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P. E. Zhidkov*

ON RADIALLY SYMMETRIC SOLUTIONS
OF THE EQUATION $-\Delta u + u = |u|^{p-1}u$.
AN **ODE** APPROACH

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*E-mail: zhidkov@theor.jinr.ru

Жидков П. Е. E5-2011-28
О радиально-симметричных решениях уравнения $-\Delta u + u = |u|^{p-1}u$.
ОДУ-подход

Вопросы существования радиально-симметричных решений уравнения, указанного в заголовке, в шаре с нулевыми граничными условиями Дирихле изучались в многочисленных публикациях, и, вообще говоря, был получен более или менее полный ответ на эти вопросы. В настоящее время известно, что если размерность пространства $d \geqslant 3$ и $1 < p < (d+2)/(d-2)$ либо если $d = 2$ и $1 < p < +\infty$, то для любого целого $l \geqslant 0$ эта задача в шаре или во всем пространстве $x \in \mathbb{R}^d$ имеет радиально-симметричное решение, которое обладает в точности l нулями как функция $r = |x|$. Если $d \geqslant 3$ и $p \geqslant (d+2)/(d-2)$, то задача во всем пространстве не имеет нетривиального решения. Сначала эта задача изучалась некоторым вариантом вариационного метода. Однако специалистам в этой области известно, что также интересно получить те же результаты с использованием методов качественной теории ОДУ. В настоящей статье представлено простое доказательство вышеуказанного результата на этом пути. Более раннее доказательство этого результата другими авторами существенно сложнее нашего.

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On Radially Symmetric Solutions of the Equation $-\Delta u + u = |u|^{p-1}u$.
An ODE Approach

Questions of the existence in a ball of radially symmetric solutions of the equation indicated in the title with the Dirichlet zero boundary conditions are studied in many publications and, generally speaking, there was obtained more or less complete answer to these questions. It is known now that if the dimension of the space $d \geqslant 3$ and $1 < p < (d+2)/(d-2)$ or if $d = 2$ and $p > 1$, then for any integer $l \geqslant 0$ this problem in a ball or in the entire space $x \in \mathbb{R}^d$ has a radially symmetric solution with precisely l zeros as a function of $r = |x|$. If $d \geqslant 3$ and $p \geqslant (d+2)/(d-2)$, then the problem in the entire space has no nontrivial solution. For the first time, this problem was studied by a variant of the variational method. However, it is known to the specialists in the field that it is also interesting to obtain the same results by using methods of the qualitative theory of ODE. In the present paper, we shall give a simple proof of the above result in this way. An earlier proof of this result of the other authors is essentially more complicated than our one.

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1. INTRODUCTION. STATEMENTS OF THE MAIN RESULTS

In this paper, for one more time, we shall deal with the existence of nontrivial solutions of the problem

$$-\Delta u + u = |u|^{p-1}u, \quad u = u(r),\ r = |x|,\ x \in \mathbb{R}^d, \tag{1}$$

$$u\big|_{|x|\to\infty} = 0. \tag{2}$$

Here $d \geqslant 2$ is integer, Δ is the Laplace operator in $\mathbb{R}^d$, and $p > 1$. Investigations of this problem have a long history, and the main results on this subject are known now. Denote $p^\star = \dfrac{d+2}{d-2}$ if $d \geqslant 3$ or $p^\star = +\infty$ for $d = 2$. Then the result on the existence known currently says that for any $p \in (1, p^\star)$ and integer $l \geqslant 0$ problem (1)–(2) has a radially symmetric solution that, being regarded as a function of $r = |x|$, has precisely l zeros in the half-line $(0, +\infty)$. If $d \geqslant 3$ and $p \geqslant p^\star$, then the problem has no nontrivial solution. We refer readers to papers [6, 11, 12, 14] for historical remarks.

To the author's knowledge, for $d = 3$ this result for the first time was obtained in its complete form by Sansone [11] (in fact, in this paper, the proofs are made for positive solutions, but they still hold for solutions that alternate sign) and, a few years later, by Macky [8]. This result for all integer $d \geqslant 3$ is reestablished, for example, in [14]. For all values of d it was proved by Schekhter [12] and later reestablished by Kiguradze and Schekhter in [2]. In [5, 6], H. Berestycki and P. L. Lions obtained a result on existence for nonlinearities $f(u)$ of a very general kind. Their theorem says that in our case when $1 < p < p^\star$ the problem has a positive solution and an infinite sequence of pairwise different solutions.

The first proofs of the results above were based on minimization methods with constraints as in [5, 6] and in [11, 14]. Methods of the qualitative theory of ODEs allowed one to obtain intermediate results not for all values $p \in (1, p^\star)$. However, in the author's opinion, applications of methods of the qualitative theory of ODEs to problem (1)–(2) and to similar ones are of a separate interest. In particular, they allow one to restore the behavior of solutions of equation (1) on the (r, u)-plane. In [7], H. Berestycki, P. L. Lions and L. A. Peletier proved the existence of a positive solution for an arbitrary value of p in this interval (and for nonlinearities of a more general kind) by methods of ODEs, but some principal steps in their approach are based on variational methods. In the important paper [12] B. L. Schekhter succeeded in proving the existence of a radially symmetric solution of (1)–(2) with an arbitrary given number of zeros by methods of ODEs: his result holds for all $p \in (1, p^\star)$ (in fact, he considered problem (3)–(4) below with arbitrary real d).

Concerning the uniqueness of a solution with a given number of zeros, to our knowledge, for problem (1)–(2) only the uniqueness of a positive solution is known. For the first time, this result is due to Kwong [3]. However, his work seems to be sufficiently intricate and complicated. A simpler and clear proof of the same result was obtained by McLeod [9].

So, to our knowledge, the first complete investigation of the existence of solutions for problem (1)–(2) by methods of the qualitative theory of ODEs was made by Schekhter [12]. However, his proofs seem to be sufficiently complicated. With the present paper, the author wanted to establish a simpler and shorter proof of the same result (for integer d only) by using the methods of the qualitative theory of ODEs.

Now, we shall establish the statements of our results. By substitution $u = u(r)$, problem (1)–(2) reduces to the following:

$$-u'' - \frac{d-1}{r}u' + u = |u|^{p-1}u, \quad u = u(r),\ r > 0, \tag{3}$$

$$u'(0) = u(+\infty) = 0, \tag{4}$$

where the prime denotes the derivative in r. In addition to problem (3)–(4), we consider

$$-u'' - \frac{d-1}{r}u' + u = |u|^{p-1}u, \quad u = u(r),\ r > \overline{r}, \tag{5}$$

$$u'(\overline{r}) = u(+\infty) = 0, \tag{6}$$

where $\overline{r} > 0$ is a parameter. Our first main result is as follows.

Theorem 1. *Let $d \geqslant 2$ be integer and $1 < p < p^\star$. Then, for any integer $l \geqslant 0$ there exist constants $C_0 = C_0(l) > 0$ and $\tilde{r} > 0$ such that for any $\overline{r} \in (0, \tilde{r})$ and an arbitrary radially symmetric solution $u(r)$ of problem (5)–(6) that possesses precisely l zeros in $(\bar{r}, +\infty)$ one has*

$$\sup_{r \in (\overline{r}, +\infty)} |u(r)| \leqslant C_0(l).$$

Earlier this result was proved in [2, 12]. In our opinion, our proof is simpler and shorter than the one in [2, 12].

Using Theorem 1, we shall prove the following.

Theorem 2. *Let $d \geqslant 2$ be integer and $1 < p < p^\star$. Then, for any integer $l \geqslant 0$ problem (1)–(2) has a radially symmetric solution that, being regarded as a function of the argument $r = |x|$, possesses precisely l zeros in $(0, +\infty)$.*

As we already noted, Theorem 2 was already known earlier, but we shall prove this result by the methods of the qualitative theory of ODEs, and with this we shall simplify such a proof presented in [2, 12].

In addition, one can obtain the following a priori estimates of solutions of equations (1) and (2) with a given number of zeros just as when proving Theorem 1, with quite elementary modifications only.

Theorem 3. *Let $d \geqslant 2$ be integer and $1 < p < p^\star$. Then, for any integer $l \geqslant 0$ there exists a constant $\overline{C}_0(l) > 0$ such that for an arbitrary radially symmetric solution $u(r)$ of problem (1)–(2) with precisely l zeros in the half-line $r \in (0, +\infty)$ one has:*

$$\sup_{r>0} |u(r)| \leqslant \overline{C}_0(l).$$

Now, let us introduce some *notation*. Let $I \subset \mathbb{R}$ be an interval and $B_R = B_R(0) := \{x \in \mathbb{R}^d : |x| < R\}$ be a ball, where $0 < R \leqslant +\infty$. By $C(I)$ and $C(B_R)$ we denote the spaces of continuous bounded functions in I and B_R, respectively, with the uniform norm. Let $L_q(B_R)$, $q \geqslant 1$, be the standard Lebesgue space with the norm

$$\|g\|_{L_q(B_R)} = \left\{ \int\limits_{B_R} |g(x)|^q dx \right\}^{\frac{1}{q}}.$$

If $g(\cdot)$ is radially symmetric, then $\|g\|^q_{L_q(B_R)} = \omega_d \int\limits_0^R |g(r)|^q dr$, where the constant $\omega_d > 0$ depends only on d. By $H^1_q(B_R)$, $q \geqslant 1$, we denote the standard Sobolev space taken with the norm

$$\|g\|_{H^1_q(B_R)} = \left\{ \int\limits_{B_R} |g(x)|^q dx \right\}^{\frac{1}{q}} + \left\{ \int\limits_{B_R} |\nabla u|^q dx \right\}^{\frac{1}{q}}.$$

If $g \in H^1_q(B_R)$ is radially symmetric, then $\|g\|^q_{H^1_q(B_R)} = \omega_d \int\limits_0^R r^{d-1}\{|g(r)|^q + |u'(r)|^q\} dr$. By $H^1_{q,r}(B_R)$ we denote the subspace of the space $H^1_q(B_R)$ that consists of radially symmetric functions. According to [4] (on this subject see also [14]), any $g \in H^1_{2,r}(\mathbb{R}^d)$ is continuous at any point $x \neq 0$. Let also $H^2_q(B_R)$ be the Sobolev space of functions in B_R equipped with the norm

$$\|g\|_{H^2_q(B_R)} = \left\{ \int\limits_{B_R} |g(x)|^q dx \right\}^{\frac{1}{q}} + \sum_{i=1}^{d} \left\{ \int\limits_{B_R} \left| \frac{\partial g}{\partial x_i} \right|^q dx \right\}^{\frac{1}{q}} +$$
$$\sum_{i,j=1}^{d} \left\{ \int\limits_{B_R} \left| \frac{\partial^2 g}{\partial x_i \partial x_j} \right|^q dx \right\}^{\frac{1}{q}}.$$

Finally, we denote by $C_0^\infty(\mathbb{R}^d)$ the linear space of infinitely differentiable functions in $\mathbb{R}^d$ with compacts supports.

We shall prove Theorem 1 in the next section 2 and Theorem 2 in section 3. As was already noted, the proof of Theorem 3 repeats our proof of Theorem 1, with quite elementary modifications only. Methods of this paper can also be adapted to a wider class of nonlinearities in (1). Results analogous to Theorems 1, 2 and 3 still hold if in place of problem (1)–(2) we consider the following similar problem in a ball $B_R(0)$, where $0 < R < +\infty$:

$$-\Delta u + u = |u|^{p-1}u, \quad u = u(r),\ r = |x|,\ x \in B_R(0),\ u\big|_{\partial B_R(0)} = 0.$$

2. PROOF OF THEOREM 1

In fact, when we derive our a priori estimates stated by Theorem 1, we use a simplified variant of a similar derivation in [13] where a system of equations is considered. Our method is based on the same ideas as in [1].

Lemma 1. *Let integer $l \geqslant 0$ be arbitrary. Then, there exists $C_1 > 0$ such that for any $\overline{r} \in (0, 1/2)$ and an arbitrary solution $u(r)$ of problem (5)–(6) that has precisely l zeros in $(0, +\infty)$ one has*

$$|u(1)| + |u'(1)| \leqslant C_1.$$

Proof. On the contrary, suppose that there exist sequences $\overline{r}^n \in (0, 1/2)$ and $u_n(r)$ of values of the parameter $\overline{r}$ and of solutions of problem (5)–(6) with l zeros such that $|u_n(1)| + |u_n'(1)| \to +\infty$ as $n \to \infty$. We establish the following three observations. The first one is as follows. Let $u(r)$ be an arbitrary solution of problem (5)–(6) and $I = (a, b) \subset (1/2, 1)$ be an interval such that $|u(r)| > D$ in I for a constant $D > 0$. If D is sufficiently large, then one has $b - a < \dfrac{1}{16(l+1)}$, for any such a solution and an interval.

The proof immediately follows by comparing equation (5) with the equation

$$-z'' - 2^{1-d}(D^{p-1} - 1)z = 0, \quad z = z(r),\ r \in I,$$

by applying the standard comparison theorem. We fix such a constant $D > 0$.

We have for an arbitrary solution $u(r)$ of equation (5):

$$E'(r) = -\frac{d-1}{r}[u'(r)]^2, \tag{7}$$

where $E(r) = \dfrac{1}{2}[u'(r)]^2 - \dfrac{1}{2}u^2(r) + \dfrac{1}{p+1}|u(r)|^{p+1}$. The second observation is that, in view of (7), $|u_n(r)| + |u_n'(r)| \to +\infty$ as $n \to \infty$ uniformly with respect to $r \in [\overline{r}^n, 1]$ (because $E_n(r) \geqslant E_n(1)$ for any $r \in [\overline{r}^n, 1]$ where by $E_n(r)$ we denoted the function $E(r)$ corresponding to the solution $u_n(r)$).

The third observation, following from the second one, is that $|u_n'(r)| \to +\infty$ as $n \to \infty$ uniformly with respect to $r \in S$, where $S = \{r \in [1/2, 1] : |u_n(r)| \leqslant D\}$.

Now, it follows from the first and third observations that for any sufficiently large n the solution $u_n(r)$ has in the interval $[1/2, 1]$ at least $(l+1)$ zeros, which is a contradiction.□

We have from (7) for an arbitrary solution $u(r)$ of equations (5) and (6) for $r > \overline{r}$:

$$-\frac{1}{2}u^2(\overline{r}) + \frac{1}{p+1}|u(\overline{r})|^{p+1} = E(\overline{r}) \geqslant E(r) \geqslant -\frac{1}{2}u^2(r) + \frac{1}{p+1}|u(r)|^{p+1} \geqslant 0$$

and $-\frac{1}{2}u^2(\overline{r}) + \frac{1}{p+1}|u(\overline{r})|^{p+1} > 0$ which imply that $|u(\overline{r})| \geqslant |u(r)|$ for any $r > \overline{r}$. Now, we shall derive the following variant of the Pohozaev identity obtained for the first time in [10].

Lemma 2. *Let $u(r)$ be a solution of problem (5)–(6) taken with $\overline{r} \in (0, 1/2]$. Then, one has*

$$\int\limits_{B_1 \setminus B_{\overline{r}}} [|\nabla u(|x|)|^2 + u^2(|x|)]dx = \int\limits_{B_1 \setminus B_{\overline{r}}} |u(|x|)|^{p+1}dx + \omega_d u(1)u'(1) \tag{8}$$

and

$$\int\limits_{B_1 \setminus B_{\overline{r}}} \left\{\frac{2-d}{2}|\nabla u(|x|)|^2 - \frac{d}{2}u^2(|x|) + \frac{d}{p+1}|u(|x|)|^{p+1}\right\} dx +$$

$$+ \frac{\omega_d \overline{r}^d}{p+1}|u(\overline{r})|^{p+1} + \frac{\omega_d}{2}u^2(1) = \frac{\omega_d}{2}[u'(1)]^2 + \frac{\omega_d \overline{r}^d}{2}u^2(\overline{r}) + \frac{\omega_d}{p+1}|u(1)|^{p+1}. \tag{9}$$

Proof. To obtain (8), multiply (1) by $u(|x|)$ and integrate the result over $B_1 \setminus B_{\overline{r}}$. To obtain (9), multiply equation (5) by $r^d u'(|x|)$ and integrate the result from $\overline{r}$ to 1.□

Lemma 3. *Given integer $l \geqslant 0$, there exists $D_l > 0$ such that for any $\overline{r} \in (0, 1/2)$ and an arbitrary solution $u(r)$ of problem (5)–(6) that possesses precisely l zeros in $(\overline{r}, +\infty)$ one has:*

$$\|u\|_{H_2^1(B_1 \setminus B_{\overline{r}})} \leqslant D_l.$$

Proof. First, suppose that $d = 2$. Then, from (9) and Lemma 1,

$$\int\limits_{B_1 \setminus B_{\overline{r}}} \left\{\frac{d}{p+1}|u(|x|)|^{p+1} - \frac{d}{2}u^2(|x|)\right\} dx \leqslant C_1, \tag{10}$$

where the constant $C_1 > 0$ does not depend on $\overline{r} \in (0, 1/2)$ and u. Therefore, since $\dfrac{d}{2}u^2 \leqslant \dfrac{d}{2(p+1)}|u|^{p+1} + C_2$ for a constant $C_2 > 0$ that does not depend on u, from (10) and (8) we obtain our claim.

Now, let $d \geqslant 3$. Multiply (8) by $\dfrac{d-2}{2}$ and add the result to (9). Then,

$$\int\limits_{B_1 \setminus B_{\overline{r}}} \left\{ -u^2(|x|) + \left(\frac{d}{p+1} + \frac{2-d}{2} \right) |u(|x|)|^{p+1} \right\} dx \leqslant C_3,$$

where $C_3 > 0$ does not depend on $\overline{r} \in (0, 1/2)$ and on u. Therefore, since $\dfrac{d}{p+1} - \dfrac{d-2}{2} > 0$ and since again $u^2 \leqslant \dfrac{1}{2}\left(\dfrac{d}{p+1} + \dfrac{2-d}{2} \right) |u|^{p+1} + C_4$, where $C_4 > 0$ does not depend on u, we deduce that

$$\|u\|_{L_{p+1}(B_1 \setminus B_{\overline{r}})} \leqslant C_5$$

for a constant $C_5 > 0$ independent of $\overline{r} \in (0, 1/2)$ and of u. Now, in view of (8), we obtain our claim.□

Lemma 4. *Let $q \in (2, p^\star + 1]$, if $d \geqslant 3$, or $q \in (2, \infty)$, if $d = 2$, be arbitrary. Then, the constant $C > 0$ in the embedding inequality*

$$\|g\|_{L_q(B_1 \setminus B_{\overline{r}})} \leqslant C\|g\|_{H^1_{2,r}(B_1 \setminus B_{\overline{r}})},$$

where $g \in H^1_{2,r}(B_1 \setminus B_{\overline{r}})$ is arbitrary, does not depend on sufficiently small $\overline{r} > 0$.

Proof. Take arbitrary $\overline{r} \in (0, 1/2)$ and $g \in H^1_{2,r}(B_1 \setminus B_{\overline{r}})$. As is known (see [4] and, in addition, [14]), g can be chosen continuous in $B_1 \setminus B_{\overline{r}}$. We set $\tilde{g}(r) = g(\overline{r})$ for $r \in (0, \overline{r})$ and $\tilde{g}(r) = g(r)$ if $r \in [\overline{r}, 1]$. Then, there exists $C_6 > 0$ independent of $g \in H^1_{2,r}(B_1 \setminus B_{\overline{r}})$ such that $\|\tilde{g}\|_{L_q(B_1)} \leqslant C_6\|\tilde{g}\|_{H^1_{2,r}(B_1)}$. The latter is equivalent to

$$\|\tilde{g}\|_{L_q(B_{\overline{r}})} + \|g\|_{L_q(B_1 \setminus B_{\overline{r}})} \leqslant C_7(\|\nabla g\|_{L_2(B_1 \setminus B_{\overline{r}})} + \|g\|_{L_2(B_1 \setminus B_{\overline{r}})} + \|\tilde{g}\|_{L_2(B_{\overline{r}})}).$$

From this, since $\|\tilde{g}\|_{L_2(B_{\overline{r}})} \leqslant C_8(\overline{r})\|\tilde{g}\|_{L_q(B_{\overline{r}})}$ for a constant $C_8(\overline{r}) > 0$ that goes to $+0$ as $\overline{r} \to +0$, we obtain our claim.□

Lemma 5. *Consider the linear problem*

$$-\Delta g = f(r) \in L_s(B_1 \setminus B_{\overline{r}}) \quad \text{in} \quad B_1 \setminus B_{\overline{r}},$$
$$g\big|_{|x|=1} = u(1), \; g'(r)\big|_{r=\overline{r}} = 0,$$

where f depends only on r, $\overline{r} \in (0, 1/2)$ is sufficiently small and $s > 1$. As is known, there exists $C_9 = C_9(s) > 0$ such that $\|g\|_{H^2_s(B_1 \setminus B_{\overline{r}})} \leqslant C_9(\|f\|_{L_s(B_1 \setminus B_{\overline{r}})} + |u(1)|)$. In fact, C_9 does not depend on sufficiently small $\overline{r} > 0$.

Proof. It is clear that g depends only on r. We define $\tilde{g}$ as earlier. Then

$$-\Delta\tilde{g} = \tilde{f} \quad \text{in} \quad B_1,$$

$$\tilde{g}\big|_{|x|=1} = u(1),$$

where $\tilde{f} = 0$ in $B_{\overline{r}}$ and $\tilde{f} = f$ in $B_1 \backslash B_{\overline{r}}$. We have $\|\tilde{g}\|_{H_s^2(B_1)} \leqslant C_{10}(s)(\|\tilde{f}\|_{L_s(B_1)} +|u(1)|)$, where the constant $C_{10} > 0$ does not depend on $\tilde{f}$ and $\overline{r}$ (here we used the assumption that $g'(\overline{r}) = 0$). The latter is equivalent to the following:

$$\|g\|_{H_s^2(B_1\backslash B_{\overline{r}})} + \|\tilde{g}\|_{L_s(B_{\overline{r}})} \leqslant C'_{10}(\|f\|_{L_s(B_1\backslash B_{\overline{r}})} + |u(1)|),$$

and we obtain our result.□

Lemma 6. *Let $\tilde{H}_s^2(B_1 \setminus B_{\overline{r}})$ denote the subspace of the space $H_s^2(B_1 \setminus B_{\overline{r}})$ that consists of radially symmetric functions g from $H_s^2(B_1 \setminus B_{\overline{r}})$ each of which is equal to 0 at $r = 1$ and is such that $g'(\overline{r}) = 0$. If $\dfrac{2}{d} - \dfrac{1}{s} + \dfrac{1}{q} \geqslant 0$ for some $q > s$, then the constant $C_{11} = C_{11}(s) > 0$ in the embedding inequality $\|g\|_{L_q(B_1\backslash B_{\overline{r}})} \leqslant C_{11}\|g\|_{H_s^2(B_1\backslash B_{\overline{r}})}$, which holds for any $g \in \tilde{H}_s^2(B_1\backslash B_{\overline{r}})$, does not depend on sufficiently small $\overline{r} > 0$. By analogy, if $\dfrac{2}{d} - \dfrac{1}{s} > 0$, then the constant $C_{12} = C_{12}(s) > 0$ in the embedding inequality $\|g\|_{C(B_1\backslash B_{\overline{r}})} \leqslant C_{12}\|g\|_{H_s^2(B_1\backslash B_{\overline{r}})}$, that holds for any $g \in \tilde{H}_s^2(B_1 \setminus B_{\overline{r}})$, does not depend on sufficiently small $\overline{r} > 0$.*

Proof. Again, if $\dfrac{2}{d} - \dfrac{1}{s} + \dfrac{1}{q} \geqslant 0$, we have for $\tilde{g}$: $\|\tilde{g}\|_{L_q(B_1)} \leqslant C_{13}\|\tilde{g}\|_{H_s^2(B_1)}$. But again,

$$\|\tilde{g}\|_{H_s^2(B_1)} \leqslant \|\tilde{g}\|_{L_s(B_{\overline{r}})} + \|g\|_{H_s^2(B_1\backslash B_{\overline{r}})} \leqslant C_{14}(\overline{r})\|\tilde{g}\|_{L_q(B_{\overline{r}})} + \|g\|_{H_s^2(B_1\backslash B_{\overline{r}})},$$

where $C_{14}(\overline{r}) > 0$ goes to 0 as $\overline{r} \to +0$. Hence, we obtain our claim (for sufficiently small $\overline{r} > 0$). The second claim can be proved by analogy.□

Now, we turn to proving Theorem 1. Let $d \geqslant 3$ (for $d = 2$ the proof can be made by analogy). According to Lemma 3, there exists $C_{15} > 0$ such that

$$\|u - |u|^{p-1}u\|_{L_{s_1}(B_1\backslash B_{\overline{r}})} \leqslant C_{15},$$

where $s_1 = \dfrac{2d}{p(d-2)}$, for any $\overline{r} > 0$ sufficiently small and for an arbitrary solution $u(r)$ of equations (5) and (6) that has precisely l zeros in $(\bar{r}, +\infty)$. Therefore, by Lemma 5,

$$\|u\|_{H_{s_1}^2(B_1\backslash B_{\overline{r}})} \leqslant C_{16}(s_1), \tag{11}$$

where $C_{16}(s_1) > 0$ does not depend on sufficiently small $\overline{r} > 0$. If $q_1 := \dfrac{2d}{d-2} > \dfrac{pd}{2}$, which occurs when $1 < p < \dfrac{4}{d-2}$, then Lemma 6 and (11) imply Theorem 1. Suppose that

$$\frac{2d}{p(d-2)} \leqslant \frac{d}{2}.$$

If $p = \dfrac{4}{d-2}$, then from (11) according to Lemma 6 the expression $u - |u|^{p-1}u$ belongs to $L_{s_2}(B_1 \setminus B_{\overline{r}})$ with arbitrary large $s_2 > 1$. Therefore, as above, the solution u is bounded in $\tilde{H}^2_{s_2}(B_1 \setminus B_{\overline{r}})$ with arbitrary large $s_2 > 1$. Thus, in this case we obtain the statement we need.

Consider the case $\dfrac{4}{d-2} < p < \dfrac{d+2}{d-2}$. In this case,

$$\|u\|_{L_{q_2}(B_1 \setminus B_{\overline{r}})} \leqslant C_{17}(s_1)\|u\|_{\tilde{H}^2_{s_1}(B_1 \setminus B_{\overline{r}})},$$

where $C_{17}(s_1) > 0$ does not depend on u and sufficiently small $\overline{r} > 0$ and $\dfrac{2}{d} + \dfrac{1}{q_2} - \dfrac{1}{s_1} = 0$. Observe that $q_2 > q_1$ and denote $s_2 = q_2/p$. Then, we have

$$\|u - |u|^{p-1}u\|_{L_{s_2}(B_1 \setminus B_{\overline{r}})} \leqslant C_{18}(s_1),$$

where $C_{18}(s_1) > 0$ does not depend on sufficiently small $\overline{r} > 0$ and u.

Continue this iteration process. Then, we obtain the sequences q_n and $s_n = q_n/p$ such that the norms of u in $L_{q_n}(B_1 \setminus B_{\overline{r}})$ and in $H^2_{s_n}(B_1 \setminus B_{\overline{r}})$ are bounded uniformly with respect to all sufficiently small $\overline{r} > 0$. Observe that each of these two sequences strictly increases and that the first of them does not have a fixed point larger than $\dfrac{2d}{d-2}$. Therefore, the first sequence goes to $+\infty$ as $n \to \infty$ so that the values s_n become unboundedly large for sufficiently large n. Therefore, there exists a number n_0 such that $\dfrac{2}{d} - \dfrac{1}{s_{n_0}} \leqslant 0$, but $\dfrac{2}{d} - \dfrac{1}{s_{n_0+1}} > 0$. Thus, in view of Lemma 6 and because the number n_0 does not depend on $\overline{r}$, we obtain the statement of Theorem 1.□

3. PROOF OF THEOREM 2

A large part of auxiliary results in this section has been already published earlier. However, we include their proofs for the completeness of our presentation.

Let us prove the existence of a solution $u(r)$ of problem (5)–(6) that has precisely a given number l of zeros in $(\overline{r}, +\infty)$. Consider the Cauchy problem

$$-u'' - \frac{d-1}{r}u' + u = |u|^{p-1}u, \quad u = u(r),\ r > \overline{r}, \tag{12}$$

$$u(\overline{r}) = A > 0, \; u'(\overline{r}) = 0, \tag{13}$$

where $A > 0$ is a parameter. It easily follows from identity (7) that an arbitrary solution of problem (12)–(13) is bounded, hence, it is global (that is, it can be continued on the entire half-line $(\overline{r}, +\infty)$). It can be proved completely as in the proof of Lemma 1 that for $A > 0$ sufficiently large the corresponding solution of problem (12)–(13) has at least $(l+1)$ zeros in $(\overline{r}, +\infty)$. If $A \in \left(0, \left(\dfrac{p+1}{2}\right)\right)^{\frac{1}{p-1}}$, then by identity (7) the corresponding solution has no zeros because in this case $E(\bar{r}) < 0$ and it must be $E(r) > 0$ at such a zero r. Denote

$$\Lambda_l = \{A > 0 : \; u(r) \text{ has no less than } (l+1) \text{ zeros in } (\overline{r}, +\infty)\}$$

and set $A_l = \inf \Lambda_l$. Then, $A \geqslant \left(\dfrac{p+1}{2}\right)^{\frac{1}{p-1}}$.

Lemma 7. *Let $u(r)$ be a non-constant solution of problem (12)–(13) such that $u(r_1) = 0$ for some $r_1 > \overline{r}$. One has $u'(r_1) \neq 0$ so that any zero of such a solution is isolated. Denote by $r_0 \geqslant \overline{r}$ the zero of this solution u smaller than r_1 and closest to r_1 or set $r_0 = \overline{r}$ if there is no such a zero. Then, $u(r)$ has precisely one point of extremum $\overline{r}_1$ in the interval $[r_0, r_1]$. In addition, $|u(\overline{r}_1)| > 1$.*

Proof. First of all, $u'(r_1) \neq 0$ by the uniqueness theorem because otherwise we would have a nontrivial solution of equation (12) taken with the initial data $u(r_1) = u'(r_1) = 0$. Now, let $r_0 < r_1$ be two closest to each other zeros of our solution $u(r)$ (the case $r_0 = \overline{r}$ can be considered by analogy). Suppose that there exist two points of extremum of $u(r)$ in $[r_0, r_1]$. Then, there exists a point of minimum $\hat{r}$ of $|u(r)|$ in this interval. But according to the maximum principle $|u(\hat{r})| \in (0, 1]$ so that $E(\hat{r}) < 0$ which contradicts (7).□

Denote by $u_l(r)$ the solution of problem (12)–(13) taken with $A = A_l$.

Lemma 8. *The solution $u_l(r)$ has at most l zeros in $(\overline{r}, +\infty)$.*

Proof. On the contrary, suppose that the solution $u_l(r)$ has more than l zeros in $(\overline{r}, +\infty)$. According to Lemma 7, $u_l'(r_k) \neq 0$ at any such zero r_k. Therefore, the solution of problem (12)–(13) taken with any $A > A_l$ sufficiently close to A_l has more than l zeros in the same interval. This contradiction proves our claim.□

Denote by $r_1 < r_2 < ... < r_k$ and $0 = \overline{r}_0 < \overline{r}_1 < ... < \overline{r}_{k-1}$ the zeros and the first k points of extremum of our solution $u_l(r)$, respectively. Then,

$$\overline{r}_0 < r_1 < \overline{r}_1 < ... < \overline{r}_{k-1} < r_k.$$

The proof of the following technical result was established in [15] and later reestablished in [14] (in fact, in these two publications a slightly different problem is considered, but the proof holds in our case).

Lemma 9. *One has $k = l$.*

Proof. According to Lemma 8 $k \leqslant l$. Take an arbitrary $A \in \Lambda_l$ sufficiently close to A_l and let $u(r)$ be the corresponding solution of problem (12)–(13). Denote by $s_1 < s_2 < ... < s_{l+1}$ and $0 = \overline{s}_0 < \overline{s}_1 < ... < \overline{s}_l$ the first $(l+1)$ zeros and the points of extremum of this solution $u(r)$ so that

$$\overline{s}_0 < s_1 < \overline{s}_1 < s_2 < ... < \overline{s}_l < s_{l+1}.$$

Let us prove that there exists $C > 0$ such that

$$\overline{s}_l \leqslant C$$

for any $A \in \Lambda_l$ sufficiently close to A_l. According to (7) $E(\overline{r}_l) > 0$, where $E(r)$ is introduced with (7) and corresponds here to the solution $u(r)$, because $E(s_{l+1}) > 0$ and $s_{l+1} > \overline{s}_l$. By (7)

$$E(r) \geqslant E(\overline{s}_l) > 0$$

for any $r \in [s_l, \overline{s}_l)$.

Denote by $\overline{z} < \overline{\overline{z}}$ two points in $(s_l, \overline{s}_l)$ such that $u(\overline{z}) = h$, where h is the point in the interval $(1, |u(\overline{s}_l)|)$ such that $\frac{1}{2}h^2 - \frac{1}{p+1}|h|^{p+1} = \frac{1}{4} - \frac{1}{2(p+1)} > 0$, and $\frac{1}{2}u^2(\overline{\overline{z}}) - \frac{1}{p+1}|u(\overline{\overline{z}})|^{p+1} = \frac{1}{8} - \frac{1}{4(p+1)} > 0$. By (7), $C_1 \leqslant |u'(r)| \leqslant C_2$ for any $r \in [\overline{z}, \overline{\overline{z}}]$, where the constants $C_1 > 0$ and $C_2 > 0$ do not depend on the above $A \in \Lambda_l$. Therefore,

$$(d-1)\int\limits_{\overline{z}}^{\overline{\overline{z}}} \frac{[u'(r)]^2}{r} dr \geqslant C_3 \overline{s}_l^{-1}.$$

Hence, by (7),

$$|u'(r)| \geqslant C_3^{\frac{1}{2}} \overline{s}_l^{-\frac{1}{2}}$$

for a constant $C_3 > 0$ independent of $A \in \Lambda_l$ sufficiently close to A_l and for any $r \in [s_l, \overline{z}]$. Since $|u''(r)| \geqslant C_4$ in $[\overline{z}, \overline{s}_l]$, we deduce that

$$\overline{s}_l - s_l \leqslant C_5 \overline{s}_l^{\frac{1}{2}} + C_6,$$

where the positive constants C_4, C_5 and C_6 do not depend on the above $A \in \Lambda_l$.

By analogy, there exist constants $C_7 > 0$ and $C_8 > 0$ such that

$$\overline{s}_i - s_i \leqslant C_8 + C_7 \overline{s}_l^{\frac{1}{2}} \quad \text{and} \quad s_i - \overline{s}_{i-1} \leqslant C_8 + C_7 \overline{s}_l^{\frac{1}{2}}, \quad i = 1, 2, ..., l \tag{14}$$

for any $A \in \Lambda_l$ sufficiently close to A_l. Now, summing estimates (14), we obtain

$$\overline{s}_l \leqslant C_{10} + C_9 \overline{s}_l^{\frac{1}{2}},$$

where the constants $C_9 > 0$ and $C_{10} > 0$ do not depend on $A \in \Lambda_l$ sufficiently close to A_l. Thus, $\overline{s}_l$ is bounded uniformly with respect to $A \in \Lambda_l$ sufficiently close to A_l. In addition, $|u'(s_k)| \geqslant C_{11}$ for a constant $C_{11} > 0$ independent of $k = 1, 2, ..., l$ and of the above $A \in \Lambda_l$. Thus, indeed, $k = l$.□

Lemma 10. *As in the proof of Lemma 9, we have that $|u_l(r)|$ achieves a maximum at some $\overline{r}_l > r_l$, where $|u_l(\overline{r}_l)| > 1$. In fact $|u_l(r)|$ decreases in $(\overline{r}_l, +\infty)$ and $u_l(+\infty) = 0$.*

Proof. First, suppose that there exists a point of minimum $\hat{r}$ of $|u_l(r)|$ in $(\overline{r}_l, +\infty)$. Then, from (12) by the maximum principle $|u_l(\hat{r})| \in (0, 1)$. But then, $E_l(\hat{r}) < 0$, where the function $E(r) = E_l(r)$ corresponds to the solution $u_l(r)$. Therefore, from (7), any solution of problem (12)–(13) with $A \in \Lambda_l$ sufficiently close to A_l cannot have more than l zeros in $(\overline{r}, +\infty)$, which is a contradiction.

Now, suppose that $u_l(r)$ does not have a point of minimum in $(\overline{s}_l, +\infty)$. Then, this function is monotone and bounded in this interval. Therefore, its graph has a horizontal asymptote $u = c$, where c is a constant. From (12), it must be $c = 0$ or $c = \pm 1$. If $c = \pm 1$, then $E_l(r) < 0$ for r sufficiently large, therefore in this case the solution of equations (12) and (13), taken with $A \in \Lambda_l$ sufficiently close to A_l cannot have more than l zeros in $(\overline{r}, +\infty)$, which is a contradiction. Thus $c = 0$ and Lemma 10 is proved.□

Note that, from the arguments below Lemma 1, $\sup\limits_{r > \overline{r}} |u_l(r)| = |u_l(\overline{r})|$. Take now a sequence $\{\overline{r}^n\}$ of values of $\overline{r}$ that goes to 0 and let $u_l^n(r)$ be a solution of problem (5)–(6) taken with $\overline{r} = \overline{r}^n$ that has precisely l zeros in $(\overline{r}^n, +\infty)$ (here $n = 1, 2, 3, ...$). Then, according to Theorem 1 there exists a constant $C > 0$ such that

$$\|u_l^n\|_{C(\mathbb{R}^d \setminus B_{\overline{r}^n})} \leqslant C$$

for any number n. In addition, it follows from (7) that $\|[u_l^n(r)]'\|_{C(\overline{r}^n, +\infty)} \leqslant C_2$. Hence, the sequence $\{u_l^n(|x|)\}$ has a subsequence still denoted by $\{u_l^n(|x|)\}$ that converges to a $u(\cdot)$ in $C(B_b \setminus B_a)$ and weakly in $H_2^1(B_b \setminus B_a)$, where $0 < a < b < +\infty$ are arbitrary. In addition, it is easily seen that $u \in H_2^1(B_R(0))$ for any $R > 0$ and that u is radially symmetric. In view of (5), we can also accept that for any $0 < a < b < +\infty$ the sequence $\{[u_l^n(r)]'\}$ converges to $u'(r)$ in $C([a, b])$.

Take an arbitrary $\varphi \in C_0^\infty(\mathbb{R}^d)$ equal to 0 in a neighborhood of the point $x = 0$, multiply equation (1), written for $u_l^n(|\cdot|)$, by φ and integrate the result over $\mathbb{R}^d$. Then, we obtain

$$\int_{\mathbb{R}^d} \left\{ \nabla u_l^n(|x|) \nabla \varphi(x) + u_l^n(|x|)\varphi(x) - |u_l^n(|x|)|^{p-1} u_l^n(|x|)\varphi(x) \right\} dx = 0$$

for all sufficiently large n because each $u_l^n(|x|)$ is a solution of equation (1) in the domain $\mathbb{R}^d \setminus B_{\overline{r}^n}$. Take in this identity the limit $n \to \infty$. Then, we get

$$\int_{\mathbb{R}^d} \left\{ \nabla u(|x|) \nabla \varphi(x) + u(|x|)\varphi(x) - |u(|x|)|^{p-1} u(|x|)\varphi(x) \right\} dx = 0.$$

Take the limit in the latter relation over a sequence of functions $\varphi_m \in C_0^\infty(\mathbb{R}^d)$, each of which is equal to zero in a neighborhood of the point $x = 0$, converging in $H_2^1(\mathbb{R}^d)$ to an arbitrary $\varphi(\cdot) \in C_0^\infty(\mathbb{R}^d)$. Then, we obtain the equality above for an arbitrary such a function $\varphi(\cdot)$. Therefore, $u(|\cdot|)$ is a weak solution of equation (1) in $\mathbb{R}^d$ bounded in $C(\mathbb{R}^d)$. Hence, as is well known, $u(|\cdot|)$ is locally Hölder continuous and thus, it is a smooth solution of equation (1). As in the proved part of Theorem 2, the solution u, regarded as a function of the argument r, has precisely l zeros in $(0, +\infty)$ and $u(+\infty) = 0$. Theorem 2 is proved.$\square$

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E-mail: publish@jinr.ru
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