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EXISTENCE OF POSITIVE RADIAL SOLUTIONS FOR SOME NONVARIATIONAL SUPERLINEAR ELLIPTIC SYSTEMS

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Существование положительных радиальных решений для некоторых невариационных суперлинейных эллиптических систем

Рассматривается система

$$-\Delta u + cu = g(u,v) + u^p, \quad u = u(x),\ x \in B \subset \mathbb{R}^N,\ u\big|_{\partial B} = 0,$$

$$-\Delta v + dv = h(u,v) + v^q, \quad v = v(x),\ v\big|_{\partial B} = 0,$$

где $c, d \geqslant 0$ — постоянные, B — шар и $1 < p, q < p^\star$ с $p^\star = (N+2)/(N-2)$, если $N \geqslant 3$, и с $p^\star = +\infty$, если $N = 1, 2$. Среди прочего предполагается, что $g(0,v) = h(u,0) = g'_u(0,v) = h'_v(u,0) = 0$ и что g и h — неубывающие функции каждого из их аргументов, подчиняющиеся определенным условиям роста на бесконечности. Доказано существование радиального решения (u,v), удовлетворяющего условию $u, v > 0$ в B.

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Existence of Positive Radial Solutions for Some Nonvariational Superlinear Elliptic Systems

The system under consideration is

$$-\Delta u + cu = g(u,v) + u^p, \quad u = u(x),\ x \in B \subset \mathbb{R}^N,\ u\big|_{\partial B} = 0,$$

$$-\Delta v + dv = h(u,v) + v^q, \quad v = v(x),\ v\big|_{\partial B} = 0,$$

where $c, d \geqslant 0$ are constants, B is a ball and $1 < p, q < p^\star$ with $p^\star = (N+2)/(N-2)$ if $N \geqslant 3$, and $p^\star = +\infty$ if $N = 1, 2$. Among others, it is assumed that $g(0,v) = h(u,0) = g'_u(0,v) = h'_v(u,0) = 0$ and that g and h are nondecreasing functions in each of their arguments obeying certain growth conditions at infinity. We prove the existence of a radial solution (u,v) satisfying $u, v > 0$ in B.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

We consider the problem

$$
\begin{gathered}
-\Delta u + cu = u^{a_1} v^{b_1} + u^p, \quad u = u(x), \\
-\Delta v + dv = u^{a_2} v^{b_2} + v^q, \quad v = v(x),\ x \in B \subset \mathbb{R}^N, \\
u\big|_{\partial B} = v\big|_{\partial B} = 0
\end{gathered}
$$

and its generalizations. Here N is a positive integer, $B = \{x = (x_1, ..., x_N) \in \mathbb{R}^N :\ |x| < 1\}$ is a ball, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + ... + \dfrac{\partial^2}{\partial x_N^2}$ is the Laplace operator, $c, d \geqslant 0$ are constants, $a_i + b_i < \overline{p} = \min\{p; q\}$, $a_1, b_2 > 1,\ a_2, b_1 \geqslant 0$ and $1 < p, q < \dfrac{N+2}{N-2}$ for $N \geqslant 3$ (to be interpreted as $1 < p, q < \infty$ if $N = 1, 2$). For this problem, we prove the existence of a solution (u, v) radial in B (that is, depending only on $r = |x|$) and satisfying $u(r),\ v(r) > 0,\ u'(r) \leqslant 0$ and $v'(r) \leqslant 0$ for $r \in (0, 1)$.

Studies of the scalar superlinear second-order elliptic equations have a long history and the basic results in this direction are well known; we mention the result by P. H. Rabinowitz [5] according to which a scalar equation similar to the system above has a positive solution and, if the nonlinearity is odd, an infinite sequence of pairwise different solutions. We also indicate monograph [7] where some results on the existence of solutions for scalar equations in the entire space $\mathbb{R}^N$ are reviewed. For systems of similar equations, to our knowledge, in the literature almost all results on the existence are established for variational problems, i. e., for systems for which there exist corresponding functionals whose critical points are solutions of these systems. For nonvariational problems, behavior of solutions (without proving the existence) is studied in a number of publications. For an information on this subject, we refer readers to the recent paper [3] and the references therein. Concerning the existence of solutions for nonvariational problems, we mention only article [8] where some interesting results in this direction are presented in a more general case than our one when a domain is not necessarily a ball and the solutions are not necessarily radial, mainly for nonlinearities of the type $au^p + bv^q$ and also $u^p - u^a v^b + v^q$. In both cases positive a, b, p and q satisfy additional restrictions (for example, for $N = 3$ it is assumed that $q > 1,\ 1 < p < 5$ in the first equation and $p > 1,\ 1 < q < 5$ in the second one).

In the present paper, we consider another system which naturally arises as a system of two scalar superlinear equations coupled by a perturbation function which is not necessarily small or bounded. As for applications, systems of this type have a lot of them, in particular, in the heat and diffusion theory, physical and chemical kinetics, etc. Because the problems of this class seem to be difficult, we deal with one of the simplest ones. It can be considered as a model problem. When the article was already prepared, its author learned about the result by W. C. Troy [6] according to which, if (u, v) is an arbitrary solution of our system and $u, v > 0$ in B, then u and v are radial functions nonincreasing in r. We establish independent proofs not based on this statement.

Finishing our introduction, we illustrate some difficulties of the analysis of systems one of which we study by the following very simple example. Consider the system

$$-\Delta u = f_1(u,v)u, \quad u = u(x),\ x \in \Omega,\ u\big|_{\partial\Omega} = 0,$$
$$-\Delta v = f_2(u,v)v, \quad v = v(x),\ v\big|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary. If one assumes that $f_1(u,v) \neq f_2(u,v)$ for all $u, v > 0$ (so that either $f_1(u,v) > f_2(u,v)$ or $f_1(u,v) < f_2(u,v)$, for all $u, v > 0$), then a simple comparison theorem applied to the first and second equations in this system shows that it has no solution (u, v) that satisfies $u, v > 0$ in Ω (multiply the first equation by v, the second one by u, subtract the results from each other and integrate the obtained relation over Ω; then, one gets a false equality). For example, if $f_1(u,v) = 2u^2$ and $f_2(u,v) = u^2$, then the system above has no component-wise positive solution.

2. PRECISE STATEMENT OF THE PROBLEM. RESULT

In fact, we consider the problem

$$-\Delta u + cu = g(u,v) + |u|^{p-1}u,\ u = u(x), \tag{1}$$
$$-\Delta v + dv = h(u,v) + |v|^{q-1}v,\ v = v(x),\ x \in B \subset \mathbb{R}^N, \tag{2}$$
$$u\big|_{\partial B} = v\big|_{\partial B} = 0, \tag{3}$$

where we changed the power terms in the right-hand sides of the equations by expressions equal to them for $u, v \geqslant 0$ and defined for all $u, v \in \mathbb{R}$. Here $N \geqslant 1$ is integer and $B = \{x \in \mathbb{R}^N :\ |x| < 1\}$. Hereafter all quantities we deal with are real. We consider classical $C^2(B) \cap C(\overline{B})$ solutions of (1)–(3). Our assumptions are the following:

(h1) c, d *are nonnegative constants in* B*;*

(h2) $1 < p, q < \dfrac{N+2}{N-2}$ *for* $N \geqslant 3$ *and* $1 < p, q < \infty$ *if* $N = 1, 2$*;*

(h3) the functions g, h *are locally Hölger continuous in* $[0, \infty) \times [0, \infty)$*;*

(h4) there exists $\kappa = \kappa(u, v) > 0$ *such that* $|g(u, v)| \leqslant \kappa(u, v)(|u|^{\overline{p}} + |v|^{\overline{p}})$ *and* $|h(u, v)| \leqslant \kappa(u, v)(|u|^{\overline{p}} + |v|^{\overline{p}})$ *for all* $u, v \geqslant 1$*, where* $\overline{p} = \min\{p; q\}$ *and* $\kappa(u, v) \to 0$ *as* $|(u, v)| \to \infty$*;*

(h5) the functions g *and* h *are nondecreasing in* $[0, \infty) \times [0, \infty)$ *in each of two arguments;*

(h6) for any $A > 0$ *there exists a function* $\gamma_A(s) \to +0$ *as* $s \to +0$ *such that* $g(u, v) \leqslant \gamma_A(u)u$ *for all* $v \in [0, A]$ *and all* $u \in (0, 1]$*, and* $h(u, v) \leqslant \gamma_A(v)v$ *for all* $u \in [0, A]$ *and for all* $v \in (0, 1]$.

In the present paper, our main result is the following.

Theorem. *Under assumptions (h1)–(h6) there exists a radial solution* $(u(r), v(r))$ *of problem (1)–(3) that satisfies* $u(r) > 0,\ v(r) > 0,\ u'(r) \leqslant 0$ *and* $v'(r) \leqslant 0$ *for all* $r \in [0, 1)$.

For brevity, we call solutions as in this theorem *positive radial nonincreasing solutions*.

This result admits a natural generalization for systems of n equations, $n \geqslant 3$. Our method of its proving is mainly based on two ideas. First, we obtain a priori estimates in $C(B)$ for positive radial nonincreasing solutions of (1)–(3). For this aim, we apply (and partly modernize) the approach by D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum [2]. In particular, we establish a derivation of the Pohozaev identity [4] for our system proceeding in the way well known in physics. Then, to prove the existence of a solution, we apply an abstract result presented in [1] and based on the concept of the index of a compact operator in a Banach space.

Everywhere in the following it is accepted that assumptions (h1)–(h6) are valid. We also continue the functions $g(s, t)$ and $h(t, s)$ for negative values of their arguments being odd in s and even in t.

3. A PRIORI ESTIMATES OF SOLUTIONS

In this and the next sections, we assume in addition to hypotheses (h1)–(h6) that the functions g and h are continuously differentiable. In the class of radial solutions, problem (1)–(3) reduces to the following:

$$-u'' - \frac{N-1}{r}u' + cu = g(u,v) + |u|^{p-1}u, \quad u = u(r), \tag{4}$$

$$-v'' - \frac{N-1}{r}v' + dv = h(u,v) + |v|^{q-1}v, \quad v = v(r),\ r \in (0,1), \tag{5}$$

$$u'(0) = v'(0) = u(1) = v(1) = 0, \tag{6}$$

where the prime denotes the derivative in r. Denote by $L_s(B)$, $s \geqslant 1$, the standard Lebesgue space with the norm $|w|_s = \left\{\int\limits_B |w(x)|^s dx\right\}^{1/s}$ and by $H_0^1(B)$ the usual Sobolev space of functions in B equal to zero, a. e. on the boundary ∂B, equipped with the norm $||w|| = \left\{\int\limits_B |\nabla w(x)|^2 dx\right\}^{1/2}$; for radial $u \in L_s(B)$ and $v \in H_0^1(B)$ one has, respectively: $|u|_s^s = \int\limits_0^1 r^{N-1}|u(r)|^s dr$ and $||v||^2 = \int\limits_0^1 r^{N-1}|v'(r)|^2 dr$, where we omit a positive coefficient C_N depending only on N. Denote by H_r^1 the subspace of $H_0^1(B)$ consisting of radial functions. We identify this space with the space of functions $u(r)$ of $r \in (0,1]$ equal to 0 at $r = 1$ with the same norm. Introduce two quantities

$$I(u,v) = \int\limits_0^1 r^{N-1}\left\{\frac{1}{2}u'^2(r) + \frac{c}{2}u^2(r) - G(u,v) - \frac{1}{p+1}u^{p+1}\right\} dr$$

and

$$J(u,v) = \int\limits_0^1 r^{N-1}\left\{\frac{1}{2}v'^2(r) + \frac{d}{2}v^2(r) - H(u,v) - \frac{1}{q+1}v^{q+1}\right\} dr,$$

where $G(u,v) = \int\limits_0^u g(s,v)ds$ and $H(u,v) = \int\limits_0^v h(u,s)ds$. As one can easily verify (and as is well known, see, for example, [7]), for any radial solution $(u_0, v_0) \in C^2(B) \cap C(\overline{B})$ of problem (1)–(3) the functional $I_1(u) = I(u, v_0(r))$ taken with the fixed $v_0(r)$ is continuously differentiable in $u \in H_r^1$ and $u_0(r)$ is its critical point in this space; by analogy, the functional $J_1(v) = J(u_0(r), v)$ taken with the fixed $u_0(r)$ is continuously differentiable in $v \in H_r^1$ and v_0 is its critical point in this space.

Now, let us take two functions $\alpha, \beta \in C^3([0,1])$ satisfying $\alpha^{(l)}(0) = \beta^{(l)}(0) = 0$ for $l = 1,2,3$ and $\alpha^{(l)}(1) = \beta^{(l)}(1) = 0$ for $l = 0,1,2,3$ and consider the following problem:

$$-(r^{N-1}(u' - \alpha'))' + cr^{N-1}(u - \alpha) = r^{N-1}[g(u,v) + u^p], \tag{7}$$

$$-(r^{N-1}(v' - \beta'))' + dr^{N-1}(v - \beta) = r^{N-1}[h(u,v) + v^q], \tag{8}$$

$$u'(0) = v'(0) = u(1) = v(1) = 0. \tag{9}$$

As above, the corresponding functionals whose critical points in H_r^1 are solutions of (7)–(9) are the following:

$$I_\alpha(u,v) = \int_0^1 r^{N-1} \left\{ \frac{1}{2}(u' - \alpha')^2 + \frac{c}{2}(u - \alpha)^2 - G(u,v) - \frac{1}{p+1} u^{p+1} \right\} dr$$

and

$$J_\beta(u,v) = \int_0^1 r^{N-1} \left\{ \frac{1}{2}(v' - \beta')^2 + \frac{c}{2}(v - \beta)^2 - H(u,v) - \frac{1}{q+1} v^{q+1} \right\} dr.$$

The statement below is a variant and an extension of several results. The first one was obtained by S. I. Pohozaev [4]. Here, we apply another method to derive it.

Lemma 1. *Let $v \in C^2([0,1])$ and α (resp., $u \in C^2([0,1])$ and β) be fixed, let $u \in C^2([0,1])$ (resp., $v \in C^2([0,1])$) be a critical point of $I_2(u) = I_\alpha(u,v)$ (resp., of $J_2(v) = J_\beta(u,v)$) in H_r^1 and $u'(r) \leqslant 0$ in $[0,1]$ (resp., $v'(r) \leqslant 0$ in $[0,1]$). Then, the following relations hold:*

$$\int_0^1 \left\{ r^{N-1} u'^2 + cr^{N-1}u^2 - cr^{N-1}\alpha u + u(r^{N-1}\alpha')' \right\} dr =$$

$$= \int_0^1 r^{N-1} \left\{ ug(u,v) + u^{p+1} \right\} dr \tag{10}$$

(respectively,

$$\int_0^1 \left\{ r^{N-1} v'^2 + dr^{N-1}v^2 - dr^{N-1}\beta v + v(r^{N-1}\beta')' \right\} dr =$$

$$= \int_0^1 r^{N-1} \left\{ vh(u,v) + v^{q+1} \right\} dr)$$

and

$$\int_0^1 \left\{ \frac{2-N}{2} r^{N-1} u'^2 + (r^{N-1}\alpha')' u - (r^N \alpha')'' u - \frac{cN}{2} r^{N-1} u^2 + c(r^N\alpha)' u + \right.$$

$$\left. + \frac{N}{p+1} r^{N-1} u^{p+1} - r^N g(u,v) u' \right\} dr = \frac{1}{2}[u'(1)]^2 \tag{11}$$

(respectively,

$$\int_0^1 \left\{ \frac{2-N}{2} r^{N-1} v'^2 + (r^{N-1}\beta')' v - (r^N \beta')'' v - \frac{dN}{2} r^{N-1} v^2 + d(r^N\beta)' v + \right.$$

$$\left. + \frac{N}{q+1} r^{N-1} v^{q+1} - r^N h(u,v) v' \right\} dr = \frac{1}{2}[v'(1)]^2).$$

Proof. We derive only the first equalities (10) and (11) because the second ones can be obtained by analogy. To obtain (10), multiply equation (7) by u and integrate the result from 0 to 1. To derive (11), continue the function $u(r)$ by 0 for $r > 1$ and consider a parameter $a \in [1,2)$ and the function $w(a,r) = u(ar)$. We also introduce $C^2([0,\infty))$ approximations u_ϵ, where $\epsilon > 0$ is sufficiently small, of u equal to $u(r)$ for $r \in [0, 1-\epsilon) \cup [1,\infty)$, positive, nonincreasing, satisfying $|u'_\epsilon(r)| \leqslant 2 \max\limits_{r\in[0,1]} |u'(r)|$ for all r and such that u''_ϵ changes sign in $(1-\epsilon, 1)$ at most three times. In addition, we set $w_\epsilon(a,r) = u_\epsilon(ar)$, $r \geqslant 0$. Then, the mappings $w_\epsilon(a,\cdot): \ [1,2) \to H^1_r$ are continuously differentiable. We have: $\left.\dfrac{\partial I_\alpha(w_\epsilon(a,\cdot),v)}{\partial a}\right|_{a=1+0} = I'_{\alpha,u}(u_\epsilon, v)(\gamma)$, where $\gamma = r u'_\epsilon$. Consider the following double limit:

$$\lim_{\epsilon\to+0} \left\{ \lim_{\epsilon\to+0} I'_{\alpha,u}(u_\epsilon, v)(\gamma) \right\} \Big|_{\gamma = r u'_\epsilon} := L,$$

where we mean that the interior limit is taken when γ is fixed. Clearly, since $I'_{\alpha,u}(u,v) = 0$, one has:

$$L = 0 = \lim_{\epsilon\to+0} I'_{\alpha,u}(u,v)(r u'_\epsilon) = \lim_{\epsilon\to+0} \int_0^1 \left\{ r^{N-1}[u'(r) u'_\epsilon(r) + r u'(r) u''_\epsilon(r)] + \right.$$

$$\left. + (r^{N-1}\alpha')' u_\epsilon - (r^N\alpha')'' u_\epsilon + c r^N u u'_\epsilon + c(r^N\alpha)' u_\epsilon - r^N u^p u'_\epsilon - r^N g(u,v) u'_\epsilon \right\} dr.$$

It is easy to see that all the terms in the integral except, maybe, the second one, go to the quantities obtained from these terms by substitution of u in place of u_ϵ. As for the second term $\int\limits_0^1 r^N u'(r)u''_\epsilon(r)dr = \int\limits_0^\infty r^N u'(r)u''_\epsilon(r)dr$, we have $u''_\epsilon(r) \sim u''(r) - u'(1)\delta(r-1)$, where δ denotes the standard delta-function, at least in the sense of distributions. In addition, u''_ϵ can be nonequal to 0 only in $[0,1)$. Thus, taking the limit $\epsilon \to +0$ in the second term, we obtain (11).□

Remark 1. If system (1)–(3) is considered in the entire space $\mathbb{R}^N$, then our derivation of relations analogous to (11) becomes simpler. In this case, it suffices to calculate the quantity $\left.\dfrac{\partial I_\alpha(w(a,\cdot),v)}{\partial a}\right|_{a=1+0}$ which is also equal to 0.

Now, let us take two functions $\alpha_0, \beta_0 \in C^3([0,1])$ positive and nonincreasing in $[0,1)$ and satisfying $\alpha_0^{(l)}(0) = \beta_0^{(l)}(0) = 0$ for $l = 1,2,3$ and $\alpha_0^{(l)}(1) = \beta_0^{(l)}(1) = 0$ for $l = 0,1,2,3$, and keep them fixed throughout the article. For each $\lambda, \mu \geqslant 0$, denote by $K_{\lambda,\mu}$ the set of such positive nonincreasing solutions (u,v) of (7)–(9) taken with $\alpha = \lambda\alpha_0$ and $\beta = \mu\beta_0$ that $\lambda\alpha_0(r) \leqslant u(r)$ and $\mu\beta_0(r) \leqslant v(r)$ for all $r \in [0,1]$. Set $K = \bigcup\limits_{\lambda,\mu\geqslant 0} K_{\lambda,\mu}$.

Lemma 2. *There exists $D_0 > 0$ such that for any functions g and h continuously differentiable and satisfying hypotheses (h1)–(h6) and for any $(u,v) \in K$ one has $|u|_{p+1} \leqslant D_0$ and $|v|_{q+1} \leqslant D_0$.*

Proof. We establish the proof only for $N \geqslant 3$ and estimate only $|u|_{p+1}$ because in all other cases the proof can be made by analogy. In view of the definition of λ, one has

$$0 \leqslant \lambda \int\limits_0^1 r^{N-1}\alpha_0(r)dr \leqslant \int\limits_0^1 r^{N-1}u(r)dr,$$

hence

$$0 \leqslant \lambda \leqslant C|u|_2 \tag{12}$$

for a positive constant C because the function α_0 is fixed and by analogy for μ.

Now, let us prove that there exists $C > 0$ such that $u'^2(1) + v'^2(1) \leqslant C$ for any $(u,v) \in K$. On the contrary, suppose that there exists a sequence $\{(u_n, v_n)\}$ of this class such that ${u'_n}^2(1) + {v'_n}^2(1) \to +\infty$ as $n \to \infty$. Then, one of the two following possibilities can occur: 1) the sequence (u_n, v_n) is bounded uniformly in $[0,1]$ and 2) this sequence is unbounded.

Let us consider case 1. It follows from (7), (8) and (12) that the sequences $\{u''_n\}$ and $\{v''_n\}$ are bounded from below in $[0,1]$. Hence,

$$(u'_n(0), v'_n(0)) = (u'_n(1), v'_n(1)) - \int\limits_0^1 (u''_n(r), v''_n(r))dr \neq 0$$

for all sufficiently large n which is a contradiction.

Then we consider case 2. The sequence (u_n, v_n) is not also uniformly bounded in $[1/2,1]$, because otherwise the sequences $\{u''_n\}$ and $\{v''_n\}$ are bounded from below in the same interval as in case 1 and therefore, one of the sequences $\{u'_n\}$ and $\{v'_n\}$ contains a subsequence that goes to $-\infty$ uniformly in $r \in [1/2, 1]$. But then, the sequence (u_n, v_n) is not bounded uniformly in $r \in [1/2,1]$ which is a contradiction. So, at least one of two sequences $\{u_n\}$ and $\{v_n\}$ contains a subsequence that goes to $+\infty$ as $n \to \infty$ uniformly with respect to $r \in [1/4, 1/2]$, because we consider monotone solutions. In addition, there exists $b > 0$ such that $\lambda \leqslant bu_n(r)$ and $\mu \leqslant bv_n(r)$ for any n and $r \in [1/4, 1/2]$. But then, in view of equations (7) and (8) and since $p, q > 1$, by the standard comparison theorem, each element of the indicated subsequence, at least beginning from the number $N_0 > 0$, achieves the maximum in $[1/4, 1/2]$ and strictly increases in a left half-neighborhood of this point of maximum. But this contradicts the fact that we consider monotone solutions. So, our claim is proved.

Now, multiply (10) by $(2-N)/2$ and subtract the result from (11). Then, in view of (12) and since α_0 and β_0 are fixed, we obtain for any $\epsilon > 0$ after simple transformations: $C_1(\epsilon)+|u|_2^2+|v|_2^2+\epsilon(|u|_{\overline{p}+1}^{\overline{p}+1}+|v|_{\overline{p}+1}^{\overline{p}+1}) \geqslant |u|_{p+1}^{p+1}+|v|_{q+1}^{q+1}$, where $C_1(\epsilon) > 0$ does not depend on $(u,v) \in K$. But by the Hölder inequality $|u|_2^2 \leqslant C_2|u|_{p+1}^2$ and $|v|_2^2 \leqslant C_2|v|_{q+1}^2$ and thus, $|u|_{p+1} \leqslant \text{Constant}.\Box$

Remark 2. Note that the constant D_0 does not depend on the choice of the functions g and h obeying hypotheses (h1)–(h6) with the same constants c, d, p, q and C.

Now, we establish the main result of this section.

Proposition 1. *Let $\{g_\lambda\}$ and $\{h_\lambda\}$, where $\lambda \in \Lambda$, be arbitrary families of continuously differentiable functions (Λ is an arbitrary set) that satisfy hypotheses (h1)–(h6) with the same constants c, d, p, q and C for all λ. Then, there exists $D > 0$ such that for any $(u_\lambda, v_\lambda) \in K_\lambda$, where $K = K_\lambda$ is the above-defined set of solutions of problem (7)–(9) corresponding to $(g, h) = (g_\lambda, h_\lambda)$, one has $u_\lambda(0) = \max\limits_{r\in[0,1]} u_\lambda(r) \leqslant D$ and $v_\lambda(0) = \max\limits_{r\in[0,1]} v_\lambda(r) \leqslant D$.*

Proof. By Lemma 2 one has $|u_\lambda|_{p+1} \leqslant D_0$ and $|v_\lambda|_{q+1} \leqslant D_0$ for any $(u_\lambda, v_\lambda) \in K_\lambda$ and for a constant $D_0 > 0$ independent of λ and of $(u_\lambda, v_\lambda) \in K_\lambda$.

By these estimates and hypothesis (h4)

$$\int\limits_B u_\lambda(|x|)g_\lambda(u_\lambda(|x|),v_\lambda(|x|))dx \le C_1 \text{ and } \int\limits_B v_\lambda(|x|)h_\lambda(u_\lambda(|x|),v_\lambda(|x|))dx \leqslant C_1$$

for a constant $C_1 > 0$ independent of λ and of $(u_\lambda, v_\lambda) \in K_\lambda$. Hence, according to Lemma 2 and (10)

$$\|u_\lambda\| \leqslant C_2 \quad \text{and} \quad \|v_\lambda\| \leqslant C_2$$

for a constant $C_2 > 0$ independent of $\lambda \in \Lambda$ and of $(u_\lambda, v_\lambda) \in K_\lambda$. Now, it can be proved completely as in [7], section 2.2, by using well-known arguments that $|u_\lambda(0)| \leqslant D$ and $|v_\lambda(0)| \leqslant D$ for a constant $D > 0$ independent of λ and of $(u_\lambda, v_\lambda) \in K_\lambda$.$\square$

4. PROOF OF THE THEOREM. THE CASE OF SMOOTH g AND h

In this section, we assume in addition to hypotheses (h1)–h6) that g and h are C^1-functions. Then, proposition 1 holds. In the following, we apply results presented by H. Amann in [1], Sections 11 and 12. Let $C_0([0,1])$ be the space of functions $s(r)$ continuous in $[0,1]$ and satisfying $s(1) = 0$, equipped with the uniform norm. Denote by X the set of functions $s(|x|)$, where $s \in C_0([0,1])$, and let R be the subset of X consisting of functions $s(|x|)$ such that the corresponding functions $s(r)$ are nonnegative and nonincreasing in $(0,1)$. Then, since R is a closed convex set in X, according to [1] R is a retract in X which means by definition that there exists a continuous function (retraction) $\theta:\ X \to R$ satisfying $\theta\big|_R = Id$, where Id denotes the identity. In addition, it is easily seen that $R \times R$ is a retract in $X \times X$ with one of the retractions $\theta \times \theta$, where θ is one of the retractions in X.

For $s, t \in X$, consider the operators $S_u(s) := (-\Delta + c)^{-1}s$, $S_v(t) := (-\Delta + d)^{-1}t$, $S = S_u \times S_v$, $(u,v) = T(s,t) := S((g(s,t) + s^p), (h(s,t) + t^q))$ and $(u_\lambda, v_\lambda) = T_\lambda(s,t) := S((\lambda g(s,t) + s^p), (\lambda h(s,t) + t^q))$, where $\lambda \in [0,1]$ is a parameter. Clearly, $T_0 = T_u \times T_v$ with $T_u(s) = (-\Delta + c)^{-1}(s^p)$ and $T_v(t) = (-\Delta + d)^{-1}(t^q)$. We denote also $B_\rho = \{u \in R:\ \|u\|_{C(B)} < \rho\}$, where $\rho > 0$. It is known that for any $s > 1$ the linear operators S_u and S_v are bounded from $C(\overline{B})$ into the Sobolev space $W_s^2(B)$ which is compactly embedded into $C^1(\overline{B})$ for all sufficiently large s. Therefore, if $T_\lambda(u,v) = (u,v)$ for some $u, v \in X$, then (u,v) is a C^2 radial solution of the system obtained from (1)–(3) by substitution of functions λg and λh in place of g and h, respectively. Also, all the operators we introduced a moment ago transform X into X and $X \times X$ into $X \times X$, respectively, because it is clear that the functions $S_u(s)$ and $S_v(t)$

are radial. Note also that the functions λg and λh, $\lambda \in [0,1]$, satisfy assumptions (h1)–(h6) with the same constants c, d, p, q and C.

Now, we consider three sets $A, B \subset R$ and $F \subset R \times R$ bounded, respectively, in X and in $X \times X$ and open in the induced topologies of R and $R \times R$, respectively. According to [1], Sections 11 and 12, if $T_u(u) \neq u$ for any $u \in \partial A$, $v \neq T_v(v)$ for any $v \in \partial B$ and $T_\lambda(u,v) \neq (u,v)$ for any $(u,v) \in \partial F$, then the indexes of the operators T_u, T_v and T_λ are determined: $i(T_u, A, R)$, $i(T_v, B, R)$ and $i(T_\lambda, F, R\times R)$, respectively. In fact, indexes take integer values and by definition, for example, $i(T_u, A, R) = \deg(Id - T_u \circ \theta, \theta^{-1}(A), X)$, where θ is an arbitrary retraction in X corresponding to the retract R (the index does not depend on the retraction θ). In addition, $i(T_0, A\times B, R\times R) = i(T_u, A, R)\cdot i(T_v, B, R)$ provided $T_u(u) \neq u$ and $T_v(v) \neq v$ for any $u \in \partial A$ and $v \in \partial B$ and if $T_\lambda(u,v) \neq (u,v)$ for all $\lambda \in [0,1]$ and for all $(u,v) \in \partial F$, then $i(T, F, R \times R) = i(T_0, F, R \times R)$.

Proposition 2. *For any $\lambda \in [0,1]$, the operator T_λ transforms $R \times R$ into $R \times R$.*

Proof. We prove this statement for $\lambda = 1$ to make the notation simpler. Let $s, t \in R$ and $(u,v) = T(s,t)$. As is noted earlier, $u, v \in C^1(\overline{B})$ and, in addition, $(u(r), v(r))$ is a solution (maybe, a weak solution) of the problem

$$\begin{gathered} -(r^{N-1}u')' + cr^{N-1}u = r^{N-1}[g(s(r),t(r)) + s^p(r)], \\ -(r^{N-1}v')' + dr^{N-1}v = r^{N-1}[h(s(r),t(r)) + t^q(r)], \\ u'(0) = v'(0) = u(1) = v(1) = 0. \end{gathered} \tag{13}$$

It is well known that, in fact, $u, v \in C^2([0,1])$ (see, for example, [7], proof of theorem II.1.1). Therefore, by the maximum principle, $u(r) \geqslant 0$ and $v(r) \geqslant 0$ for all $r \in [0,1]$. We have also to prove that $u'(r), v'(r) \leqslant 0$ in $[0,1)$. On the contrary, suppose that, for example, $u'(r_1) > 0$ at some $r_1 \in (0,1)$. Denote by $[r_0, r_1] \subset [0, r_1]$ the maximal left half-interval such that $u'(r) \geqslant 0$ for all $r \in [r_0, r_1]$. Then, $u''(r_0) \geqslant 0$ and hence, in view of the differential equation $cu(r_0) \geq g(s(r_0), t(r_0)) + s^p(r_0)$. Therefore, it is clear that $cu(r_1) > g(s(r_1), t(r_1)) + s^p(r_1)$ and so, it is easy to see from the equation that $u'(r) \geqslant 0$ everywhere in $[r_1, 1]$, which contradicts the boundary condition $u(1) = 0$. So, our proposition is proved.□

Remark 3. In view of propositions 1 and 2 and the arguments at the beginning of this section, $a(\alpha_0, \beta_0) + T_\lambda(u,v) \neq (u,v)$ for any $\lambda \in [0,1]$, $a \geqslant 0$ and $(u,v) \in \partial(B_{2D} \times B_{2D})$ and $T_u(u) \neq u$ and $T_v(v) \neq v$ for any $u, v \in \partial B_{2D}$, where the boundaries $\partial(B_{2D} \times B_{2D})$ and ∂B_{2D} are taken in the topological spaces $R \times R$ and R with the induced topologies, respectively.

Proposition 3. *For any* $\lambda \in [0,1]$ *one has* $i(T_\lambda, B_\rho \times B_\rho, R \times R) = i(T_u, B_\rho, R) = i(T_v, B_\rho, R) = 1$ *if* $\rho > 0$ *is sufficiently small and* $i(T_\lambda, B_{2D} \times B_{2D}, R \times R) = i(T_u, B_{2D}, R) = i(T_v, B_{2D}, R) = 0$.

Proof. This result is, in fact, a variant of Lemma 12.1 in [1]. In view of remark 3, the second claim immediately follows from this result. Further, since according to hypothesis (h6) $[\lambda g(s,t) + s^p] = o(s)$ as $s \to +0$ uniformly in $t \in [0, 2D]$ and $[\lambda h(s,t) + t^q] = o(t)$ as $t \to +0$ uniformly in $s \in [0, 2D]$ and in $\lambda \in [0,1]$, for any $\lambda \in [0,1]$, sufficiently small $\rho > 0$ and $s, t \in B_\rho$ one has $\|u_\lambda\|_{C(B)} < \rho/3$ and $\|v_\lambda\|_{C(B)} < \rho/3$, where $(u_\lambda, v_\lambda) = T_\lambda(s,t)$. Hence, by Lemma 12.1 in [1], $i(T, B_\rho \times B_\rho, R \times R) = i(T_\lambda, B_\rho \times B_\rho, R \times R) = i(T_u, B_\rho, R) \cdot i(T_v, B_\rho, R) = 1$ for the same ρ and λ.□

In $R \times R$, consider the set

$$A = ((((B_{2D} \times B_{2D}) \setminus \overline{[(B_{2D} \setminus B_\rho) \times B_\rho]}) \setminus \overline{[B_\rho \times (B_{2D} \setminus B_\rho)]}) \setminus \overline{(B_\rho \times B_\rho)}),$$

where $\rho > 0$ is sufficiently small. By proposition 3, the arguments above and the results in [1], Section 11, one has: $i(T, (B_{2D} \setminus B_\rho) \times B_\rho, R \times R) = i(T_0, (B_{2D} \setminus B_\rho) \times B_\rho, R \times R) = i(T_u, B_{2D} \setminus B_\rho, R) \cdot i(T_v, B_\rho, R) = (0-1) \cdot (+1) = -1$. By analogy, $i(T, B_\rho \times (B_{2D} \setminus B_\rho), R \times R) = -1$. So, we have: $i(T, A, R \times R) = 0 - (-1) - (-1) - (+1) = 1$. Therefore, the operator T has a fixed point $(u,v) \in A$. By construction $u \not\equiv 0$ and $v \not\equiv 0$, $u(r)$ and $v(r)$ are nonnegative and nonincreasing functions in $(0,1)$ and $(u(|x|), v(|x|))$ is a $C^2(B) \cap C(\overline{B})$ solution of problem (1)–(3). By standard arguments $u(r) > 0$ and $v(r) > 0$ in $[0,1)$ (because otherwise $u(r_0) = u'(r_0) = 0$ or $v(r_0) = v'(r_0) = 0$ at some point $r_0 \in (0,1)$, but this relations imply $u(r) \equiv 0$ (resp., $v(r) \equiv 0$) in $(0,1)$ by the uniqueness theorem). Our theorem is proved in the case when g and h are continuously differentiable functions.□

5. PROOF OF THE THEOREM. THE CASE OF NON-SMOOTH g AND h

Take two sequences $\{g_n\}$ and $\{h_n\}$ of smooth functions g_n and h_n converging to g and h, respectively, uniformly in $[0,\infty) \times [0,\infty)$ and satisfying hypotheses (h1)–(h6) uniformly in n (that is, with the same constants c, d, p, q and C and the same functions γ_A for all n). Note that the constant $D > 0$ introduced in proposition 1 can be chosen the same for all g_n and h_n. For each n, by (u_n, v_n) we denote an arbitrary positive radial nonincreasing solution of (1)–(3) taken with $g = g_n$ and $h = h_n$. Then, as earlier, the sequence $\{(u_n, v_n)\}$ contains a subsequence still denoted $\{(u_n, v_n)\}$ that converges in $C^1(\overline{B}) \times C^1(\overline{B})$. Denote by (u,v) its limit. Let us prove that $u \not\equiv 0$ and $v \not\equiv 0$. On the contrary, suppose that, for example, $u \equiv 0$ in $[0,1]$. Then, $u_n \to 0$ in $C^1(\overline{B})$. But from hypothesis (h6) and (10) by the Sobolev embedding we have

$$C_1|u_n|^2_{p+1} \leqslant C_2\gamma(u_n(0))|u_n|^2_{p+1} + |u_n|^{p+1}_{p+1},$$

with constants $C_1, C_2 > 0$ independent of n. This relation easily implies $|u_n|_{p+1} \geqslant C_3 > 0$ with a constant C_3 independent of sufficiently large n, and we get a contradiction. So, it is proved that $u \not\equiv 0$ and $v \not\equiv 0$. In addition, obviously u and v are radial functions, $u(r), v(r) \geqslant 0$ and $u'(r), v'(r) \leqslant 0$ in $(0,1)$ and $(u(|x|), v(|x|))$ is a fixed point of T in $R \times R$, hence, a $C^2(B) \cap C(\overline{B})$ solution of system (1)–(3) as earlier. The fact that $u(|x|), v(|x|) > 0$ in B is well known and can be proved as in the previous section. So, our theorem is completely proved.□

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