the d_c as a function of the anisotropy parameters are presented. The difference between our results and those of Ref. [13] can be seen immediately. Figs. 1 and 2 show that in three dimensions the Kolmogorov scaling regime is unstable in the limit $\alpha_{1,2} \rightarrow -1$ and for large enough values of parameter α_1 together with negative or relatively small positive values of the parameter α_2. Our conclusion is the following: to destroy stability of the Kolmogorov scaling regime in three-dimensional space by the uniaxial anisotropy, which is in our model represented by the parameters α_1 and α_2, it is necessary to apply anisotropy with rather specific values of these parameters.

5. NUMERICAL METHODS

One possible way how to solve the problem of the IR fixed point of the four differential RG equations (25) and (26), with β functions (23), and corresponding integrals (34) of Appendix I is based on the analytical calculations of the integrals. The integrands of (34) have the form of fractions of two polynomial $P_i(x^2)/Q(x^2)$ ($i = 1, \dots, 4$), with different numerators $P_i(x^2) = b_i(x^2)$ but with the same denominator $Q_i(x^2) = (M_1 M_2 M_3)^3$. It is possible to expand the expression $1/(M_1 M_2 M_3)^3$ into a sum of partial fractions of the type $R(x^2)/(a + x^2)^j$, where $R(x^2)$ is a polynomial, a is, in general, complex function of parameters $\chi_i, i = 1, 2, 3$, and $j = 1, 2, 3$. Now using the following result (see, e. g., Ref. [30]):

$$\int_{-1}^{1} dx \frac{(1 - x^2)^{\frac{d-3}{2}} x^{2n}}{(a + x^2)^j} = \frac{(1 + (-1)^{2n})\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2} + n)}{2a^j \Gamma(\frac{d}{2} + n)} {}_2F_1 \times \times \left(1, \frac{1}{2} + n; \frac{d}{2} + n; -\frac{1}{a}\right) , \quad (29)$$

one can represent the integrals in the form of a combination of hypergeometric functions ${}_2F_1(a, b; c; z)$ defined as ${}_2F_1(a, b; c; z) = 1 + \frac{ab}{1!c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \dots$ Eq. (29) is held when $Re[d] > 1$ ($Re[x]$ means the real part of x), $Re[n] > -1/2$, and $Arg[a] \neq \pi$. In our case, these conditions are fulfilled because $d \in (2, 3\rangle$, n is 0 or positive integer, and it can be shown that the last condition is also held.

By using of this representation of integrals (34) it is possible to find the IR fixed point of differential equations (25) and (26) by solving of the system of equations (27) together with the matrix of the first derivatives (28) to test the IR stability of the fixed point. But this way is rather complicated and we shall not use it here.

The most comfortable way how to find the IR fixed point of the system of four differential RG equations (25) and (26) with (23) is to solve it numerically using some appropriate numerical method. In what follows, we work with the fourth-order Runge–Kutta method with the adaptive choice of the integration step. It is convenient to transform the system of differential equations (25) and (26) into an autonomic system by the substitution $t = \mathrm{e}^{-s}$. Using this transformation one obtains

$$\frac{d\bar{g}}{ds} = -\beta_g(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d) , \quad (30)$$

$$\frac{d\bar{\chi}_i}{ds} = -\beta_{\chi_i}(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d) , \quad i = 1, 2, 3, \quad (31)$$

where $s \in \langle 0, \infty)$. The initial conditions correspond to $s = 0$, and the IR fixed point is found in the limit $s \to \infty$. The first step for the variable s was taken as $\Delta s = 10^{-3}$. The initial values of the parameters can be chosen arbitrary but the most convenient choice is to take them to be the fixed point of the 3D isotropic model (see Ref. [13]).

As was already discussed our differential equations are made of the linear combinations of the following integrals:

$$I = \int_{-1}^{1} dx \frac{(1-x^2)^{\frac{d-3}{2}} x^{2n}}{(M_1 M_2 M_3)^3} \,, \tag{32}$$

where explicit forms of the functions $M_i, i = 1, 2, 3$ are given in Appendix I. Therefore, the first step to solve the problem is the necessity to guarantee their convergence within the interval $x \in \langle -1, 1 \rangle$, i. e., to determine the allowed values of the parameters $\chi_1,\ \chi_2,\ \chi_3$. That is why, let us first discuss the conditions under which the integrals will be well defined.

Assume that one (or more) of the expressions $M_i, (i = 1, 2, 3)$ (defined in Appendix I) is vanished in respect to the variable x within the interval $\langle -1, 1 \rangle$. Let us denote as x_1 the point in which one of the M_i is equal to zero. Then, from the convergence point of view, the integral (32) is equivalent to the integral of the type $\int_{-1}^{1} 1/\left(x - x_1\right)^3 dx$. Hence, if one of the expressions $M_i, i = 1, 2, 3$ vanish in the interval $\langle -1, 1 \rangle$ then the integrand will be nonintegrable. Thus, to guarantee the convergence of the integrals, we are looking for such conditions on variables $\chi_1,\ \chi_2,\ \chi_3$ which give nonzero values for M_i within the corresponding interval. The necessary and sufficient conditions of the convergence of the integrals (32) are as follows:

$$\chi_1 > -1 \,, \quad \chi_2 > -1 \,, \quad \chi_3 > -(\sqrt{1+\chi_1} + \sqrt{1+\chi_2})^2. \tag{33}$$

The detail proof of these conditions can be found in Appendix II. They are also important in the numerical solution of our system of differential equations and they must be tested on each step of the Runge–Kutta method.

An important question is related to the choice of a numerical method for calculation of the integrals. It can be shown that the most appropriate method is the using of the Chebyshev quadrature formula. The question of the number of divisions of the integration interval is another important one. In our calculations, we used the division to 1024 subintervals, which was found as the best choice from the point of view of the accuracy and needed time of the calculation. On the other hand, in Ref. [13], the division to the 128 subintervals was used. We suppose that this fact could lead to the difference between our and their results because the division to 128 subintervals can be not sufficient in some critical situations.

To find the borderline dimension it is enough to use the bisection method. Our results were calculated with the accuracy of 0.005. The same accuracy was supposed in Ref. [13] but, as was already discussed, this accuracy was not achieved even in the isotropic limit where exact result exists (see discussion in the previous section).

From numeric calculations point of view, the problem is rather time-consuming, i. e., the calculations take relatively long time. Therefore, the question of using of a modern computational methods arises. In what follows, we shall analyze possible speed up of calculations based on the utilization of the parallel programming methods using of the Message Passing Interface (MPI) (see, e. g., [31, 32]). Let us discuss this problem in more detail. First of all, we have to calculate our system of differential equations at large number of points. For example, Fig. 1 was obtained by using of results on the lattice of the size 16×16 in the plane of anisotropy parameters α_1 and α_2, i. e., we had to repeat the calculational procedure 256 times.

Suppose first that we have to calculate the borderline dimension d_c for one concrete value of the parameters α_1 and α_2. How can the MPI help in this situation? To find the borderline dimension with a precision Δd when the starting interval for d has the length l one has to carry out prescribed number of calculations n. By using of the one-processor computer the best way how to find the d_c is to use the bisection method. In this case, the result is obtained after $n = \lceil \log_2(l/\Delta d) \rceil$ calculations ($\lceil x \rceil$ means the smallest integer greater or equal to x). On the other hand, in the case of the multiprocessor computer with m processors, one can divide the interval into $m+1$ subintervals and carry out the calculations in m points of the division at the same time. Thus, the result is obtained after $n = \lceil \log_{m+1}(l/\Delta d) \rceil$ serial calculations (of course, in this case, the total number of calculations is larger but the total time of the calculations is shorter). Let us demonstrate it by an example. Suppose that $l = 1$ and $\Delta d = 0.005$. The results are shown in Table 2. The table shows the effective numbers of processors which are $1, 2, 3, 5, 14$, and 199. If we suppose that the calculations take the same time for all values of dimension d then n is directly related to the time of calculation. For example, the calculation with three processors (the same holds also for four processors) is two times shorter than calculation with one processor, see Table 2. On the other hand, the calculation takes the same time for the computations with three, and four processors. The same is held for the computers with number of processors from the intervals $m = [5, 13]$, $m = [14, 198]$, and $m = [199, \infty)$. Therefore, our conclusion is the following: if one needs to do only one computational process (in our case, it means to find one borderline dimension d_c for concrete value of the parameters of the model) then it is appropriate to use the advantage of the parallel computing.

Table 2. The number of needed serial calculations n as a function of the number of processors m

m	1	2	3, 4	5–13	14–198	$\geqslant 199$
n	8	5	4	3	2	1

Now let us analyze the situation when one needs to calculate the borderline dimension d_c as a function of the anisotropy parameters (it is our case). Thus, it is necessary to carry out two or more independent calculations for different values of the parameters of the model. The simplest situation occurs when the number of independent calculations are much more larger than the number of processors. Because this is our case, we shall analyze it in detail. The situation is shown in Fig. 3, where total number of computational processes N is shown as a function of the number of processors m and of the desired precision Δd. It is seen immediately that the most effective utilization of the processors is to give to each processor to calculate independent borderline dimension d_c alone.

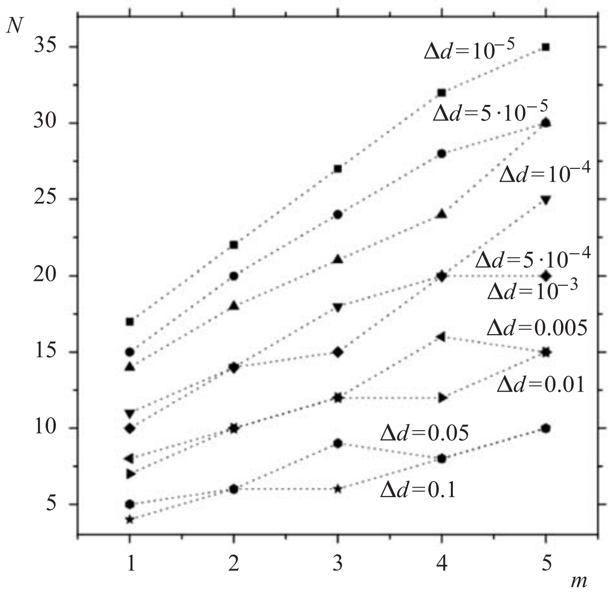


Fig. 3. The dependence of the total number of calculational processes as a function of the number of processors m and the precision Δd. The length of the initial interval is $l = 1$

We have analyzed two special cases, namely, the case with one independent calculation, and the case with the large number of independent calculations. The situation becomes more complicated when one needs to carry out the number of calculations which is comparable to the number of processors. But each such situation needs special analysis and we shall not analyze it here.

In concrete calculations we used advantage of the parallel programming. Our situation is the case with a large number of independent calculations, therefore, each processor has calculated borderline dimension for defined values of the parameters of the model.

CONCLUSION

By using of the field-theoretic RG method the influence of the uniaxial anisotropy on the stability of the Kolmogorov scaling regime in fully developed turbulence was investigated. The stability of the regime is defined by the very existence of the IR stable fixed point. The fixed point was found numerically by the solving of the corresponding differential RG equations. It was shown that the earlier results obtained in Ref. [13] as well as their conclusions about the dependence of the borderline dimension d_c as a function of the anisotropy parameters $\alpha_{1,2}$ are not precise enough. We have found that the stability of the three-dimensional scaling regime is destroyed only in the case of rather large (in the sense of the absolute value) and special values of the anisotropy parameters. We have also analyzed the optimal way how to calculate the numerical problem by using of the parallel programming methods.

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APPENDIX I

The explicit form of the functions $a_i,\ i = 1, \ldots, 4$ from the divergent parts of the one-loop diagram of the model (see Sec. 2) is as follows:

$$a_i = \frac{1}{4}\int_{-1}^{1} dx \frac{(1-x^2)^{(d-3)/2}}{(M_1M_2M_3)^3}\, b_i, \quad i = 1, \ldots, 4, \tag{34}$$

with

$$\begin{aligned} b_1 \quad &= \quad c_{1,1} + 4M_2M_3x^2x_1^2(c_{1,2} + c_{1,3}) \\ &+ \quad M_1x^2x_1\Big(c_{1,4} + M_2M_3^2r_4x_1c_{1,5} + M_2^3c_{1,6} - 2M_2^2M_3^2c_{1,7}\Big), \end{aligned} \tag{35}$$

$$\begin{aligned} b_2 = c_{2,1} - M_1^2M_2^3r_1(c_{2,2} + c_{2,3}) - 4M_2M_3x_1(f_3 + x_1)c_{2,4} - \\ - M_1\Big(- 2M_3^3r_4^2w_1x_1^2(f_3 + x_1)y_4 + M_2^3(2x_1^2c_{2,5} - 2f_3c_{2,6} + d_1M_3r_1r_4^2x_1y_7) + \\ + 2M_2^2M_3^2(2\alpha_2M_3c_{2,7} + c_{2,8}) + M_2M_3^2r_4(c_{2,9} + c_{2,10})\Big), \end{aligned} \tag{36}$$

$$b_3 = c_{3,1} + M_1^2 M_2^3 r_1 c_{3,2} + 4M_2M_3(c_{3,3} + M_2M_3r_4(c_{3,4} + x_1c_{3,5}) + \\ + M_2^2(c_{3,6} + x_1c_{3,7})) + M_1\Big(c_{3,8} - 2M_2^2M_3^2(c_{3,9} + 2f_3c_{3,10} + c_{3,11}) + \\ + M_2M_3^2r_4(-x_1c_{3,12} - 4f_3c_{3,13}) + M_2^3(-d_1c_{3,14} + 2f_3c_{3,15} - 2x_1^2c_{3,16})\Big), \tag{37}$$

$$\frac{b_4}{2} = c_{4,1} - 2c_{4,2}M_1^2M_2^3r_1 + 2M_2M_3(-c_{4,7} + (c_{4,5} - c_{4,6}d_1)M_2^2 + \\ + (c_{4,3} - c_{4,4}d_1)M_2M_3r_4) + M_1\Big((d_1(c_{4,10} + c_{4,11}f_1) + c_{4,9}f_2 + 3c_{4,8}f_3)M_2^3 + \\ + 2(d_1(c_{4,20} + 2c_{4,19}f_1) + c_{4,18}f_2 + 3c_{4,17}f_3)M_2^2M_3^2 + \\ + (d_1(c_{4,15} + 2c_{4,16}f_1) + c_{4,14}f_2 + 3c_{4,13}f_3)M_2M_3^2r_4 - c_{4,12}M_3^3r_4^2x_1y_4\Big), \tag{38}$$

where

$$\begin{aligned}
c_{1,1} &= 4M_1^2M_2^3r_1x^2x_1(-d_1M_3r_4 + (-\chi_2M_3 + r_4(1+3M_3))x_1) \\
&+ M_1^3M_2^3r_1(d_1M_3(d_{-1} - 3x_1) + 2(1+2M_3)x_1^2)\,, \\
c_{1,2} &= -M_3^2r_4^2x_1(r_1 + \alpha_2x_1) + M_2^2(-4\alpha_2M_3^2 + \\
&+ r_1r_4(2M_3 - r_4(1+w_1)x_1))\,, \\
c_{1,3} &= M_2M_3r_4(4\alpha_2M_3x_1 + r_1(2M_3 - r_4(2+w_1)x_1))\,, \\
c_{1,4} &= -2M_3^3r_4^2w_1x_1^2(r_1 + \alpha_2x_1)\,, \\
c_{1,5} &= \alpha_2M_3x_1(4(w_1 + h_2x_1) + r_4y_1) + \\
&+ r_1(4\chi_2M_3x_1 + r_4(-4w_1x_1 + M_3y_1))\,, \\
c_{1,6} &= d_1M_3(4\alpha_2M_3^2 + r_1r_4(-2M_3 + r_4x_1)) \\
&- 2x_1(8\alpha_2M_3^3 + r_1r_4(-4M_3^2 + r_4x_1 - 2M_3(1 + w_1 - y_2)))\,, \\
c_{1,7} &= d_1r_4(2\alpha_2M_3x_1 + r_1(M_3 - r_4x_1)) \\
&- 4x_1(r_1r_4(M_3 - y_2) + \alpha_2M_3(r_4x_1 + y_2))\,, \\
c_{2,1} &= 2M_1^3M_2^3r_1(f_3 - d_1f_1M_3 + 2f_3M_3)\,, \\
c_{2,2} &= -4f_3M_3r_4(-2 + 3x^2) + d_1M_3r_4(d_{-1} - 4f_1 + 3f_1x^2 - 5x_1)\,, \\
c_{2,3} &= 4f_3(-\chi_2M_3 + r_4)x_1 + 4(-\chi_2M_3 + r_4 + 2M_3r_4)x_1^2\,, \\
c_{2,4} &= -4\alpha_2M_2M_3^2(M_2 - r_4x_1) \\
&- r_4(M_2r_1(-2M_1M_3 + r_4(M_1 + M_3 + M_1w_1)x_1) + M_3^2r_4x_1y_4)\,,
\end{aligned}$$

$$
\begin{aligned}
c_{2,5} &= -4\alpha_2 M_3^3 + r_1 r_4(-r_4(M_3 x_4 + x_1) + 2M_3(1 + M_3 + w_1 + \chi_2 x_1))\,, \\
c_{2,6} &= 4\alpha_2 M_3^3 x_2 + r_1 r_4(r_4(x_1^2 + M_3 y_3) - 2M_3(M_3 x_2 + x_1(1 + y_5)))\,, \\
c_{2,7} &= x_1^2(-2\chi_2 x_1 + r_4(x_4 + x_1)) \\
&+ f_3(-2\chi_2 x_1^2 + r_4(1 + 4x^4 + x_1 - 2x^2(3 + x_1)))\,, \\
c_{2,8} &= 2f_3 r_1 r_4(M_3 x_2 + 2\chi_2 x_1^2 - r_4 y_3) \\
&+ r_4 x_1(2r_1 x_1(M_3 - r_4 x_4 + 2\chi_2 x_1) + d_1 y_6 y_7)\,, \\
c_{2,9} &= -2r_1(-2\chi_2 M_3 x_1^3 + r_4(x_1^2(M_3 x_4 + 2w_1(f_3 + x_1)) + f_3 M_3 y_3))\,, \\
c_{2,10} &= M_3 x_1\Big(\alpha_2(4f_3 w_1 x_1 - 2r_4 x_4 x_1^2 - 2f_3 r_4 y_3 + 4x_1^2 y_5) \\
&\quad + y_4(4f_3\chi_2 x_1 + d_1 r_4 y_7)\Big)\,, \\
c_{3,1} &= M_1^3 M_2^3 r_1(-d_1 f_1 M_3 + 2f_3(1 + \chi_1 + 2M_3) + 2\chi_1 x_1^2)\,, \\
c_{3,2} &= -4(-\chi_2 M_3 + r_4 + 3M_3 r_4 + \chi_3 M_3 x^2 - \chi_1 r_4 x^2)x_1^2 \\
&- d_1(d_{-1} M_3 r_4 + f_1 M_3 r_4(-2 + x^2) + (-5M_3 r_4 - 4\chi_3 M_3 x^2 + \\
&+ 4\chi_1 r_4 x^2)x_1) \\
&+ 4f_3(M_3(-\chi_3 x^2 + \chi_2 x_1) + r_4(-3M_3 x_1 + y_8))\,, \\
c_{3,3} &= -M_3^2 r_4^2 x_1 y_4(-d_1\chi_1 x^2 x_1 + f_3 y_8 + x_1^2 y_9)\,, \\
c_{3,4} &= f_3(4\alpha_2 M_3 x_1 y_8 + r_1(-r_4 x_1(-2 - w_1 + x^2(2 + 2\chi_1 + w_1 + w_2)) + \\
&+ 2M_3 y_8))\,, \\
c_{3,5} &= -d_1 x^2(2\chi_1(2\alpha_2 M_3 x_1 + r_1(M_3 - r_4 x_1)) - r_1 r_4 x_1 w_2) \\
&+ x_1(4\alpha_2 M_3 x_1 y_9 + r_1(r_4 x_1(w_1 - x^2 w_2 - 2y_9) + 2M_3 y_9))\,, \\
c_{3,6} &= -f_3(4\alpha_2 M_3^2 y_8 + r_1 r_4(r_4 x_1(-1 - w_1 + x^2(1 + \chi_1 + w_1 + w_2)) - \\
&- 2M_3 y_8))\,, \\
c_{3,7} &= d_1 x^2(\chi_1(4\alpha_2 M_3^2 + r_1 r_4(-2M_3 + r_4 x_1)) + r_1 r_4^2 x_1 w_2) \\
&+ x_1(-4\alpha_2 M_3^2 y_9 + r_1 r_4(r_4 x_1(w_1 - x^2 w_2 - y_9) + 2M_3 y_9))\,, \\
c_{3,8} &= -2M_3^3 r_4^2 x_1(f_3(-w_1 x_1 + x^2 w_2) - x_1(w_1 x_1 + x^2(d + x^2)w_2))y_4\,, \\
c_{3,9} &= 2x_1^2\Big(r_1 r_4(3M_3 - 2r_4 x_3 + 2\chi_2 x_1 - 2\chi_3 x^2 x_1) \\
&\quad + \alpha_2 M_3(-2(\chi_2 - \chi_3 x^2)x_1 + r_4(2x_3 + 3x_1))\Big)\,, \\
c_{3,10} &= \alpha_2 M_3(2x_1(\chi_3 x^2 - \chi_2 x_1) + r_4(2 + 4x^4 + 3x_1 - 2x^2(4 + x_1))) \\
&+ r_1 r_4(3M_3 - 2M_3 x^2 - 2\chi_3 x^2 x_1 + 2\chi_2 x_1^2 - 2r_4 y_{10})\,, \\
c_{3,11} &= d_1\Big(x_1(\alpha_2 M_3(r_4(-x_5 - 4x_1) - 4\chi_3 x^2 x_1) + \\
&+ r_1 r_4(-4M_3 + r_4 x_5 + 4\chi_3 x^2 x_1)) + d_{-1} M_3 r_4 y_4 + \\
&+ f_1 M_3 r_4(-2 + x^2)y_4\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
c_{3,12} &= d_1\Big(\alpha_2 M_3 x_1(r_4x_5+4x^2(\chi_3x_1+w_2)) \\
&\quad +r_1(4\chi_3M_3x^2x_1+r_4(M_3x_5-4x^2x_1w_2))\Big) \\
&- 4x_1\Big(\alpha_2M_3x_1(-w_1+r_4x_3-\chi_2x_1+\chi_3x^2x_1+x^2w_2) \\
&\quad +r_1(M_3(-\chi_2+\chi_3x^2)x_1+r_4(M_3x_3+w_1x_1-x^2x_1w_2))\Big)\,, \\
c_{3,13} &= \alpha_2M_3x_1(w_1x_1-r_4y_{10}-x^2(\chi_3x_1+w_2)) \\
&- r_1(\chi_3M_3x^2x_1+r_4(w_1x_1^2+M_3y_{10}-x^2x_1w_2))+\chi_2M_3x_1^2y_4\,, \\
c_{3,14} &= 2d_{-1}M_3^2(-2\alpha_2M_3+r_1r_4) \\
&+ 2f_1M_3^2(-2\alpha_2M_3+r_1r_4)(-2+x^2)+x_1\Big(16\alpha_2M_3^3 \\
&+ r_1r_4(r_4(M_3x_5-2\chi_1x^2x_1)+4M_3(-2M_3 \\
&+ x^2(\chi_1+\chi_3x_1+w_2)))\Big)\,, \\
c_{3,15} &= 4\alpha_2M_3^3(3-2x^2)+r_1r_4\Big(M_3^2(-6+4x^2) \\
&\quad +2M_3(\chi_1x^2-x_1+\chi_3x^2x_1-w_1x_1-\chi_2x_1^2+r_4y_{10}+x^2w_2)- \\
&- r_4x_1y_8\Big)\,, \\
c_{3,16} &= -12\alpha_2M_3^3+r_1r_4\Big(2M_3(3M_3+w_1+\chi_2x_1-\chi_3x^2x_1-x^2w_2-y_9) \\
&\quad -r_4(2M_3x_3-x_1y_9)\Big)\,, \\
c_{4,1} &= M_1^3M_2^3(3f_3\chi_1+f_2(1+\chi_1+2M_3))r_1\,, \\
c_{4,2} &= -3f_3(2\chi_2M_3-\chi_3M_3x^2+r_4(-5M_3+x_8)) \\
&+ f_2(M_3(\chi_3x^2+\chi_2x_7)-r_4(M_3(-5+3x^2)+x_9)) \\
&+ d_1(f_1(M_3(\chi_2+\chi_3-2\chi_3x^2)+r_4(-2M_3+x_{10}))+\chi_3M_3x_1- \\
&- \chi_1r_4x_1)\,, \\
c_{4,3} &= 3f_3(4\alpha_2M_3x_8x_1+r_1(2M_3x_8+r_4x_1(2w_1-2x_8-x^2w_2))) \\
&+ f_2(4\alpha_2M_3x_9x_1+r_1(2M_3x_9-r_4x_1(-2(2+w_1) \\
&+ x^2(2+2\chi_1+w_1+w_2))))\,, \\
c_{4,4} &= x_1(-2\chi_1(M_3r_1+2\alpha_2M_3x_1-r_1r_4x_1)+r_1r_4x_1w_2) \\
&+ f_1(4\alpha_2M_3x_{10}x_1+r_1(2M_3x_{10}+r_4x_1(2+2\chi_1-4\chi_1x^2+ \\
&+ w_1+x_2w_2)))\,, \\
c_{4,5} &= 3f_3(-4\alpha_2M_3^2x_8+r_1r_4(2M_3x_8-r_4x_1(-2w_1+x_8+x^2w_2))) \\
&+ f_2(-4\alpha_2M_3^2x_9+r_1r_4(2M_3x_9-r_4x_1(-2(1+w_1)+ \\
&+ x^2(1+\chi_1+w_1+w_2))))\,,
\end{aligned}
$$

$$
\begin{aligned}
c_{4,6} &= x_1(\chi_1(4\alpha_2 M_3^2 + r_1 r_4(-2M_3 + r_4 x_1)) + r_1 r_4^2 x_1 w_2) \\
&+ f_1(-4\alpha_2 M_3^2 x_{10} + r_1 r_4(2M_3 x_{10} + r_4 x_1(-x_{10} + w_1 + x_2 w_2)))\,, \\
c_{4,7} &= M_3^2 r_4^2 x_1(-d1 f_1 x_{10} + 3f_3 x_8 + f_2 x_9 + d1\chi_1 x_1) y_4\,, \\
c_{4,8} &= 16\alpha_2 M_3^3 \\
&+ r_1 r_4(r_4(M_3 x_6 - x_8 x_1) + 2M_3(-4M_3 - 2w_1 + x_8 - 2\chi_2 x_1 + \\
&+ \chi_3 x^2 x_1 + x^2 w_2))\,, \\
c_{4,9} &= -8\alpha_2 M_3^3 x_7 + r_1 r_4\Big(r_4(-x_9 x_1 + M_3 y_{13}) \\
&\quad +2M_3(\chi_1 x^2 + (1 + 2M_3 + w_1)x_7 + \chi_3 x^2 x_1 - \chi_2 y_{11} + x^2 w_2)\Big)\,, \\
c_{4,10} &= r_1 r_4(d_{-1} M_3 r_4 + x_1(2\chi_1 M_3 - 5M_3 r_4 + 2\chi_3 M_3 x_1 - \\
&- \chi_1 r_4 x_1 + 2M_3 w_2))\,, \\
c_{4,11} &= -8\alpha_2 M_3^3 + r_1 r_4\Big(4M_3^2 - r_4(x_1 + \chi_1 y_{12}) \\
&\quad +2M_3(2\chi_3 x^4 - x_{10} + r_4 x_{11} + w_1 + \chi_3 x_3 + \chi_2 x_1 + x_2 w_2)\Big)\,, \\
c_{4,12} &= -6f_3 w_1 + f_2 w_1 x_7 + f_2 x^2 w_2 + 3f_3 x^2 w_2 + d1(x_1 w_2 + \\
&+ f_1(w_1 + x_2 w_2))\,, \\
c_{4,13} &= \alpha_2 M_3 x_1(-4w_1 + r_4 x_6 + 2x^2(\chi_3 x_1 + w_2)) \\
&+ r_1(2\chi_3 M_3 x^2 x_1 + r_4(M_3 x_6 + 4w_1 x_1 - 2x^2 x_1 w_2)) - 4\chi_2 M_3 x_1 y_4\,, \\
c_{4,14} &= \alpha_2 M_3(-2w_1 y_{11} + r_4 x_1 y_{13} + 2x^2 x_1(\chi_3 x_1 + w_2)) \\
&+ r_1(2\chi_3 M_3 x^2 x_1 + r_4(M_3 y_{13} - 2x_1(w_1 x_7 + x^2 w_2))) - 2\chi_2 M_3 y_{11} y_4\,, \\
c_{4,15} &= d_{-1} M_3 r_4 y_4 - x_1(r_1 r_4(5M_3 + 2x_1 w_2) + \\
&+ M_3 x_1(5\alpha_2 r_4 - 2\alpha_2 w_2 - 2\chi_3 y_4))\,, \\
c_{4,16} &= r_1 r_4(M_3 x_{11} - x_1(w_1 + x_2 w_2)) \\
&+ M_3(\alpha_2 r_4 x_{11} x_1 + \alpha_2 w_1 x_1 + \alpha_2 y_{12} w_2 + \chi_2 x_1 y_4 + \chi_3 y_{12} y_4)\,, \\
c_{4,17} &= r_1 r_4(-4M_3 + r_4 x_6 - 4\chi_2 x_1 + 2\chi_3 x^2 x_1) \\
&- \alpha_2 M_3(r_4 x_6 - 4\chi_2 x_1 + 4r_4 x_1 + 2\chi_3 x^2 x_1)\,, \\
c_{4,18} &= r_1 r_4(2M_3 x_7 + 2\chi_3 x^2 x_1 - 2\chi_2 y_{11} + r_4 y_{13}) \\
&- \alpha_2 M_3(2\chi_3 x^2 x_1 + 2r_4 x_2 x_1 - 2\chi_2 y_{11} + r_4 y_{13})\,, \\
c_{4,19} &= r_1 r_4(M_3 + r_4 x_{11} + \chi_2 x_1 + \chi_3 y_{12}) - \\
&- \alpha_2 M_3(r_4 x_{11} + \chi_2 x_1 - r_4 x_1 + \chi_3 y_{12})\,, \\
c_{4,20} &= (d_{-1} r_4 + x_1(-5r_4 + 2\chi_3 x_1)) y_6\,,
\end{aligned}
$$

and

$$
\begin{aligned}
M_1 &= 2(1+\chi_1 x^2)+(\chi_2+x^2\chi_3)(1-x^2)\,,\\
M_2 &= 1+\chi_1 x^2+(\chi_2+x^2\chi_3)(1-x^2)\,,\\
M_3 &= 1+\chi_1 x^2\,,\\
r_1 &= 1+\alpha_1 x^2\,,\\
r_4 &= \chi_2+\chi_3 x^2\,,\\
f_1 &= x^2 d-1\,,\\
f_2 &= -(d+2)x^4+(d+3)x^2-1\,,\\
f_3 &= (d+4)(d+2)x^4-6(d+2)x^2+3\,,\\
w_1 &= 1+\chi_2+\chi_3 x^4\,,\\
w_2 &= \chi_1-\chi_2+\chi_3(1-2x^2)\,,\\
y_1 &= -3+d+8x^2\,,\\
y_2 &= r_4(1-2x^2)-\chi_2(1-x^2)\,,\\
y_3 &= 1-6x^2+4x^4\,,\\
y_4 &= r_1+\alpha_2 x_1\,,\\
y_5 &= w_1+\chi_2 x_1\,,\\
y_6 &= r_1 r_4-\alpha_2 M_3\,,\\
y_7 &= 1-3x^2+f_1 x_1\,,\\
y_8 &= -1+(1+\chi_1)x^2\,,\\
y_9 &= -1+\chi_1 x^2\,,\\
y_{10} &= 1-4x^2+2x^4\,,\\
y_{11} &= 2-3x^2+x^4\,,\\
y_{12} &= 1-3x^2+2x^4\,,\\
y_{13} &= 3-12x^2+4x^4\,,\\
d_1 &= d+1\,,\\
d_{-1} &= d-1\,,\\
x_1 &= 1-x^2\,,\\
x_2 &= 1-2x^2\,,\\
x_3 &= 1-3x^2\,,\\
x_4 &= 1-4x^2\,,\\
x_5 &= 1-5x^2\,,\\
x_6 &= 3-10x^2\,,
\end{aligned}
$$

$$
\begin{aligned}
x_7 &= -2 + x^2\,, \\
x_8 &= -2 + \chi_1 x^2\,, \\
x_9 &= -2 + (1+\chi_1)x^2\,, \\
x_{10} &= -1 + \chi_1(-1+2x^2)\,, \\
x_{11} &= -3 + 5x^2\,.
\end{aligned}
$$

APPENDIX II

In this Appendix we shall prove the necessary and sufficient conditions needed for the convergence of integrals (32). First of all, we prove the following theorem:

Theorem 1: *If the expressions $M_i, (i = 1, 2, 3)$ (see Appendix I) are nonzero at each point $x \in \langle -1, 1\rangle$ then they are positive on whole interval.*

Proof: The expressions $M_i, i = 1, 2, 3$ are continuous functions in respect to x on the interval $x \in \langle -1, 1\rangle$. On the other hand, $M_3(0) = 1 > 0$. If one supposes that there exists $y \in \langle -1, 1\rangle$ such that $M_3(y) < 0$ then according to the property of continuity there must exist a point at which the function is vanished. But it is a contradiction with the assumption of the theorem. Thus, $M_3(x) > 0$ for all $x \in \langle -1, 1\rangle$.

Because $M_2(1) = 1 + \chi_1 = M_3(1) > 0$, then using the same arguments we come to the same conclusion, namely, $M_2(x) > 0$ for all $x \in \langle -1, 1\rangle$. Finally, because $M_1 = M_2 + M_3$, then $M_1(x)$ is also positive for all $x \in \langle -1, 1\rangle$. This is what we had to prove.

Now we are able to prove the necessary and sufficient conditions of the convergence which are the contents of the following theorem.

Theorem 2: (Necessary and sufficient conditions of the convergence of integrals (32)). *Expressions $M_i, (i = 1, 2, 3)$ are nonzero for each $x \in \langle -1, 1\rangle$ if and only if the following conditions are fulfilled*:

i) $\chi_1 \in (-1, \infty)$;

ii) $\chi_2 \in (-1, \infty)$;

iii) $\chi_3 \in \Big(-\big(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\big)^2, \infty\Big)$.

Proof: First we shall prove the statement that if the expressions M_i are nonzero for each $x \in \langle -1, 1\rangle$ then the conditions i), ii), and iii) are fulfilled.

Suppose that the implication is not true, i. e., the expressions M_i are nonzero and at the same time some of the conditions i), ii), and iii) are not fulfilled. We shall show in items a), b), and c) that when the conditions for parameters χ_1, χ_2, χ_3 are not held then one comes to a conflict.

a) Let us suppose that $\chi_1 \leqslant -1$. It is enough to take $\hat{x} = \sqrt{-1/\chi_1}$, and it is evident that $0 < \sqrt{-1/\chi_1} \leqslant 1$, therefore $\hat{x} \in (0,1\rangle \subseteq \langle -1,1\rangle$. But

$$M_3(\hat{x}) = M_3\left(\sqrt{-\frac{1}{\chi_1}}\right) = 1 + \chi_1\left(\sqrt{-\frac{1}{\chi_1}}\right)^2 = 0,$$

which is in conflict with assumption of the theorem.

b) Suppose that $\chi_2 \leqslant -1$, then $M_2(0) = 1 + \chi_2 \leqslant 0$, and according to the Theorem 1 it is in conflict with the assumption of the theorem.

c) In the end, suppose that $\chi_3 \leqslant -\left(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\right)^2$. Therefore, $\chi_3 + \left(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\right)^2 \leqslant 0$. At the same time (as was already proven in items a) and b)) $\chi_1 > -1$ and $\chi_2 > -1$.

Let us take

$$0 < \hat{x} = \sqrt{\frac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1} + \sqrt{1+\chi_2}}} < 1,\ \hat{x} \in (0,1) \subseteq \langle -1,1\rangle\,.$$

Then

$$\begin{aligned} M_2(\hat{x}) &= \left(1+\chi_1\hat{x}^2\right) + \left(\chi_2 + \chi_3\hat{x}^2\right)\left(1-\hat{x}^2\right) = \\ &= 1 + \chi_1\frac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1}+\sqrt{1+\chi_2}} + \\ &+ \left(\chi_2 + \chi_3\frac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1}+\sqrt{1+\chi_2}}\right)\left(1 - \frac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1}+\sqrt{1+\chi_2}}\right), \end{aligned}$$

and after some manipulations we have

$$M_2(x) = \frac{\sqrt{1+\chi_1}\sqrt{1+\chi_2}\left[\left(\sqrt{1+\chi_1}+\sqrt{1+\chi_2}\right)^2 + \chi_3\right]}{\left(\sqrt{1+\chi_1}+\sqrt{1+\chi_2}\right)^2} \leqslant 0,$$

and according to the Theorem 1 it is again in conflict with the assumptions.

Now we have to prove the second part of the theorem, namely: If the conditions i), ii), and iii) are fulfilled then the expressions $M_i, i = 1,2,3$ are nonzero for each $x \in \langle -1,1\rangle$.

a) Suppose that the conditions i)–iii) are fulfilled and, at the same time, there exists a point $\tilde{x}$ from the interval $\langle -1,1\rangle$ such that $M_3(\tilde{x},\chi_1) = 1 + \chi_1\tilde{x}^2 = 0$. Because $M_3(0,\chi_1) = 1$ then $\tilde{x} \neq 0$, and

$$M_3(\tilde{x},\chi_1) = 1 + \chi_1\tilde{x}^2 = 0 \overset{(\tilde{x}\neq 0)}{\Rightarrow} \chi_1 = -\frac{1}{\tilde{x}^2} \Rightarrow \chi_1 \leqslant -1,$$

which is in conflict with assumption i). Thus, $M_3(x) > 0$ for all $x \in \langle -1, 1\rangle$.

b) As in the item a), suppose that conditions i)–iii) are fulfilled and at the same time suppose the existence of $\tilde{x} \in \langle -1, 1\rangle$ such that $M_2(\tilde{x}, \chi_1, \chi_2, \chi_3) = 0$. But $M_2(0, \chi_1, \chi_2, \chi_3) = 1+\chi_2 > 0$ and $M_2(\pm 1, \chi_1, \chi_2, \chi_3) = 1+\chi_1 > 0$. Thus, if $M_2(\tilde{x}, \chi_1, \chi_2, \chi_3) = 0$, then $\tilde{x} \neq 0 \ \wedge \ \tilde{x} \neq \pm 1$. As a result $\tilde{x}^2 \in (0,1)$. Then

$$\begin{aligned} M_2(\tilde{x}, \chi_1, \chi_2, \chi_3) &= 1 + \chi_1 \tilde{x}^2 + \left(\chi_2 + \chi_3 \tilde{x}^2\right)\left(1 - \tilde{x}^2\right) = \\ &= \tilde{x}^2\left(1 - \tilde{x}^2\right)\left[\frac{1+\chi_1}{1-\tilde{x}^2} + \frac{1+\chi_2}{\tilde{x}^2} + \chi_3\right] = 0\,. \end{aligned}$$

Because $\tilde{x} \neq 0 \ \wedge \ \tilde{x} \neq \pm 1$ then

$$\frac{1+\chi_1}{1-\tilde{x}^2} + \frac{1+\chi_2}{\tilde{x}^2} + \chi_3 = 0,$$

which is equivalent to

$$\chi_3 = -\frac{1+\chi_1}{1-\tilde{x}^2} - \frac{1+\chi_2}{\tilde{x}^2}.$$

Further, the maximum of the function $f\,(t) = -\dfrac{1+\chi_1}{1-t} - \dfrac{1+\chi_2}{t}$ within the interval $t \in (0,1)$ is obtained at the point $t^\star = \dfrac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1}+\sqrt{1+\chi_2}}$ and its value is

$$f(t^\star) = -\left(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\right)^2.$$

Therefore

$$\chi_3 \leqslant -\left(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\right)^2,$$

which is in conflict with assumption iii). As a result $M_2(x) > 0$ for every x from the interval $\langle -1, 1\rangle$.

c) In the end, it is evident that

$$M_1(x, \chi_1, \chi_2, \chi_3) = M_2(x, \chi_1, \chi_2, \chi_3) + M_3(x, \chi_1)$$

and according to a) and b) one obtains

$$M_1(x, \chi_1, \chi_2, \chi_3) = M_2(x, \chi_1, \chi_2, \chi_3) + M_3(x, \chi_1) > 0$$

for all $x \in \langle -1, 1\rangle$. This is what we had to prove.

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141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@pds.jinr.ru
www.jinr.ru/publish/