E2-2007-112

Kh. M. Beshtoev

EXAMINATION OF UNITARITY CONDITION (POSITIVE DEFINITENESS OF EXPRESSION FOR TRANSITION PROBABILITIES) AT THREE NEUTRINO OSCILLATIONS IN VACUUM

Бештоев Х. М. E2-2007-112

Проверка условия унитарности (положительной определенности выражения для вероятности переходов) при трехнейтринных осцилляциях в вакууме

Показано, что при строгом выполнении условия $\Delta m_{13}^2 = \Delta m_{12}^2 + \Delta m_{23}^2$ выражение для вероятности $P_{\nu_e \to \nu_e}(t)$ при $\nu_e \to \nu_e$ переходе является положительно определенной величиной при любых значениях θ и β, хотя при любых сколь-нибудь малых отклонениях от этого условия она становится отрицательной величиной. Тогда, для того чтобы сделать это выражение для вероятности переходов положительно определенной величиной, необходимо наложить ограничение на угол β при установленном значении для угла $\theta = 32{,}45°$ (т. е. значение для угла смешивания β должно быть $\beta \leqslant 15 \div 17°$).

Работа выполнена в Лаборатории физики частиц ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна, 2007

Beshtoev Kh. M. E2-2007-112

Examination of Unitarity Condition (Positive Definiteness of Expression for Transition Probabilities) at Three Neutrino Oscillations in Vacuum

This work has shown that at strict fulfilment of condition $\Delta m_{13}^2 = \Delta m_{12}^2 + \Delta m_{23}^2$ the expression for probability of $\nu_e \to \nu_e$ transitions $P_{\nu_e \to \nu_e}(t)$ is positively defined at every value of θ and β, while at any arbitrarily small deviation from this condition it becomes negative. In order to make this expression positively defined for probability transitions, it is necessary to put a limitation on angle mixing β at fixed value of $\theta = 32.45°$ (i. e., the value for β must be $\beta \leqslant 15 \div 17°$).

The investigation has been performed at the Laboratory of Particle Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna, 2007

INTRODUCTION

The suggestion that in analogy with $K^0, \bar{K}^0$ oscillations there could be neutrino–antineutrino oscillations ($\nu \to \bar{\nu}$), was considered by Pontecorvo [1] in 1957. It was subsequently considered by Maki et al. [2] and Pontecorvo [3] that there could be mixings (and oscillations) of neutrinos of different flavors (i. e., $\nu_e \to \nu_\mu$ transitions).

In the general case there can be two schemes (types) of neutrino mixings (oscillations): mass mixing schemes and charge mixings scheme (as it takes place in the vector dominance model or vector boson mixings in the standard model of electroweak interactions) [4].

In the scheme of charge mixings the oscillation parameters are expressed through weak interaction couple constants (charges) and neutrino masses [4].

In the both cases the neutrino mixing matrix V can be given [4] in the following convenient form proposed by Maiani [6] ($\theta = \theta_{12}, \beta = \theta_{13}, \gamma = \theta_{23}$):

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & s_\gamma \\ 0 & -s_\gamma & c_\gamma \end{pmatrix} \begin{pmatrix} c_\beta & 0 & s_\beta \exp(-i\delta) \\ 0 & 1 & 0 \\ -s_\beta \exp(i\delta) & 0 & c_\beta \end{pmatrix} \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where

$$c_{e\mu} = \cos\theta, \quad s_{e\mu} = \sin\theta, \quad c_{e\mu}^2 + s_{e\mu}^2 = 1;$$

$$c_{e\tau} = \cos\beta, \quad s_{e\tau} = \sin\beta, \quad c_{e\tau}^2 + s_{e\tau}^2 = 1;$$

$$c_{\mu\tau} = \cos\gamma, \quad s_{\mu\tau} = \sin\gamma, \quad c_{\mu\tau}^2 + s_{\mu\tau}^2 = 1;$$

$$\exp(i\delta) = \cos\delta + i\sin\delta.$$

Using the above matrix V, we can connect the wave functions of physical neutrino states $\Psi_{\nu_e}, \Psi_{\nu_\mu}, \Psi_{\nu_\tau}$ with the wave functions of intermediate neutrino states $\Psi_{\nu_1}, \Psi_{\nu_2}, \Psi_{\nu_3}$ and write it down in a component-wise form [5]:

$$\Psi_{\nu_l} = \sum_{k=1}^{3} V_{\nu_l \nu_k}^* \Psi_{\nu_k},$$

$$\Psi_{\nu_k} = \sum_{k=l}^{3} V_{\nu_k \nu_l} \Psi_{\nu_l}, \qquad l = e, \mu, \tau, \qquad k = 1 \div 3, \tag{2}$$

where Ψ_{ν_k} is a wave function of neutrino with momentum p and mass m_k. We suppose that neutrino mixings (oscillations) are virtual if neutrinos have different masses (if we suppose that these transitions are real, as it is supposed in the standard theory of neutrino oscillations, then it is necessary to accept that expression (2) is based on a supposition that masses difference of ν_k neutrinos is so small that coherent neutrino states are formed in the weak interactions (computation has shown that this condition is not fulfilled, i. e., neutrino as wave packet is unstable and decays)).

$$\Psi_{\nu_k}(t) = \mathrm{e}^{-iE_k t}\Psi_{\nu_k}(0). \tag{3}$$

Then

$$\Psi_{\nu_l(t)} = \sum_{k=1}^{3} \mathrm{e}^{-iE_k t} V^*_{\nu_l \nu_k} \Psi_{\nu_k}(0). \tag{4}$$

Using unitarity of matrix V or expression (2) we can rewrite expression (4) in the following form:

$$\Psi_{\nu_l}(t) = \sum_{l'=e,\mu,\tau} \sum_{k=1}^{3} V_{\nu_{l'}\nu_k} \mathrm{e}^{-iE_k t} V^*_{\nu_l \nu_k} \Psi_{\nu_{l'}(0)}, \tag{5}$$

and introducing symbol $b_{\nu_l \nu_{l'}}(t)$

$$b_{\nu_l \nu_{l'}}(t) = \sum_{k=1}^{3} V_{\nu_{l'}\nu_k} \mathrm{e}^{-iE_k t} V^*_{\nu_l \nu_k}, \tag{6}$$

we obtain

$$\Psi_{\nu_l}(t) = \sum_{l'=e,\mu,\tau} b_{\nu_l \nu_{l'}}(t) \Psi_{\nu_{l'}}(0), \tag{7}$$

where $b_{\nu_l \nu_{l'}}(t)$ is the amplitude of transition probability $\Psi_{\nu_l} \to \Psi_{\nu_{l'}}$.
And the corresponding expression for transition probability $\Psi_{\nu_l} \to \Psi_{\nu_{l'}}$ is

$$P_{\nu_l \nu_{l'}}(t) = \mid \sum_{k=1}^{3} V_{\nu'_l \nu_k} \mathrm{e}^{-iE_k t} V^*_{\nu_l \nu_k} \mid^2 . \tag{8}$$

Using expression (8) in work [6] the amplitudes and expression for probability of three neutrino transitions (oscillations) for all interesting cases (i. e., for $\nu_e \to \nu_e, \nu_\mu, \nu_\tau$, $\nu_\mu \to \nu_e, \nu_\mu, \nu_\tau$, $\nu_\tau \to \nu_e, \nu_\mu, \nu_\tau$) were obtained. In these works, an examination was done: is the expression for probability of $\nu_e \leftrightarrow \nu_e$ neutrino transitions $P_{\nu_e \nu_e}$

$$P_{\nu_e \to \nu_e}(t) = 1 - \cos^4(\beta)\sin^2(2\theta)\sin^2(-t(E_1 - E_2)/2) -$$

$$-\cos^2(\theta)\sin^2(2\beta)\sin^2(-t(E_1 - E_3)/2)- \qquad (9)$$
$$-\sin^2(\theta)\sin^2(2\beta)\sin^2(-t(E_2 - E_3)/2)$$

positive definite value at all values for $\beta, \theta, E_1, E_2, E_3$ and t? It was shown that at arbitrary values of the parameters this expression for probability is not a positive definite value. Then, to make this expression for probability positively defined, it is necessary to put limitation on values of these parameters.

This work continues the study of this question.

1. EXAMINATION OF POSITIVE DEFINITENESS OF THE EXPRESSION FOR PROBABILITY $P_{\nu_e \to \nu_e}(t)$ TRANSITIONS

The aim of this work is to examine positive definiteness of the expression for probability $P_{\nu_e \to \nu_e}(t)$ transitions (expr. (9)). For this purpose, we will fulfill graphical modelling of this function. This expression contains 7 parameters — θ, β, Δm^2_{12}, Δm^2_{23}, Δm^2_{13}, t, $E_{\nu_e} \simeq p_{\nu_e} c$. There is one connection between neutrino masses $\Delta m^2_{13} = \Delta m^2_{12} + \Delta m^2_{23}$, therefore this expression contains only 6 independent parameters. Obtaining the extremums of this expression in the general case is a serious problem. Instead of it we decided to simplify this problem and used the following values for parameters obtained in the experiments:

$$\sin^2(2\theta_{\nu_e \nu_\mu}) \cong 0.83, \quad \theta = 32.45^\circ, \quad \Delta m^2_{12} = 8.3 \cdot 10^{-5} \text{eV}^2, \qquad (10)$$

in [7] and

$$\sin^2(2\gamma_{\nu_\mu \nu_\tau}) \cong 1, \quad \gamma \cong \frac{\pi}{4}, \quad \Delta m^2_{23} = 2.5 \cdot 10^{-3} \text{eV}^2, \qquad (11)$$

in [8]. Value for E_{ν_e} is free. We expressed all lengths of oscillations through the length of oscillations determined by ν_1, ν_2 masses. The values of $\Delta m^2_{13} = \Delta m^2_{12} + \Delta m^2_{23}$ are $2.583 \cdot 10^{-3} \text{eV}^2$, and $2.417 \cdot 10^{-3} \text{eV}^2$.

Thus, in expression (9) there is remained dependence only of two parameters β and t, and then for the first case $\Delta m^2_{13} = 2.583 \cdot 10^{-3} \text{eV}^2$ this expression can be rewritten in the following form:

$$P_{\nu_e \to \nu_e}(t) = 1 - \cos^4(\beta)\sin^2(2\theta)\sin^2(R/L_{12})-$$
$$-\cos^2(\theta)\sin^2(2\beta)\sin^2(30.120 R/L_{12})- \qquad (12)$$
$$-\sin^2(\theta)\sin^2(2\beta)\sin^2(31.120 R/L_{12}),$$

where $L_{12} = 1.27 \dfrac{c p_{\nu_e}}{\Delta m^2_{12}}$ is the length of oscillations determined by ν_1, ν_2 neutrino masses, and then lengths of oscillations determined by ν_1, ν_3 and ν_2, ν_3

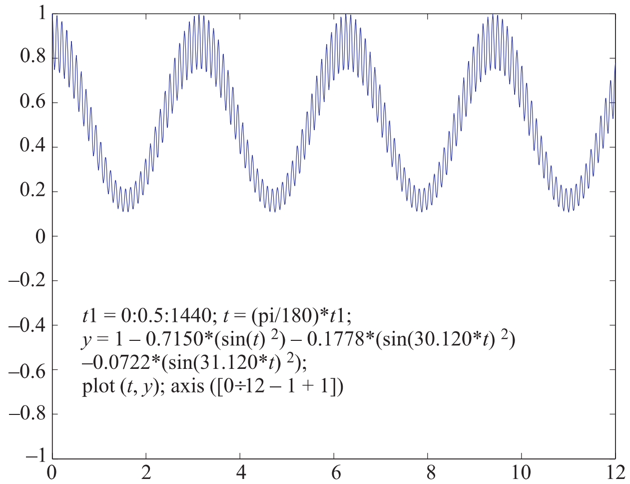

Fig. 1. $P_{\nu_e \to \nu_e}(t)$ at $\theta = 32.45^\circ, \beta = 15^\circ$

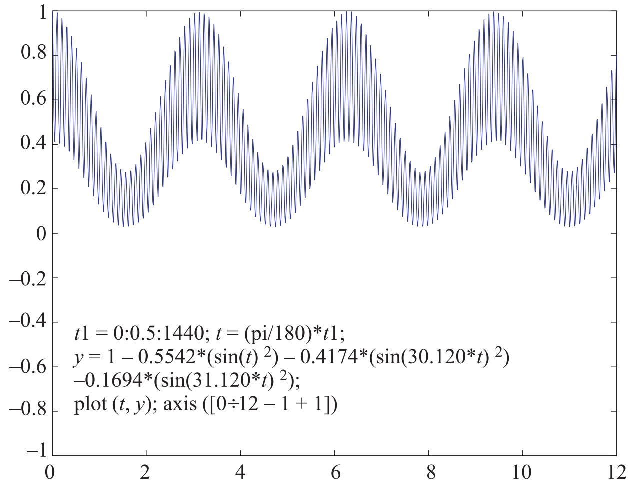

Fig. 2. $P_{\nu_e \to \nu_e}(t)$ at $\theta = 32.45^\circ, \beta = 25^\circ$

neutrino masses are expressed through L_{12} oscillations length of ν_1, ν_2 neutrinos ($2.510^{-3}/8.310^{-5} = 30.120$, $2.58310^{-3}/8.310^{-5} = 31.120$), and R is a distance from the neutrino source.

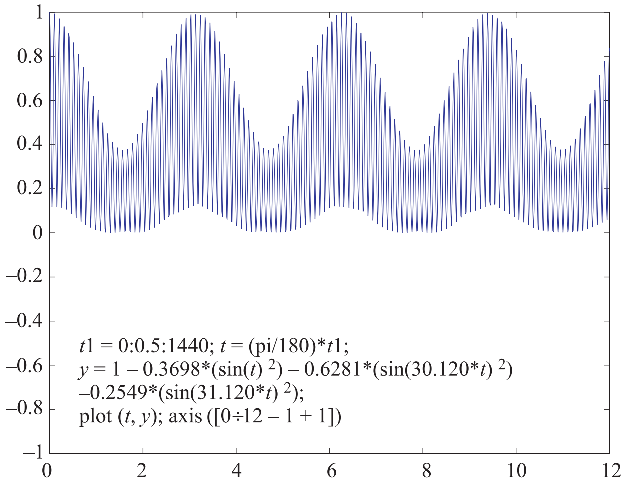

Fig. 3. $P_{\nu_e\to\nu_e}(t)$ at $\theta = 32.45^\circ, \beta = 35^\circ$

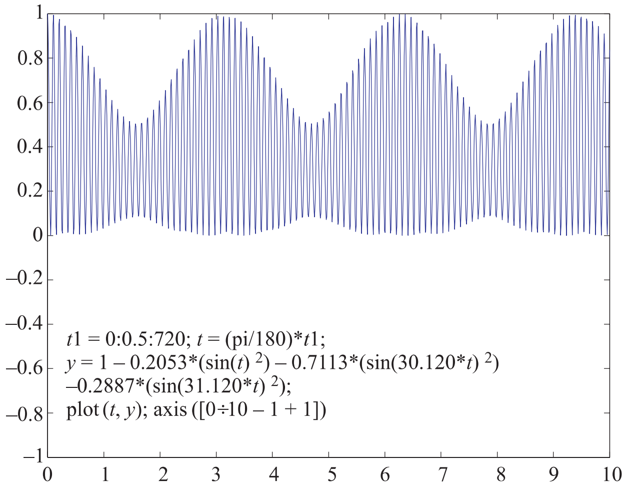

Fig. 4. $P_{\nu_e\to\nu_e}(t)$ at $\theta = 32.45^\circ, \beta = 45^\circ$

We fulfilled modelling of this expression for $\beta = 10$, 15, 20, 25, 30, 35, 40, 45, 50, 55° for $t = 0 \div 4\pi$ for the two above-mentioned cases when $\Delta m^2_{13} = 2.583 \cdot 10^{-3}\mathrm{eV}^2$, and $2.417 \cdot 10^{-3}\mathrm{eV}^2$. The check has confirmed that for all the above cases the unitarity condition is fulfilled (i. e., $P_{\nu_e\to\nu_e}(t) \geqslant 0$).

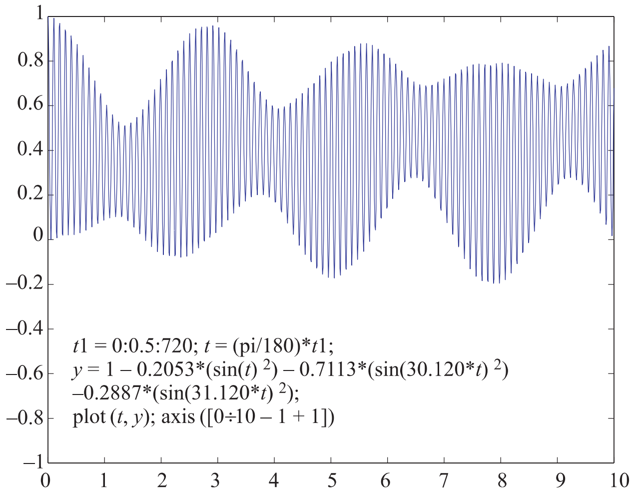

Fig. 5. $P_{\nu_e\to\nu_e}(t)$ at $\theta = 32.45^\circ, \beta = 45^\circ$, (31.120 $\to$ 31.320)

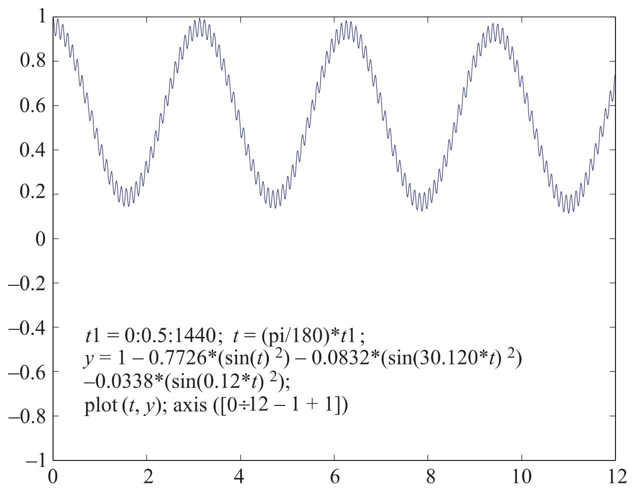

Fig. 6. $P(\nu_e \to \nu_e)(t)$ at $\theta = 32.45^\circ, \beta = 10^\circ, (L_{13}/L_{12} = 0.12)$

Figures 1–4 show modelling of probability of $P_{\nu_e\to\nu_e}(t)$ for $\beta = 15, 25, 35,$ 45° at $\Delta m^2_{13} = 2.583 \cdot 10^{-3}\mathrm{eV}^2$. From these figures it is seen that the unitarity condition is fulfilled when we take precision values (relations) for lengths of

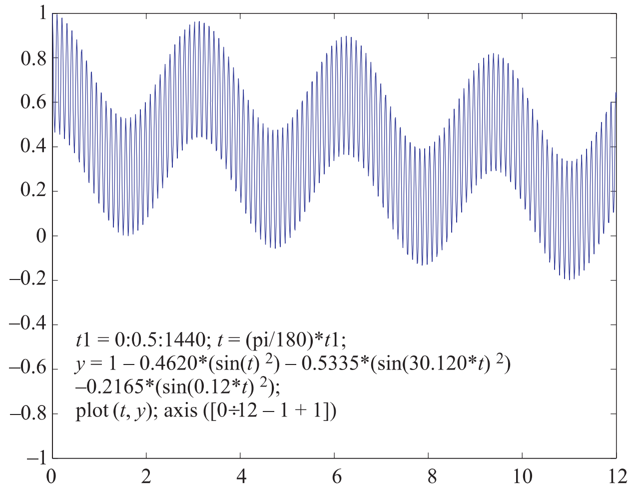

Fig. 7. $P(\nu_e \to \nu_e)(t)$ at $\theta = 32.45^\circ, \beta = 30^\circ, (L_{13}/L_{12} = 0.12)$

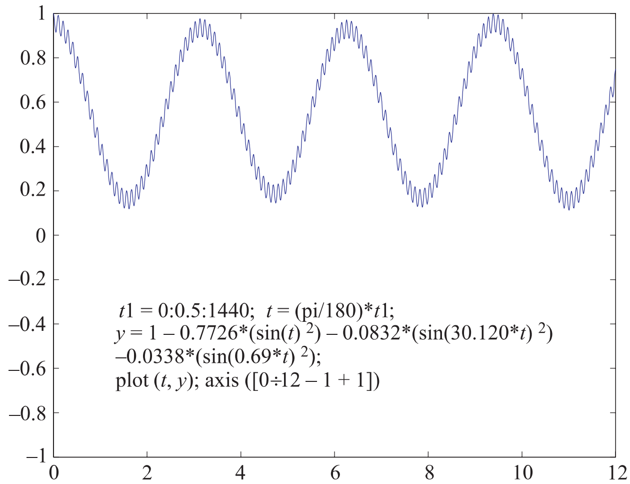

Fig. 8. $P(\nu_e \to \nu_e)(t)$ at $\theta = 32.45^\circ, \beta = 10^\circ, (L_{13}/L_{12} = 0.69)$

oscillations (i. e., the condition $\Delta m_{13}^2 = \Delta m_{12}^2 + \Delta m_{23}^2$ is strongly fulfilled). At an arbitrary small deviation from the precision values (relations) for lengths of oscillations the unitarity condition is violated as it is well seen in Fig. 5 ($\beta = 45^\circ$, $31.120 \to 31.320$), and then to fulfil this condition, it is necessary to apply the limitation on angle β.

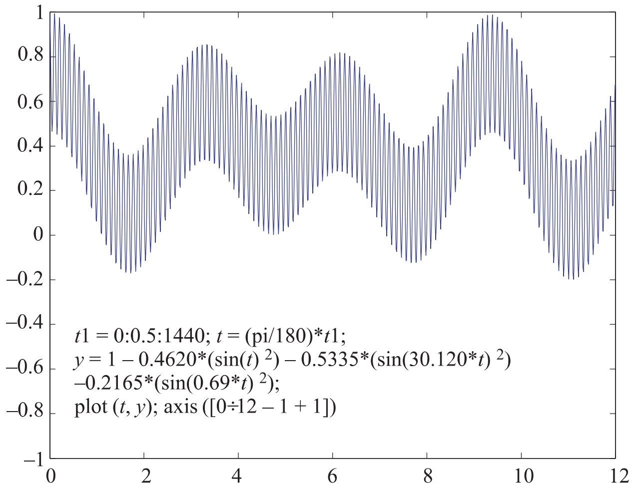

Fig. 9. $P(\nu_e \rightarrow \nu_e)(t)$ at $\theta = 32.45^\circ, \beta = 30^\circ, (L_{13}/L_{12} = 0.69)$

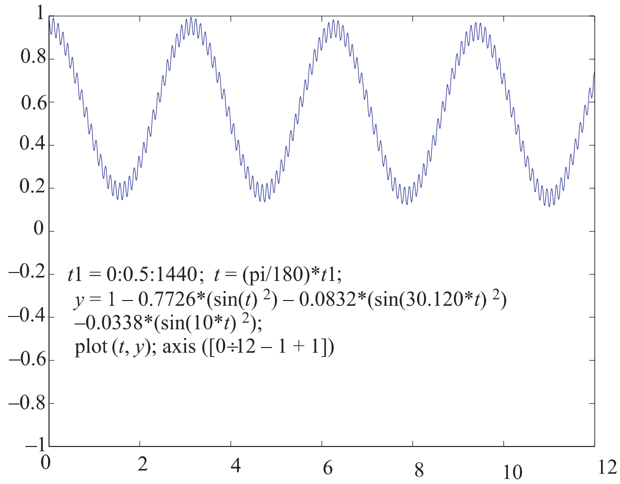

Fig. 10. $P(\nu_e \rightarrow \nu_e)(t)$ at $\theta = 32.45^\circ, \beta = 10^\circ, (L_{13}/L_{12} = 10)$

In the previous work [9] there was graphical modelling of this function (9) by using the following values for [7] $\theta = 32.45^\circ, \Delta m_{12}^2 = 8.3 \cdot 10^{-5}\text{eV}^2$ and for [8] $\Delta m_{23}^2 = 2.5 \cdot 10^{-3}\text{eV}^2$, for the cases when $\Delta m_{13}^2 = 10^{-5}\text{eV}^2$

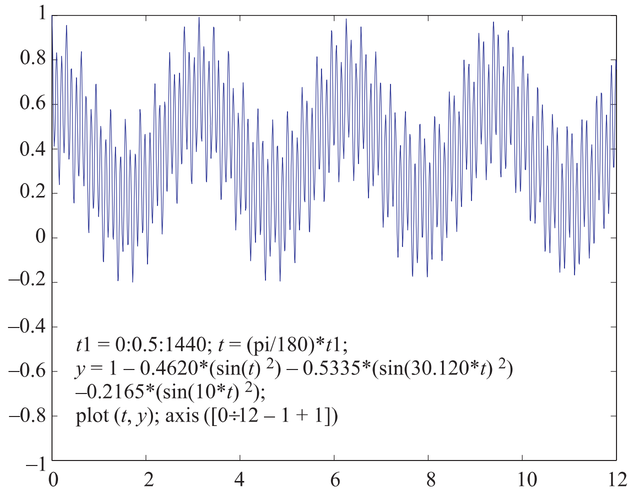

Fig. 11. $P(\nu_e \to \nu_e)(t)$ at $\theta = 32.45°, \beta = 30°, (L_{13}/L_{12} = 10)$

$(L_{13}/L_{12} = 0.12)$, $5.7{\cdot}10^{-5}\text{eV}^2$ $(L_{13}/L_{12} = 0.69)$, $8.3{\cdot}10^{-4}\text{eV}^2$ $(L_{13}/L_{12} = 10)$ (for checking) for different values of $\beta = 10 \div 45°$ and it was established that the value for $P_{\nu_e \to \nu_e}(t)$ becomes a positively defined value at $\beta \leqslant 15 \div 17°$ ($P_{\nu_e \to \nu_e}(t) \simeq 0$ at some values of t). Figures 6–11 show results of modelling for $\beta = 10°$, $\beta = 30°$ for the above-indicated three cases. From these figures we see that at $\beta = 10°$ the unitarity condition is fulfilled, while this condition is violated at $\beta = 30°$.

So, we see that at strict fulfillment of condition $\Delta m^2_{13} = \Delta m^2_{12} + \Delta m^2_{23}$, the expression for probability of $\nu_e \to \nu_e$ transitions $P(\nu_e \to \nu_e)(t)$ is positively defined, while at any deviation from this condition in order to make this expression for probability positively defined, it is necessary to put a limitation on angle mixing β (i. e., the value for β must be $\beta \leqslant 15 \div 17°$).

CONCLUSION

The numeral value of sum Sum of coefficients in expression (9) connected with mixings

$$\text{Sum} = \cos^4(\beta)\sin^2(2\theta) + \cos^2(\theta)\sin^2(2\beta) + \sin^2(\theta)\sin^2(2\beta)$$

is larger than one (for example, for $\theta = 32.45°, \beta = 45°, \text{Sum} = 1.2053$). In order to do the values of $P_{\nu_e \to \nu_e}(t)$ positively defined, the maxima of oscillation components of this expression do not have to coincide. Examination has shown that for all the considered cases this condition is fulfilled.

This work has shown that at strict fulfillment of condition $\Delta m_{13}^2 = \Delta m_{12}^2 + \Delta m_{23}^2$ the expression for probability of $\nu_e \to \nu_e$ transitions $P_{\nu_e \to \nu_e}(t)$ is positively defined at every value of θ and β, while at any arbitrarily small deviation from this condition it becomes negative, and then in order to make this expression for probability positively defined it is necessary to put a limitation on angle mixing β at fixed value of $\theta = 32.45°$ (i.e., the value for β must be $\beta \leqslant 15 \div 17°$).

REFERENCES

1. *Pontecorvo B. M.* // Sov. J. JETP. 1957. V. 33. P. 549; JETP. 1958. V. 34. P. 247.
2. *Maki Z. et al.* // Prog. Theor. Phys. 1962. V. 28. P. 870.
3. *Pontecorvo B. M.* // Sov. J. JETP. 1967. V. 53. P. 1717.
4. *Beshtoev Kh. M.* JINR Commun. E2-2004-58. Dubna, 2004, hep-ph/0406124;
   JINR Commun. E2-2005-123. Dubna, 2005, hep-ph/0506248;
5. *Bilenky S. M., Pontecorvo B. M.* // Phys. Rep. C. 1978. V. 41. P. 225;
   Boehm F., Vogel P. Physics of Massive Neutrinos: Cambridge: Cambridge Univ. Press, 1987. P. 27, 121;
   Bilenky S. M., Petcov S. P. // Rev. Mod. Phys. 1977. V. 59. P. 631;
   Gribov V., Pontecorvo B. M. // Phys. Lett. B. 1969. V. 28. P. 493.
6. *Maiani L.* // Proc. Intern. Symp. on Lepton–Photon Interaction, Hamburg: DESY, P. 867.
7. *Eguchi K. et al.* // Phys. Rev. Lett. 2003. V. 90. P. 021802;
   T. Mitsui // 28th Intern. Cosmic Ray Conf., Japan, 2003. V. 1. P. 1221;
   G. Gtatta // Report on the XXI Intern. Conf. on Neutrino Physics and Astrophysics, Paris, France, June 2004.
8. *Habig A.* // Proc. of Intern. Cosmic Ray Conf., Japan, 2003. V. 1. P. 1255;
   (*SUPER-KAMIOKANDE Collaboration*) // Report on Intern. Conf. «Neutrino 2004» / Ed. Kearns. Paris, 2004.
9. *Beshtoev Kh. M.* // JINR Commun. E2-2006-16. Dubna, 2006; hep-ph/0508122, 2005.

Received on July 18, 2007.

Редактор *В. В. Булатова*

Подписано в печать 28.09.2007.
Формат 60 × 90/16. Бумага офсетная. Печать офсетная.
Усл. печ. л. 0,81. Уч.-изд. л. 1,04. Тираж 415 экз. Заказ № 55906.

Издательский отдел Объединенного института ядерных исследований
141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@jinr.ru
www.jinr.ru/publish/