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POLYHEDRAL APPROACH TO INTEGER PARTITIONS

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Полиэдральный подход к разбиениям чисел

Развивается полиэдральный подход к задаче о разбиениях чисел. Множество разбиений натурального числа n рассматривается как политоп $P_n \subset \mathbb{R}^n$. Его вершины составляют класс разбиений, образующих базис множества всех разбиений n. Показано, что существует подкласс вершин, из которых с помощью двух комбинаторных операций можно построить все остальные вершины. Вычисления говорят о том, что при каждом переходе к классу базисных вершин число разбиений значительно уменьшается. Основное внимание уделено задаче распознавания вершин P_n. Доказано, что вершины образуют идеал разбиений предложенной Г. Эндрюсом решетки разбиений. Это позволяет строить вершины P_n методом лифтинга, исследуя только часть разбиений n. Критерий, является ли заданное разбиение выпуклой комбинацией двух других, связывает вершины с рюкзачными разбиениями, множествами без сумм, множествами Сидона и введенными в работе мультимножествами Сидона. Несмотря на то, что проверка условия критерия является NP-полной задачей, с его помощью удается распознать почти все разбиения-невершины для малых n. Доказаны также несколько легко проверяемых необходимых условий для того, чтобы разбиение являлось вершиной.

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Shlyk V. A. E5-2010-115
Polyhedral Approach to Integer Partitions

This paper develops the polyhedral approach to integer partitions. We consider the set of partitions of an integer n as a polytope $P_n \subset \mathbb{R}^n$. Vertices of P_n form the class of partitions that provide the first basis for the whole set of partitions of n. Moreover, we show that there exists a subclass of vertices, from which all others can be generated with the use of two combinatorial operations. Numerical experiments demonstrate considerable decrease in the cardinality of these classes of basic partitions. We focus on the vertex enumeration problem for P_n. We prove that vertices of all partition polytopes form a partition ideal of Andrews' partition lattice. This allows us to construct vertices of P_n by a lifting method, which requires examining only certain partitions of n. A criterion of whether a given partition is a convex combination of two others connects vertices with knapsack partitions, sum-free sets, Sidon sets, and Sidon multisets introduced in the paper. Albeit verifying the criterion condition was proved to be an NP-hard problem, it recognizes almost all nonvertices for small n's. We also prove several easy-to-check necessary conditions for a partition to be a vertex.

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1. INTRODUCTION

We develop the polyhedral approach to integer partitions proposed in [7]. A partition of a positive integer n is any finite nondecreasing sequence of positive integers $n_1, n_2, \ldots, n_r$ such that

$$\sum_{j=1}^{r} n_j = n. \tag{1.1}$$

Informally, a partition of n is its representation of form (1.1). The integers $n_1, n_2, \ldots, n_r$ are called the parts of partition [1].

The polyhedral approach is based on the n-dimensional geometrical interpretation of integer partitions [14] that is common in Diophantine analysis but seldom used in the partition theory. A partition of n is referred to as a nonnegative integer point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ whose components x_i, $i = 1, \ldots, n$, indicate the numbers of times the parts i enter the partition. So, x satisfies equation $x_1 + 2x_2 + \ldots + nx_n = n$. We keep on writing $x \vdash n$ to indicate that $x \in \mathbb{R}^n$ is a partition of n. For example, the partition $8 = 4 + 2 + 1 + 1$ with three distinct parts 1, 2, 4 is considered as the point $x = (2, 1, 0, 1, 0, 0, 0, 0) \in \mathbb{R}^8$.

The polytope $P_n \subset \mathbb{R}^n$ of partitions of n is defined as the convex hull of the set T_n of all partitions of n:

$$P_n = \operatorname{conv} T_n = \operatorname{conv}\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x \vdash n\}.$$

The conversion from set to polytope brings geometry into arithmetic of partitions and raises new problems concerned with the geometrical structure of integer partition polytopes. The other well-known 2-dimensional interpretation of partitions as Young tables proved to be extremely useful for studying connections between individual partitions but it hardly provides tools to treat the set of partitions of an integer as a whole.

There are two ways to describe any polytope: 1) to enumerate its facets, i.e., faces of the maximal dimension, and 2) to identify its vertices. Facets of P_n were described in [7] as all but one coordinate hyperplanes and certain solutions of some systems of subadditive inequalities and equalities. This was done with the use of a representation of P_n as a polytope on a partial algebra and a technique borrowed from the group theoretical approach to the integer linear programming problem and generalized for the case.

This article focuses on vertices of partition polytopes. As for any polytope, a point $x \in P_n$ is its vertex if it cannot be expressed as convex combination $x = \sum_{j=1}^{k} \lambda_j y^j$, $\sum_{j=1}^{k} \lambda_j = 1$, $\lambda_j > 0$, of some other points $y^j \in P_n$, $j = 1, \ldots, k$. Vertices of P_n are of importance because they form a kind of basis for T_n as each $x \vdash n$ is a convex combination of some vertices. To the best of our knowledge this is the first attempt to reduce the set of all partitions of n to its subset.

The first partition that is not a vertex appears for $n = 4$. There are five partitions of 4: $x_1 = (4, 0, 0, 0)$, $x_2 = (2, 1, 0, 0)$, $x_3 = (1, 0, 1, 0)$, $x_4 = (0, 2, 0, 0)$, and $x_5 = (4, 0, 0, 0)$. P_4 is a tetrahedron with vertices x_1, x_3, x_4, x_5 since $x_2 = \frac{1}{2}(x_1 + x_4)$ is not a vertex. For greater n's, the vertex recognition problem for P_n 'Is a given partition $x \vdash n$ a vertex of P_n?' cannot be solved the same easily. However, calculation shows that the gap between the number of vertices and the number of partitions rapidly increases as n grows.

Another motif to study vertices of P_n is linked to optimization problems on partitions: these are vertices which provide their optimal solutions in the linear case. Anyway, we believe that any result on the topic is of interest for its own sake.

The paper is organized as follows. Section 2 contains notation and a few quotes of some previous results. In Sec. 3, we prove that vertices of partition polytopes form a partition ideal of the partition lattice introduced by Andrews [1]. This property is crucial for constructing vertices of P_n by a lifting method: they should be selected from only those partitions of n that are induced by certain vertices of some polytopes P_j, $j < n$. The main result of Sec. 4 is a criterion of whether a given partition can be expressed as a convex combination of two others. It generalizes all known necessary conditions for vertices and provides some new ones. In particular, it yields the exact bound on the number of distinct parts of partitions-vertices of P_n. A great amount of partitions that are not vertices can be recognized and rejected with the help of this criterion. However, for $n = 15, 21$, and many others, there exist nonvertices, which this criterion is incapable to capture because they need three partitions for convex representation.

In Sec. 5, we show that partition polytopes possess an uncommon feature: there exists a subset of vertices (support vertices), from which all others can be generated with the use of two operations of merging parts. Numerical data testify that the subset of support vertices is small by comparison with the set of all vertices. Some results of Secs. 4 and 5 were announced in preliminary publications [8, 9].

We suggest two hypotheses in two subsequent sections. The first flows from the discussion of how the vertex recognition problem for P_n is related to analogous problems concerning knapsack partitions, sum-free and Sidon sets, and the combinatorial problem partition [4]. It claims that the main problem of the article is polynomially insoluble. The second states that the number of vertices of P_n is inversely dependent on the number of divisors of n. In the concluding section, we sketch the most promising directions for the future study.

2. PRELIMINARIES

In this paper, $\mathbb{Z}_+$ denotes the set of nonnegative integers; $[1, m]$ denotes the set of integers $\{1, 2, \dots, m\}$, $0 < m \in \mathbb{Z}$; $|M|$ denotes cardinality of a set M. For $\alpha \in \mathbb{R}$, $\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$ denote the greatest (respectively, the least) integer not greater (respectively, not less) than α. We denote by $S(x)$ the set $\{i \in [1, n] \mid x_i > 0\}$ of distinct parts of x, write $\operatorname{vert} P$ for the set of vertices of a polytope P and 0^k for the sequence of k zeroes. Symbol $\uplus$ denotes the union of disjoint sets.

Recall some results from [7]. We study the polytope P_n in $\mathbb{R}^n$, though, in fact, it is $(n-1)$-dimensional since it belongs to the hyperplane $x_1 + 2x_2 + \ldots + nx_n = n$. It is not hard to see that P_n is a pyramid with the point $(0^{n-1}, 1)$ as the apex and the base lying in the hyperplane $x_n = 0$.

Define transformations φ_i: $\mathbb{R}^{n-i} \to \mathbb{R}^n$, $i = 1, 2, \dots, n-1$, as

$$\varphi_i(y_1, y_2, \dots y_{n-i}) = (y_1, y_2, \dots, y_{i-1}, y_i + 1, y_{i+1}, \dots, y_{n-i}, 0^i).$$

Each φ_i is a composition of translation by 1 along the i-axis and embedding of $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$. It is easy to see that if $y \vdash n - i$, then $\varphi_i(y) \vdash n$. Conversely, if $x \vdash n$ with some $x_i > 0$, $i < n$, then the preimage $\varphi_i^{-1}(x)$ is well-defined and $\varphi_i^{-1}(x) \vdash n - i$.

Some necessary and some sufficient conditions for $x \in \operatorname{vert} P_n$ were obtained in [7]. One of those is as follows.

Theorem 1 [7]. Let $1 = i_1 < i_2 < \ldots < i_k \leqslant n$ be an increasing sequence of integers. Define $n_k = n$, $x_{i_k} = \left\lfloor \dfrac{n_k}{i_k} \right\rfloor$; $n_{k-1} = n_k - x_{i_k} i_k$, $x_{i_{k-1}} = \left\lfloor \dfrac{n_{k-1}}{i_{k-1}} \right\rfloor$; $\ldots$; $n_1 = n_2 - x_{i_2} i_2$, $x_1 = x_{i_1} = \left\lfloor \dfrac{n_1}{i_1} \right\rfloor = n_1$; and $x_i = 0$ for $i \neq i_1, i_2, \dots, i_k$. Then $x = (x_1, x_2, \dots, x_n)$ is a vertex of P_n.

One can see that the theorem holds for the case $i_1 > 1$, $\dfrac{n_1}{i_1}$ integer.

Partitions of n with parts in some subset $M \subset [1, n]$ are often studied. Denote by $P_n(M)$ the polytope of such partitions: $P_n(M) = \operatorname{conv} \{x \vdash n \mid S(x) \subseteq M\}$.

Theorem 2 [7]. A vertex x of P_n is a vertex of $P_n(M)$ if and only if $x_i = 0$ for all $i \notin M$.

3. VERTEX IDEALS AND GENERATING VERTICES

In this section, we show how one can construct all vertices of P_n provided certain vertices of some polytopes P_j, $j < n$, are known. This method is based on the property of vertices to form a partition ideal in Andrews' lattice of partitions of all numbers.

It is shown in [1] that the set of all partitions of all numbers forms a lattice $\mathcal{P}$ relative to the partial order that can be defined as follows. Let $x \vdash n$ and $y \vdash m$, where $n, m \in \mathbb{N}$, $n \geqslant m$. Consider $y \preceq x$ if $y_i \leqslant x_i$ for all $i \in [1, m]$. Then the lower bound $u \wedge v$ of two partitions $u, v \in \mathcal{P}$ is the partition with parts in $S(u) \cap S(v)$ that contains each part i $\min(u_i, v_i)$ times. The upper bound $u \vee v$ is the partition with parts in $S(u) \cup S(v)$ that contains each part i $\max(u_i, v_i)$ times. It is easy to check that these operations satisfy the lattice identities.

Recall that for any lattice $\mathcal{L}$ with the partial order $\preceq_L$, a subset $\mathcal{M} \subset \mathcal{L}$ is called an ideal of $\mathcal{L}$ if $\mathcal{M}$ contains the lower bound of any its two elements and satisfies the condition: $m \in \mathcal{M}$, $l \in \mathcal{L}$, $l \leqslant_{\mathcal{L}} m$ imply $l \in \mathcal{M}$ [2]. Sometimes the terms «the order ideal» and «semi-ideal» are used [12]. We follow Andrews [1] and, in the case $\mathcal{L} = \mathcal{P}$ we deal with, call such $\mathcal{M}$ a partition ideal.

For any integer $k \geqslant 2$, denote by $\mathcal{V}_k$ (respectively, $\mathcal{V}_{\leqslant k}$) the set of partitions $x \vdash n$ of all $n \in \mathbb{N}$ that cannot be expressed as convex combinations of exactly k (respectively, at most k) partitions of n.

Proposition 1. $\mathcal{V}_k$ and $\mathcal{V}_{\leqslant k}$ are partition ideals of $\mathcal{P}$ for $k \geqslant 2$.

Proof. Prove the theorem for the case of $\mathcal{V}_k$. Let $x \in \mathcal{V}_k$, $y \in \mathcal{P}$, and $y \preceq x$. It is sufficient to show that x without any its part i belongs to $\mathcal{V}_k$. Then one can apply this claim consequently for all parts $i \in S(x)$ that are extra relative to y and conclude that $y \in \mathcal{V}_k$.

Deleting a part i from x gives us partition $z = \varphi_i^{-1}(x)$ of $n - i$. Suppose $z \notin \mathcal{V}_k$. Then z is a convex combination $z = \sum\limits_{t=1}^{k} \lambda_t z^t$, $\sum\limits_{t=1}^{k} \lambda_t = 1$, $\lambda_t > 0$, of some k partitions $z^t \vdash n - i$, $1 \leqslant t \leqslant k$. Define the integer points $x^t \in \mathbb{R}^n$, $1 \leqslant t \leqslant k$, with the components: $x_i^t = z_i^t + 1$; $x_j^t = z_j^t$, $1 \leqslant j \leqslant n - i$, $j \neq i$; $x_j^t = 0$, $n - i < j \leqslant n$. It is clear that $x^t \vdash n$ for all t and we have the convex representation $x = \sum\limits_{t=1}^{k} \lambda_t x^t$ since

$$\sum_t \lambda_t x_i^t = \sum_t \lambda_t (z_i^t + 1) = 1 + \sum_t \lambda_t z_i^t = 1 + z_i = 1 + x_i - 1 = x_i,$$

$$\sum_t \lambda_t x_j^t = \sum_t \lambda_t z_j^t = z_j = x_j \quad \text{for } 1 \leqslant j \leqslant n - i, \quad j \neq i,$$

$$\sum_t \lambda_t x_j^t = 0 \quad \text{for } n - i < j \leqslant n.$$

This contradicts $x \in \mathcal{V}_k$, yields $z \in \mathcal{V}_k$, and ends the proof of the first claim. The case of $\mathcal{V}_{\leqslant k}$ can be considered similarly with the only difference that some k_1, $2 \leqslant k_1 \leqslant k$, should be used for the number of partitions of $n-i$.

Denote by $\mathcal{V}$ the set of vertices of all integer partition polytopes: $\mathcal{V} = \bigcup_{n\in\mathbb{N}}$ vert P_n.

Theorem 3. The following statements are true:

(i) $\mathcal{V}$ is a partition ideal of $\mathcal{P}$;

(ii) $\mathcal{V} = \bigcap_{k\geqslant 2} \mathcal{V}_k$;

(iii) $\mathcal{V} = \lim_{k\to\infty} \mathcal{V}_{\leqslant k}$.

Proof. (ii) follows from definition of vertices of P_n: these are those partitions $x \vdash n$ that cannot be expressed as convex combinations of any $k \geqslant 2$ partitions of n. Inclusion $\mathcal{V}_{\leqslant k+1} \subseteq \mathcal{V}_{\leqslant k}$, $k \geqslant 2$, implies (iii). Statement (i) can be proved directly but it follows from (ii) and the fact that the intersection of any two ideals is an ideal.

We will see in Sec. 4 that $\mathcal{V}_{\leqslant 2} \neq \mathcal{V}_{\leqslant 3}$, so that $\mathcal{V}_{\leqslant 3} \subset \mathcal{V}_2$. However, it is not known yet whether $\mathcal{V}_{\leqslant k} \neq \mathcal{V}_{\leqslant k+1}$ for any $k > 2$.

Corollary 1. $\mathcal{V}$ is a lower sublattice of $\mathcal{P}$ but not its sublattice.

Proof. The statement (i) implies $u \wedge v \in \mathcal{V}$ for all $u, v \in \mathcal{V}$. The instance $u = (1,1,0) \vdash 3$ and $v = (2,0) \vdash 2$ shows that $\mathcal{V}$ is not a lattice since $u \vee v = (2,1,0,0) \vdash 4$ is a half-sum of two partitions of 4, $(2,1,0,0) = \dfrac{1}{2}\left((0,2,0,0) + (4,0,0,0)\right)$, whence $u \vee v \notin \mathcal{V}$.

The next corollary extends Theorem 2 [7].

Corollary 2. For any $x \in \operatorname{vert} P_n$ with a part $i < n$, the inclusion $\varphi_i^{-1}(x) \in \operatorname{vert} P_{n-i}$ holds.

Now we show that the most complicated case of the vertex recognition problem for P_n is that when all parts of the partition are small.

Proposition 2. The following assertions are true:

(i) a partition $x \vdash n$ with some $x_i > 0$, $\left\lfloor \dfrac{n}{2} \right\rfloor < i < n$, is a vertex of P_n if and only if $y = \varphi_i^{-1}(x)$ is a vertex of P_{n-i};

(ii) for any even n, the partition $n = \dfrac{n}{2} + \dfrac{n}{2}$ is the unique vertex of P_n with $x_{\frac{n}{2}} > 0$.

Proof. In view of Corollary 2, to prove (i), it remains to prove only one implication: if $x \vdash n$, $x_i > 0$, $\left\lfloor \dfrac{n}{2} \right\rfloor < i < n$, and $y \in \operatorname{vert} P_{n-i}$, then $x \in \operatorname{vert} P_n$. At first, note that in this case $x_i = 1$. Suppose $x \notin \operatorname{vert} P_n$. Then x is a convex combination of some $x^1, \ldots, x^k \vdash n$ with $x_i^t = 1$, $t = 1, \ldots, k$. Further, $x_j^t = 0$ for all $j > n-i$, $j \neq i$, since $x_j^t > 0$ would imply $x_j^t j + x_i^t i > n$. Therefore, $y^1 = \varphi_i^{-1}(x^1), \ldots, y^k = \varphi_i^{-1}(x^k)$ are well-defined and are partitions of $n-i$. One can check that y is a convex combination of $y^1, \ldots, y^k$, and hence $y \notin \operatorname{vert} P_{n-i}$, as in the proof of Proposition 1. The contradiction completes the proof of (i).

To prove (ii), notice that if $x = (x_1, x_2, \ldots, x_{\frac{n}{2}} = 1, \ldots, x_n) \in \operatorname{vert} P_n$, then $x_{\frac{n}{2}+1} = \ldots = x_n = 0$ and x is the half-sum of partitions $(2x_1, 2x_2, \ldots, 2x_{\frac{n}{2}-1}, x_{\frac{n}{2}} = 0, 0^{\frac{n}{2}})$ and $(0^{\frac{n}{2}-1}, x_{\frac{n}{2}} = 2, 0^{\frac{n}{2}})$. Thus, any $x \in \operatorname{vert} P_n$ with $x_{\frac{n}{2}} > 0$ satisfies $x_{\frac{n}{2}} = 0$. On the other hand, $(0^{\frac{n}{2}-1}, x_{\frac{n}{2}} = 2, 0^{\frac{n}{2}})$ is obviously a vertex of P_n. Theorem is proved.

For $m \in \mathbb{N}$, denote $T_m[\geqslant i] = \{x \vdash m \mid x_j = 0, j = 1, \ldots, i-1\}$ and consider the polytope $P_m[\geqslant i] = \operatorname{conv} T_m[\geqslant i]$ of partitions of m with all parts $\geqslant i$.

Theorem 4. The set T_n of partitions of n and the set of vertices of P_n satisfy the following recurrence relations:

$$T_n = \left(\biguplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \varphi_i(T_{n-i}[\geqslant i]) \right) \bigcup (0^{n-1}, 1), \tag{3.1}$$

$$\operatorname{vert} P_n \subseteq \left(\biguplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \varphi_i(\operatorname{vert} P_{n-i}[\geqslant i]) \right) \bigcup (0^{n-1}, 1). \tag{3.2}$$

Proof. Note that $(0^{n-1}, 1) \in \operatorname{vert} P_n$. Let $x \neq (0^{n-1}, 1)$ be a partition of n, and let i be its least part. Then $i \leqslant \left\lfloor \dfrac{n}{2} \right\rfloor$ and $x = \varphi_i(y)$ for some $y \vdash n - i$. This implies inclusion $\subseteq$ in (3.1). The opposite inclusion is obvious, as well as disjointness, so (3.1) is proved. The proof of (3.2) is similar. For any $(0^{n-1}, 1) \neq x \in \operatorname{vert} P_n$ with the least part $i \leqslant \left\lfloor \dfrac{n}{2} \right\rfloor$, Theorem 3 (i) implies $x = \varphi_i(y)$ for some $y \in \operatorname{vert} P_{n-i}$ with $y_j = 0$, $j = 1, \ldots, i-1$. By Theorem 2, y is a vertex of $P_{n-i}[\geqslant i]$.

Relation (3.2) can be used as the base for the lifting method to construct vertices of P_n. It states that the set of partitions for which the vertex recognition problem should be solved can be reduced to the set of φ_i images of all vertices of the polytopes P_{n-i}, $i = 1, 2, \ldots, \left\lfloor \dfrac{n}{2} \right\rfloor$, with parts $\geqslant i$. In subsequent sections we consider how to treat this problem.

4. CRITERION FOR REPRESENTATION OF A PARTITION AS CONVEX COMBINATION OF TWO OTHERS

In this section we characterize partitions that are convex combinations of two partitions of the same number and deduce new easy-to-check necessary conditions for a partition to be a vertex. The case considered seems to be the simplest, however, in Sec. 6 we argue that recognizing such partitions is a hard problem.

Theorem 5 [8]. A partition $x \vdash n$ is a convex combination of two partitions of n (whence $x \notin \operatorname{vert} P_n$) if and only if there exist two disjoint subsets S_1, S_2 of parts of x and two tuples of integers $u = \langle u_j \in \mathbb{N}; j \in S_1 \rangle$, $v = \langle v_k \in \mathbb{N}; k \in S_2 \rangle$, $u \neq v$, satisfying relations

$$\sum_{j \in S_1} u_j j = \sum_{k \in S_2} v_k k, \quad u_j \leqslant x_j, \quad v_k \leqslant x_k. \tag{4.1}$$

Proof. Given the subsets $S_1, S_2 \subset S(x)$ and the tuples u and v, one can build partitions $y, z \vdash n$, of which x is the half-sum, by setting

$$\begin{array}{lll} y_j = x_j + u_j,\ j \in S_1; & y_k = x_k - v_k,\ k \in S_2; & y_i = x_i,\ i \notin S_1 \cup S_2; \\ z_j = x_j - u_j,\ j \in S_1; & z_k = x_k + v_k,\ k \in S_2; & z_i = x_i,\ i \notin S_1 \cup S_2. \end{array}$$

Conversely, if $x \vdash n$ is a convex combination $x = z + \lambda(y - z)$, $0 < \lambda < 1$, of two partitions $y, z \vdash n$, then λ is rational and we can consider that $\lambda = \dfrac{p}{q}$, with p and q coprime. Then q divides all components of $y - z$ and x is the half-sum of partitions $z + \dfrac{p-1}{q}(y - z)$ and $z + \dfrac{p+1}{q}(y - z)$ of n. So, we can consider that $x = \dfrac{1}{2}(y + z)$. Define three subsets: $S = \{i \in S(x) \mid x_i \neq y_i\}$, $S_1 = \{j \in S(x) \mid x_j < y_j\}$, and $S_2 = \{k \in S(x) \mid x_k > y_k\}$. It is easy to see that $S_1, S_2 \subset S \subseteq S(x)$, $S_1 \cap S_2 = \emptyset$, $S_1 = \{j \in S(x) \mid x_j > z_j\}$, $S_2 = \{k \in S(x) \mid x_k < z_k\}$. The tuples u and v can be constructed by setting $u_j = y_j - x_j$, $j \in S_1$, and $v_k = x_k - y_k$, $k \in S_2$. Equality $x = \dfrac{1}{2}(y + z)$ and nonnegativity of x, y, z imply $u_j < x_j$ and $v_k < x_k$, and $x, y \vdash n$ implies equality (4.1).

Corollary 3. For a given $x \in \operatorname{vert} P_n$, no integer $k < n$ of the form $k = \sum_{i \in S(x)} \alpha_i i$, $\alpha_i \in \mathbb{Z}_+$, $\alpha_i \leqslant x_i$, except for the trivial case $k = 1 \cdot i$, is a part of x.

Corollary 4. Replacing requirement $S_1 \cap S_2 = \emptyset$ by $S_1 \neq S_2$ transforms Theorem 5 to an equivalent form.

Proof. Disjoint sets are nonequal, so the new version of the statement follows from the original one. To prove the opposite implication, note that if some $S_1 \neq S_2$ satisfy (4.1) and $i \in S_1 \cap S_2$, $u_i \leqslant v_i$, then i can be excluded from S_1 and either (a) left in S_2 while v_i replaced by $v_i - u_i$ if $u_i < v_i$, or (b) excluded from S_2 as well if $u_i = v_i$.

Theorem 5 has a simple interpretation. Given some $x \vdash n$, consider that for every $i \in S(x)$ one has x_i weights of i grams each. Then (4.1) means that there exists some weight that can be weighed in two different ways with the use of the given weights.

Criterion (4.1) successfully determines all partitions of $n \leqslant 20$ that are not vertices of P_n, except one for $n = 15$. The deviant partition $15 = 2\cdot3+4+5$ with parts 3, 4, 5 is convex combination of three partitions $\frac{1}{3}(3\cdot5)+\frac{1}{3}(3+3\cdot4)+\frac{1}{3}(5\cdot3)$ but not of any two. For $n = 21$, there are three partitions of this kind: $1+4+7+9$, $3+5+6+7$, and $3\cdot3+5+7$. The next proposition shows that these partitions are not exclusions.

Proposition 3. For any integer $k \geqslant 0$, partitions $1+(4+k)+(7+2k)+(9+3k)$ and $3+(5+k)+(6+k)+(7+k)$ of the numbers $n = 21+6k$ and $n = 21+3k$, respectively, are convex combinations of three partitions of n but not of any two ones.

Proof. One can check straightforwardly that each partition of the two series above can be expressed as convex combination of three partitions: $3\cdot(7+2k)$, $3\cdot(4+k)+1\cdot(9+3k)$, $3\cdot1+2\cdot(9+3k)$ for the first, and $3\cdot(7+k)$, $1\cdot3+3\cdot(6+k)$, $2\cdot3+3\cdot(5+k)$, for the second. Not harder is it to see that any partition does not satisfy condition (4.1), whence not any two partitions are sufficient for its convex representation.

Theorem 6. $\mathcal{V}_2 \neq \mathcal{V}_3$ and $\mathcal{V} \subset \mathcal{V}_2$.

Proof follows from Proposition 3 and Theorem 3.

Theorem 5 induces new necessary conditions for vertices of P_n.

Theorem 7 [8]. Every $x \in \operatorname{vert} P_n$ satisfies the conditions:

(i) $\prod\limits_{i\in S(x)} (x_i+1) \leqslant n+1$;

(ii) the number of distinct parts of x is not greater than $\lfloor \log(n+1) \rfloor$ and this bound is sharp.

Proof. Let $x \in \operatorname{vert} P_n$. Then x does not satisfy (4.1) and all sums

$$\sum_{i\in S(x)} u_i i, \quad 0 \leqslant u_i \leqslant x_i, \tag{4.2}$$

are pairwise different. The number of such sums is $\prod\limits_{i\in S(x)} (x_i+1)$, all of them are less than or equal to n, where the least sum is zero. By the Pigeonhole principle, this implies (i).

Denote the number of distinct parts of $x \in \operatorname{vert} P_n$ by d. The inequality $\prod\limits_{i\in S(x)} (x_i+1) \geqslant 2^d$ obviously holds, so the estimate in (ii) follows from (i). By Theorem 1, the partition $1+2+2^2+\ldots+2^m = n = 2^{m+1}-1$ is a vertex of P_n, thus the bound is sharp.

5. MERGING PARTS AND SUPPORT VERTICES

We show in this section that all vertices of each P_n can be generated from some subset of support vertices using two combinatorial operations of merging parts of partitions. This means that support vertices of P_n constitute an even smaller basis of the set of all partitions of n. At the end, we compare the numbers of vertices and support vertices of some partition polytopes with the total numbers of partitions of corresponding integers. Define these operations.

Operation $\mu_{u,v}$. Let $x \vdash n$ and let $u, v \in S(x)$, $u \neq v$, be two distinct parts of x; assume that $1 \leqslant x_u \leqslant x_v$. Build the point $y = \mu_{u,v}(x) \in \mathbb{Z}_+^n$ with components $y_u = 0$, $y_v = x_v - x_u$, $y_{u+v} = x_{u+v} + x_u$, and $y_j = x_j$ for $1 \leqslant j \leqslant n$, $j \neq u, v, u+v$.

Operation μ_u. Let $x \vdash n$ and a part $u \in S(x)$ enter x more than once, i.e., $x_u > 1$. Build the point $y = \mu_u(x) \in \mathbb{Z}_+^n$ with components $y_u = 0$, $y_{x_u u} = x_{x_u u} + 1$, and $y_j = x_j$ for $1 \leqslant j \leqslant n$, $j \neq u, x_u u$.

Theorem 8. Let a vertex x of the polytope P_n have two distinct parts $u, v \in S(x)$, $u \neq v$. Then $y = \mu_{u,v}(x)$ is a vertex of P_n.

Proof. At first, we show that $y \vdash n$. Indeed,

$$\sum_{i=1}^{n} y_i i = \sum_{\substack{i=1,\\ j\neq u,\, v,\, u+v}}^{n} y_j j + (x_v - x_u)v + (x_{u+v} + x_u)(u+v) =$$

$$= \sum_{\substack{i=1,\\ j\neq u,\, v,\, u+v}}^{n} x_j j + x_v v - x_u v + x_{u+v}(u+v) + x_u(u+v) = \sum_{i=1}^{n} x_i i = n.$$

Now prove that $y \in \operatorname{vert} P_n$. Note that by Corollary 3, $x_{u+v} = 0$. Suppose, on the contrary, that $y \notin \operatorname{vert} P_n$. Then y is convex combination $y = \sum\limits_{t=1}^{k} \lambda_t y^t$, $\sum\limits_{t=1}^{k} \lambda_t = 1$, $\lambda_t > 0$, of some partitions $y^t \vdash n$, $1 \leqslant t \leqslant k$. It follows from $y_u = 0$ that $y_u^t = 0$ for all t. Define integer points $x^t \in \mathbb{R}^n$, $1 \leqslant t \leqslant k$, with the components

$$x_u^t = y_{u+v}^t; \quad x_v^t = y_{u+v}^t + y_v^t; \quad x_{u+v}^t = 0; \quad x_j^t = y_j^t,\ j \neq u, v, u+v.$$

One can check that all x^t are partitions of n. Since $\sum\limits_t \lambda_t x_u^t = x_u$; $\sum\limits_t \lambda_t x_v^t = x_v$; $\sum\limits_t \lambda_t x_{u+v}^t = x_{u+v}$; $\sum\limits_t \lambda_t x_j^t = x_j$ for $j \neq u, v, u+v$, we have convex representation $x = \sum\limits_{t=1}^{k} \lambda_t x^t$, which contradicts x being a vertex of P_n. Therefore, $y \in \operatorname{vert} P_n$.

The next theorem for μ_u can be proved similarly.

Theorem 9. Let a vertex x of the polytope P_n have a part $u \in S(x)$ with $x_u > 1$. Then $y = \mu_u(x)$ is a vertex of P_n.

Theorems 8, 9 provide new sufficient conditions for $x \in \operatorname{vert} P_n$.

Let us illustrate application of operations of merging parts using polytope P_6 as an example. There are 7 vertices of P_6 [7]: $x^1 = (6,0,0,0,0,0)$, $x^2 = (2,0,0,1,0,0)$, $x^3 = (1,0,0,0,1,0)$, $x^4 = (0,3,0,0,0,0)$, $x^5 = (0,1,0,1,0,0)$, $x^6 = (0,0,2,0,0,0)$, and $x^7 = (0,0,0,0,0,1)$. We have $\mu_{4,1}(x^2) = x^3$ and $\mu_{1,5}(x^3) = \mu_{2,4}(x^5) = x^7$. Further, $\mu_1(x^2) = x^5$ and $\mu_2(x^4) = \mu_3(x^6) = x^7$. On the other hand, none of the vertices x^1, x^2, x^3, x^6 can be obtained from any other with the use of these operations. Therefore, all vertices of P_6 can be obtained from 4 vertices x^1, x^2, x^3, x^6 and this is a minimal set of this kind, relative to inclusion. The next definition is natural.

Definition 1. A vertex of a partition polytope is called support vertex if it does not result from any other vertex of the same polytope with the use of operations $\mu_{u,v}$ or μ_u.

Thus, we have seen that x^1, x^2, x^3, x^6 are support vertices of P_6. Denote the numbers of partitions, vertices, and support vertices of P_n by $p(n)$, $v(n)$, and $s(n)$, respectively. The values of these functions for $6 \leqslant n \leqslant 21$ are presented in the Table. One can observe that while the part of support vertices for $n = 6$ constitutes 36% of $p(n)$, it decreases to 21% for $n = 10$ and falls down to 5% for $n = 20$. The ratio $s(n)/v(n)$ also definitely decreases as n grows.

Numbers of partitions, vertices and support vertices

n	$p(n)$	$v(n)$	$s(n)$	n	$p(n)$	$v(n)$	$s(n)$
6	11	7	4	14	135	41	12
7	15	11	5	15	176	57	15
8	22	12	5	16	231	56	17
9	30	17	8	17	297	84	20
10	42	19	9	18	385	75	19
11	56	29	9	19	490	117	28
12	77	25	10	20	627	99	27
13	101	41	14	21	792	146	42

6. ADDITIVE STRUCTURES RELATED TO VERTICES OF P_n

Theorem 5 reveals relations of vertices and the vertex recognition problem for integer partition polytopes (PPVR problem) with several structures of additive combinatorics. Ehrenborg and Readdy [3] independently came to a class of knapsack partitions. These are partitions, all collections of parts of which give different sums. Theorem 5 states that knapsack partitions are just those that cannot be expressed as convex combinations of two other partitions of the same number. Therefore, knapsack partitions form the class $\mathcal{V}_2$ and we can refer to the problem 'Does a given partition belong to the $\mathcal{V}_2$ class?' as the decision problem *Knapsack Partition*: 'Is a given partition knapsack partition?' Ehrenborg and Readdy displayed the numbers $k(n)$ of knapsack partitions for $n \leqslant 50$ in the

On-Line Encyclopedia of Integer Sequences [6]. Comparison of $v(n)$ with $k(n)$ shows that $v(n) = k(n)$ for all $n < 24$, $n \neq 15, 21$, and $v(15) = k(15) - 1$, $v(21) = k(21) - 3$, where the differences are caused by appearance of partitions that need three partitions for their convex representations.

Other structures related to vertices are Sidon and sum-free sets, well-known in additive combinatorics [13]. We will define Sidon sets in the way that looks most accepted now, though the original Sidon's definition [11] was slightly different and the term B_2-sequence could be more appropriate [5]. So, let a Sidon set be a set $A \subset \mathbb{N}$ such that

$$a_1 + a_2 \neq a_3 + a_4 \tag{6.1}$$

for all $a_i \in A$ unless $\{a_1, a_2\} = \{a_3, a_4\}$. Some authors, see [5,13] for references, consider Sidon sets of order $h > 2$ (or B_h-sequences). In their definition, instead of pairs, both sums in (6.1) engage all possible h-tuples, h fixed. Considering inequality $a_1 + a_2 \neq a_3$ instead of (6.1) brings us to sum-free sets $A \subset \mathbb{N}$, see [13]. Note that repetitions of numbers are allowed in all cases.

Theorem 5 states that $x \in \mathcal{V}_2$ (respectively, if $x \in \operatorname{vert} P_n$) if and only if (respectively, then) neither two h- and k-tuples of elements of $S(x)$, with each element i engaged at most x_i times, have equal sums; here h and k are arbitrary integers $< \sum\limits_{a \in A} x_a$. So, knapsack partitions differ from Sidon sets in that

(a) the lengths h and k of the tuples can be different and

(b) repetitions of any $a \in A$ in either tuple are restricted by x_a.

We introduce the notion of Sidon multiset. Recall that a multiset is a pair $\langle A, x\rangle$ of a set A and a positive integer-valued multiplicity function $x : A \to \mathbb{N}$, whose values $x_a, a \in A$, can be considered as the numbers of copies of a in the multiset.

Definition 2. We call a multiset $\langle A, x\rangle$, $A \subset \mathbb{N}$, Sidon multiset if all its submultisets $\langle B, y\rangle$, where $B \subseteq A$ and $y_b \leqslant x_b$ for all $b \in B$, have distinct sums $\sum\limits_{b \in B} y_b b$ of their elements.

Note that all types of Sidon sets and sum-free sets can be represented as Sidon multisets with some additional restrictions (e.g., $x_a \geqslant 2$, $a \in A$, and $\sum\limits_{a \in B} y_a = 2$ for Sidon sets). Sidon multisets satisfy conditions (a) and (b) a priori. Moreover, any knapsack partition corresponds to a Sidon multiset and vice versa: a knapsack partition $x \vdash n$ automatically defines a multiset $\langle S(x), x\rangle$, while the subsets $S_1, S_1 \subset S(x)$ and the tuples u, v from Theorem 5 correspond to its submultisets $\langle S_1, u\rangle$ and $\langle S_2, v\rangle$, so that (4.1) does not hold for Sidon multisets by definition. The problem *Knapsack Partition* is then equivalent to the decision problem *Sidon Multiset*: 'Is a given multiset Sidon multiset?' (Polynomial equivalence of the sizes of these problems as functions of n is proved in [10]).

Let us summarize what we know about vertices and the additive structures considered. First, $\mathcal{V} \subset \mathcal{V}_2$. Further, we have that $\mathcal{V}_2$ is a subclass of the classes of

Sidon sets of any order and sum-free sets. It coincides with the class of knapsack partitions and is in one-to-one correspondence with the class of Sidon multisets. We proved

Theorem 10 [10]. Decision problems *Knapsack Partition* and *Sidon Multiset* are co-NP-complete.

Theorem implies that these problems cannot be solved in polynomial time unless $P = NP$, though it does not clarify the complexity level of the PPVR problem. Nevertheless, the difficulties we encountered while determining whether a given partition is a convex combination of arbitrary ($\geqslant 2$) number of partitions along with Theorem 10 move us to conjecture that the PPVR problem is also co-NP-hard.

To conclude, we should say that co-NP-completeness of the problem *Knapsack Partition* was proved by reducing to its complement the well-known NP-complete problem *Partition* [4]:

Partition. For a given set M with 'weights' $w(m) \in \mathbb{N}$ of its elements, decide whether there exists a subset $M' \subset M$ such that $\sum\limits_{m\in M'} w(m) = \sum\limits_{m\in M\setminus M'} w(m)$.

The problem *Knapsack Partition* differs from the problem *Partition* in that the subsets S_1 and S_2 are not required to split the set A.

7. CONCLUDING REMARKS

One of the goals of the polyhedral approach to integer partitions is to avoid enumeration of the enormous amount of partitions by exploring the geometrical structure of the polytope they form up. Should one know vertices of P_n, the whole set of partitions of n could be built as their integer-valued convex combinations. Moreover, one can concentrate on the subset of support vertices since all vertices can be built from these with the use of recursive application of two special operations of merging parts. The criterion of Theorem 5, and more easy-to-check necessary and sufficient conditions allowed us to calculate almost all vertices of P_n for small n's. The data demonstrate that the gaps between the numbers of support vertices, vertices, and partitions are considerable. Theorem 5 revealed connections of vertices with several other additive structures. We hope our discussion of these connections helps to assess the complexity level of the vertex recognition problem. Two new notions introduced in the paper — support vertex and Sidon multiset, deserve, in our opinion, further attention.

This work draws forth new questions. The first group includes the problems of characterizing vertices, support vertices, knapsack partitions (Sidon multisets), as well as those of obtaining necessary and/or sufficient conditions for a partition to belong to these classes. Exploring connections between vertices and facets would be also valuable.

The targets of the second group include asymptotic behaviour of functions $k(n)$, $v(n)$, $s(n)$, estimates on their values, and dependence on the properties of n. Similar functions can be considered for some special classes of partitions, e.g., knapsack partitions with distinct parts and so forth. These problems are most likely to be hard — we cannot refer to any beneficial results, only a few close in subject can be found in [13] for Sidon sets. Yet, known values of $v(n)$ and $k(n)$ demonstrate their definite dependence on the evenness of n: for all odd n's, except a few very small, the intriguing inequalities $k(n) > k(n+1)$ and $v(n) > v(n+1)$ hold. Lots of by hand and computer calculations, as well as some more formal arguments, impel us to suggest a stronger hypothesis: the values of $v(n)$ and $k(n)$ are inversely dependent on the number of divisors of n. If true these facts might have divergent consequences.

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