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F. V. Ignatovich*, V. K. Ignatovich

AN EXPERIMENT ON A BALL-LIGHTNING MODEL

*Lumetrics Inc, 150 Lucius Gordon Dr, ste 117, West Henrietta, N. Y. 14586

Игнатович Ф. В., Игнатович В. К. E4-2010-83
Эксперимент по исследованию модели шаровой молнии

Обсуждается полное внутреннее отражение от границы раздела между стеклом и активной газообразной средой. Предлагается эксперимент по мощному усилению света, относящийся к исследованию модели шаровой молнии.

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We discuss total internal reflection (TIR) from an interface between glass and gainy gaseous media and propose an experiment for strong light amplification related to investigation of a ball-lightning model.

The investigation has been performed at the Frank Laboratory of Neutron Physics, JINR.

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INTRODUCTION

It was the paper by A. Siegman [1] published in the magazine Optics & Photonics News (OPN) that inspired us for this work. In [1] it was claimed that reflectivity at total internal reflection (TIR) is less than unity for both cases of lossy and gainy reflecting media. Below we will show that it is not true. Some comments on this were already published in [2], though OPN rejected detailed explanation of our disagreement. During preparation of the paper with careful analysis of reflectivities from lossy or gainy media we were stricken by an idea that this matter can be easily checked by an experiment, if one uses multiple reflections of light in whispering gallery mode (WGM) in a glass sphere immersed in an active medium. In that case the intensity of WGM light will grow up with number of reflections exponentially. At the same time, bearing in mind a model of the ball lightning [3], as a spherical bubble with thin walls filled with photons, we found that such an experiment can be a first step to investigation of such a model.

Below we remind the elements of Maxwell electrodynamics and of electromagnetic waves, show how reflection and refraction amplitudes for an interface between two dielectric media are obtained, show why the claims of [1] cannot be true, and estimate outcome of the proposed experiment. We also add some comments on spherical harmonics analysis and on usual definition of energy flux with the help of the Pointing vector.

1. MAXWELL EQUATIONS AND ELECTROMAGNETIC WAVES

Light reflection from an interface between two media is determined by the wave equation and the boundary conditions, which follow from Maxwell's equations. We consider Maxwell's equations in media without free charges, with zero conductivities σ and time-independent permittivities ϵ, μ:

$$-[\nabla \times \mathbf{E}(\mathbf{r},t)] = \mu\frac{\partial}{\partial t}\mathbf{H}(\mathbf{r},t), \tag{1}$$

$$[\nabla \times \mathbf{H}(\mathbf{r},t)] = \epsilon \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t), \tag{2}$$

$$\nabla \cdot \epsilon\mathbf{E}(\mathbf{r},t) = 0, \quad \nabla \cdot \mu\mathbf{H}(\mathbf{r},t) = 0. \tag{3}$$

In a homogeneous medium the parameters ε and μ are constant in space, and the last two equations simplify to

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0, \quad \nabla \cdot \mathbf{H}(\mathbf{r},t) = 0. \tag{4}$$

With the help of (1), (2) and (4) we can obtain the wave equations for the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ as follows.

The time derivative of both parts of (2), taking into account (1), gives

$$-[\nabla \times [\nabla \times \mathbf{E}(\mathbf{r},t)]] = \mu\epsilon \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t), \tag{5}$$

which, with the help of (4), is reduced to the well-known wave equation for the electric field $\mathbf{E}(\mathbf{r},t)$:

$$\Delta\mathbf{E}(\mathbf{r},t) = -\mu\epsilon \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t). \tag{6}$$

Similarly, the time derivative of both parts of (1) along with (2) and (4) gives wave equation for the magnetic field $\mathbf{H}(\mathbf{r},t)$:

$$\Delta\mathbf{H}(\mathbf{r},t) = -\mu\epsilon \frac{\partial^2}{\partial t^2}\mathbf{H}(\mathbf{r},t). \tag{7}$$

Both equations have solutions in the form of a plain wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}\exp(i\mathbf{kr} - i\omega t), \quad \mathbf{H}(\mathbf{r},t) = \mathbf{H}\exp(i\mathbf{kr} - i\omega t). \tag{8}$$

By placing these solutions into the wave equations we find that $k^2 = \epsilon\mu\omega^2 = \omega^2/c^2$, where $c = 1/\sqrt{\epsilon\mu}$ is the speed of light in the medium.

The two wave equations are independent, and for a given frequency and direction of propagation, the total electromagnetic field is representable by the wave function

$$\psi(\mathbf{r},t) = (\mathbf{E} + \mathbf{H})\exp(i\mathbf{kr} - i\omega t). \tag{9}$$

However, the wave equations are derived from Maxwell's equations, and substitution of (8) for $\mathbf{E}(\mathbf{r},t)$ into (1) shows that

$$\mathbf{H} = \frac{1}{\mu\omega}[\mathbf{k} \times \mathbf{E}], \tag{10}$$

or similarly, substitution of (8) for $\mathbf{H}(\mathbf{r},t)$ into (2) gives

$$\mathbf{E} = -\frac{1}{\epsilon\omega}[\mathbf{k} \times \mathbf{H}]. \tag{11}$$

Therefore, $\mathbf{E}$ and $\mathbf{H}$ are not independent and are orthogonal to $\mathbf{k}$ and to each other. Moreover, if $|\mathbf{E}| = 1$, the length of $\mathbf{H}$ is $|\mathbf{H}| = \sqrt{\varepsilon/\mu} = 1/Z$, where $Z = \sqrt{\mu/\varepsilon}$ is called a medium impedance.

In the next section we consider the reflection of the light wave from an interface between two media, when the reflecting medium is lossy or gainy, find peculiarities of the reflection amplitudes, prove that the TIR reflection coefficient for gainy reflecting medium is larger than unity and propose an experiment for strong enhancement of light field.

2. WAVE REFLECTION AND REFRACTION AT AN INTERFACE

If space is not completely homogeneous but consists of two halves with different ϵ and μ, we cannot replace (3) with (4), because the permittivities now depend on the coordinates. However, each half is homogeneous and each part has its own wave equation with its own plain wave solution. The transmission through an interface is similar to travelling through a border between two countries where certain rules are imposed. In case of two media, the rules are imposed by Maxwell's equations. According to these rules, the transmission is allowed only if (according to (2) and (1)) components $\mathbf{E}_{\|}(\mathbf{r}, t)$, $\mathbf{H}_{\|}(\mathbf{r}, t)$ of the fields $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{H}(\mathbf{r}, t)$ parallel to the interface are the same, and (according to (3)) the products $\epsilon(\mathbf{n} \cdot \mathbf{E}(\mathbf{r}, t))$, $\mu(\mathbf{n} \cdot \mathbf{H}(\mathbf{r}, t))$, where $\mathbf{n}$ is a unit vector normal to the interface, are the same on both sides of the border/interface. Because of these restrictions only a fraction (refracted, or transmitted) of the incident wave is permitted to go through the border, and the remaining part (reflected) is ordered to go back. So, before starting our journey we should calculate how much of the incident wave (9) is transmitted and how much is reflected at an interface between the half-space 1 at $z < 0$ and the half-space 2 at $z > 0$, which have different electromagnetic constants $\epsilon_{1,2}$ and $\mu_{1,2}$. We can expect that the total wave function in the presence of the interface is

$$\psi(\mathbf{r}, t) = \Theta(z < 0)(\exp(i\mathbf{k}_1\mathbf{r} - i\omega t)\psi_1 + \exp(i\mathbf{k}_r\mathbf{r} - i\omega t)\psi_r\rho) + \\ + \Theta(z > 0)\exp(i\mathbf{k}_2\mathbf{r} - i\omega t)\psi_2\tau, \quad (12)$$

where the term $\exp(i\mathbf{k}_1\mathbf{r} - i\omega t)\psi_1$ with the wave vector $\mathbf{k}_1$ describes the plain wave incident on the interface from medium 1, factors $\psi_i = \mathbf{E}_i + \mathbf{H}_i$ $(i = 1, r, 2)$ denote sum of electric and magnetic polarization vectors, $\mathbf{k}_{r,2}$ are wave vectors of the reflected and transmitted waves, ρ, τ are the reflection and transmission amplitudes respectively, and $\Theta(z)$ is the step function, which is equal to unity when inequality in its argument is satisfied, and to zero otherwise.

The wave vectors $\mathbf{k}_{r,2}$ of the reflected and refracted waves are completely determined by the wave vector $\mathbf{k}_1$ of the wave incident on the interface from medium 1. It is determined uniquely by the constants ϵ_i, μ_i, and by the fact that the frequency ω and the part $\mathbf{k}_\parallel$ of the wave vectors parallel to the interface must be identical for all the waves. In the following we assume that medium 1 is lossless, i.e., $\epsilon_1\mu_1$ is real. In this case, all the components of the wave vector $\mathbf{k}_1$ are also real.

Frequency of all the waves is the same because reflection and refraction are elastic scattering processes. Vector $\mathbf{k}_\parallel$ is constant because the space along the interface is homogeneous and there are no points where $\mathbf{k}_\parallel$ could change. Thus, the length of the refracted wave vector is $|\mathbf{k}_2| = \omega\sqrt{\epsilon_2\mu_2}$, which can also be written as

$$|\mathbf{k}_2| = \omega\sqrt{\epsilon_1\mu_1}\sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}} = k_1 n = k_1\sqrt{\epsilon}, \tag{13}$$

where n is the relative refractive index,

$$n = \sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}} \tag{14}$$

and where $\epsilon = n^2$.

Because $\mathbf{k}_\parallel$ is the same for all the waves, and medium 1 is isotropic, the reflection is specular. Therefore, the normal component of the reflected wave vector is $k_{r\perp} = -k_{1\perp} = -\sqrt{k_1^2 - \mathbf{k}_\parallel^2}$. The normal component of the refracted wave vector, when the reflecting medium is also isotropic, can be represented as

$$\begin{aligned} k_{2\perp} = \sqrt{\epsilon_2\mu_2\omega^2 - \mathbf{k}_\parallel^2} = \sqrt{\epsilon_1\mu_1\omega^2 - \mathbf{k}_\parallel^2 - (\epsilon_1\mu_1 - \epsilon_2\mu_2)\omega^2} = \\ = \sqrt{k_{1\perp}^2 - (\epsilon_1\mu_1 - \epsilon_2\mu_2)\omega^2}, \end{aligned} \tag{15}$$

or

$$k_{2\perp} = \sqrt{\epsilon k_1^2 - \mathbf{k}_\parallel^2}. \tag{16}$$

From the last expression it follows that for lossless media when $0 < \epsilon < 1$ is real, the incident wave, for which $\mathbf{k}_\parallel$ is within $nk_1 \leqslant |\mathbf{k}_\parallel| \leqslant k_1$, is totally reflected from the interface. This happens because

$$k_{2\perp} = iK''_{2\perp} \equiv i\sqrt{k_\parallel^2 - \epsilon k_1^2}, \tag{17}$$

the factor $\exp(ik_{2\perp}z) = \exp(-K''_{2\perp}z)$ of the wave $\exp(i\mathbf{k}_2\mathbf{r})$ exponentially decays, and the refracted wave becomes an evanescent one. Therefore, the energy does not flow inside medium 2, and due to energy conservation it must be totally reflected back into medium 1.

If medium 2 is lossy or gainy, the constant ϵ is a complex quantity $\epsilon = \epsilon' \pm i\epsilon''$, with positive ϵ' and ϵ''. In this case, far outside the total internal reflection (TIR) region ($|\mathbf{k}_\parallel|^2 \ll \epsilon' k_1^2$) for small ϵ'' ($\epsilon'' k_1^2 \ll \epsilon' k_1^2 - |\mathbf{k}_\parallel|^2$) we have

$$k_{2\perp} = \sqrt{\epsilon' k_1^2 - |\mathbf{k}_\parallel|^2 \pm i\epsilon'' k_1^2} = k'_{2\perp} \pm ik''_{2\perp}, \tag{18}$$

where

$$k'_{2\perp} \approx \sqrt{\epsilon' k_1^2 - |\mathbf{k}_\parallel|^2}, \qquad k''_{2\perp} \approx \epsilon'' \frac{k_1^2}{2k'_{2\perp}}. \tag{19}$$

The sign of the square root in (18) is positive, because $k'_{2\perp}$ must be positive to describe propagation of the refracted wave away from the interface. At the same time we see that for lossy media the imaginary part should have positive sign to get the exponential decay of the refracted wave; and for gainy media it should have negative sign to get exponential growth of the refracted wave. In this case, the exponential growth is determined by value of the gain ϵ''.

At the TIR regime, $k'_{2\perp}$ in (18) transforms into $iK''_{2\perp}$, where $K''_{2\perp} \approx \sqrt{|\mathbf{k}_\parallel|^2 - \epsilon' k_1^2}$, and $k''_{2\perp}$ in (19) transforms to

$$k''_{2\perp} \to -iK'_{2\perp} = \epsilon'' \frac{k_1^2}{2iK''_{2\perp}}. \tag{20}$$

Therefore, at TIR

$$k_{2\perp} = \sqrt{-(|\mathbf{k}_\parallel|^2 - \epsilon' k_1^2) \pm i\epsilon'' k_1^2} = \pm K'_{2\perp} + iK''_{2\perp}, \tag{21}$$

where

$$K'_{2\perp} = \epsilon'' \frac{k_1^2}{2K''_{2\perp}}, \quad K''_{2\perp} \approx \sqrt{|\mathbf{k}_\parallel|^2 - \epsilon' k_1^2}. \tag{22}$$

Because of (12) the refracted wave function becomes

$$\propto \exp(i\mathbf{k}_\parallel \mathbf{r}_\parallel) \exp(-K''_{2\perp} z \pm iK'_{2\perp} z). \tag{23}$$

The sign before the square root in (21) defines the exponential decay of the refracted wave away from the interface for both lossy and gainy media cases. However, the real parts of $k_{2\perp}$ have opposite signs. The positive value of $K'_{2\perp}$ for lossy medium means that the reflection coefficient at TIR is less than one, because part of the energy flux proportional to $K'_{2\perp}$ enters medium 2 and decays there. The negative value of $K'_{2\perp}$ for gainy medium means that the reflection coefficient at TIR is larger than one, because part of the energy flux proportional to $K'_{2\perp}$ exits medium 2 and adds to the TIR wave.

Here we can show what is not correct in [1]. For gainy media A. Siegman instead of (21) took

$$k_{2\perp} = \sqrt{-(|\mathbf{k}_\parallel|^2 - \epsilon' k_1^2) - i\epsilon'' k_1^2} = K'_{2\perp} - iK''_{2\perp}, \tag{24}$$

because he suggested that $K'_{2\perp}$ gives the flux inside the medium, and this flux exponentially increases in the gainy medium. However, exponential growth $\exp(K''_{2\perp}z)$ of the field does not depend on value of ϵ'', and even for infinitesimal ϵ'' the refracted wave function at a distance ~ 1 mm from the reflecting interface for light with wave length ~ 1 μm becomes of the order $\sim \exp(10^3) \gg 10^{100}$ which is astronomically large and completely violates the energy conservation law.

2.1. Reflection and Refraction Amplitudes. The procedure for calculating the reflection amplitude in general case is well explained in [4], so here we only briefly recall it. The polarization $\mathbf{E}_1$ of the incident wave can be arbitrary (except it must be perpendicular to the wave vector $\mathbf{k}_1$). It can be decomposed as $\mathbf{E} = \mathbf{E}_{1s} + \mathbf{E}_{1p}$, where $\mathbf{E}_{1s}$ is the component parallel to the interface and perpendicular to the plane of incidence (plane of vectors $\mathbf{k}_1$ and the normal $\mathbf{n}$ to the interface), and where $\mathbf{E}_{1p}$ lies in the plane of incidence. Reflection of each component is different and can be found independently.

Let's find the reflection of the wave $\mathbf{E}_{1s}$. It is usually called s-wave or TE wave. The field $\mathbf{E}_{1s}$ is accompanied by the field $\mathbf{H}_{1p}$, which lies in the incidence plane. The total wave function of the TE wave according to (12) can be represented as $\exp(i\mathbf{k}_\parallel \mathbf{r}_\parallel - i\omega t)\psi(z)$, where

$$\psi(z) = \Theta(z<0)[\psi_{1s}\exp(ik_{1\perp}z) + \psi_{rs}\rho_s\exp(-ik_{1\perp}z)] + \\ + \Theta(z>0)\psi_{2s}\tau_s\exp(ik_{2\perp}z), \tag{25}$$

and for $i = 1, r, 2$ we introduced notations

$$\psi_{is} = \mathbf{E}_{1s} + \mathbf{H}_{ip}, \quad \mathbf{H}_{ip} = \frac{1}{\mu_i\omega}[\mathbf{k}_i \times \mathbf{E}_{1s}], \quad \mu_r = \mu_1. \tag{26}$$

The corresponding wave vectors are

$$\mathbf{k}_1 = \mathbf{k}_\parallel + \mathbf{n}k_{1\perp}, \quad \mathbf{k}_r = \mathbf{k}_\parallel - \mathbf{n}k_{1\perp}, \quad \mathbf{k}_2 = \mathbf{k}_\parallel + \mathbf{n}k_{2\perp}. \tag{27}$$

Maxwell's equations require continuity of the electric field $\mathbf{E}_s$ at the interface, which leads to the equation $1 + \rho_s = \tau_s$. The same requirement for the component $\mathbf{H}_{\parallel p}$ of the magnetic field parallel to the interface leads to the equation $(1-\rho_s)k_{\perp 1}/\mu_1 = \tau_s k_{\perp 2}/\mu_2$. The third requirement for the continuity of the quantity $\mu(\mathbf{n}\cdot\mathbf{H}_p)$ leads to the same equation $1 + \rho_s = \tau_s$ as the one obtained from the continuity of $\mathbf{E}_s$. Therefore, we have only two independent equations, from which we obtain the well-known Fresnel formulas

$$\rho_s = \frac{\mu_2 k_{1\perp} - \mu_1 k_{2\perp}}{\mu_2 k_{1\perp} + \mu_1 k_{2\perp}}, \quad \tau_s = \frac{2\mu_2 k_{1\perp}}{\mu_2 k_{1\perp} + \mu_1 k_{2\perp}}. \tag{28}$$

Similar considerations of the TH wave with $\mathbf{E}_{1p}$ polarization gives the other two expressions

$$\rho_p = \frac{\epsilon_2 k_{1\perp} - \epsilon_1 k_{2\perp}}{\epsilon_2 k_{1\perp} + \epsilon_1 k_{2\perp}}, \quad \tau_p = \frac{2\epsilon_2 k_{1\perp}}{\epsilon_2 k_{1\perp} + \epsilon_1 k_{2\perp}}. \tag{29}$$

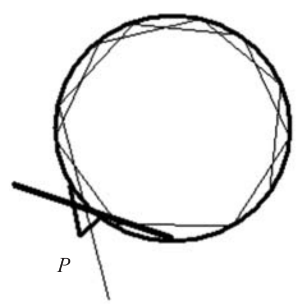
For simplicity, we limit ourselves only to TE case and assume that $\mu_2 = \mu_1$, so the formulas (28) are reduced to

$$\rho_s = \frac{k_{1\perp} - k_{2\perp}}{k_{1\perp} + k_{2\perp}}, \quad \tau_s = \frac{2k_{1\perp}}{k_{1\perp} + k_{2\perp}}. \tag{30}$$

Substitution of (21) for the gainy medium shows that at TIR the reflection coefficient from a gainy medium is larger than one, and it increases with gain. The growth of the reflection coefficient is the result of the induced by the evanescent field emission of photons toward the interface. The induced photon at TIR regime cannot propagate inside the reflecting medium for the same reasons as the refracted photon cannot propagate in it. Therefore, the increase in the reflected flux is due to the sub-barrier induction of the photon, which tunnels from the gainy medium into medium 1 and coherently adds to the reflected primary photon. The larger is the gain, the larger is the probability of such a process.

3. AN EXPERIMENT TO STRONGLY ENHANCE THE LIGHT TRAPPED IN A GLASS SPHERE

The increase of the reflection coefficient at TIR from a gainy medium can be used to develop a curious experiment for storage and amplification of light. Imagine a glass sphere with a coupler P, as shown in the figure. The sphere has thin walls (it is also possible to use a homogeneous glass sphere) and is surrounded by an excited gas (or other active media). The ray of light, shown by the thin line, enters the glass walls through the coupler and then undergoes TIR multiple times. At every reflection the light is amplified according to the analysis in the previous section. In the end the ray escapes the sphere, as shown by the thick line. The amplification depends on the number of reflections and on the gain coefficient of the active medium. The number of the reflections is very sensitive to the angle of the incident ray. If the overall amplification is sufficiently high, the glass will melt into a liquid bubble with a thin skin filled with the light, similar to the ball lightning [3, 5].



Schematics of the experiment

We can estimate the magnitude of the light enhancement in such a sphere. Assume that the outside gas has $\epsilon_2 \approx 1 - i\alpha$, and ϵ_1 of the glass has a real value. We suppose that radius R of the sphere is much larger than the wave length of the light, so we can neglect curvature of the surface at the reflection points and use plane geometry for calculation of reflectivity. For the TE mode, the reflection amplitude at TIR according to (30) can be written as

$$\rho_s = \frac{k_{1\perp} - i\sqrt{(\epsilon_1 - 1 + i\alpha)k^2 - k_{1\perp}^2}}{k_{1\perp} + i\sqrt{(\epsilon_1 - 1 + i\alpha)k^2 - k_{1\perp}^2}} \approx \frac{k_{1\perp} - iK_{2\perp} + \alpha k^2/2K_{2\perp}}{k_{1\perp} + iK_{2\perp} - \alpha k^2/2K_{2\perp}}, \tag{31}$$

where $K_{2\perp} = \sqrt{(\epsilon_1 - 1)k^2 - k_{1\perp}^2}$, and the last equality is valid for small α. From (31) it follows that the reflectivity is

$$|\rho_s|^2 = \frac{[k_{1\perp} + \alpha k^2/2K_{2\perp}]^2 + K_{2\perp}^2}{[k_{1\perp} - \alpha k^2/2K_{2\perp}]^2 + K_{2\perp}^2} \approx 1 + 2\alpha \frac{k_{1\perp}k^2}{K_{2\perp}(k_{1\perp}^2 + K_{2\perp}^2)} = \\ = 1 + 2\alpha \frac{k_{1\perp}}{K_{2\perp}(\epsilon_1 - 1)}. \tag{32}$$

For the TH mode,

$$\rho_p = \frac{\epsilon_2 k_{1\perp} - \epsilon_1 k_{2\perp}}{\epsilon_2 k_{1\perp} + \epsilon_1 k_{2\perp}} \approx \frac{(1 - i\alpha)k_{1\perp} - i\epsilon_1 K_{2\perp} + \alpha\epsilon_1 k^2/2K_{2\perp}}{(1 - i\alpha)k_{1\perp} + i\epsilon_1 K_{2\perp} - \alpha\epsilon_1 k^2/2K_{2\perp}}, \tag{33}$$

and

$$|\rho_p|^2 = \frac{[k_{1\perp} + \alpha\epsilon_1 k^2/2K_{2\perp}]^2 + (\alpha k_{1\perp} + \epsilon_1 K_{2\perp})^2}{[k_{1\perp} - \alpha\epsilon_1 k^2/2K_{2\perp}]^2 + (\alpha k_{1\perp} - \epsilon_1 K_{2\perp})^2} \approx \\ \approx 1 + 2\alpha\epsilon_1 \frac{k_{1\perp}}{K_{2\perp}} \frac{k^2 + 2K_{2\perp}^2}{k_{1\perp}^2 + \epsilon_1^2 K_{2\perp}^2}. \tag{34}$$

For estimating purposes we can assume that each reflection amplifies the light by $\approx 1 + 2\alpha$. It means that after n reflection the intensity will increase $\propto (1 + 2\alpha)^N \approx \exp(2N\alpha)$. In sphere of radius R the flight time of light between 2 consecutive collisions with the wall is $t_1 = 2R\sin\theta/c$, where θ is the grazing angle. Therefore, the exponential growth $\exp(2N\alpha)$ of intensity I of the light can be also represented as

$$I/I_0 = \exp[(tc/R)\alpha/\sin\theta] = \exp(t/\tau), \tag{35}$$

where I_0 is the primary intensity, and $1/\tau = c\alpha/R\sin\theta$.

The following analysis is used to estimate α. Enhancement of a laser wave along a path l inside a gainy media is described by the exponent $\exp(2k''l)$, where k'' is the imaginary part of the wave number of the wave, and value $g = 2k''$ is called the gain coefficient. In medium with $\epsilon = 1 - i\alpha$, the gain coefficient is $g \approx \alpha k = 2\pi\alpha/\lambda$, where λ is the wavelength. For N_2,CO_2 gas lasers, the gain coefficient is approximately 10^{-2} cm^{-1} [6]. For $\lambda/2\pi \simeq 10^{-4}$ cm we obtain $\alpha = 10^{-6}$. Therefore, the number of the reflections off the interface should be larger than $N = 10^6$ to obtain any practical light amplification.

In the past, many experiments were performed with the whispering gallery mode resonators (WGMR) of small dimensions ($R \sim 1$ mm) and large Q-factors (up to $Q \sim 10^{10}$) [7], where light undergoes large number $N \sim Q$ total internal reflections. If a sphere of radius $R = 10$ cm is submerged into an active medium with $\alpha \sim 10^{-7}$, then at $\theta = 0.1$ we obtain $1/\tau = 3 \cdot 10^3$ s^{-1} in (35). Therefore, the initial energy of say 10^{-19} J after $t = 20$ ms will grow up to 10 MJ. The stored photons will heat and melt the resonator, but the electrostriction will hold the melted substance together. One can expect to see many interesting nonlinear phenomena in such systems.

3.1. Derivation of the Energy Increase with the Help of Spherical Harmonics. In [7, 8] and many other works with microspheres an analysis with spherical harmonics is used. It means that TE or TH field in and out of the sphere is represented by

$$\psi(\mathbf{r}, t) = \exp(-i\omega t) Y_{l,m}(\theta, \phi) F_l(r), \tag{36}$$

where $Y_{l,m}(\theta, \phi)$ are the usual spherical harmonics, and the radial function $F_l(r)$ can be represented as

$$F_l(r) = \Theta(r < R) j_l(nk_0 r) + \Theta(r > R) \frac{j_l(nk_0 R)}{h_l^{(1)}(n'k_0 R)} h_l^{(1)}(n'k_0 r), \tag{37}$$

where $j_l(kr)$, $h_l^{(1)}(k'r)$ are spherical Bessel and Hankel functions respectively, n, n' are refraction indices inside and outside the sphere, $k_0 = \omega/c$ and the factor before $h_l^{(1)}(n'k_0 r)$ provides continuity of the function $F_l(r)$ at $r = R$.

The second boundary condition, say, for TE field is

$$\frac{d}{dr} j_l(nk_0 r)_{r=R} = \frac{j_l(nk_0 R)}{h_l^{(1)}(n'k_0 R)} \frac{d}{dr} h_l^{(1)}(n'k_0 r)_{r=R}. \tag{38}$$

This condition determines values of k_0 for which solution in the form (37) is possible.

However, such an approach is not appropriate for trapped light, because outside function must be evanescent, while spherical Hankel functions are not. For description of the trapped light in WG mode, which is distributed closely to the sphere radius R and corresponds to $l \gg 1$, we can use expansion [9]

$$\frac{l^2}{r^2} \approx \frac{l^2}{R^2} - 2(r - R)\frac{l^2}{R^3}, \tag{39}$$

treat the linear term as a perturbation, and then in the simplest approximation the radial equation becomes

$$\left(\frac{d^2}{dr^2}+\epsilon k^2-\frac{l^2}{R^2}\right)F_l(r)=0. \tag{40}$$

Its solution is

$$F_l(r)=\Theta(r<R)\sin(k_r r)+\Theta(r>R)\sin(k_r R)\exp(-K_r(r-R)), \tag{41}$$

where $k_r\approx\sqrt{\epsilon k_0^2-l^2/R^2}$, $K_r=\sqrt{(\epsilon-\epsilon')k_0^2-k_r^2}$. To get WGM we should have $\epsilon'<\epsilon$, and sufficiently large l for arguments of both square roots to be positive.

The second boundary condition analogous to (38) will give limitations (or quantization) for k_0^2. If the medium outside the sphere is gainy one, then ϵ' contains negative imaginary part $-i\epsilon''$, and the second boundary condition will make k_0^2 or ω complex numbers with positive imaginary part $i\omega''$. It means that the factor $\exp(-i\omega t)$ in (36) provides exponential growth $\sim\exp(\omega'' t)$.

We do not follow this way, because approximation (40) is equivalent to reflection in plane geometry, and representation of wave function in the form (36) means that distribution of field in the sphere is periodic or all the rays in WG mode are closed. In general, it is not so like for the rays shown in the figure. Therefore, if $Rk_0\gg 1$, the analysis of trapped light in WG mode with spherical harmonics is not appropriate.

3.2. On Definition of the Energy Flux. We want to note here that the widely spread belief that the energy flux is given by the Pointing vector $\mathbf{J}=[\mathbf{E}\times\mathbf{H}]$ is in general not correct. The energy flux at least in isotropic media is given by

$$\mathbf{J}=c\frac{\mathbf{k}}{k}\frac{\epsilon E^2+\mu H^2}{8\pi}, \tag{42}$$

i.e., it is equal to energy density times the light speed, and it has a direction along the wave vector $\mathbf{k}$. For a plain wave in isotropic media this definition coincides with the Pointing vector. However, the latter can be defined for wider variety of vectors $\mathbf{E}$ and $\mathbf{H}$, including stationary fields and evanescent waves where it has no relation to the energy flux.

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Издательский отдел Объединенного института ядерных исследований
141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@jinr.ru
www.jinr.ru/publish/